MATH 4604 Spring 2017, Final exam
Handout date: Thursday 11 May 2017
Instructor: Scot Adams

PRINT YOUR NAME:

SOLUTIONS

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Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let V and W be finite dimensional vector spaces, and let $f : V \dashrightarrow W$. Let $p \in V$. Then $\mathrm{LINS}_p(f) = \cdots$

$$\{L \in \mathcal{L}(V, W) \mid f_p^T - L \in \mathcal{o}_{VW}(1)\}$$

B. (5 pts) Let V and W be vector spaces, $f : V \dashrightarrow W$. Let $x \in \mathrm{dom}[f]$. Let $v \in V$. Then $SS^f_{x,v} : DSS^f_{x,v} \to W$ is defined by $SS^f_{x,v}(h) = \cdots$

$$\frac{[f(x+hv)] - [f(x)]}{h}$$

C. (5 pts) Let V and W be finite dimensional vector spaces and let $p \geqslant 0$. Then $\mathcal{O}_{V,W}(p) := \cdots$

$$\mathrm{ELT}\{\hat{\mathcal{O}}_{VW\|\cdot\|}(p) \text{ s.t. } \|\cdot\| \in \mathcal{N}(V)\}$$

D. (5 pts) Let $n \in \mathbb{N}$ and let $h : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$. Then h is **near constant** (or **half-Lipschitz**) means ...

$\forall p, q \in \mathrm{dom}[h]$,

$\left| [h(p)] - [h(q)] \right|_2 \leq \frac{1}{2} |p - q|_2$

E. (5 pts) Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$. Then A is **positive definite** means: ...

$\forall x \in \mathbb{R}^n \setminus \{0_n\}$,

$(x^H A x^V)_{11} > 0$

F. (5 pts) Let $d \in \mathbb{N}$. Let $S \subseteq \mathbb{R}^d$ be bounded. Then $\overline{\mu}_d(S) := \cdots$

$\inf \{ \mu_d(C) \mid C \in \mathcal{R}_d, \ C \supseteq S \}$

II. True or false (no partial credit):

a. (5 pts) Let $f : \mathbb{R}^5 \dashrightarrow \mathbb{R}^8$. Assume, for all $i \in \{1, \ldots, 5\}$, that $\partial_i f \in o_{5,8}(1)$. Then $f \in o_{5,8}(2)$.

False $\qquad C^1_{\mathbb{R}^5} \notin \check{\mathcal{O}}_{5,8}(2)$

b. (5 pts) $\mathcal{O} \circ o \subseteq o$.

False $\qquad C^1_{\mathbb{R}} \circ C^0_{\mathbb{R}} = C^1_{\mathbb{R}} \notin \check{\mathcal{O}}$

c. (5 pts) Let V be a finite dimensional vector space and let $n := \dim V$. Then V is isomorphic to $\mathbb{R}^n$.

$\forall \mathcal{B} \in \mathcal{OB}(V)$,

True $\qquad L_{\mathcal{B}} : \mathbb{R}^n \to V$ is a VS isomorphism.

d. (5 pts) Let V be a fdVS. Let $| \bullet |, \| \bullet \| \in \mathcal{N}(V)$. Then $| \bullet | \approx \| \bullet \|$.

True $\qquad$ On a fdVS, all norms are comparable.

e. (5 pts) Let $\mathcal{S}$ be a semiring of sets. Then $\langle \mathcal{S} \rangle_{\text{fin} \sqcup}$ is a ring of sets.

True. $\qquad$ A semiring generates a ring.

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I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Let V and W both be finite dimensional vector spaces. Let $P \in \mathcal{P}_4(V,W)$. Let $F \in \mathrm{SM}_4(V,W)$ denote the polarization of P. Let $x, u \in V$. Find a formula for $(D_x P)(u)$, in terms of F, x and u.

$$P^T_x(u) = [P(x+u)] - [P(x)]$$

$$= [F(x+u, x+u, x+u, x+u)] - [F(x,x,x,x)]$$

$$= 4[F(x,x,x,u)] + 6[F(x,x,u,u)]$$

$$+ 4[F(x,u,u,u)] + [F(u,u,u,u)]$$

$$(D_x P)(u) = 4[F(x,x,x,u)]$$

2. (10 pts) Find a function $f \in \mathcal{O}(1)$ such that $0 \notin \mathrm{dom}[f']$.

Let $f := |\cdot| : \mathbb{R} \longrightarrow \mathbb{R}$.

Then $f \in \hat{\mathcal{O}}(1)$

and $0 \notin \mathrm{dom}[f']$.

3. (10 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume $f \in o(1)$. Show: $0 \in \mathrm{dom}[f']$.

<u>Want</u>: $f'(0) = 0$

Let $I := id_{\mathbb{R}} : \mathbb{R} \longrightarrow \mathbb{R}$.

$\forall h \in \mathbb{R} \backslash \{0\}, \quad \left(\frac{|\bullet|}{I}\right)(h) = \frac{|h|}{h} \in \{1, -1\}$

$\therefore \frac{|\bullet|}{I} \in \hat{\mathcal{O}}^{\times}$

$f \in \check{o}(1) \quad \therefore \frac{f}{|\bullet|} \in \check{o}^{\times}$

$f \in \check{o}(1) \quad \therefore f \in \check{o} \quad \therefore f(0) = 0$

$\forall h \in \mathbb{R}, \quad SS_0^f(h) = \frac{[f(0+h)] - [f(0)]}{h}$

$= \frac{f(h)}{h} = \left(\frac{f}{I}\right)(h)$

$SS_0^f = \frac{f}{I} = \frac{f}{|\bullet|} \cdot \frac{|\bullet|}{I} \in (\check{o}^{\times}) \cdot (\hat{\mathcal{O}}^{\times}) \subseteq \check{o}^{\times}$

$\therefore \lim_0 SS_0^f = 0$

$f'(0) = \lim_0 SS_0^f = 0.$ $\blacksquare$

4. (15 pts.) Let $\|\bullet\| := |\bullet|_2 \in \mathcal{N}(\mathbb{R}^2)$. Let $f : \mathbb{R}^2 \dashrightarrow \mathbb{R}$. Assume $(0,0) \in \mathrm{dom}[f]$ and $f(0,0) = 0$. Define $P \in \mathcal{Q}(\mathbb{R}^2, \mathbb{R})$ by $P(x,y) = xy$. Assume $f - P \in \circ_{2,1}(2)$. Show f does not have a local semi-minimum at $(0,0)$. That is, show, for all $\delta > 0$, that there exists $z \in \mathbb{R}^2$ such that: both ($\|z\| < \delta$) and ($f(z) < 0$).

Given $\delta > 0$. Want: $\exists z \in \mathbb{R}^2$ s.t. $[(\|z\| < \delta)$ and $(f(z) < 0)]$

$f - P \in \check{o}_{21}(2)$, so $\dfrac{|f-P|}{\|\cdot\|^2} \in \check{o}_{21}^{\times}$

Choose $\eta > 0$ s.t. $\eta < \delta\sqrt{2}$, s.t. $B(0_2, \eta) \subseteq \mathrm{dom}[f-P]$ and s.t., $\forall z \in (B(0_2,\eta)) \setminus \{0_2\}$, $\dfrac{|[f(z)] - [P(z)]|}{\|z\|^2} < \dfrac{1}{4}$

Let $x := \eta/2$, $z := (x, -x)$. Want: $(\|z\| < \delta)$ and $(f(z) < 0)$

$\|z\|^2 = 2x^2$ $\therefore \|z\|^2 = \eta^2/2$ $\therefore \|z\| = \eta/\sqrt{2}$ $\therefore \|z\| < \delta$. Want: $f(z) < 0$

$\|z\| = \eta/\sqrt{2} < \eta$ and $z = (x,-x) \neq 0_2$, so $z \in (B(0_2,\eta)) \setminus \{0_2\}$.

Then $|[f(z)] - [P(z)]| < \frac{1}{4}\|z\|^2$

Then $[P(z)] - \frac{1}{4}\|z\|^2 < f(z) < [P(z)] + \frac{1}{4}\|z\|^2$

Then $f(z) < [P(z)] + \frac{1}{4}\|z\|^2 = [P(x,-x)] + \frac{1}{4}[2x^2]$

$= -x^2 + \frac{1}{2}x^2 = -\frac{1}{2}x^2 < 0$ ∎