Writeup on integration for MATH 4604 (Advanced Calculus II)
Spring 2015

We adopt here all the notation from the handout on measures. This handout is a continuation of that one.

Definition: Let $n \in \mathbb{N}$, let $D \subseteq \mathbb{R}^n$ and let $f : D \to \mathbb{R}$. Then the **ordinate set** of f is defined to be $\mathcal{O}_f := \{(x,y) \in D \times \mathbb{R} \,|\, 0 < y < f(x)\} \subseteq \mathbb{R}^{n+1}$.

Definition: Let $n \in \mathbb{N}$, let $D \subseteq \mathbb{R}^n$ and let $f : D \to \mathbb{R}$. Then f is **integrable** means: the sets $\mathcal{O}_f$ and $\mathcal{O}_{-f}$ are both contented in $\mathbb{R}^{n+1}$. If f is integrable, then the **integral** of f is defined to be $\displaystyle\int_D f \quad := \quad [\,v_{n+1}(\mathcal{O}_f)\,] \;-\; [\,v_{n+1}(\mathcal{O}_{-f})\,]$.

Definition: For any $n \in \mathbb{N}$, for any nonempty $S \subseteq \mathbb{R}^n$, $\quad \mathrm{diam}\, S := \displaystyle\sup_{x,y\in S} |x-y|$.

Definition: For any $n \in \mathbb{N}$, for any nonempty $\mathcal{F} \subseteq 2^{\mathbb{R}^n}\backslash\{\emptyset\}$, $\quad \mathrm{mesh}\,\mathcal{F} := \displaystyle\max_{F\in\mathcal{F}} \mathrm{diam} F$.

Recall that $\mathcal{I} \subseteq 2^{\mathbb{R}}$ denotes the set of all intervals in the real number line, $\mathbb{R}$.

Fact (chopOnion): Let $n \in \mathbb{N}$, let $S \in \mathcal{C}_n\backslash\{\emptyset\}$ and let $\delta > 0$. Then there exists a finite nonempty pairwise-disjoint $\mathcal{F} \in \mathcal{C}_n\backslash\{\emptyset\}$ such that $\cup\mathcal{F} = S$ and such that $\mathrm{mesh}\,\mathcal{F} < \delta$.

Proof: Choose $m \in \mathbb{N}$ such that $\sqrt{n}/m < \delta$. Let $\mathcal{I}_0 := \{\,[\,j/m\,,\,(j+1)/m)\,)\mid j \in \mathbb{Z}\}$. Then $\mathcal{I}_0 \subseteq \mathcal{I} \subseteq 2^{\mathbb{R}}$. Let $\mathcal{G} := \{I_1 \times \cdots \times I_n \,|\, I_1,\ldots,I_n \in \mathcal{I}_0\}$. Then $\mathcal{G} \subseteq \mathcal{B}_n \subseteq 2^{\mathbb{R}^n}$. Also, $\mathcal{G}$ is pairwise-disjoint and $\cup\mathcal{G} = \mathbb{R}^n$. Also, for all $G \in \mathcal{G}$, we have $\mathrm{diam}\, G = \sqrt{n}/m < \delta$. We define $\mathcal{F} := \{S \cap G \,|\, G \in \mathcal{G}, S \cap G \neq \emptyset\}$. QED

Lemma (inOutApprox2): Let $n \in \mathbb{N}$, let $S \subseteq \mathbb{R}^n$ be bounded and let $\sigma \geq 0$. Then:
$(+)$ $\quad [\;(\;S$ is contented $)$ and $(\;v_n(S) = \sigma\;)\;]$

$\Leftrightarrow$
$(*)$ $\forall\varepsilon > 0, \exists K, L \in \mathcal{K}_n$ s.t. $K \subseteq S \subseteq L$ and s.t. $\sigma - \varepsilon \leq v_n^{\mathcal{K}}(K) \leq v_n^{\mathcal{K}}(L) \leq \sigma + \varepsilon$.

Proof: Proof of $\Rightarrow$: Assume $(+)$. Given $\varepsilon > 0$. We wish to prove
$(*')$ $\exists K, L \in \mathcal{K}_n$ s.t. $K \subseteq S \subseteq L$ and s.t. $\sigma - \varepsilon \leq v_n^{\mathcal{K}}(K) \leq v_n^{\mathcal{K}}(L) \leq \sigma + \varepsilon$.
Since $\underline{v}_n(S) = v_n(S) = \sigma$, choose $K \in \mathcal{K}_n$ such that $K \subseteq S$ and such that $\sigma - \varepsilon \leq v_n^{\mathcal{K}}(K)$. Since $\overline{v}_n(S) = v_n(S) = \sigma$, choose $L \in \mathcal{K}_n$ such that $S \subseteq L$ and such that $v_n^{\mathcal{K}}(L) \leq \sigma + \varepsilon$. Then $K \subseteq S \subseteq L$, and it remains only to show that $v_n^{\mathcal{K}}(K) \leq v_n^{\mathcal{K}}(L)$. However, $K \subseteq L$, and so, by monotonicity of v_n, we get $v_n^{\mathcal{K}}(K) \leq v_n^{\mathcal{K}}(L)$, as desired. *End of proof of $\Rightarrow$.*

Proof of $\Leftarrow$: Assume $(*)$. We wish to prove $\underline{v}_n(S) = \overline{v}_n(S) = \sigma$. Given $\varepsilon > 0$. We wish to prove $\sigma - \varepsilon \leq \underline{v}_n(S) \leq \overline{v}_n(S) \leq \sigma + \varepsilon$.

By $(*)$, choose $K, L \in \mathcal{K}_n$ such that $K \subseteq S \subseteq L$ and $\sigma - \varepsilon \leq v_n^{\mathcal{K}}(K) \leq v_n^{\mathcal{K}}(L) \leq \sigma + \varepsilon$. By definition of $\underline{v}_n$, we have $v_n^{\mathcal{K}}(K) \leq \underline{v}_n(S)$. By (a) of Fact (innerOutr) in the handout on measures, we have $\underline{v}_n(S) \leq \overline{v}_n(S)$. By definition of $\overline{v}_n$, we have $\overline{v}_n(S) \leq v_n^{\mathcal{K}}(L)$. Then

$$\sigma - \varepsilon \quad\leq\quad v_n^{\mathcal{K}}(K) \quad\leq\quad \underline{v}_n(S) \quad\leq\quad \overline{v}_n(S) \quad\leq\quad v_n^{\mathcal{K}}(L) \quad\leq\quad \sigma + \varepsilon,$$

so $\sigma - \varepsilon \le \underline{v}_n(S) \le \overline{v}_n(S) \le \sigma + \varepsilon$, as desired. *End of proof of* $\Leftarrow$. QED

Note that Lemma (inOutApprox2) may be used to give a quick proof of Lemma (inOutApprox) in the writeup on measures.

Lemma (prodMsr): Let $m, n \in \mathbb{N}$, let $S \in \mathcal{C}_m$ and let $T \in \mathcal{C}_n$. Then $S \times T \in \mathcal{C}_{m+n}$ and $v_{m+n}(S \times T) = [v_m(S)][v_n(T)]$.

Proof: Let $\sigma := v_m(S)$ and let $\tau := v_n(T)$. We wish to prove: $S \times T \in \mathcal{C}_{m+n}$ and $v_{m+n}(S \times T) = \sigma\tau$. Given $\varepsilon > 0$. By Lemma (inOutApprox2), we wish to show that there exist $A, B \in \mathcal{K}_{m+n}$ such that

$$A \subseteq S \times T \subseteq B \qquad \text{and} \qquad \sigma\tau - \varepsilon \le v_{m+n}^{\mathcal{K}}(A) \le v_{m+n}^{\mathcal{K}}(B) \le \sigma\tau + \varepsilon.$$

Choose $\varepsilon_0 > 0$ such that $\varepsilon_0 \le 1$ and $(\sigma + \tau + 1)\varepsilon_0 \le \varepsilon$. By Lemma (inOutApprox2), choose $K, L \in \mathcal{K}_m$ such that

$$K \subseteq S \subseteq L \qquad \text{and} \qquad \sigma - \varepsilon_0 \le v_m^{\mathcal{K}}(K) \le v_m^{\mathcal{K}}(L) \le \sigma + \varepsilon_0.$$

By Lemma (inOutApprox2), choose $P, Q \in \mathcal{K}_n$ such that

$$P \subseteq T \subseteq Q \qquad \text{and} \qquad \tau - \varepsilon_0 \le v_n^{\mathcal{K}}(P) \le v_n^{\mathcal{K}}(Q) \le \tau + \varepsilon_0.$$

Let $A := K \times P$ and $B := L \times Q$. Then $A \subseteq S \times T \subseteq B$, and it remains to show that $\sigma\tau - \varepsilon \le v_{m+n}^{\mathcal{K}}(A) \le v_{m+n}^{\mathcal{K}}(B) \le \sigma\tau + \varepsilon$.

We leave it as an exercise to prove: $v_{m+n}^{\mathcal{K}}(B) = (v_m^{\mathcal{K}}(L))(v_n^{\mathcal{K}}(Q))$. Then

$$\begin{aligned} v_{m+n}^{\mathcal{K}}(B) &= (v_m^{\mathcal{K}}(L))(v_n^{\mathcal{K}}(Q)) \\ &\le (\sigma + \varepsilon_0)(\tau + \varepsilon_0) \\ &= \sigma\tau + (\sigma + \tau + \varepsilon_0)\varepsilon_0 \\ &\le \sigma\tau + (\sigma + \tau + 1)\varepsilon_0 \\ &\le \sigma\tau + \varepsilon. \end{aligned}$$

Also, as $A \subseteq B$, we get $v_{m+n}^{\mathcal{K}}(A) \le v_{m+n}^{\mathcal{K}}(B)$. It remains to prove: $v_{m+n}^{\mathcal{K}}(A) \ge \sigma\tau - \varepsilon$.

We leave it as an exercise to prove: $v_{m+n}^{\mathcal{K}}(A) = (v_m^{\mathcal{K}}(K))(v_n^{\mathcal{K}}(P))$. Then

$$\begin{aligned} v_{m+n}^{\mathcal{K}}(A) &= (v_m^{\mathcal{K}}(K))(v_n^{\mathcal{K}}(P)) \\ &\ge (\sigma - \varepsilon_0)(\tau - \varepsilon_0) \\ &= \sigma\tau - (\sigma + \tau - \varepsilon_0)\varepsilon_0 \\ &\ge \sigma\tau - (\sigma + \tau + 1)\varepsilon_0 \\ &\ge \sigma\tau - \varepsilon, \end{aligned}$$

as desired. QED

Lemma (ctdContin): Let $n \in \mathbb{N}$ and let $S \subseteq \mathbb{R}^n$. Assume, for all $\varepsilon > 0$ that there exist $R, T \in \mathcal{C}_n$ such that $R \subseteq S \subseteq T$ and such that $v_n(T \backslash R) \le \varepsilon$. Then $S \in \mathcal{C}_n$.

Proof: Choose $T_0 \in \mathcal{C}_n$ such that $S \subseteq T_0$. Since $T_0 \in \mathcal{C}_n$, T_0 is bounded. So, since $S \subseteq T_0$, S is bounded. Given $\varepsilon > 0$. By Lemma (inOutApprox) from the handout on measures, we wish to prove that there exist $K, L \in \mathcal{K}_n$ such that $K \subseteq S \subseteq L$ and $v_n^{\mathcal{K}}(L \backslash K) \leq \varepsilon$.

Choose $R, T \in \mathcal{C}_n$ such that $R \subseteq S \subseteq T$ and such that $v_n(T \backslash R) \leq \varepsilon/3$. By definition of $\underline{v}_n$, choose $K \in \mathcal{K}_n$ such that $K \subseteq R$ and such that $v_n^{\mathcal{K}}(K) \geq (\underline{v}_n(R)) - (\varepsilon/3)$. Since $R \in \mathcal{C}_n$, we have $\underline{v}_n(R) = v_n(R)$. Then $v_n^{\mathcal{K}}(K) \geq (v_n(R)) - (\varepsilon/3)$. By definition of $\overline{v}_n$, choose $L_0 \in \mathcal{K}_n$ such that $T \subseteq L_0$ and such that $v_n^{\mathcal{K}}(L_0) \leq (\overline{v}_n(T)) + (\varepsilon/3)$. Since $T \in \mathcal{C}_n$, we have $\overline{v}_n(T) = v_n(T)$. Then $v_n^{\mathcal{K}}(L_0) \leq (v_n(T)) + (\varepsilon/3)$. Since $T \in \mathcal{C}_n$, it follows that T is bounded. Then choose a bounded $B \in \mathcal{B}_n$ such that $T \subseteq B$. Let $L := L_0 \cap B$. Since $L_0 \in \mathcal{K}_n$ and since $B \in \mathcal{B}_n \subseteq \mathcal{K}_n$, it follows that $L \in \mathcal{K}_n$. We have $K \subseteq R \subseteq S \subseteq T \subseteq L_0 \cap B = L$, so $K \subseteq S \subseteq L$. It remains to show that $v_n^{\mathcal{K}}(L \backslash K) \leq \varepsilon$.

Since $L \subseteq L_0$, we get $v_n^{\mathcal{K}}(L) \leq v_n^{\mathcal{K}}(L_0)$. Since B is bounded and $K \subseteq L \subseteq B$, it follows that K and L are bounded. Then $v_n^{\mathcal{K}}(K) = v_n(K)$ and $v_n^{\mathcal{K}}(L) = v_n(L)$. Then $v_n(K) = v_n^{\mathcal{K}}(K) \geq (v_n(R)) - (\varepsilon/3)$ and $v_n(L) = v_n^{\mathcal{K}}(L) \leq v_n^{\mathcal{K}}(L_0) \leq (v_n(T)) + (\varepsilon/3)$. Then $v_n(R \backslash K) = (v_n(R)) - (v_n(K)) \leq \varepsilon/3$ and $v_n(L \backslash T) = (v_n(L)) - (v_n(T)) \leq \varepsilon/3$. Also, recall that $v_n(T \backslash R) \leq \varepsilon/3$. Then, as $L \backslash K = (L \backslash T) \cup (T \backslash R) \cup (R \backslash K)$, we see that $v_n(L \backslash K) \leq (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3) = \varepsilon$, as desired. QED

Remark (supInfDiff): Let D be a set, let $g : D \to \mathbb{R}$, let $F \subseteq D$ and let $\varepsilon_0 > 0$. Assume, for all $x, y \in F$, that $|(g(x)) - (g(y))| < \varepsilon_0$. Then $\big(\sup_F g\big) - \big(\inf_F g\big) \leq \varepsilon_0$.

Proof: Let $M := \sup_F g$, let $m := \inf_F g$ and let $v := M - m$. We wish to show that $v \leq \varepsilon_0$. Assume, for a contradiction, that $v > \varepsilon_0$.

Let $\delta := v - \varepsilon_0$. Then $\delta > 0$. Choose $y \in F$ such that $g(y) \geq M - (\delta/2)$. Choose $x \in F$ such that $g(x) \leq m + (\delta/2)$. Then

$$v \;=\; M - m \;\leq\; [(g(y)) + (\delta/2)] - [(g(x)) - (\delta/2)] \;=\; (g(y)) - (g(x)) + \delta.$$

We have $v > \varepsilon_0 > 0$, so $v = |v|$. Then $v = |v| \leq |(g(y)) - (g(x))| + \delta < \varepsilon_0 + \delta = v$, and so we get $v < v$, contradiction. QED

Lemma (contInt): Let $n \in \mathbb{N}$, let $D \in \mathcal{C}_n$ be closed and let $g : D \to \mathbb{R}$ be continuous. Then g is integrable.

Proof: We will only prove that $\mathcal{O}_g \in \mathcal{C}_{n+1}$, and will leave it as an (unassigned) exercise to the reader to prove that $\mathcal{O}_{-g} \in \mathcal{C}_{n+1}$. Since $D \in \mathcal{C}_n$, D is bounded. Since D is closed and bounded, D is compact. So, since g is continuous, it follows that $g : D \to \mathbb{R}$ is bounded and uniformly continuous. Choose $K > 0$ such that $g(D) \subseteq [-K, K]$. Then $\mathcal{O}_g \subseteq D \times (0, K]$. Then $\mathcal{O}_g$ is bounded. Given $\varepsilon > 0$. By Lemma (ctdContin), it suffices to prove that there exist $R, T \in \mathcal{C}_{n+1}$ such that $R \subseteq \mathcal{O}_g \subseteq T$ and $v_{n+1}(T \backslash R) \leq \varepsilon$.

Choose $\varepsilon_0 > 0$ such that $[v_n(D)]\varepsilon_0 \leq \varepsilon$. By uniform continuity of g, choose $\delta > 0$ such that, for all $x, y \in D$, we have:

$$\big(\; |x - y| \,<\, \delta \;\big) \qquad \Rightarrow \qquad \big(\; |(g(x)) - (g(y))| \,<\, \varepsilon_0 \;\big).$$

By Fact (chopOnion), choose a finite, pairwise-disjoint $\mathcal{F} \subseteq \mathcal{C}_n \backslash \{\emptyset\}$ such that $\cup \mathcal{F} = D$ and such that $\operatorname{mesh} \mathcal{F} < \delta$.

For all $F \in \mathcal{F}$, let $m_F := \inf_F g$ and $M_F := \sup_F g$; then $m_F \leq M_F$. For all $F \in \mathcal{F}$, for all $x, y \in F$, we have $|x - y| \leq \operatorname{diam} F \leq \operatorname{mesh} \mathcal{F} < \delta$, so $|(g(x)) - (g(y))| < \varepsilon_0$.

By Remark (supInfDiff), for all $F \in \mathcal{F}$, we have $M_F - m_F \leq \varepsilon_0$. For all $F \in \mathcal{F}$, define

$$R_F := \begin{cases} F \times (0, m_F], & \text{if } m_F > 0 \\ \emptyset, & \text{if } m_F \leq 0 \end{cases} \qquad \text{and} \qquad T_F := \begin{cases} F \times (0, M_F], & \text{if } M_F > 0 \\ \emptyset, & \text{if } M_F \leq 0; \end{cases}$$

then $R_F \subseteq \mathcal{O}_g \cap (F \times \mathbb{R}) \subseteq T_F$ and $T_F \backslash R_F \subseteq F \times [m_F, M_F]$.

Let $R := \bigcup_{F \in \mathcal{F}} R_F$ and $T := \bigcup_{F \in \mathcal{F}} T_F$. Since $\cup \mathcal{F} = D$ and since $\mathcal{O}_g \subseteq D \times \mathbb{R}$, we conclude that $\bigcup_{F \in \mathcal{F}} \mathcal{O}_g \cap (F \times \mathbb{R}) = \mathcal{O}_g \cap (D \times \mathbb{R}) = \mathcal{O}_g$. Then

$$R \subseteq \mathcal{O}_g \subseteq T \qquad \text{and} \qquad T \backslash R \subseteq \bigcup_{F \in \mathcal{F}} (F \times [m_F, M_F]).$$

It remains to show that $v_{n+1}(T \backslash R) \leq \varepsilon$.

We have

$$\begin{aligned} v_{n+1}(T \backslash R) \quad & \leq \quad \sum_{F \in \mathcal{F}} (v_{n+1}(F \times [m_F, M_F])) \\ & \leq \quad \sum_{F \in \mathcal{F}} ([v_n(F)][M_F - m_F]) \\ & \leq \quad \sum_{F \in \mathcal{F}} ([v_n(F)] \varepsilon_0) \\ & \leq \quad \left[\sum_{F \in \mathcal{F}} [v_n(F)] \right] \varepsilon_0 \\ & = \quad [v_n(\cup \mathcal{F})] \, \varepsilon_0 \quad = \quad [v_n(D)] \varepsilon_0 \quad \leq \quad \varepsilon, \end{aligned}$$

as desired. QED

Lemma (puffUpBox): Let $n \in \mathbb{N}$, let $B \in \mathcal{B}_n$ be bounded and let $\varepsilon > 0$. Then there exists an open, bounded $C \in \mathcal{B}_n$ such that both $B \subseteq C$ and $v_n^{\mathcal{B}}(C) < (v_n^{\mathcal{B}}(B)) + \varepsilon$.

Proof: If $B = \emptyset$, the result follows with $C := \emptyset$. We therefore assume that $B \neq \emptyset$. Choose $I_1, \ldots, I_n \in \mathcal{I}$ such that $B = I_1 \times \cdots \times I_n$. Since B is bounded, $I_1, \ldots, I_n$ are all bounded. Define a polynomial $f : \mathbb{R} \to \mathbb{R}$ of degree n by $f(t) = [(\ell(I_1)) + t] \cdots [(\ell(I_n)) + t]$. Then $f(0) = [\ell(I_1)] \cdots [\ell(I_n)] = v_n^{\mathcal{B}}(B)$. So, by continuity of f at 0, choose $\eta > 0$ such that, for all $t \in (-\eta, \eta)$, we have $|(f(t)) - (v_n^{\mathcal{B}}(B))| < \varepsilon$. Let $\delta := \eta/2$. Then $\delta \in (-\eta, \eta)$, so $|(f(\delta)) - (v_n^{\mathcal{B}}(B))| < \varepsilon$, so $(f(\delta)) - (v_n^{\mathcal{B}}(B)) < \varepsilon$, so $f(\delta) < (v_n^{\mathcal{B}}(B)) + \varepsilon$.

For all integers $k \in [1, n]$, choose an open bounded $J_k \in \mathcal{I}$ such that $I_k \subseteq J_k$ and such that $\ell(J_k) \leq (\ell(I_k)) + \delta$. Let $C := J_1 \times \cdots \times J_n$. Then C is open, C is bounded, $C \in \mathcal{B}_n$ and $B = I_1 \times \cdots \times I_n \subseteq J_1 \times \cdots \times J_n \subseteq C$. It remains to show that $v_n^{\mathcal{B}}(C) < (v_n^{\mathcal{B}}(B)) + \varepsilon$.

We have $[\ell(J_1)]\cdots[\ell(J_n)] \leq [(\ell(I_1))+\delta]\cdots[(\ell(I_n))+\delta]$. That is, $v_n^{\mathcal{B}}(C) \leq f(\delta)$. Recall that $f(\delta) < (v_n^{\mathcal{B}}(B)) + \varepsilon$. Then $v_n^{\mathcal{B}}(C) \leq f(\delta) < (v_n^{\mathcal{B}}(B)) + \varepsilon$, as desired. QED

Lemma (puffUpNegl): Let $n \in \mathbb{N}$, let $Z \subseteq \mathbb{R}^n$ be negligible and let $\varepsilon > 0$. Then there exists an open, bounded $U \in \mathcal{K}_n$ such that both $Z \subseteq U$ and $v_n(U) < \varepsilon$.

Proof: Since Z is negligible, it follows that Z is bounded. Choose a bounded $B \in \mathcal{B}_n$ such that $Z \subseteq B$. We have $\overline{v}_n(Z) = 0$. By definition of $\overline{v}_n$, choose $K_0 \in \mathcal{K}_n$ such that both $Z \subseteq K_0$ and $v_n^{\mathcal{K}}(K_0) \leq \varepsilon/2$. Let $K := K_0 \cap B$. Then $K \in \mathcal{K}_n$ and $Z \subseteq K$. Also, $v_n^{\mathcal{K}}(K) \leq v_n^{\mathcal{K}}(K_0) \leq \varepsilon/2$. Since B is bounded, K is bounded. Any bounded kidset is contented, so $K \in \mathcal{C}_n$. Moreover, $v_n(K) = v_n^{\mathcal{K}}(K)$. Then $v_n(K) \leq \varepsilon/2$.

By definition of kidset, choose a finite, pairwise-disjoint $\mathcal{F} \subseteq \mathcal{B}_n$ such that $\cup\mathcal{F} = K$. For all $F \in \mathcal{F}$, we have $F \subseteq K$; so, as K is bounded, F is bounded. Let $m := \#\mathcal{F}$ be the number of elements in $\mathcal{F}$. Choose $\varepsilon_0 > 0$ such that $m\varepsilon_0 \leq \varepsilon/2$.

By Lemma (puffUpBox), for all $F \in \mathcal{F}$, choose an open, bounded $C_F \in \mathcal{B}_n$ such that $F \subseteq C_F$ and $v_n^{\mathcal{B}}(C_F) < (v_n^{\mathcal{B}}(F)) + \varepsilon_0$. Let $U := \bigcup_{F\in\mathcal{F}} C_F$. Then U is open and bounded. Moreover, $U \in \mathcal{K}_n$ and $Z \subseteq K = \cup\mathcal{F} \subseteq \bigcup_{F\in\mathcal{F}} C_F = U$. It remains to show that $v_n(U) \leq \varepsilon$.

We have $v_n(\cup F) = v_n(K) \leq \varepsilon/2$. Recall that $m\varepsilon_0 < \varepsilon/2$. Then

$$\begin{aligned} v_n(U) &\leq \sum_{F\in\mathcal{F}} (v_n(C_F)) \\ &< \sum_{F\in\mathcal{F}} [(v_n(F)) + \varepsilon_0] \\ &\leq \left[\sum_{F\in\mathcal{F}} (v_n(F))\right] + [\#\mathcal{F}]\varepsilon_0 \\ &= [v_n(\cup\mathcal{F})] + m\varepsilon_0 \quad\leq\quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad=\quad \varepsilon, \end{aligned}$$

as desired. QED

Definition: Let $n \in \mathbb{N}$, let $D \subseteq \mathbb{R}^n$, let $f : D \to \mathbb{R}$ and let $A \subseteq D$. Then f **is integrable on** A means: $f|A : A \to \mathbb{R}$ is integrable.

Remark (approxInt): Let $n \in \mathbb{N}$, let $D \subseteq \mathbb{R}^n$ be bounded and let $f : D \to \mathbb{R}$ be bounded. Assume, for all $\varepsilon > 0$ that there exists $U \in \mathcal{C}_n$ such that $v_n(U) \leq \varepsilon$ and such that f is integrable on $D\backslash U$. Then f is integrable.

Proof: We will only prove that $\mathcal{O}_f \in \mathcal{C}_{n+1}$, and will leave it as an (unassigned) exercise to the reader to prove that $\mathcal{O}_{-f} \in \mathcal{C}_{n+1}$. Choose $K > 0$ such that $f(D) \subseteq [-K, K]$. Then $\mathcal{O}_f \subseteq D \times (0, K]$. Then $\mathcal{O}_f$ is bounded. Given $\varepsilon > 0$. By Lemma (ctdContin), it suffices to prove that there exist $R, T \in \mathcal{C}_{n+1}$ such that $R \subseteq \mathcal{O}_f \subseteq T$ and $v_{n+1}(T\backslash R) \leq \varepsilon$.

Choose $\varepsilon_0 > 0$ such that $\varepsilon_0 K \leq \varepsilon$. Choose $U \in \mathcal{C}_n$ such that $v_n(U) \leq \varepsilon_0$ and such that f is integrable on $D\backslash U$. Let $R := \mathcal{O}_{f|(D\backslash U)}$. Then $R = \mathcal{O}_f \cap ((D\backslash U) \times \mathbb{R})$. Then $R \subseteq \mathcal{O}_f$. Let $Y := U \times (0, K]$. Then $Y \supseteq \mathcal{O}_f \cap (U \times \mathbb{R})$. Let $T := R \cup Y$. Then $T \supseteq \mathcal{O}_f \cap (D \times \mathbb{R})$.

So, since $\mathcal{O}_f \subseteq D \times \mathbb{R}$, we see that $T \supseteq \mathcal{O}_f$. Then $R \subseteq \mathcal{O}_f \subseteq T$, and it remains to show that $v_{n+1}(T\backslash R) \leq \varepsilon$.

Since $T\backslash R = (R \cup Y)\backslash R \subseteq Y$, we get $v_{n+1}(T\backslash R) \leq v_{n+1}(Y)$. Since $Y = U \times (0, K]$, it follows that $v_{n+1}(Y) = [v_n(U)][K - 0]$. So, as $v_n(U) \leq \varepsilon_0$, we see that $v_{n+1}(Y) \leq \varepsilon_0 K$. Then $v_{n+1}(T\backslash R) \leq v_{n+1}(Y) \leq \varepsilon_0 K \leq \varepsilon$, as desired. QED

Definition: Let $n \in \mathbb{N}$, let $D \subseteq \mathbb{R}^n$ and let $f : D \to \mathbb{R}$. By the **discontinity set** of f, we mean $\mathrm{DSC}\, f := \{x \in D \,|\, f \text{ is discontinuous at } x\}$.

Theorem (Rnint): Let $n \in \mathbb{N}$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded. Assume both that $f^{-1}(\mathbb{R}\backslash\{0\})$ is bounded and that $\mathrm{DSC}\, f$ is negligible. Then f is integrable.

Proof: We wish to prove that $\mathcal{O}_f, \mathcal{O}_{-f} \in \mathcal{C}_{n+1}$. Because $f^{-1}(\mathbb{R}\backslash\{0\})$ is bounded, choose a compact $D_0 \in \mathcal{B}_n$ such that $f^{-1}(\mathbb{R}\backslash\{0\}) \subseteq D_0$. Let $f_0 := f|D_0 : D_0 \to \mathbb{R}$. Then we have both $\mathcal{O}_f = \mathcal{O}_{f_0}$ and $\mathcal{O}_{-f} = \mathcal{O}_{-f_0}$, and so it suffices to prove that $\mathcal{O}_{f_0}, \mathcal{O}_{-f_0} \in \mathcal{C}_{n+1}$, *i.e.*, that $f_0 : D_0 \to \mathbb{R}$ is integrable.

Since D_0 is compact, D_0 is bounded. Since f is bounded, f_0 is bounded. Given $\varepsilon > 0$. By Remark (approxInt), it suffices to show that there exists $U \in \mathcal{C}_n$ such that $v_n(U) \leq \varepsilon$ and such that f_0 is integrable on $D_0\backslash U$.

Let $Z := \mathrm{DSC}\, f$. Then Z is negligible. By Lemma (puffUpNegl), choose an open, bounded $U \in \mathcal{K}_n$ such that both $Z \subseteq U$ and $v_n(U) < \varepsilon$. As U is a bounded kidset, we see that U is contented, *i.e.*, that $U \in \mathcal{C}_n$. It remains to show that f_0 is integrable on $D_0\backslash U$. Let $D := D_0\backslash U$ and let $g := f_0|D : D \to \mathbb{R}$. We wish to show that $g : D \to \mathbb{R}$ is integrable.

Because D_0 is compact, D_0 is closed and bounded. So, since U is open, we see that $D_0\backslash U$ is closed. That is, D is closed. We have $D_0 \in \mathcal{B}_n \subseteq \mathcal{K}_n$, so D_0 is a bounded kidset, so $D_0 \in \mathcal{C}_n$. So, since $U \in \mathcal{C}_n$, we see that $D_0\backslash U \in \mathcal{C}_n$. That is, $D \in \mathcal{C}_n$. Then, by Lemma (contInt), it suffices to show that $g : D \to \mathbb{R}$ is continuous.

We have $D = D_0\backslash U \subseteq D$. Also, $f_0 = f|D_0$ and $g = f_0|D$. Then $g = f|D$. We have $D_0 \subseteq \mathbb{R}^n$ and $\mathrm{DSC}\, f = Z \subseteq U$, so $D_0\backslash U \subseteq \mathbb{R}^n\backslash(\mathrm{DSC}\, f)$. That is, $D \subseteq \mathbb{R}^n\backslash(\mathrm{DSC}\, f)$. Then $f|D : D \to \mathbb{R}$ is continuous. That is, $g : D \to \mathbb{R}$ is continuous, as desired. QED

Remark (Rnint): Let $n \in \mathbb{N}$, let $D \subseteq \mathbb{R}^n$ and let $g : D \to \mathbb{R}$. Define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) \qquad = \qquad \begin{cases} g(x), & \text{if } x \in D \\ 0, & \text{if } x \notin D. \end{cases}$$

Then

(i) $\mathrm{DSC}\, f \quad \subseteq \quad (\mathrm{DSC}\, g) \;\cup\; (\partial D)$ and
(ii) $\mathcal{O}_f \;=\; \mathcal{O}_g$ and $\mathcal{O}_{-f} \;=\; \mathcal{O}_{-g}$ and
(iii) $f^{-1}(\mathbb{R}\backslash\{0\}) \quad \subseteq \quad g^{-1}(\mathbb{R}\backslash\{0\})$.

Proof: We leave (ii) and (iii) as exercises and prove only (i).

Let $x \in \mathrm{DSC}\, f$. Assume that $x \notin \mathrm{DSC}\, g$. We wish to show that $x \in \partial D$.

Recall that $\partial D = \overline{D}\backslash D^\circ$. Because $\mathbb{R}^n\backslash\overline{D}$ is an open set and because $f = 0$ on $\mathbb{R}^n\backslash\overline{D}$, it follows that f is continuous on $\mathbb{R}^n\backslash\overline{D}$. Then $x \notin \mathbb{R}^n\backslash\overline{D}$, *i.e.*, $x \in \overline{D}$. It remains to prove that $x \notin D^\circ$. Assume, for a contradiction, that $x \in D^\circ$.

As $x \notin \mathrm{DSC}\, g$, we know that g is continuous at x. However $f = g$ on D and $D^\circ \subseteq D$, so $f = g$ on D°. So, since D° is open and since g is continuous at x, it follows that f is continuous at x. However, $x \in \mathrm{DSC}\, f$, *i.e.*, f is discontinuous at x, contradiction. QED

Theorem (intCriterion): Let $n \in \mathbb{N}$, let $D \subseteq \mathbb{R}^n$ and let $g : D \to \mathbb{R}$ be bounded. Assume

(a) ∂D is negligible,
(b) $g^{-1}(\mathbb{R}\backslash\{0\})$ is bounded and
(c) $\mathrm{DSC}\, g$ is negligible.

Then g is integrable.

Proof: Define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) \qquad = \qquad \begin{cases} g(x), & \text{if } x \in D \\ 0, & \text{if } x \notin D. \end{cases}$$

Since $g : D \to \mathbb{R}$ is bounded, $f : \mathbb{R}^n \to \mathbb{R}$ is bounded. Since $g^{-1}(\mathbb{R}\backslash\{0\})$ is bounded, by (iii) of Remark (Rnint), we see that $f^{-1}(\mathbb{R}\backslash\{0\})$ is bounded. Since $\mathrm{DSC}\, g$ is negligible and since ∂D is negligible, by (i) of Remark (Rnint), we see that $\mathrm{DSC}\, f$ is negligible. Then, by Theorem (Rnint), we conclude that f is integrable, *i.e.*, that $\mathcal{O}_f, \mathcal{O}_{-f} \in \mathcal{C}_{n+1}$. Then, by (ii) of Remark (Rnint), we see that $\mathcal{O}_g, \mathcal{O}_{-g} \in \mathcal{C}_{n+1}$, *i.e.*, that g is integrable. QED

Since $\partial\mathbb{R}^n = \emptyset$ is negligible, Theorem (intCriterion) implies Theorem (Rnint). Also, recall that any contented set has negligible boundary. So, we get the following:

Corollary: Let $n \in \mathbb{N}$, let $D \subseteq \mathbb{R}^n$ be contented and let $g : D \to \mathbb{R}$ be bounded. Assume both that $g^{-1}(\mathbb{R}\backslash\{0\})$ is bounded and that $\mathrm{DSC}\, g$ is negligible. Then g is integrable.