PRINT YOUR NAME:

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Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.
I. Definitions

A. (5 pts) Let \( X : \mathbb{R}^3 \to \mathbb{R}^3 \) be the identity. Let \( Q : \mathbb{R}^3 \to \mathbb{R} \) be a function. We say “\( Q \) is homogeneous quadratic” if . . .

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B. (5 pts) Let \( V \) and \( W \) be finite dimensional vector spaces, let \( f : V \to W \) be a function and let \( p \in V \). We say that \( f \) is differentiable at \( p \) if . . .

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C. (5 pts) Let \( (X,d) \) be a metric space and let \( f : X \to X \). Then we say that \( f \) is a half-contraction if . . .
D. (5 pts) Let $U, V$ and $W$ be vector spaces and let $B : U \times V \to W$. We say that $B$ is bilinear if ...

E. (5 pts) Let $V$ be a vector space, $B \in \text{SBF}(V, \mathbb{R})$. Then $B$ is positive definite if ...

F. (5 pts) Let $B$ denote the set of all open disks in $\mathbb{R}^2$. Let $U \subseteq \mathbb{R}^2$. We say that $U$ is open (in the standard topology) if ...
II. True or false (no partial credit):

a. (5 pts) Let $V$ and $W$ be finite-dimensional vector spaces. Then $\text{SL}(V,W) \subseteq \text{SQ}(V,W)$.

b. (5 pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and assume that $0_2 \in \text{dom}[f'']$. Assume that $f'_0 = 0_{\mathbb{R}^2,\mathbb{R}}$ and that $f''_0 \in \text{SBF}(\mathbb{R}^2,\mathbb{R})$ is positive definite. Then $f$ has a local strict minimum at $0_2$.

c. (5 pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$. Assume that $0_2 \in \text{dom}[\partial_1 \partial_2 f]$ and that $0_2 \in \text{dom}[\partial_2 \partial_1 f]$. Then $(\partial_1 \partial_2 f)(0_2) = (\partial_2 \partial_1 f)(0_2)$.

d. (5 pts) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be $C^\infty$ and assume, for all $v \in \mathbb{R}^2$, that $[f'_v] \in \mathbb{R}^{2 \times 2}$ is an invertible matrix. Then $f$ is one-to-one.

e. (5 pts) Let $V$ and $W$ be finite dimensional vector spaces. Let $Q := Q(V,W)$ and let $S := \text{SQ}(V,W)$. Then $Q \cap S = \{0_{V,W}\}$.
III. Hand-graded problems. Show work.

1. (10 pts) Let \( f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) both be differentiable. Let \( p := (1, 2) \) and \( q := (3, 4) \).
Assume \( f(p) = q, \quad f(q) = p, \quad g(p) = p \) and \( g(q) = q \).

Assume \[ f'_p = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}, \quad f'_q = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad g'_p = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad g'_q = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}. \]

Compute the two \( 2 \times 2 \) matrices \((f \circ g)'_p\) and \((g \circ f)'_q\).
2. (10 pts.) Let $V$ and $W$ be finite dimensional vector spaces. Let $f, g : V \to W$ both be differentiable. Assume, for all $v \in V$, that $f'_v = g'_v$. Using the Mean Value Theorem, prove that $f - g \in C(V, W)$. 
3. a. (5 pts) Give an example a polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ such that:

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\begin{align*}
  f(0, 0) &= 7, \\
  (\partial_1 f)(0, 0) &= 0, \\
  (\partial_2 f)(0, 0) &= 0, \\
  (\partial_1 \partial_1 f)(0, 0) &= 2, \\
  (\partial_1 \partial_2 f)(0, 0) &= 4, \\
  (\partial_2 \partial_2 f)(0, 0) &= 10.
\end{align*}
\]

b. (5 pts) Determine whether your $f$ has a local strict minimum at $0, 0$. 

4. (15 pts) Let $V := \mathbb{R}^{2 \times 2}$. Define $f, T : V \to \mathbb{R}$ by $f(M) = \det(M)$ and $T(M) = \text{tr}(M)$.

So, for all $w, x, y, z \in \mathbb{R}$, we have: $f \begin{bmatrix} w & x \\ y & z \end{bmatrix} = wz - xy$ and $T \begin{bmatrix} w & x \\ y & z \end{bmatrix} = w + z$.

Let $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ be the identity. Show that $f'_I = T$.

*Note:* You may use, without proof, that $f \in \mathcal{Q}(V, \mathbb{R})$ and that $T \in \mathcal{L}(V, \mathbb{R})$. 
