MATH 4604 Spring 2017, Midterm #1
Handout date: Thursday 23 February 2017
Instructor: Scot Adams

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SOLUTIONS

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Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let X be a topological space, $f : X \dashrightarrow \mathbb{R}$ and $a \in \mathrm{dom}[f]$. Then f **has a local strict max near** a **in** X means ...

$\exists$ pnbd P in X of a

s.t. $f_*(P) < f(a)$

B. (5 pts) (Fill in **BOTH** "..."s in the following.) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Let $a \in \mathrm{dom}[f]$. Then $SS_a^f : \cdots \to \mathbb{R}$ is defined by ...

$$[(\mathrm{dom}[f]) - a] \setminus \{0\}$$

$$SS_a^f(h) = \frac{[f(a+h)] - [f(a)]}{h}$$

C. (5 pts) We defined $\mathcal{O} := \cdots$.

$$\left\{ \varepsilon : \mathbb{R} \dashrightarrow \mathbb{R} \;\middle|\; \begin{array}{c} \mathrm{dom}[\varepsilon] \text{ is a nbd in } \mathbb{R} \text{ of } 0 \\ \varepsilon(0) = 0 \\ \varepsilon \text{ is continuous at } 0 \end{array} \right\}$$

D. (5 pts) We defined $\mathcal{O}(3) := \cdots$.

$$\hat{\mathcal{O}} \cdot |\cdot|^3$$

E. (5 pts) Let V and W be vector spaces. Then $SBF(V, W) := \cdots$.

$$\{B \in \mathcal{B}(V, V, W) \mid \forall u, v \in V \quad B(u, v) = B(v, u)\}$$

F. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathbb{R}$. Then f is **differentiable on** S means ...

$$S \subseteq dd[f]$$

II. True or false (no partial credit):

a. (5 pts) Let V be a vector space, $L, M \in \mathcal{L}(V)$. Then $LM \in \mathcal{Q}(V)$.

True $\qquad \deg[LM] = (\deg L) + (\deg M)$

b. (5 pts) $\mathcal{O}(1) \subseteq o(1)$.

False $\qquad |\bullet| \in [\mathcal{O}(1)] \setminus [o(1)]$

c. (5 pts) If V and W are isomorphic vector spaces and V is finite dimensional, then W is also finite dimensional.

True $\qquad$ VS isomorphisms preserve and reflect linear algebra properties

d. (5 pts) Let V be a finite dimensional vector space and let $n := \dim V$. Then V is isomorphic to $\mathbb{R}^n$.

True $\qquad \forall \mathcal{B} \in \mathcal{OB}(V),\ L_{\mathcal{B}} : \mathbb{R}^n \to V$ is a VS isomorphism

e. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Assume f is differentiable on $\mathbb{R}$ and f is semi-increasing on $\mathbb{R}$. Then, for all $x \in \mathbb{R}$, $f'(x) \geqslant 0$.

True $\qquad \forall x, h \in \mathbb{R},\ SS_x^f(h) \geqslant 0$

$\therefore \forall x \in \mathbb{R},\ f'(x) \geqslant 0$

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PLEASE DO NOT WRITE BELOW THE LINE

I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Give an example of a polynomial $f : \mathbb{R} \to \mathbb{R}$ such that
   - f is strictly increasing on $\mathbb{R}$ and
   - $f'(0) = f'(3) = 0$.

$\forall x \in \mathbb{R}$, $\quad x^2(x-3)^2 = x^2(x^2 - 6x + 9)$

$$= x^4 - 6x^3 + 9x^2$$

Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{x^5}{5} - \frac{3x^4}{2} + 3x^3$$

2. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$, and assume that f is differentiable on $\mathbb{R}$. Assume that $f(0) = 0$ and that $f' \in \mathcal{O}$. Show that $f \in \mathcal{O}(1)$.

Let $\alpha := f/|\bullet|$ Want: $\alpha \in \mathcal{O}^{\times}$

Since $f' \in \mathcal{O}$, choose $\delta > 0$ s.t. $[((-\delta,\delta) \subseteq \mathrm{dom}[f'])$ & $(f'_*((-\delta,\delta))$ bdd in $\mathbb{R})]$

Let $P := (-\delta,\delta)\backslash\{0\}$. Want: $[(P \subseteq \mathrm{dom}[\alpha])$ & $(\alpha_*(P)$ bdd in $\mathbb{R})]$

$(-\delta,\delta) \subseteq \mathrm{dom}[f'] \subseteq \mathrm{dom}[f]$

$P = (-\delta,\delta)\backslash\{0\} \subseteq (\mathrm{dom}[f])\backslash\{0\} = \mathrm{dom}[\alpha]$

Want: $\alpha_*(P)$ bdd in $\mathbb{R}$. Know: $f'_*((-\delta,\delta))$ bdd in $\mathbb{R}$

Choose $K \geqslant 0$ s.t. $\forall x \in (-\delta,\delta)$, $|f'(x)| \leqslant K$

Want: $\forall x \in P$, $|\alpha(x)| \leqslant K$

Given $x \in P$. Want: $|\alpha(x)| \leqslant K$

By MVT, choose $c \in (\langle 0,x \rangle)$ s.t. $f'(c) = \dfrac{[f(x)]-[f(0)]}{x - 0}$

$(x \in P \subseteq (-\delta,\delta))$ & $(c \in (\langle 0,x \rangle))$, so $c \in (-\delta,\delta)$, so $|f'(c)| \leqslant K$

$$|\alpha(x)| = \left|\frac{f(x)}{|x|}\right| = \frac{|f(x)|}{|x|} = \left|\frac{f(x)}{x}\right| = \left|\frac{[f(x)]-[f(0)]}{x - 0}\right|$$

$$= |f'(c)| \leqslant K. \quad \blacksquare$$

3. (10 pts) Let $f \in \mathcal{O}(2)$ and let $g \in \mathcal{O}(3)$. Show that $g \circ f \in \mathcal{O}(6)$.

NOTE: You may use, without proof, that $\mathcal{O} \circ o \subseteq \mathcal{O}$, that $|\mathcal{O}| \subseteq \mathcal{O}$ and that $\mathcal{O} \cdot \mathcal{O} \subseteq \mathcal{O}$.

Choose $\alpha, \beta \in \mathcal{O}$ s.t. $f = \alpha \cdot |\bullet|^2$ & $g = \beta \cdot |\bullet|^3$

$$\forall x,\ g(f(x)) = [\beta(f(x))] \cdot |f(x)|^3$$
$$= [\beta(f(x))] \cdot \left| [\alpha(x)] \cdot |x|^2 \right|^3$$
$$= [\beta(f(x))] \cdot |\alpha(x)|^3 \cdot |x|^6$$

so $g \circ f = [\beta \circ f] \cdot |\alpha|^3 \cdot |\bullet|^6$

$\beta \circ f \in \mathcal{O} \circ [\mathcal{O}(2)] \subseteq \mathcal{O} \circ o \subseteq \mathcal{O}$

$|\alpha|^3 \in |\mathcal{O}|^3 \subseteq \mathcal{O}^3 \subseteq \mathcal{O}$

$g \circ f \in \mathcal{O} \cdot \mathcal{O} \cdot |\bullet|^6 \subseteq \mathcal{O} \cdot |\bullet|^6 = \mathcal{O}(6)$

4. (15 pts.) Let $S \subseteq \mathbb{R}$. Recall that S is **bounded in** $\mathbb{R}$ means: there exist $p, q \in \mathbb{R}$ such that $p \leqslant S \leqslant q$.

Let $f_1, f_2, f_3, \ldots : \mathbb{R} \to \mathbb{R}$ be a sequence of functions. Assume that $\{f_1(0), f_2(0), f_3(0), \cdots\}$ is bounded in $\mathbb{R}$. Assume, for all $j \in \mathbb{N}$, for all $x \in \mathbb{R}$, that f_j is differentiable at x and that $|f_j'(x)| \leqslant 3$. Show that $\{f_1(1), f_2(1), f_3(1), \ldots\}$ is bounded in $\mathbb{R}$.

Choose $p, q \in \mathbb{R}$ s.t., $\forall j \in \mathbb{N}$, $p \leq f_j(0) \leq q$

<u>Want</u>: $\forall j \in \mathbb{N}$, $p - 3 \leqslant f_j(1) \leq q + 3$

Given $j \in \mathbb{N}$. <u>Want</u>: $p - 3 \leq f_j(1) \leq q + 3$

<u>Know</u>: $p \leqslant f_j(0) \leq q$

By MVT, choose $c \in (0,1)$ s.t. $f_j'(c) = \dfrac{[f_j(1)] - [f_j(0)]}{1 - 0}$

$$\left| [f_j(1)] - [f_j(0)] \right| = \left| \frac{[f_j(1)] - [f_j(0)]}{1 - 0} \right| = |f_j'(c)| \leq 3$$

$$[f_j(0)] - 3 \leq f_j(1) \leqslant [f_j(0)] + 3$$

$$p - 3 \leq f_j(1) \leqslant q + 3 \qquad \blacksquare$$