MATH 4604 Spring 2017, Midterm #2
Handout date: Thursday 23 March 2017
Instructor: Scot Adams

PRINT YOUR NAME:

SOLUTIONS

PRINT YOUR @umn.edu EMAIL ADDRESS:

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let V and W be vector spaces. Then $\mathcal{Q}(V, W) := \cdots$

$$\{ Q_B \mid B \in SBF(V, W) \}$$

B. (5 pts) Let V and W be vector spaces. Then $\mathrm{SBF}(V, W) := \cdots$

$$\{ B \in \mathcal{B}(V, V, W) \mid \forall u, v \in V, \ B(u, v) = B(v, u) \}$$

C. (5 pts) Let V be a finite dimensional vector space, $n := \dim V$, $\mathcal{B} \in \mathrm{OB}(V)$ an ordered basis. Then $\mathcal{B}$ determines a topology, $\mathcal{T}_{\mathcal{B}} := \cdots$.

$$(L_{\mathcal{B}})_{**}(\mathcal{T}_n)$$

D. (5 pts) Let V be a vector space and let $\|\bullet\| : V \to [0,\infty)$. Then $\|\bullet\|$ is a **norm on** V if ...

$$\left(\forall v \in V, \quad \left[(\|v\| = 0) \Leftrightarrow (v = 0_V) \right] \right) \&$$
$$\left(\forall c \in \mathbb{R}, \forall v \in V, \quad \left[\|cv\| = |c| \cdot \|v\| \right] \right) \&$$
$$\left(\forall v, w \in V \quad \left[\|v + w\| \leq \|v\| + \|w\| \right] \right)$$

E. (5 pts) Let V be a vector space and let $B \in \mathrm{SBF}(V, \mathbb{R})$. Then B **is positive definite** means: ...

$$\forall v \in V \setminus \{0_V\}, \qquad B(v, v) > 0$$

F. (5 pts) Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. Then $A \geqslant 0$ means: ...

$$\forall x \in \mathbb{R}^n, \qquad \underset{\|}{(x^H A x^V)} \geqslant 0$$

II. True or false (no partial credit):

a. (5 pts) Let $F : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ denote the cofactor map, so, $\forall A \in \mathbb{R}^{2\times 2}$, $F(A)$ is the cofactor matrix of A. Then $F \in \mathcal{P}_3(\mathbb{R}^{2\times 2}, \mathbb{R}^{2\times 2})$.

False $\quad F \in \mathcal{P}_1(\mathbb{R}^{2\times 2}, \mathbb{R}^{2\times 2})$

b. (5 pts) Let V be a finite dimensional vector space. Then, for all $\mathcal{A}, \mathcal{B} \in \mathrm{OB}(V)$, we have $\mathcal{T}_{\mathcal{A}} = \mathcal{T}_{\mathcal{B}}$.

True $\quad \mathcal{T}_{\mathcal{A}} = \mathcal{T}_V = \mathcal{T}_{\mathcal{B}}$

c. (5 pts) We have $[\mathcal{O}(2)] \cdot [\mathcal{O}(4)] \subseteq \mathcal{O}(8)$.

False $\quad |\cdot|^2 \cdot |\cdot|^4 = |\cdot|^6 \notin \hat{\mathcal{O}}(8)$

d. (5 pts) There exists a vector space isomorphism $\mathbb{R}^5 \to \mathbb{R}^6$.

False $\quad \dim \mathbb{R}^5 = 5 \neq 6 = \dim \mathbb{R}^6$

e. (5 pts) Let V and W be vector spaces and let $B \in \mathcal{B}(V, V, W)$. Then there exists $S \in \mathrm{SBF}(V, W)$ such that, $\forall v \in V$, $S(v, v) = B(v, v)$.

True $\quad S := \mathrm{Sym}[B]$

THE BOTTOM OF THIS PAGE IS FOR TOTALING SCORES
PLEASE DO NOT WRITE BELOW THE LINE

I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Find $f \in o$ and $g \in \mathcal{O}$ such that $g \circ f \notin o$.

Let $f := C^0_{\mathbb{R}}$, $g := C^1_{\mathbb{R}}$

Then $f \in \check{o}$, $g \in \check{\mathcal{O}}$

and $g \circ f = C^1_{\mathbb{R}} \circ C^0_{\mathbb{R}} = C^1_{\mathbb{R}} \notin \check{o}$

2. (10 pts) Let $n \in \mathbb{N}$. Let $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denote the dot product. Let $A \in \mathbb{R}^{n\times n}$. Recall that $A^t \in \mathbb{R}^{n\times n}$ denotes the transpose of A. Let $v, w \in \mathbb{R}^n$. Show that $v \cdot [L_A(w)] = [L_{A^t}(v)] \cdot w$.

Let $I := \{1, \ldots, n\}$, $B := A^t$. Then

$$v \cdot [L_A(w)] = \sum_{i \in I} v_i [L_A(w)]_i$$

$$= \sum_{i\in I} v_i [A_{i\bullet} \cdot w] = \sum_{i \in I} v_i \left[\sum_{j \in I} A_{ij} w_j\right]$$

$$= \sum_{i \in I} \sum_{j \in I} v_i A_{ij} w_j = \sum_{j \in I} \sum_{i \in I} v_i A_{ij} w_j$$

$$= \sum_{j \in I} \left[\sum_{i \in I} B_{ji} v_i\right] w_j = \sum_{j \in I} [B_{j\bullet} \cdot v] w_j$$

$$= \sum_{j \in I} [L_B(v)]_j w_j = [L_B(v)] \cdot w$$

$$= [L_{A^t}(v)] \cdot w \quad \blacksquare$$

3. (10 pts) Let V, W and X be vector spaces. Let $B \in \mathcal{B}(V, W, X)$. Show: $B \in \mathcal{Q}(V \times W, X)$. That is, show: $\exists A \in \mathrm{SBF}(V \times W, X)$ such that $\forall v \in V$, $\forall w \in W$, $A((v, w), (v, w)) = B(v, w)$.

Define $A \in SBF(V \times W, X)$

by $$A((v,w),(v',w')) = \frac{[B(v,w')] + [B(v',w)]}{2}$$

<u>Want</u>: $\forall v \in V$, $\forall w \in W$, $A((v,w),(v,w)) = B(v,w)$

Given $v \in V$, $w \in W$.

<u>Want</u>: $A((v,w),(v,w)) = B(v,w)$

$$A((v,w),(v,w)) = \frac{[B(v,w)] + [B(v,w)]}{2}$$

$$= B(v,w) \qquad \blacksquare$$

4. (15 pts.) Let $\psi \in o(7)$ and $\phi \in \mathcal{O}(1)$. Show that $\psi \circ \phi \in o(7)$.

NOTE: You may use without proof, all of the following facts:

$$|\mathcal{O}| \subseteq \mathcal{O}, \qquad \mathcal{O} \cdot o \subseteq o, \qquad o \circ o \subseteq o, \qquad \mathcal{O}(1) \subseteq o,$$
$$\forall n \in \mathbb{N},\ \mathcal{O}^n \subseteq \mathcal{O}.$$

Choose $\varepsilon \in \check{o}$ s.t. $\psi = |\bullet|^7 \cdot \varepsilon$

Choose $\alpha \in \hat{\mathcal{O}}$ s.t. $\phi = |\bullet| \cdot \alpha$

$$\forall x,\ (\psi \circ \phi)(x) = \psi(\phi(x)) = (|\bullet|^7 \cdot \varepsilon)(\phi(x))$$
$$= |\phi(x)|^7 \cdot [\varepsilon(\phi(x))]$$
$$= |(|\bullet| \cdot \alpha)(x)|^7 \cdot [\varepsilon(\phi(x))]$$
$$= \big||x| \cdot [\alpha(x)]\big|^7 \cdot [\varepsilon(\phi(x))]$$
$$= |x|^7 \cdot |\alpha(x)|^7 \cdot [\varepsilon(\phi(x))]$$

$$\psi \circ \phi = |\bullet|^7 \cdot |\alpha|^7 \cdot [\varepsilon \circ \phi] \in |\bullet|^7 \cdot |\hat{\mathcal{O}}|^7 \cdot [\check{o} \circ (\hat{\mathcal{O}}(1))]$$
$$\subseteq |\bullet|^7 \cdot \hat{\mathcal{O}}^7 \cdot [\check{o} \circ \check{o}] \subseteq |\bullet|^7 \cdot \hat{\mathcal{O}} \cdot \check{o}$$
$$\subseteq |\bullet|^7 \cdot \check{o} = \check{o}(7) \qquad \blacksquare$$