MATH 4604 Spring 2018, Midterm #1
Handout date: Thursday 22 February 2018
Instructor: Scot Adams

PRINT YOUR NAME:

SOLUTIONS

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Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Then f is **semiincreasing** means ...

$\forall s, t \in \mathrm{dom}[f]$,

$$(s \leqslant t) \Rightarrow (f(s) \leqslant f(t))$$

B. (5 pts) Let W be a vector space. Let $f : \mathbb{R} \dashrightarrow W$. Let $p \in \mathrm{dom}[f]$. Then $SS_f^p : \mathbb{R} \dashrightarrow \mathbb{R}$ is defined by $(SS_f^p)(h) = \cdots$

$$\frac{[f(p+h)] - [f(p)]}{h}$$

C. (5 pts) Let X be a topological space, $S \subseteq X$. Then $\mathrm{LP}_X S = \cdots$.

$$[\mathrm{Cl}_X S] \setminus [\mathrm{Isol}_X S]$$

D. (5 pts) Let X be a topological space. Let $f : X \dashrightarrow \mathbb{R}$ and let $p \in X$. Then f has a **local maximum** at p in X means: ...

$\exists V \in \mathcal{N}_X(p)$ s.t.

$\big(V \subseteq \mathrm{dom}[f] \big)$ & $\big(f_*(V) \leq f(p) \big)$

E. (5 pts) Let X be a topological space. Then X is **sequentially compact** means: ...

$\forall s \in X^{\mathbb{N}}$, s is subconvergent in X

F. (5 pts) Let S be a set. Let $f : S \to \mathbb{R}$ and let $p \in S$. Then f is **has an extremum at** p means ...

[(f has a maximum at p)

OR (f has a minimum at p)]

II. True or false (no partial credit):

a. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and assume that $f'(1) = 5$. Then there exists a neighborhood U of 1 in $\mathbb{R}$ such that f is strictly increasing on U.

False   Define $g : \mathbb{R} \dashrightarrow \mathbb{R}$ by $g(x) = x + [x^2][\sin(x^{-3})]$
Let $f := \mathrm{adj}_0^0(g)$

b. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Assume f has a maximum at 0. Then f has a local maximum at 0.

True   Rmk 33.10, p. 252

c. (5 pts) Let X and Y be topological spaces, and let $f : X \to Y$. Let $w \in X$ and let $z \in Y$. Assume: $f \to z$ near w. Then $\lim_w f = z$.

False $X := [1,2] \cup \{3\}$, $Y := \mathbb{R}$, $f := C_X^4$, $w := 3$, $z := 1000$

d. (5 pts) Let $f : \mathbb{R} \hookrightarrow\!\!> \mathbb{R}$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then $f^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous.

True   Th 30.6, p. 228

e. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, and assume that f is strictly increasing. Then $\mathrm{im}[DQ_f] > 0$.

True $\forall p, q \in \mathbb{R}$, $(DQ_f)(p, q)$ is: $[0]$ or $\left[\frac{pos}{pos}\right]$ or $\left[\frac{neg}{neg}\right]$

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PLEASE DO NOT WRITE BELOW THE LINE

I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Give an example of two functions $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ such that:
[$f'(0) = 3$] and [$g'(0) = 4$] and [$(f + g)'(0) = ☹$].

Define $f : (-\infty, 0] \longrightarrow \mathbb{R}$

by $f(x) = 3x$

and $g : [0, \infty) \longrightarrow \mathbb{R}$

by $g(x) = 4x.$

2. (10 pts) Let $\| \bullet \| \in \mathcal{N}(\mathbb{R}^2)$ be defined by $\|(x,y)\| = \sqrt{x^2+y^2}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 7x - 8y$. Let $q := (3,2) \in \mathbb{R}^2$. Show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall p \in \mathbb{R}^2$,

$$[\, \|p - q\| < \delta \,] \quad \Rightarrow \quad [\, |[f(p)] - [f(q)]| < \varepsilon \,].$$

DON'T use theorems about continuity and limits.
Instead, follow templates: Give a method for finding δ, given ε, then show that it works.

Pf:

Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall p \in \mathbb{R}^2$, $\left([\|p-q\| < \delta] \Rightarrow [|[f(p)] - [f(q)]| < \varepsilon] \right)$

$\delta := \frac{\varepsilon}{15}$. Want: $\forall p \in \mathbb{R}^2$, $\left([\|p-q\| < \delta] \Rightarrow [|[f(p)] - [f(q)]| < \varepsilon] \right)$

Given $p \in \mathbb{R}^2$. Want: $[\|p-q\| < \delta] \Rightarrow [|[f(p)] - [f(q)]| < \varepsilon]$

Assume $\|p-q\| < \delta$. Want: $|[f(p)] - [f(q)]| < \varepsilon$

Choose $x, y \in \mathbb{R}$ s.t. $p = (x,y)$

$\|p-q\| = \|(x,y) - (3,2)\| = \|(x-3, y-2)\| = \sqrt{(x-3)^2 + (y-2)^2}$

$\therefore \left(\|p-q\| \geq \sqrt{(x-3)^2} = |x-3| \right) \,\&\, \left(\|p-q\| \geq \sqrt{(y-2)^2} = |y-2| \right)$

$|[f(p)] - [f(q)]| = |[f(x,y)] - [f(3,2)]| = |[7x - 8y] - [7 \cdot 3 - 8 \cdot 2]|$

$= |7 \cdot [x-3] - 8 \cdot [y-2]| \leq |7| \cdot |x-3| + |-8| \cdot |y-2|$

$\leq 7 \cdot \|p-q\| + 8 \cdot \|p-q\| = 15 \cdot \|p-q\| < 15\,\delta = \varepsilon$ ∎

3. (10 pts) Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(x) = (2x + 4, 5x - 1)$. Let $p \in \mathbb{R}$. Prove that $f'(p) = (2, 5)$.

You may use without proof: For any topological spaces X and Y, for any $w \in \mathrm{LP}_X X$, for any $z \in Y$, $\lim\limits_w C_X^z = z$.

You may also use without proof: For any topological spaces X and Y, for any $f, g : X \dashrightarrow Y$, for any $w \in X$,

$$[\ \exists U \in \mathcal{N}_X^\times(w) \text{ s.t. } (\ f = g \text{ on } U\ )\ ] \Rightarrow [\ \lim_w f = \lim_w g\ ].$$

Pf: $z := (2,5)$ Want: $f'(p) = z$

$\forall h \in \mathbb{R}_0^\times$, $(SS_f^p)(h) = \frac{1}{h}([f(p+h)] - [f(p)])$

$= \frac{1}{h}((2 \cdot (p+h) + 4, 5 \cdot (p+h) - 1) - (2p+4, 5p-1))$

$= \frac{1}{h}(2p + 2h + 4 - 2p - 4, 5p + 5h - 1 - 5p + 1)$

$= \frac{1}{h}(2h, 5h) = (2,5) = z = C_{\mathbb{R}}^z(h)$

$\therefore SS_f^p = C_{\mathbb{R}}^z$ on $\mathbb{R}_0^\times$ $\therefore \lim\limits_0 SS_f^p = \lim\limits_0 C_{\mathbb{R}}^z$

$0 \in \mathbb{R} = LP_{\mathbb{R}}\mathbb{R}$ $\therefore \lim\limits_0 C_{\mathbb{R}}^z = z$

$f'(p) = \lim\limits_0 SS_f^p = \lim\limits_0 C_{\mathbb{R}}^z = z$ ∎

4. (15 pts.) Let $f : \mathbb{R} \to \mathbb{R}$. Assume $f' = f$. Let $P : \mathbb{R} \to \mathbb{R}$ be defined by $P(x) = x^4$. Let $g := P \circ f$. Let $a \in \mathbb{R}$. Show: $g'(a) = 4 \cdot [g(a)]$.

If you invoke the Chain Rule, be sure to verify its limit point hypothesis and don't forget that its conclusion is a *contingent* equality. You may use, without proof, that: $\forall x \in \mathbb{R}, \quad P'(x) = 4x^3$.

Pf:

$$\mathrm{LPD}_{\mathbb{R}}(P \circ f) = \mathrm{LP}_{\mathbb{R}}(\mathrm{dom}\,[P \circ f]) = \mathrm{LP}_{\mathbb{R}}\,\mathbb{R} = \mathbb{R}$$

$$a \in \mathbb{R} = \mathrm{LPD}_{\mathbb{R}}(P \circ f) \therefore (P \circ f)'(a) \overset{*}{=} [P'(f(a))] \cdot [f'(a)]$$

$$P'(f(a)) = 4 \cdot [f(a)]^3$$

$$f' = f \therefore \quad f'(a) = f(a)$$

$$g(a) = (P \circ f)(a) = P(f(a)) = [f(a)]^4$$

$$g'(a) = (P \circ f)'(a) \overset{*}{=} [P'(f(a))] \cdot [f'(a)]$$
$$= 4 \cdot [f(a)]^3 \cdot [f(a)] = 4 \cdot [f(a)]^4 = 4 \cdot [g(a)]$$

$$a \in \mathbb{R} = \mathrm{dom}\,[P \circ f] = \mathrm{dom}\,[g]$$

$$g(a) \in \mathrm{im}\,[g] = \mathrm{im}\,[P \circ f] \subseteq \mathbb{R}$$

$$4 \cdot [g(a)] \in \mathbb{R} \quad \therefore \quad 4 \cdot [g(a)] \neq \odot$$

$$g'(a) = 4 \cdot [g(a)] \neq \odot \quad \therefore \quad g'(a) = 4 \cdot [g(a)] \quad \blacksquare$$