Homework for MATH 4604 (Advanced Calculus II)
Spring 2018

Homework 14: Due on Tuesday 1 May

14-1. Let W be a normed vector space, let $f : \mathbb{R} \dashrightarrow W$ and let $p \in \mathbb{R}$. Show: $(f_p^T)'(0) = f'(p)$.

14-2. Let V, W be finite dimensional vector spaces, $\alpha, \beta : V \dashrightarrow W$, $p \geqslant 0$. Assume: $\alpha = \beta$ near 0_V and $\alpha \in \breve{\mathcal{o}}_p(V, W)$. Show: $\beta \in \breve{\mathcal{o}}_p(V, W)$.

14-3. Let $\delta > 0$, let $I := (-\delta, \delta)$ and let $\alpha : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume: $\forall x \in I$, $\alpha(x) \in [0|x]$. Show that $\alpha \in \widehat{\mathcal{O}}_1(\mathbb{R}, \mathbb{R})$.

14-4. Let $k \in \mathbb{N}_0$ and let $f \in \breve{\mathcal{o}}_k(\mathbb{R}, \mathbb{R})$. Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume that $g' = f$ near 0. Assume that $g(0) = 0$. Show that $g \in \breve{\mathcal{o}}_{k+1}(\mathbb{R}, \mathbb{R})$.

14-5. Let V, W be finite dimensional vector spaces. Let $f, g : V \dashrightarrow W$. Let $p \in V$. Assume: $f = g$ near p. Show: $\mathrm{LINS}_p f \subseteq \mathrm{LINS}_p g$.

Homework 13: Due on Tuesday 24 April

13-1. Let V, W be finite dimensional VSs. Show: $L(V, W) \subseteq \widehat{\mathcal{O}}_1(V, W)$.

13-2. Let $m, n \in \mathbb{N}$. Let $V := (\mathbb{R}^m, |\bullet|_{m,1})$ and let $W := (\mathbb{R}^n, |\bullet|_{n,1})$. Let Z be a normed vector space. Let $* \in B(V, W, Z)$. Show: $\exists K \geqslant 0$ such that, $\forall v \in V$, $\forall w \in W$, $\quad |v * w|_Z \leqslant K \cdot |v|_V \cdot |w|_W$.

13-3. Let $\ell, m, n \in \mathbb{N}$. Let $U := (\mathbb{R}^\ell, |\bullet|_{\ell,1})$, let $V := (\mathbb{R}^m, |\bullet|_{m,1})$ and let $W := (\mathbb{R}^n, |\bullet|_{n,1})$. Let Z be a normed vector space. Let $F \in T(U, V, W, Z)$. Show: $\exists K \geqslant 0$ such that, $\forall u \in U$, $\forall v \in V$, $\forall w \in W$,

$$|F(u, v, w)|_Z \quad \leqslant \quad K \cdot |u|_U \cdot |v|_V \cdot |w|_W.$$

13-4. Let U, V and W be finite dimensional normed vector spaces. Let Z be a normed vector space. Let $F \in T(U, V, W, Z)$. Show: $\exists K \geqslant 0$ such that, $\forall u \in U$, $\forall v \in V$, $\forall w \in W$,

$$|F(u, v, w)|_Z \quad \leqslant \quad K \cdot |u|_U \cdot |v|_V \cdot |w|_W.$$

13-5. Let S V, W and Z all be finite dimensional vector spaces, and let $* \in B(V, W, Z)$. Show: $[\widehat{\mathcal{O}}(S, V)] \underset{S}{*} [\widehat{\mathcal{O}}(S, W)] \subseteq \widehat{\mathcal{O}}(S, Z)$.

Homework 12: Due on Tuesday 17 April

12-1. Let V and W both be finite dimensional vector spaces, and let $|\bullet|, \|\bullet\| \in \mathcal{N}(W)$. Show: $\widehat{\mathcal{O}}(V,W,|\bullet|) \subseteq \widehat{\mathcal{O}}(V,W,\|\bullet\|)$.

12-2. Let V and W both be finite dimensional vector spaces, and let $\alpha : V \dashrightarrow W$. Assume that $\operatorname{dom}[\alpha] \in \mathcal{N}_V(0_V)$. Assume that α is continuous at 0_V. Show: $\alpha \in \widehat{\mathcal{O}}(V,W)$.

12-3. Let V and W both be finite dimensional vector spaces, and let $|\bullet|, \|\bullet\| \in \mathcal{N}(V)$, and let $p > 0$. Show:

(1) $\breve{\mathcal{O}}_p(V,W,|\bullet|) \subseteq \breve{\mathcal{O}}_p(V,W,\|\bullet\|)$ and

(2) $\widehat{\mathcal{O}}_p(V,W,|\bullet|) \subseteq \widehat{\mathcal{O}}_p(V,W,\|\bullet\|)$.

12-4. Let V and W both be finite dimensional vector spaces, and let $p,q \geqslant 0$. Assume $p < q$. Show: $\breve{\mathcal{O}}_p(V,W) \supseteq \widehat{\mathcal{O}}_q(V,W)$.

12-5. Let $p,q \geqslant 0$. Show:

$$(\, [\, (\breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R})) \cdot (\breve{\mathcal{O}}_q(\mathbb{R},\mathbb{R})) \subseteq \breve{\mathcal{O}}_{p+q}(\mathbb{R},\mathbb{R}) \,] \quad \text{and}$$
$$[\, (\breve{\mathcal{O}}_q(\mathbb{R},\mathbb{R})) \circ (\breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R})) \subseteq \breve{\mathcal{O}}_{qp}(\mathbb{R},\mathbb{R}) \,] \,).$$

Homework 11: Due on Tuesday 10 April

11-1. Let $S \subseteq \mathbb{R}$. Assume that S has a minimum. (That is, assume: $\exists a \in S$ s.t. $a \leqslant S$.) Show that $\inf S = \min S \in S$.

11-2. Let U be a vector space, let $|\bullet|, \|\bullet\| \in \mathcal{N}(V)$ and let $S \subseteq U$. Let $V := (U, |\bullet|)$ and let $W := (U, \|\bullet\|)$. Assume both that $|\bullet| << \|\bullet\|$, and that S is bounded in W. Show that S is bounded in V.

11-3. Let V and W be normed vector spaces, and let $T \in L(V,W)$. Assume that $T : V \to W$ is bounded below. Show that T is 1-1.

11-4. Let $T \in L(\mathbb{R}^4,\mathbb{R}^4)$ be defined by $Tx = (6x_1, 5x_2, 8x_3, 7x_4)$, and define $V := (\mathbb{R}^4, |\bullet|_{4,2})$. Show that $\breve{T}_{VV} = 5$ and that $\widehat{T}_{VV} = 8$.

11-5. Let V and W be normed vector spaces and let $T : V \hookrightarrow\!\!> W$ be a vector space isomorphism. Assume $T : V \to W$ is bounded below. Show that $T^{-1} : W \to V$ is bounded.

Homework 10: Due on Tuesday 3 April

10-1. Let V and W be normed vector spaces, $T \in L(V,W)$ and $\varepsilon > 0$. Show: $[\, T$ is ε-bounded below $] \Leftrightarrow [\, \forall x \in V,\ |Tx|_W \geqslant \varepsilon |x|_V \,]$.

10-2. Let $m \in \mathbb{N}$. Show: $|\bullet|_{m,\infty} \leqslant |\bullet|_{m,2} \leqslant |\bullet|_{m,1} \leqslant m \cdot |\bullet|_{m,\infty}$.

10-3. Let V be a normed vector space, let Z be a topological space, let $\mu : V \dashrightarrow Z$ and let $p \in V$. Let $\lambda := \mu(p + \bullet)$. Show: $\lim_{0_V} \lambda = \lim_{p} \mu$.

10-4. Let X be a metric space, let $D \subseteq X$ and let $p \in \mathrm{Int}_X D$. Show that there exists $B \in \mathcal{B}_X(p)$ such that $B \subseteq D$.

10-5. Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$. Let $p, q \in \mathbb{R}$. Assume g has a local unique min at p in $\mathbb{R}$. Show: $g + C^q_{\mathbb{R}}$ has a local unique min at p in $\mathbb{R}$.

Homework 9: Due on Tuesday 27 March

9-1. Let V and W be normed vector spaces, $T \in L(V, W)$ and $K \geqslant 0$. Show: $[\ T \text{ is } K\text{-bounded}\ ] \quad \Leftrightarrow \quad [\ \forall x \in V,\ \ |Tx|_W \leqslant K \cdot |x|_V\ ]$.

9-2. Let V and W be normed vector spaces and let $K \geqslant 0$. Let $T \in L(V, W)$ be K-bounded. Show: T is K-Lipschitz. That is, show:

$$\forall x, y \in V, \qquad |\,[T(x)] \,-\, [T(y)]\,|_W \ \leqslant\ K \cdot |\,x \,-\, y\,|_V.$$

9-3. Let Y and Z be metric spaces and let $K > 0$. Let $f : Y \dashrightarrow Z$ be a K-Lipschitz function. Let $x \in \mathrm{dom}\,[f]$ and let $r > 0$. Show that: $f_*(B_Y(x, r)) \subseteq B_Z(f(x), Kr)$.

9-4. Let V and W be normed vector spaces and let $T \in L(V, W)$. Assume that T is continuous at 0_V. Show that T is bounded.

9-5. Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume: $\exists M \in \mathbb{R}$ s.t. $(M, \infty) \subseteq \mathrm{dom}\,[g'/f']$. Assume: $\lim_{\infty} f = 0 = \lim_{\infty} g$. Show: $\lim_{\infty} (g/f) \ =^* \ \lim_{\infty} (g'/f')$.

Homework 8: Due on Tuesday 20 March

8-1. Let W be a vector space and let $u, v, v' \in W$. Assume that $\mathbb{R}v = \mathbb{R}v'$ and that $u \| v$. Show that $u \| v'$.

8-2. Let $u, v \in \mathbb{R}^2$. Show:

$$[\ u \| v\ ] \quad \Leftrightarrow \quad [\ (u = 0_2) \text{ or } (v = 0_2) \text{ or } (\mathrm{sl}\, u = \mathrm{sl}\, v)\ ].$$

8-3. Let $u, v \in \mathbb{R}^2$. Show:

$$(\ u \| v\ ) \quad \Leftrightarrow \quad \left(\mathrm{Det} \begin{bmatrix} u \\ v \end{bmatrix} = 0 \right).$$

8-4. Let V and W be normed vector spaces. Let $x : \mathbb{R} \dashrightarrow V$ and $y : \mathbb{R} \dashrightarrow W$. Let $p \in \mathbb{R}$. Assume that x and y are both continuous at p. Show that (x, y) is continuous at p.

8-5. Let V and W be normed vector spaces. Let $x : \mathbb{R} \dashrightarrow V$ and $y : \mathbb{R} \dashrightarrow W$. Let $p \in \mathrm{LPD}_{\mathbb{R}}(x, y)$. Show: $(x, y)'(p) \;\; =^* \;\; (\, x'(p)\, ,\, y'(p)\,)$.

Homework 7: Due on Tuesday 6 March

7-1. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$ and $\delta > 0$. Assume

(1) $(p - \delta, p + \delta) \;\subseteq\; \mathrm{dom}\,[f]$,
(2) f is strictly decreasing on $(p - \delta, p]$ and
(3) f is strictly increasing on $[p, p + \delta)$.

Show that f has a local unique minimum at p in $\mathbb{R}$.

7-2. Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$ and $p \in \mathrm{IntD}_{\mathbb{R}} g$. Assume both that $g(p) = 0$ and that $g'(p) > 0$. Show: $\exists \delta > 0$ such that all three of the following hold:

(A) $(p - \delta, p + \delta) \;\subseteq\; \mathrm{dom}\,[g]$,
(B) $g < 0$ on $(p - \delta, p)$ and
(C) $g > 0$ on $(p, p + \delta)$.

7-3. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathrm{IntD}_{\mathbb{R}}(f')$. Assume both that $f'(p) = 0$ and that $f''(p) > 0$. Show: f has a local unique minimum at p in $\mathbb{R}$.

7-4. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathrm{IntD}_{\mathbb{R}}(f')$. Assume both that $f'(p) = 0$ and that $f''(p) < 0$. Show: f has a local unique maximum at p in $\mathbb{R}$.

7-5. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Asume both that $0 \in \mathrm{IntD}_{\mathbb{R}} f$ and that $f(0) = 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^4$. Assume that $f/g \to 1$ near 0. Show that f has a local unique minimum at 0 in $\mathbb{R}$.

Homework 6: Due on Tuesday 27 February

6-1. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ be continuous, and let $I \subseteq \mathrm{dom}\,[f]$. Assume that I is an interval. Show: $f_*(I)$ is an interval.

6-2. Let $X, Y \subseteq \mathbb{R}$. Let $f : X \hookrightarrow\!\!> Y$ be continuous. Let $X_0 := \mathrm{Int}_{\mathbb{R}} X$. Show that f^{-1} is continuous on $f_*(X_0)$.

6-3. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ be str. increasing. Show: f^{-1} is str. increasing.

6-4. Let $X, Y \subseteq \mathbb{R}$. Let $f : X \hookrightarrow\!\!> Y$ be strictly increasing. Let $a \in \mathbb{R}$. Assume: $X = [a, \infty)$. Show: f^{-1} is continuous at $f(a)$.

6-5. Let $X, Y \subseteq \mathbb{R}$. Let $f : X \hookrightarrow\!\!> Y$ be strictly increasing. Let $a \in \mathbb{R}$. Assume: $\exists b \in (a, \infty)$ s.t. $X = [a, b)$. Show: f^{-1} is continuous at $f(a)$.

Homework 5: Due on Tuesday 20 February

5-1. Show: $\forall w, x \in \mathbb{R}, \quad \cos(w+x) = (\cos w)\cdot(\cos x) - (\sin w)\cdot(\sin x)$.

5-2. Show: $\forall x \in \mathbb{R}$,

$$\begin{aligned} \sin(2x) &= 2 \cdot (\sin x) \cdot (\cos x) \qquad \text{and} \\ \cos(2x) &= (\cos^2 x) - (\sin^2 x). \end{aligned}$$

5-3. Let $f : \mathbb{R} \to \mathbb{R}$, $a := f(0)$. Assume $f' = f$. Show: $f = a \cdot \exp$.

5-4. Show: $\forall w, x \in \mathbb{R}, \quad \exp(w + x) = (\exp w) \cdot (\exp x)$.

5-5. Let $f : \mathbb{R} \to \mathbb{R}$. Assume that $f' = f$. Let $g := f^2$. Show: $g' = 2g$.

Homework 4: Due on Tuesday 13 February

4-1. Let $m \in \mathbb{R}$. Define $\lambda : \mathbb{R} \to \mathbb{R}$ by $\lambda(x) = mx$. Show: $\lambda' = C_{\mathbb{R}}^m$.

4-2. Let $S \subseteq \mathbb{R}$. Show: $-(\mathrm{LP}_{\mathbb{R}} S) = \mathrm{LP}_{\mathbb{R}}(-S)$.

4-3. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$. Define $f_0 : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f_0(x) = f(-x)$. Let $p_0 := -p$. Assume that f has a local maximum at p in $\mathbb{R}$. Show that f_0 has a local maximum at p_0 in $\mathbb{R}$.

4-4. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$. Define $f_1 : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f_1(x) = -[f(x)]$. Assume that f has a local minimum at p in $\mathbb{R}$. Show that f_1 has a local maximum at p in $\mathbb{R}$.

4-5. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathrm{dom}\,[f']$. Assume that f has a local extremum at p. Show that $f'(p) = 0$.

Homework 3: Due on Tuesday 6 February

3-1. Let X be a topological space, let $S \subseteq X$ and let $p \in X$. Show:

$$[\ p \in \mathrm{Int}_X S\ ] \quad \Leftrightarrow \quad [\ S \in \mathcal{N}_X(p)\ ].$$

3-2. Let $\phi : \mathbb{R} \dashrightarrow \mathbb{R}$, let $p \in \mathbb{R}$ and let $q > 0$. Assume: $\phi \to q$ near p. Show: $\exists U \in \mathcal{N}_{\mathbb{R}}^{\times}(p)$ s.t. $\phi_*(U) > 0$.

3-3. Let $\alpha, \beta, \gamma : \mathbb{R} \dashrightarrow \mathbb{R}$. Let $S \subseteq \mathrm{dom}\,[(\alpha, \beta, \gamma)]$. Let $p \in \mathrm{Int}_{\mathbb{R}} S$ and let $q \in \mathbb{R}$. Assume that $\alpha \leqslant \beta \leqslant \gamma$ on S. Assume that $\alpha(p) = \gamma(p)$. Assume that $\alpha'(p) = q = \gamma'(p)$. Show: $\beta'(p) = q$.

3-4. Let W be a normed vector space. Let $f, g : \mathbb{R} \dashrightarrow W$. Let U be an open subset of $\mathbb{R}$. Assume: $U \subseteq \mathrm{dom}\,[(f, g)]$. Assume: $f = g$ on U. Show: $f' = g'$ on U.

3-5. Define $f : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f(x) = [x^2][\sin(x^{-3})]$. Let $\beta := \mathrm{adj}_0^0 f$. Show: $\beta'(0) = 0$.

Homework 2: Due on Tuesday 30 January

2-1. Let X and Y be topological spaces. Let $X_0 \subseteq X$ and let $Y_0 \subseteq Y$. Let $f : X_0 \dashrightarrow Y_0$. Show:

$$[\ f \text{ is } (X, Y)\text{-continuous }] \quad \Rightarrow \quad [\ f \text{ is } (X_0, Y_0)\text{-continuous }].$$

2-2. Let X, Y and Z be topological spaces. Let $f : X \dashrightarrow Y$ and let $g : X \dashrightarrow Z$. Let $a \in X$, $b \in Y$ and $c \in Z$. Assume:

(1) $f \to b$ in Y near a in X and
(2) $g \to c$ in Z near a in X.

Show: $(f, g) \to (b, c)$ in $Y \times Z$ near a in X.

2-3. Show: $\forall z \in \mathbb{R}$, $\exists! x \in \mathbb{R}$ s.t. $x^5 + x^3 = z$.

2-4. Let X and Y be topological spaces. Let $f : X \dashrightarrow Y$. Let $X_0 \subseteq \mathrm{dom}\,[f]$. Assume that f is (X, Y)-continuous on X_0. Show that $f|X_0$ is (X_0, Y)-continuous.

2-5. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Let $\alpha, \beta \in \mathrm{dom}\,[f]$. Assume that f is semimonotone on $[\alpha|\beta]$. Show: $f_*([\alpha|\beta]) \subseteq [f(\alpha)|f(\beta)]$.

Homework 1: Due on Tuesday 23 January

1-1. Let X and Y be metric spaces, and let $f : X \to Y$. Assume that f is not uniformly continuous. Show: $\exists \varepsilon > 0$, $\exists p, q \in X^{\mathbb{N}}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ d_X\,(\,p_j\,,\,q_j\,)\ <\ 1/j\ ] \quad \text{and} \quad [\ d_Y\,(\,f(p_j)\,,\,f(q_j)\,)\ \geqslant\ \varepsilon\ ].$$

1-2. Let K be a sequentially compact metric space, let Y be a metric space, and let $f : K \to Y$. Assume that f is not uniformly continuous.

Show: $\exists \varepsilon > 0$, $\exists s, t \in K^{\mathbb{N}}$, $\exists u \in K$ s.t.

$$(\quad [\, s_\bullet \to u \text{ in } K \,] \quad \text{and} \quad [\, t_\bullet \to u \text{ in } K \,] \quad \text{and} \\ [\, \forall j \in \mathbb{N}, \qquad d_Y\, (\, f(s_j)\, ,\, f(t_j)\,) \;\geqslant\; \varepsilon \,] \qquad).$$

1-3. Let X be a metric space, and let $s \in X^{\mathbb{N}}$. Assume that $s_\bullet$ is convergent in X. Show that $s_\bullet$ is Cauchy in X.

1-4. Let X be a metric space, and let $s \in X^{\mathbb{N}}$. Assume that $s_\bullet$ is Cauchy and subconvergent in X. Show that $s_\bullet$ is convergent in X.

1-5. Let X be a metric space, and let $s \in X^{\mathbb{N}}$. Assume that $s_\bullet$ is Cauchy in X. Show that $s_\bullet$ is bounded in X.
