55. Let $X$ be a set, let $S \subseteq 2^X$ and let $m : S \to [0, \infty]$. Assume, for all $S, T \in S$, that $S \cup T \in S$, and that $T \setminus S \in S$. Assume, for all $S, T \in S$, that

$$[S \cap T = \emptyset] \implies [m(S \cup T) = m(S) + m(T)].$$

Show:

(i) For all $S, T \in S$, $[S \subseteq T] \implies [m(S) \leq m(T)].$

(ii) For all $S, T \in S$, $[(S \subseteq T) \land (m(T) < \infty)] \implies [m(T \setminus S) = m(T) - m(S)].$

(iii) For all $S, T \in S$, $m(S \cup T) \leq (m(S) + m(T)).$

56. Let $X$ be a set and let $S \subseteq 2^X$. Assume that $X \in S$. Assume, for all $S, T \in S$, that $T \setminus S \in S$. Show that $S$ is an algebra on $X$.

Hint: Show, for all $S, T \subseteq X$, that $S \cap T = T \setminus (X \setminus S)$.

57. Let $n \in \mathbb{N}$. Let $S, T \in \mathbb{R}^n$. Assume that $S$ and $T$ are both bounded. Show:

(a) $\overline{v}_n(S \cup T) \leq (\overline{v}_n(S)) + (\overline{v}_n(T))$

and

(b) $[S \cap T = \emptyset] \implies [\overline{v}_n(S \cup T) \geq (\overline{v}_n(S)) + (\overline{v}_n(T))]$.

58. Let $Z := \{(s, t) \in \mathbb{R}^2 \mid s^2 + t^2 = 1, 0 \leq s \leq t\}$. Show that $Z$ is negligible.

Hint: Define $f : [0, 1/\sqrt{2}] \to \mathbb{R}$ by $f(t) = \sqrt{1 - t^2}$. For all $m \in \mathbb{N}$, for all integers $j \in [1, m]$, let $A_j^{(m)} := [(j - 1)/(m\sqrt{2}), j/(m\sqrt{2})]$ and let $B_j^{(m)} := [f(j/(m\sqrt{2})), f((j - 1)/(m\sqrt{2}))]$.

For all $m \in \mathbb{N}$, let $K_m := \bigcup_{j=1}^{m} (A_j^{(m)} \times B_j^{(m)})$. Argue, for all $m \in \mathbb{N}$, that $\overline{v}_2(Z) \leq v_2^K(K_m)$.

Using the Mean Value Theorem, show that there exists $C > 0$ such that, for all $m \in \mathbb{N}$, for all integers $j \in [1, m]$, we have $l(B_j^{(m)}) \leq C/m$. Then show that $\lim_{m \to \infty} [v_2^K(K_m)] = 0$.

59. Let $n \in \mathbb{N}$ and let $F \subseteq 2^{\mathbb{R}^n}$. Assume that $F$ is finite. Show both of the following.

$$\text{(a) } \bigcup F = \bigcup_{F \in F} F \quad \text{and} \quad \text{(b) } (\bigcup F) \supseteq \bigcup_{F \in F} F^\circ$$

Homework 12: Due on Tuesday 28 April

51. Let $X$ be a set and let $S$ be a near algebra on $X$. Let $\mathcal{A} := \langle S \rangle_{\text{fin}}^\emptyset$.

a. Show: For all $A, B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$.

b. Show: For all $B \in \mathcal{A}$, $X \setminus B \in \mathcal{A}$. 

c. Show: For all $A, B \in \mathcal{A}$, $A \setminus B \in \mathcal{A}$.

d. Show: For all $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$.

52. Let $A \in \mathbb{R}$ be finite and nonempty. Show that $\ell : \mathcal{I} \to [0, \infty]$ is $\mathcal{I}_A$-additive.

53. Let $A, B \subseteq \mathbb{R}$ be two finite nonempty sets. Show that the function $v_2^B : B_2 \to [0, \infty]$ is $(\mathcal{I}_A * \mathcal{I}_B)$-additive.

54. Let $F \subseteq B_2$ and assume that $F$ is finite. Show that there exist subsets $A, B \subseteq \mathbb{R}$ such that $A$ and $B$ are both finite and nonempty and such that $F \subseteq \langle \mathcal{I}_A * \mathcal{I}_B \rangle_{\text{fin}}^\theta$.

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No homework is due on Tuesday 21 April

Homework 11: Due on Tuesday 14 April

49. Let $a, b, c, d, e, f, g, h, i \in \mathbb{R}$. Let $M := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Compute the $(3, 2)$ and $(3, 3)$ entries of $[\text{cof} M] \cdot M$.

50. Let $n \in \mathbb{N}$. Show that the map $M \mapsto \|M\| : \mathbb{R}^{n \times n} \to \mathbb{R}$ is uniformly continuous with respect to the $L^2$ norm $| \cdot |$ on $\mathbb{R}^{n \times n}$.

Hint: You must show: for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $A, B \in \mathbb{R}^{n \times n}$,

$$| A - B | < \delta \quad \Rightarrow \quad | \|A\| - \|B\| | < \varepsilon \] .$$

Let $S := \{v \in \mathbb{R}^n \mid |v| = 1\}$ be the unit sphere in $\mathbb{R}^n$. Recall, for all $M \in \mathbb{R}^{n \times n}$, that

$$|M| := \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} M_{jk}^2}, \quad \text{that} \quad \|M\| := \max \left\{ |L_M(u)| \mid u \in S \right\},$$

and that $\|M\| \leq n \cdot |M|$. Using the triangle inequality for $\| \cdot \|$, show, for all $A, B \in \mathbb{R}^{n \times n}$, that:

$$|A| - \|B\| \leq \|A - B\|.$$

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Homework 10: Due on Tuesday 7 April

46. Let $m, n \in \mathbb{N}$. Let $M \in \mathbb{R}^{n \times m}$. Show:

(a) For all $v \in \mathbb{R}^m$, $|L_M(v)| \leq \|M\| \cdot |v|$.

(b) For all $v \in \mathbb{R}^m$, for all $x \in \mathbb{R}^n$, $|x \cdot (L_M(v))| \leq |x| \cdot \|M\| \cdot |v|$.

47. Let $m, n \in \mathbb{N}$. Let $D \subseteq \mathbb{R}^m$. Let $h : D \to \mathbb{R}^n$. Let $v, w \in \mathbb{R}^m$ and let $x \in \mathbb{R}^n$. Assume that $[v, w] \subseteq D$. Assume that $h$ is differentiable on $[v, w]$. Define $f : [0, 1] \to \mathbb{R}$ by $f(t) = x \cdot [h((1-t)v + tw)]$. Show, for all $t \in (0, 1)$, that $f'(t) = x \cdot [L_{h'((1-t)v + tw)}(-v + w)]$.

Hint: Define $\alpha : [0, 1] \to D$ by $\alpha(t) = (1-t)v + tw$. Show, for all $t \in (0, 1)$, that $\dot{\alpha}(t) = -v + w$. Define $\beta : \mathbb{R}^n \to \mathbb{R}$ by $\beta(y) = x \cdot y$. Show, for all $y \in \mathbb{R}^n$, that $(\nabla \beta)(y) = x$. Apply the Chain Rule to $f = \beta \circ h \circ \alpha$. 

48. Let $n \in \mathbb{N}$, let $D \subseteq \mathbb{R}^n$, let $g : D \to \mathbb{R}^n$, let $S \subseteq D$ and let $x_0, y_0 \in \mathbb{R}^n$. Assume that $g$ is an approximate identity on $S$. Define $\gamma : D - x_0 \to \mathbb{R}^n$ by $\gamma(x) = [g(x + x_0)] - y_0$. Show that $\gamma$ is an approximate identity on $S - x_0$.

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Homework 8: Due on Tuesday 24 March

37. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $C^2$ on $\mathbb{R}^2$. Define $f' : \mathbb{R}^2 \to \mathbb{R}^{1 \times 2}$ and $f'' : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$ by

$$f'(x) = [(\partial_1 f)(x) \quad (\partial_2 f)(x)] \quad \text{and} \quad f''(x) = \begin{bmatrix} (\partial_{11} f)(x) & (\partial_{12} f)(x) \\ (\partial_{21} f)(x) & (\partial_{22} f)(x) \end{bmatrix}.$$ 

Define $C : \mathbb{R}^2 \to \mathbb{R}$ by $C(x) = f(0, 0)$. Let $L := L_{f'(0,0)} : \mathbb{R}^2 \to \mathbb{R}$ and $Q := [1/(2!)] [Q_{f''(0,0)}] : \mathbb{R}^2 \to \mathbb{R}$.

So, for all $a, b \in \mathbb{R}$, we have:

$$L(a, b) = [(\partial_1 f)(0, 0)]a + [(\partial_2 f)(0, 0)]b \quad \text{and} \quad Q(a, b) = [1/2][(\partial_{11} f)(0, 0)]a^2 + [(\partial_{12} f)(0, 0)]ab + [1/2][(\partial_{22} f)(0, 0)]b^2.$$ 

Let $P := C + L + Q : \mathbb{R}^2 \to \mathbb{R}$. Show that $f$ agrees with $P$ to order 2, in increasing derivatives, at $(0, 0)$. That is, prove all of the following:

$$f(0, 0) = P(0, 0), \quad (\partial_1 f)(0, 0) = (\partial_1 P)(0, 0), \quad (\partial_2 f)(0, 0) = (\partial_2 P)(0, 0),$$

$$(\partial_{11} f)(0, 0) = (\partial_{11} P)(0, 0), \quad (\partial_{12} f)(0, 0) = (\partial_{12} P)(0, 0), \quad (\partial_{22} f)(0, 0) = (\partial_{22} P)(0, 0).$$

38. Let $m, n \in \mathbb{N}$. Let $U \subseteq \mathbb{R}^m$ be an open set. Let $f : U \to \mathbb{R}^n$ be a function. Assume that $0_m \in U$. Assume that, in the sense of limits, $f$ agrees with the 0 function to order 1.
at $0_m$. Show that there exists a function $\tilde{f} : U \to \mathbb{R}^n$ such that $\tilde{f}(0_m) = 0_n$, such that $\tilde{f}$ is continuous at $0_m$ and such that, for all $x \in U$, we have: $f(x) = [\tilde{f}(x)][|x|]$.  

39. Let $m,n,k \in \mathbb{N}$. Let $U \subseteq \mathbb{R}^m$ be an open set. Let $\alpha : U \to \mathbb{R}^n$ be a function. Let $\mu : \mathbb{R}^n \to \mathbb{R}^k$ be a homogeneous linear function. Assume that $0_m \in U$. Assume that, in the sense of limits, $\alpha$ agrees with the $0$ function to order $1$ at $0_m$. Show that, in the sense of limits, $\mu \circ \alpha$ agrees with the $0$ function to order $1$ at $0_m$. 

*Hint:* Using Problem 38, choose $\alpha : U \to \mathbb{R}^n$ continuously vanishing at $0_m$ such that, for all $x \in U$, we have: $\alpha(x) = [\tilde{\alpha}(x)][|x|]$. It then follows, by linearity, for all $x \in U$, that we have: $\mu(\alpha(x)) = [\mu(\tilde{\alpha}(x))][|x|]$. Now divide by $|x|$ and take $\lim_{x \to 0_m}$.  

40. Let $m,n \in \mathbb{N}$. Let $\lambda : \mathbb{R}^m \to \mathbb{R}^n$ be linear. Show that there exists $C \geq 0$ such that, for all $x \in \mathbb{R}^m$, we have: $|\lambda(x)| \leq C|\lambda||x|$.  

41. Let $m,n,k \in \mathbb{N}$. Let $U \subseteq \mathbb{R}^m$ be open. Let $V \subseteq \mathbb{R}^n$ be open. Let $\alpha : U \to V$ and $\beta : V \to \mathbb{R}^k$ be functions. Let $\lambda : \mathbb{R}^m \to \mathbb{R}^n$ be a homogeneous linear function. Assume that $0_m \in U$ and that $0_n \in V$. Assume that, in the sense of limits, $\alpha$ agrees with the $0$ function to order $1$ at $0_m$. Assume that, in the sense of limits, $\beta$ agrees with the $0$ function to order $1$ at $0_n$. Show that, in the sense of limits, $\beta \circ (\lambda + \alpha)$ agrees with the $0$ function to order $1$ at $0_m$. 

*Hint:* Using Problem 38, choose $\alpha : U \to \mathbb{R}^n$ continuously vanishing at $0_m$ such that, for all $x \in U$, we have: $\alpha(x) = [\tilde{\alpha}(x)][|x|]$. Using Problem 38, choose $\beta : V \to \mathbb{R}^k$ continuously vanishing at $0_n$ such that, for all $y \in V$, we have: $\beta(y) = [\tilde{\beta}(y)][|y|]$. Choose $C$ as in Problem 40. Show that there exists $\delta > 0$ such that, for all $x \in (B_\delta(0_m)) \setminus \{0_m\}$, we have both $x \in U$ and $\frac{|[\lambda(x)] + [\tilde{\alpha}(x) ][|x|]|}{|x|} < C + 1$. For all $x \in U$,  

$$ |(\beta \circ (\lambda + \alpha))(x)| = \left[ |\tilde{\beta}([\lambda(x)] + [\alpha(x)])| \right] \left[ |[\lambda(x)] + [\tilde{\alpha}(x)][|x|]| \right]. $$

Now divide by $|x|$ and take $\lim_{x \to 0_m}$.  

42. Let $l \in \mathbb{N} \cup \{0\}$. Let $m,n \in \mathbb{N}$. Let $P,Q : \mathbb{R}^m \to \mathbb{R}^n$ be polynomials of degree $\leq l$. Let $x_0 \in \mathbb{R}^m$. Assume, in the sense of limits, $P$ agrees with $Q$ to order $l$ at $x_0$. Show $P = Q$. 

*Hint:* Given an integer $k \in \{1, n\}$. We wish to show that $P_k = Q_k$. 

Let $r := P_k - Q_k : \mathbb{R}^m \to \mathbb{R}$. We wish to show that $r$ is equal to the $0$ function. Because $|r| = ||(P - Q)|_k| \leq |P - Q|$, show that, in the sense of limits, $r$ agrees with the $0$ function to order $l$ at $x_0$. Define $R : \mathbb{R}^m \to \mathbb{R}$ by $R(x) = r(x + x_0)$. Then $R$ is a polynomial of degree $\leq l$. We wish to show that $R$ is equal to the $0$ function. By continuity of $R$, it suffices to show, for all $x_1 \in \mathbb{R}^m \setminus \{0_m\}$, that $R(x_1) = 0$. Given $x_1 \in \mathbb{R}^m \setminus \{0_m\}$. We wish to show that $R(x_1) = 0$. 

Show that, in the sense of limits, $R$ agrees with the $0$ function to order $l$ at $0_m$. Define $\rho : \mathbb{R} \to \mathbb{R}$ by $\rho(t) = R(tx_1)$. We wish to show that $\rho(1) = 0$. It therefore suffices to show that $\rho$ is equal to the $0$ function. As $\rho$ is a polynomial of degree $\leq l$, by a result from class, it suffices to show: in the sense of limits, $\rho$ agrees with the $0$ function to order $l$ at $0$. 


Using the kind of argument found in the solution to Problem 38, choose \( \tilde{R} : \mathbb{R}^m \to \mathbb{R} \) continuously vanishing at 0\( _m \) such that, for all \( x \in \mathbb{R}^m \), we have: \( R(x) = [\tilde{R}(x)][|x|^l] \). Then, for all \( t \in \mathbb{R} \), we have: \( \rho(t) = [\tilde{R}(tx_1)][|t|^l][|x_1|^l] \). Now divide by \( |t|^l \) and take \( \lim_{t \to 0} \).

43. Let \( m, n \in \mathbb{N} \). Let \( P : \mathbb{R}^m \to \mathbb{R}^n \) be a polynomial. Let \( s, t \in [1, m] \) be integers. Show that \( \partial_{st}P = \partial_{ts}P \).

**Hint:** By looking at the polynomials \( P_1, \ldots, P_n : \mathbb{R}^m \to \mathbb{R} \), we reduce to the case of polynomials \( \mathbb{R}^m \to \mathbb{R} \). Since every polynomial is a linear combination of monomials, we reduce to the case of monomials \( \mathbb{R}^m \to \mathbb{R} \). Given a monomial \( Q : \mathbb{R}^m \to \mathbb{R} \). We wish to show that \( \partial_{st}Q = \partial_{ts}Q \).

With the convention that \( 0^0 = 1 \), choose integers \( e_1, \ldots, e_m \in \mathbb{N} \cup \{0\} \) such that, for all \( x_1, \ldots, x_m \in \mathbb{R} \), we have: \( Q(x_1, \ldots, x_m) = [x_1^{e_1}] \cdots [x_m^{e_m}] \). Now find formulas for \( (\partial_{st}Q)(x_1, \ldots, x_m) \) and \( (\partial_{ts}Q)(x_1, \ldots, x_m) \).

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**Homework 7:** Due on Tuesday 10 March

35. Let \( m, n \in \mathbb{N} \). Let \( U \) be an open set in \( \mathbb{R}^m \). Let \( V \) be an open set in \( \mathbb{R}^n \). Show that \( U \times V \) is an open set in \( \mathbb{R}^{m+n} \).

36. Let \( A \subseteq \mathbb{R} \), let \( f : A \to \mathbb{R} \), let \( \rho > 0 \) and let \( l \in \mathbb{N} \cup \{0\} \). Assume that \( (-\rho, \rho) \subseteq A \). Assume that \( \text{dom}[f^{(l+1)}] \subseteq (-\rho, \rho) \). (That is, assume that the \( (l+1) \)st derivative of \( f \) is defined at every point of the interval \( (-\rho, \rho) \).) Define \( P : \mathbb{R} \to \mathbb{R} \) by

\[
P(x) = [f(0)] + [f'(0)]x + [f''(0)]\left[\frac{x^2}{2!}\right] + \cdots + [f^{(l)}(0)]\left[\frac{x^l}{l!}\right].
\]

Let \( h \in (0, \rho) \). Show that there exists \( \eta \in (0, h) \) such that

\[
f(h) = [P(h)] + [f^{(l+1)}(\eta)]\left[\frac{h^{l+1}}{(l+1)!}\right].
\]

**Hint:** Let \( C := \frac{([f(h)] - [P(h)])((l+1)!)}{h^{l+1}} \). Then \( f(h) = [P(h)] + C\left[\frac{h^{l+1}}{(l+1)!}\right] \). We wish to show that there exists \( \eta \in (0, h) \) such that \( f^{(l+1)}(\eta) = C \).

Let the function \( R : A \to \mathbb{R} \) be defined by \( R(x) = [f(x)] - [P(x)] - C\left[\frac{x^{l+1}}{(l+1)!}\right] \). Then show that \( R(0) = R'(0) = R''(0) = \cdots = R^{(l)}(0) = 0 \) and, also, show that \( R(h) = 0 \).

By repeated applications of Rolle’s Theorem, choose \( \eta \in (0, h) \) such that \( R^{(l+1)}(\eta) = 0 \). Then show that \( f^{(l+1)}(\eta) = C \).

**Note:** A similar argument shows that, for each \( h \in (-\rho, 0) \), there exists \( \eta \in (h, 0) \) such that the same formula, namely \( f(h) = [P(h)] + [f^{(l+1)}(\eta)]\left[\frac{h^{l+1}}{(l+1)!}\right], \) holds. So, for any \( h \in (-\rho, \rho) \setminus \{0\} \), we may choose \( \eta \) strictly between 0 and \( h \) such that the formula holds. Finally, if \( h = 0 \) and \( \eta = 0 \), then the formula is easily verified, though it’s not quite correct to say that \( \eta \) is strictly between 0 and \( h \) in that case.
29. Let \( n, k \in \mathbb{N} \). Let \( f : \mathbb{R}^n \to \mathbb{R}^k \). Let \( x_0 \in \mathbb{R}^n \) and let \( x_1, x_2, x_3, \ldots \) be a sequence in \( \mathbb{R}^n \). Assume that \( f \) is continuous at \( x_0 \) and that \( x_1, x_2, x_3, \ldots \to x_0 \). Prove that \( f(x_1), f(x_2), f(x_3), \ldots \to f(x_0) \).

30. Let \( X \) be a set, let \( A \subseteq 2^X \) and let \( X_0 \subseteq X \). Show:
\[
[ X_0 \in \langle A \rangle \cup ] \quad \text{iff} \quad \forall x \in X_0, \exists A \in A \text{ s.t. } x \in A \subseteq X_0.
\]

31. Let \( \mathcal{B} \) be the set of all open disks in \( \mathbb{R}^2 \). Let \( \mathcal{T} := \langle \mathcal{B} \rangle \cup \). Show, for all \( U \subseteq \mathbb{R}^2 \), that:
\[
[ U \in \mathcal{T} ] \quad \text{iff} \quad \forall x \in U, \exists \delta > 0 \text{ s.t. } B_\delta(x) \subseteq U.
\]

32. Let \( \mathcal{B} \) and \( \mathcal{T} \) be as in Problem 31. Show, for all \( B, C \in \mathcal{B} \), that \( \mathbb{R}^2 \in \mathcal{T} \).

33. Let \( \mathcal{B} \) and \( \mathcal{T} \) be as in Problem 31. Show that \( \mathbb{R}^2 \in \mathcal{T} \).

34. Let \( X \) be a set and let \( A \subseteq 2^X \). Let \( \mathcal{T} := \langle \langle A \rangle^X_{\text{fin}} \rangle \cup \). Show that \( \langle \mathcal{T} \rangle^X_{\text{fin} \cap} = \mathcal{T} \).

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Homework 5: Due on Tuesday 24 February

22. Compute: \( \text{sgn} \begin{bmatrix} R & R \\ R & R \\ R & R \end{bmatrix} \) and \( \text{sgn} \begin{bmatrix} R & R \\ R & R \\ R & R \end{bmatrix} \) and \( \text{sgn} \begin{bmatrix} R & R \\ R & R \\ R & R \end{bmatrix} \).

23. Can we go from \( \begin{bmatrix} R & R \\ R & R \\ R & R \end{bmatrix} \) to \( \begin{bmatrix} R & R \\ R & R \\ R & R \end{bmatrix} \) using an even number of rook transpositions? If so, how?

24. Can we go from \( \begin{bmatrix} R & R \\ R & R \\ R & R \end{bmatrix} \) to \( \begin{bmatrix} R & R \\ R & R \\ R & R \end{bmatrix} \) using an even number of rook transpositions? If so, how?

25. Let \( v := (2, 3) \) and let \( w := (4, 5) \). Find the area of the parallelogram whose corners are: \( 0, v, w, v+w \).

26. Let \( u := (2, 4, 1) \), let \( v := (-3, -8, 6) \) and let \( w := (1, 5, 9) \). Find the volume of the parallelepiped whose corners are: \( 0, u, v, w, u+v, u+w, v+w, u+v+w \).

27. Let \( X := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) and let \( Y := \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} \). Find \( k \in \mathbb{N} \) and near identity matrices \( A_1, \ldots, A_k \in \mathbb{R}^{4 \times 4} \) such that \( X \oplus Y = A_1 \cdots A_k \).
Hint: Let \( E_1 := \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \), let \( E_2 := \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \), let \( E_3 := \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \), let \( E_4 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), let \( E_5 := \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \), let \( E_6 := \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \) and let \( E_7 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \). Then we have both \( X = E_1 E_2 E_3 \) and \( Y = E_4 E_5 E_6 E_7 \).

28. Let \( X := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) and let \( Y := \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} \). Show that \( \det(X \oplus Y) = (\det X)(\det Y) \).

Homework 4: Due on Tuesday 17 February

17. Find all fully reduced \( 3 \times 4 \) matrices.

18. Find the inverse (if it exists) of \( \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \).

19. Let \( M := \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \). Find an invertible \( E \in \mathbb{R}^{3 \times 3} \) and an invertible \( F \in \mathbb{R}^{2 \times 2} \) such that \( EMF \) is fully reduced.

20. Let \( M := \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \), as in the preceding problem. Determine whether \( L_M : \mathbb{R}^2 \to \mathbb{R}^3 \) is one-to-one, onto, both or neither.

21. Let \( k, n \in \mathbb{N} \) and let \( M \in \mathbb{R}^{k \times n} \). Show that \( \ker(L_M) = [\text{im}(L_M^t)]^\perp \).

Hint: We have \( L_M : \mathbb{R}^n \to \mathbb{R}^k \) and \( L_M^t : \mathbb{R}^k \to \mathbb{R}^n \). Recall, for all \( v \in \mathbb{R}^n \), for all \( w \in \mathbb{R}^k \), that we have: \( [L_M(v)] \cdot w = v \cdot [L_M^t(w)] \).

Homework 3: Due on Tuesday 10 February

11. Define \( Q_2 : \mathbb{R}^2 \to \mathbb{R} \) by \( Q_2(x, y) = 5x^2 + 2y^2 - 6xy \). Show that \( Q_2 \) is positive definite.

(Hint: First show, for all \( x, y \in \mathbb{R} \), that \( Q_2(x, y) = [(3/\sqrt{2})x - \sqrt{2}y]^2 + (1/2)x^2 \).

12. Let \( V \) be a vector space and let \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \) be an inner product on \( V \). Let \( j \in \mathbb{N} \). Let \( w_1, \ldots, w_j \in V \setminus \{0\} \) be orthogonal. Let \( v \in V \). Show that there exist \( c_1, \ldots, c_j \in \mathbb{R} \) such that, for all integers \( i \in [1, j] \), \( \langle v - (c_1 w_1 + \cdots + c_j w_j), w_i \rangle = 0 \).

Hint: For all integers \( i \in [1, j] \), solve \( \langle v, w_i \rangle - c_i \langle w_i, w_i \rangle = 0 \) for \( c_i \).

13. Let \( V \) be a vector space and let \( v \in V \). Let \( j \in \mathbb{N} \), let \( w_1, \ldots, w_j \in V \) and let \( c_1, \ldots, c_j \in \mathbb{R} \). Let \( w := v - (c_1 w_1 + \cdots + c_j w_j) \). Show that \( \langle v, w_1, \ldots, w_j \rangle = \langle w, w_1, \ldots, w_j \rangle \).

14. Let \( v = \mathbb{R}^4 \) and define \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \) to be the dot product: \( \langle v, w \rangle = v \cdot w \). Let \( v_1 := (0, 5, 0, 0) \), \( v_2 := (2, 4, 1, 0) \) and \( v_3 := (5, 4, 5, 6) \).

(i) Find \( w_1, w_2, w_3 \in V \) orthogonal and with rational components such that the flag of \( w_1, w_2, w_3 \) is the same as the flag of \( v_1, v_2, v_3 \).
15. Let $V := C([0, 1], \mathbb{R})$ and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be defined by $\langle f, g \rangle = \int_0^1 [f(x)][g(x)] \, dx$.

Define $p_0, p_1, p_2, \ldots \in V$ by $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2$, $\ldots$.

(i) Find $q_0, \ldots, q_4 \in V$ polynomials with rational coefficients such that $q_0, \ldots, q_4$ are orthogonal and such that the flag of $q_0, \ldots, q_4$ is the same as the flag of $p_0, \ldots, p_4$.

(ii) Find $r_0, \ldots, r_4 \in V$ orthonormal such that the flag of $r_0, \ldots, r_4$ is the same as the flag of $p_0, \ldots, p_4$.

16. Let $V := C([-\pi, \pi], \mathbb{R})$. Define $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ by $\langle f, g \rangle = \int_{-\pi}^{\pi} [f(x)][g(x)] \, dx$.

Define $f_0, f_1, f_2, f_3, f_4 \in V$ by $f_0(x) = 1$, $f_1(x) = \sin x$, $f_2(x) = \cos x$, $f_3(x) = \sin(2x)$ and $f_4(x) = \cos(2x)$.

(i) Show that $f_0, \ldots, f_4$ is orthogonal.

(ii) Find $g_0, \ldots, g_4 \in V$ orthonormal such that the flag of $g_0, \ldots, g_4$ is the same as the flag of $f_0, \ldots, f_4$.

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**Homework 2:** Due on Tuesday 3 February

6. Let $V$ be a vector space, let $S_0 \subseteq V$ and let $v \in S_0$. Show:

$$v \in \langle S_0 \setminus \{v\} \rangle \iff \langle S_0 \rangle = \langle S_0 \setminus \{v\} \rangle.$$ 

7. Let $V$ be a vector space, let $S_0 \subseteq V$ and let $w \in \langle S_0 \rangle$. Show that there exist $k \in \mathbb{N}$ and distinct $v_1, \ldots, v_k \in S_0$ such that $w \in \langle v_1, \ldots, v_k \rangle$.

8. Let $V$ be a vector space and let $S_0 \subseteq V$. Show that $S_0$ is linearly dependent iff there exist $k \in \mathbb{N}$ and distinct $v_1, \ldots, v_k \in S_0$ such that $v_1, \ldots, v_k$ are linearly dependent.

9. (X I.2.3, p. 8.) Compute the dimension of $\langle (0, 1, 0, 1), (1, 0, 1, 0), (1, 1, 1, 1) \rangle$.

10. Let $M := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Show, for all $v \in \mathbb{R}^2$, for all $w \in \mathbb{R}^3$, that

$$v \cdot (L_M(w)) = (L_M(v)) \cdot w.$$ 

(The first “.” is the dot product in $\mathbb{R}^2$. The second “.” is the dot product in $\mathbb{R}^3$.)

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**Homework 1:** Due on Tuesday 27 January

1. (X I.1.2, p. 4.) Let $n \in \mathbb{N}$. Let $R, S \subseteq \mathbb{R}^n$. Assume that $R$ and $S$ are subspaces of $\mathbb{R}^n$. Show that $R \cap S$ is a subspace of $\mathbb{R}^n$.

2. (X I.1.4, p 4.) Let $S := \{(x, y, z) \in \mathbb{R}^3 \mid (x + 2y = 0) \& (x + y = 3z)\}$. Show that $S$ is a subspace of $\mathbb{R}^3$.
3. Let $S := \{(x, y, z) \in \mathbb{R}^3 \mid (x + 2y = 0) \& (x + y^2 = 3z)\}$. Show that $S$ is NOT a subspace of $\mathbb{R}^3$.

4. Let $V = \mathbb{R}^\mathbb{R}$. Let $D := \{f \in V \mid f$ is differentiable\}. Show that $D$ is a subspace of $V$.

5. Let $S_0 := \{(2, 3, 4), (3, 4, 5), (-2, 3, 5)\}$. Show that $\langle S_0 \rangle = \mathbb{R}^3$. 