MATH 4604 Spring 2018, Midterm #2
Handout date: Thursday 22 March 2018
Instructor: Scot Adams

PRINT YOUR NAME:

SOLUTIONS

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Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{R}$. Then $f$ is **c/d** on $S$ means: ...

($f$ is continuous on $S$)

and ($f$ is differentiable on $\mathrm{Int}_{\mathbb{R}} S$).

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B. (5 pts) Let $X$ and $Y$ be topological spaces, and let $f : X \dashrightarrow Y$. Then $f$ is **closed** means: ...

$$f_{**}(\mathcal{T}_X^{c}) \subseteq \mathcal{T}_Y^{c}$$

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C. (5 pts) Let $X$ be a topological space, let $p \in X$, and let $f : X \dashrightarrow \mathbb{R}$. Then $f$ has a **local maximum** at $p$ in $X$ means: ...

$\exists V \in \mathcal{N}_X(p)$ s.t.

$(V \subseteq \mathrm{dom}[f])$ & $(f_*(V) \leq f(p))$

D. (5 pts) Let $V$ be a vector space and let $\| \bullet \| : V \to [0, \infty)$. Then $\| \bullet \|$ is a **norm on** $V$ means: ...

$$\left( \forall x \in V, \quad \left[ \left( \|x\| = 0 \right) \Longleftrightarrow \left( x = 0_V \right) \right] \right)$$
$$\& \left( \forall a \in \mathbb{R}, \; \forall x \in V \qquad \left( \|ax\| = |a| \cdot \|x\| \right) \right)$$
$$\& \left( \forall x, y \in V \qquad \left( \|x+y\| \leq \|x\| + \|y\| \right) \right)$$

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E. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Then $f$ is **semimonotone** means: ...

($f$ is semiincreasing)
or ($f$ is semidecreasing)

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F. (5 pts) Let $X$ and $Y$ be topological spaces, and let $f : X \dashrightarrow Y$. Then $\mathrm{dct}[f] := \cdots$.

$$\{ p \in X \mid f \text{ is continuous at } p \}$$

II. True or false (no partial credit):

a. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ be 1-1 and continuous. Then $f^{-1}$ is continuous.

$\exists f : [1,2) \cup [3,4) \hookrightarrow\!\!\!\!\rightarrow [5,7)$

False

s.t. ($f$ contin) & ($f^{-1}$ not contin. at 6)

b. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be 1-1 and continuous. Then $f$ is strictly monotone.

True

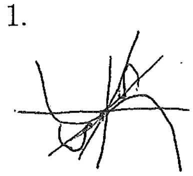
Th 30.1, p. 227

c. (5 pts) Let $f : [1,2] \to [3,4]$ be continuous. Then $f$ is closed.

True

Th 36.7, p. 275

d. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Assume that $f'(0) = 1$. Then there exists $\delta > 0$ such that $f$ is strictly increasing on $(-\delta, \delta)$.



False

Lem 34.2, p. 242

e. (5 pts) Let $f, g : \mathbb{R} \to \mathbb{R}$ both be differentiable. Assume that $f' = g'$. Assume that $f(0) = g(0)$. Then $f = g$.

True

By Cor 34.11, p. 263 $f-g$ const

$\therefore f-g = C^{0}_{\mathbb{R}}$ $\therefore f = g$

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PLEASE DO NOT WRITE BELOW THE LINE

I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 2$, $f'(0) = 3$, $f''(0) = 4$ and $f'''(0) = 6$.

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = 2 + 3x + 2x^2 + x^3.$$

2. (10 pts) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^5 + x^3 + x$. Let $g := f^{-1}$. Note that $g(3) = 1$. Compute $g'(3)$.

$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(1)}$$

$$= \frac{1}{5 \cdot 1^4 + 3 \cdot 1^2 + 1} = \frac{1}{9}.$$

3. (10 pts) Let $S := (0, \infty)$. Recall: $\exp : \mathbb{R} \hookrightarrow\!\!\!\to S$ is defined by

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

Let $\ln := \exp^{-1} : S \to \mathbb{R}$. Let $q \in S$. Show: $\ln'(q) = 1/q$.

*Hint:* Recall that $\exp' = \exp$.

Pf: By TIFT for intervals,
$\ln$ is continuous
&, $\therefore$, is continuous at $q$.

So, by DIFT, $\ln'(q) = \dfrac{1}{\exp'(\ln q)}$

$\exp' = \exp$

$\exp(\ln q) = \exp(\exp^{-1}(q)) = q$

Then $\ln'(q) = \dfrac{1}{\exp'(\ln q)}$

$= \dfrac{1}{\exp(\ln q)}$

$= \dfrac{1}{q}$ ∎

4. (15 pts.) Let $f : \mathbb{R} \to \mathbb{R}$. Assume: $\forall t \in \mathbb{R}, \quad f''(t) = -\dfrac{f(t)}{1 + [f(t)]^2}$.

Define $P, K : \mathbb{R} \to \mathbb{R}$ by $P(t) = \ln(1 + [f(t)]^2)$ and $K(t) = [f'(t)]^2$. Show that $P + K$ is constant.

Pf: Want: $\forall t \in \mathbb{R}, \quad (P+K)'(t) = 0$

Given $t \in \mathbb{R}$. Want: $(P+K)'(t) = 0$

Define $s, u : \mathbb{R} \to \mathbb{R}$ by $s(x) = x^2$, $u(x) = 1 + x^2$

$P = \ln \circ u \circ f$ and $K = s \circ (f')$

$$(P+K)'(t) \overset{*}{=} [P'(t)] + [K'(t)] \overset{*}{=}$$

$$[\ln'(u(f(t)))] \cdot [u'(f(t))] \cdot [f'(t)] + [s'(f'(t))] \cdot [f''(t)] =$$

$$\left[\frac{1}{1 + [f(t)]^2}\right] \cdot [2 \cdot (f(t))] \cdot [f'(t)] + [2 \cdot (f'(t))] \cdot \left[-\frac{f(t)}{1 + [f(t)]^2}\right] =$$

$$\frac{2 \cdot [f(t)] \cdot [f'(t)]}{1 + [f(t)]^2} - \frac{2 \cdot [f'(t)] \cdot [f(t)]}{1 + [f(t)]^2} = 0 \neq \textcircled{n}$$

$\therefore \quad (P+K)'(t) = 0 \quad \blacksquare$