MATH 4604 Spring 2018, Final exam
Handout date: Saturday 12 May 2018
Instructor: Scot Adams

PRINT YOUR NAME:

SOLUTIONS

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Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let X be a metric space and let $s \in X^{\mathbb{N}}$. Then $s_\bullet$ is **Cauchy** means ...

$\forall \varepsilon > 0, \quad \exists M \in \mathbb{N}$ s.t., $\forall j, k \in \mathbb{N},$

$(j, k \geqslant M) \implies (|s_j - s_k| < \varepsilon)$

B. (5 pts) Let X and Y be metric spaces, let $f : X \to Y$ and let $K \geqslant 0$. Then f is K-**Lipschitz** means ...

$\forall p, q \in X, \quad d_Y(f(p), f(q)) \leq K \cdot [d_X(p,q)]$

C. (5 pts) Let V and W be finite dimensional vector spaces and let $p \geqslant 0$. Let $\| \bullet \| \in \mathcal{N}(V)$. Then $\breve{o}_p(V, W, \| \bullet \|) := \cdots$

$[\|\bullet\|^p] \cdot [\breve{o}(V, W)]$

D. (5 pts) Let V and W be finite dimensional vector spaces. and let $f : V \dashrightarrow W$. Let $p \in V$. Then $\mathrm{LINS}_p^{V,W} f = \cdots$

$$\{ L \in \mathcal{L}(V,W) \mid f_p^T - L \in \mathcal{O}_1^V(V,W) \}$$

E. (5 pts) Let V and W be finite dimensional vector spaces and let $f : V \dashrightarrow W$. Then $Df : V \dashrightarrow \mathcal{L}(V,W)$ is defined by ...

$$(Df)(p) = D_p f$$

F. (5 pts) Let $\mathcal{S}$ be a set of sets. Then $\mathcal{S}$ is a **ring of sets** means: ...

$$(\langle \mathcal{S} \rangle_{\mathrm{fin}\cup} \subseteq \mathcal{S})$$

$$\& \ (\langle \mathcal{S} \rangle_{\mathrm{fin}\cap} \subseteq \mathcal{S})$$

$$\& \ (\forall A, B \in \mathcal{S}, \quad A \backslash B \in \mathcal{S})$$

II. True or false (no partial credit):

a. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Assume: $\forall x \in \mathbb{R}$, $f'(x) > 0$. Then f is strictly increasing.

True (3) of Cor 34.10 p. 260

b. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $f'(0) > 0$. Then there exists $\delta > 0$ such that f is increasing on $(0, \delta)$.

False Def $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x + [x^2][\sin(\frac{1}{x^3})]$

Let $f := \mathrm{adj}_0^0(g)$

c. (5 pts) Let $f : [1,2] \hookrightarrow \mathbb{R}$. Assume that f is continuous. Then f^{-1} is continuous.

True Th 36.5 p. 274

d. (5 pts) $\forall f, g \in \hat{O}_3(\mathbb{R}, \mathbb{R}), \quad fg \in \hat{O}_9(\mathbb{R}, \mathbb{R})$.

False Let $f := |\bullet|^3$ and $g := |\bullet|^3$

e. (5 pts) Let $\mathcal{I}$ denote the set of all intervals. Then $\mathcal{I}$ is a ring of sets.

False Let $I := [1,2]$ and $J := [3,4]$

Then $I, J \in \mathcal{I}$ but $I \cup J \notin \mathcal{I}$

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PLEASE DO NOT WRITE BELOW THE LINE

I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Find a function $f \in \hat{\mathcal{O}}_2(\mathbb{R}, \mathbb{R})$ such that $0 \notin \mathrm{dom}[f'']$.

Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x^2 & \text{if} \quad x < 0 \end{cases}$$

2. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and assume that $\mathrm{dom}[f''] = \mathbb{R}$, *i.e.*, that f is twice differentiable on $\mathbb{R}$. Assume that $f(0) = f'(0) = f''(0) = 0$. Show that $f \in \breve{o}_2(\mathbb{R}, \mathbb{R})$.

Let $g := |\bullet|^2$. Want: $f \in g \cdot [\breve{o}(\mathbb{R}, \mathbb{R})]$

Let $\alpha := \mathrm{adj}_0^0(f/g)$.

$f = g\alpha$ on $\mathbb{R}_0^\times$ & at 0 $\therefore$ on $\mathbb{R}$, so: $f = g\alpha$

Want: $\alpha \in \breve{o}(\mathbb{R}, \mathbb{R})$.

$[\mathrm{dom}[\alpha] = \mathbb{R} \in \mathcal{N}_{\mathbb{R}}(0)]$ & $[\alpha(0) = 0]$

Want: α contin at 0. Want: $\alpha \to 0$ near 0.

$\alpha = f/g$ on $\mathbb{R}_0^\times$. Want: $f/g \to 0$ near 0. Want: $\lim\limits_0 \frac{f}{g} = 0$.

$\forall x \in \mathbb{R}$, $g(x) = |x|^2 = x^2$

$\therefore \forall x \in \mathbb{R}$, $g'(x) = 2x$ and $g''(x) = 2$

By l'H's Rule, $\lim\limits_0 \frac{f}{g} \overset{*}{=} \lim\limits_0 \frac{f'}{g'}$

By Easy l'H's Rule, $\lim\limits_0 \frac{f'}{g'} \overset{**}{=} \frac{f''(0)}{g''(0)}$

Then $\lim\limits_0 \frac{f}{g} \overset{*}{=} \frac{f''(0)}{g''(0)} = \frac{0}{2} \neq \text{ⓐ}$

Then $\lim\limits_0 \frac{f}{g} = \frac{0}{2} = 0$. ∎

3. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Assume that $\mathrm{dom}[f'] = \mathbb{R}$. Assume that, for all $x \in \mathbb{R}$, we have $|f'(x)| \leqslant 5$. Show that f is 5-Lipschitz.

Want: $\forall p, q \in \mathbb{R}, \quad |[f(q)] - [f(p)]| \leq 5 \cdot |q - p|$

Given $p, q \in \mathbb{R}$. Want: $|[f(q)] - [f(p)]| \leq 5 \cdot |q - p|$

$p = q$ ① or $p \neq q$ ②

Case 1: $f(p) = f(q)$

$|[f(q)] - [f(p)]| = |0| \leq 5 \cdot |0| = 5 \cdot |q - p|$

Case 2: f is c/d on $[p|q]$

By MVT, choose $x \in (p|q)$ s.t. $f'(x) = (DQ_f)(p, q)$

$$\frac{|[f(q)] - [f(p)]|}{|q - p|} = |(DQ_f)(p, q)| = |f'(x)| \leq 5$$

$|[f(q)] - [f(p)]| \leq 5 \cdot |q - p|$

∎

4. (15 pts.) Let $* \in SB(\mathbb{R}^2, \mathbb{R})$ and let $f := \Delta_* : \mathbb{R}^2 \to \mathbb{R}$. That is, let $* : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be symmetric and bilinear, and define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(u) = u * u$. Show: there exists $K \geqslant 0$ such that, for all $x, y \in \mathbb{R}$, we have $|f(x,y)| \leqslant K \cdot [x^2 + y^2]$.

Hint: You may use, without proof: For all $x, y \in \mathbb{R}$,

$$\begin{aligned} x^2 &\leqslant x^2 + y^2, \\ 2 \cdot |x| \cdot |y| &\leqslant x^2 + y^2 \qquad \text{and} \\ y^2 &\leqslant x^2 + y^2. \end{aligned}$$

Let $e_1 := (1,0)$, $e_2 := (0,1)$

Let $A := e_1 * e_1$, $B := e_1 * e_2$, $C := e_2 * e_2$

Let $K := |A| + |B| + |C|$.

Want: $\forall x, y \in \mathbb{R}$, $|f(x,y)| \leqslant K \cdot [x^2 + y^2]$

Given $x, y \in \mathbb{R}$. Want: $|f(x,y)| \leqslant K \cdot [x^2 + y^2]$

$$\begin{aligned} f(x,y) &= f(xe_1 + ye_2) = (xe_1 + ye_2) * (xe_1 + ye_2) \\ &= Ax^2 + 2Bxy + Cy^2 \end{aligned}$$

$$\begin{aligned} |f(x,y)| &\leqslant |A| \cdot x^2 + |B| \cdot 2 \cdot |x| \cdot |y| + |C| \cdot y^2 \\ &\leqslant |A| \cdot [x^2 + y^2] + |B| \cdot [x^2 + y^2] + |C| \cdot [x^2 + y^2] \\ &= K \cdot [x^2 + y^2] \end{aligned}$$

∎