MATH 4604 Spring 2018, Final exam
Handout date: Saturday 12 May 2018
Instructor: Scot Adams

PRINT YOUR NAME:

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Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let X be a metric space and let $s \in X^{\mathbb{N}}$. Then $s_\bullet$ is **Cauchy** means ...

B. (5 pts) Let X and Y be metric spaces, let $f : X \to Y$ and let $K \geqslant 0$. Then f is K-**Lipschitz** means ...

C. (5 pts) Let V and W be finite dimensional vector spaces and let $p \geqslant 0$. Let $\| \bullet \| \in \mathcal{N}(V)$. Then $\breve{\mathcal{O}}_p(V, W, \| \bullet \|) := \cdots$

D. (5 pts) Let V and W be finite dimensional vector spaces. and let $f : V \dashrightarrow W$. Let $p \in V$. Then $\mathrm{LINS}_p^{V,W} f = \cdots$

E. (5 pts) Let V and W be finite dimensional vector spaces and let $f : V \dashrightarrow W$. Then $Df : V \dashrightarrow L(V, W)$ is defined by ...

F. (5 pts) Let $\mathcal{S}$ be a set of sets. Then $\mathcal{S}$ is a **ring of sets** means: ...

II. True or false (no partial credit):

a. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Assume: $\forall x \in \mathbb{R}$, $f'(x) > 0$. Then f is strictly increasing.

b. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $f'(0) > 0$. Then there exists $\delta > 0$ such that f is increasing on $(0, \delta)$.

c. (5 pts) Let $f : [1, 2] \hookrightarrow \mathbb{R}$. Assume that f is continuous. Then f^{-1} is continuous.

d. (5 pts) $\forall f, g \in \widehat{\mathcal{O}}_3(\mathbb{R}, \mathbb{R}), \quad fg \in \widehat{\mathcal{O}}_9(\mathbb{R}, \mathbb{R})$.

e. (5 pts) Let $\mathcal{I}$ denote the set of all intervals. Then $\mathcal{I}$ is a ring of sets.

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PLEASE DO NOT WRITE BELOW THE LINE

I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Find a function $f \in \widehat{\mathcal{O}}_2(\mathbb{R}, \mathbb{R})$ such that $0 \notin \mathrm{dom}[f'']$.

2. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and assume that $\mathrm{dom}[f''] = \mathbb{R}$, *i.e.*, that f is twice differentiable on $\mathbb{R}$. Assume that $f(0) = f'(0) = f''(0) = 0$. Show that $f \in \breve{\mathcal{O}}_2(\mathbb{R}, \mathbb{R})$.

3. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Assume that $\text{dom}[f'] = \mathbb{R}$. Assume that, for all $x \in \mathbb{R}$, we have $|f'(x)| \leqslant 5$. Show that f is 5-Lipschitz.

4. (15 pts.) Let $* \in SB(\mathbb{R}^2, \mathbb{R})$ and let $f := \Delta_* : \mathbb{R}^2 \to \mathbb{R}$. That is, let $* : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be symmetric and bilinear, and define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(u) = u * u$. Show: there exists $K \geqslant 0$ such that, for all $x, y \in \mathbb{R}$, we have $|f(x,y)| \leqslant K \cdot [x^2 + y^2]$.

Hint: You may use, without proof: For all $x, y \in \mathbb{R}$,

$$\begin{aligned} x^2 &\leqslant x^2 + y^2, \\ 2 \cdot |x| \cdot |y| &\leqslant x^2 + y^2 \qquad \text{and} \\ y^2 &\leqslant x^2 + y^2. \end{aligned}$$