Quantified equivalences

(1) Axiom of Extensionality / Set-equality equivalency: Let A and B be sets. Then "$A = B$" is equivalent to the quantified statement "$\forall x$, $[\,(x \in A) \Leftrightarrow (x \in B)\,]$".
(2) Subset equivalency: Let A and B be sets. Then "$A \subseteq B$" is equivalent to the quantified statement "$\forall x \in A$, $[\,x \in B\,]$".
(3) Union equivalency / Intersection equivalency: Let $\mathcal{S}$ be a set of sets. Then, $\forall z$,

$$\left(\,\left[\,z \in \bigcup \mathcal{S}\,\right] \quad \Leftrightarrow \quad [\,\exists X \in \mathcal{S} \text{ s.t. } z \in X\,]\,\right) \qquad \&$$

$$\left(\,\left[\,z \in \bigcap \mathcal{S}\,\right] \quad \Leftrightarrow \quad [\,\forall X \in \mathcal{S},\, z \in X\,]\,\right).$$

(4) Function-equality equivalency: Let f and g be functions and let S be a set. Assume that $\mathrm{dom}[f] \subseteq S$ and that $\mathrm{dom}[g] \subseteq S$. Then the statement "$f = g$" is equivalent to the quantified statement "$\forall x \in S$, $[\,f(x) = g(x)\,]$".
(5) Union-closure equivalency: Let $\mathcal{S}$ be a set of sets, X a set. Then "$X \in \langle \mathcal{S} \rangle_\cup$" is equivalent to the quantified statement "$\forall z \in X$, $\exists A \in \mathcal{S}$ s.t. $z \in A \subseteq X$". In other words, "some of the sets in $\mathcal{S}$ will cover X exactly" is equivalent to "every point of X can be covered by a set from $\mathcal{S}$ without going outside X".
(6) Base equivalency: Let $\mathcal{B}$ be any set of sets. Then the statement "$\mathcal{B}$ is a base for a topology" is equivalent to "$\langle \mathcal{B} \rangle_\cup$ is a topology", which is equivalent to the quantified statement "$\forall U, V \in \mathcal{B}$, $U \cap B \in \langle \mathcal{B} \rangle_\cup$". In other words, a set of sets is a base for a topology iff "each pairwise-intersection is a union".
(7) Forward-image containment equivalency: Let f be a function. Let S, T be sets. Then "$f_*(S) \subseteq T$" is equivalent to the quantified statement "$\forall x \in \mathrm{dom}[f]$, $(\,[\,x \in S\,] \Rightarrow [\,f(x) \in T\,]\,)$".
(8) Open equivalency: Let X be a topological space, $W \subseteq X$. Then "W is open in X" is equivalent to the quantified statement "$\forall q \in W$, $\exists V \in \mathcal{N}_X(q)$ s.t. $V \subseteq W$". In other words, "the set is open" is equivalent to "every point of the set is covered by a neighborhood that stays inside the set".
(9) Closure point equivalency: Let Z be a topological space, $S \subseteq Z$, $q \in Z$. Then the statement "$q \in \mathrm{Cl}_Z(S)$" is equivalent to the quantified statement "$\forall U \in \mathcal{N}_Z(q)$ s.t. $U \cap S \neq \varnothing$".

(10) Interior point equivalency: Let Z be a topological space, $S \subseteq Z$, $q \in Z$. Then the statement "$q \in \mathrm{Int}_Z(S)$" is equivalent to the quantified statement "$\exists U \in \mathcal{N}_Z(q)$ s.t. $U \subseteq S$".

(11) Boundary point equivalency: Let Z be a topological space, let $S \subseteq Z$ and let $q \in Z$. Then "$q \in \partial_Z(S)$" is equivalent to the quantified statement "$\forall U \in \mathcal{N}_Z(q)$, $U \cap S \neq \varnothing \neq U \backslash S$".

(12) Limit point equivalency: Let Z be a topological space, let $S \subseteq Z$ and let $q \in Z$. Then "$q \in \mathrm{LP}_Z(S)$" is equivalent to the quantified statement "$\forall P \in \mathcal{N}_Z^\times(q)$, $P \cap S \neq \varnothing$".

(13) Topological space function-limit equivalency: Let Y and Z be topological spaces, $f : Y \dashrightarrow Z$, $a \in Y$, $b \in Z$. Then: "$f \to b$ in Z near a in Y" is equivalent to the quantified statement "$\forall V \in \mathcal{N}_Z(b)$, $\exists U \in \mathcal{N}_Y^\times(a)$ s.t., $\forall x \in \mathrm{dom}[f]$,

$$[\ x \in U\ ] \quad \Rightarrow \quad [\ f(x) \in V\ ]\text{''.}$$

(14) Metric space function-limit equivalency: Let Y and Z be metric spaces and let $f : Y \dashrightarrow Z$ Let $a \in Y$, $b \in Z$. Then: "$f \to b$ in Z near a in Y" is equivalent to the quantified statement "$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathrm{dom}[f]$,

$$[\ 0 < d_Y(x,a) < \delta\ ] \quad \Rightarrow \quad [\ d_Z\,(\,f(x)\,,\,b\,) < \varepsilon\ ]\text{''.}$$

(15) Normed vector space function-limit equivalency: Let Y and Z be normed vector spaces, $f : Y \dashrightarrow Z$, $a \in Y$, $b \in Z$. Then: "$f \to b$ in Z near a in Y" is equivalent to the quantified statement "$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathrm{dom}[f]$,

$$[\ 0 < |\,x - a\,|_Y < \delta\ ] \quad \Rightarrow \quad [\ |\,[f(x)] - b\,|_Z < \varepsilon\ ]\text{''.}$$

(16) Topological space sequence-limit equivalency: Let Z be a topological space. Let $s \in Z^{\mathbb{N}}$, $b \in Z$. Then: "$s_\bullet \to b$ in Z" is equivalent to "$\forall V \in \mathcal{N}_Z(b)$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ j \geqslant K\ ] \quad \Rightarrow \quad [\ s_j \in V\ ]\text{''.}$$

(17) Metric space sequence-limit equivalency: Let Z be a metric space. Let $s \in Z^{\mathbb{N}}$, $b \in Z$. Then "$s_\bullet \to b$ in Z" is equivalent to the quantified statement "$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ j \geqslant K\ ] \quad \Rightarrow \quad [\ d_Z\,(\,s_j\,,\,b\,) < \varepsilon\ ]\text{''.}$$

(18) Normed vector space sequence-limit equivalency: Let Z be a normed vector space, $s \in Z^{\mathbb{N}}$, $b \in Z$. Then: "$s_\bullet \to b$ in Z" is equivalent to "$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ j \geqslant K\ ] \quad \Rightarrow \quad [\ |s_j - b|_Z < \varepsilon\ ]\text{''}.$$

(19) Topological space continuity equivalency: Let Y and Z be topological spaces, let $f : Y \dashrightarrow Z$ and let $a \in \mathrm{dom}[f]$. Then the statement "f is (Y, Z)-continuous at a" is equivalent to the statement "$\forall V \in \mathcal{N}_Z(f(a))$, $\exists U \in \mathcal{N}_Y(a)$ s.t., $\forall x \in \mathrm{dom}[f]$,

$$[\ x \in U\ ] \quad \Rightarrow \quad [\ f(x) \in V\ ]\text{''}.$$

(20) Metric space continuity equivalency: Let Y and Z be metric spaces, let $f : Y \dashrightarrow Z$ and let $a \in \mathrm{dom}[f]$. Then the statement "f is (Y, Z)-continuous at a" is equivalent to the quantified statement "$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathrm{dom}[f]$,

$$[\ d_Y(\,x\,,\,a\,) < \delta\ ] \quad \Rightarrow \quad [\ d_Z(\,f(x)\,,\,f(a)\,) < \varepsilon\ ]\text{''}.$$

(21) Normed vector space continuity equivalency: Let Y and Z be normed vector spaces, let $f : Y \dashrightarrow Z$ and let $a \in \mathrm{dom}[f]$. Then the statement "f is (Y, Z)-continuous at a" is equivalent to the quantified statement "$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathrm{dom}[f]$,

$$[\ |\,x - a\,|_Y < \delta\ ] \quad \Rightarrow \quad [\ |\,[f(x)] - [f(a)]\,|_Z < \varepsilon\ ]\text{''}.$$

(22) K-bounded quantified equivalence: Let V, W be normed vector spaces, $T \in L(V, W)$, $K \geqslant 0$. Then:

$$[\ T \text{ is } K\text{-bounded}\ ] \Leftrightarrow [\ \forall u \in S_V,\ \ |T(x)|_W \leqslant K\ ].$$

(23) Another K-bounded quantified equivalence: Let V and W be normed vector spaces, $T \in L(V, W)$ and $K \geqslant 0$. Then:

$$[\ T \text{ is } K\text{-bounded}\ ] \quad \Leftrightarrow \quad [\ \forall x \in V,\ \ |T(x)|_W \leqslant K \cdot |x|_V\ ].$$

(24) δ-bounded below quantified equivalence: Let V, W be normed vector spaces, $T \in L(V, W)$, $\delta > 0$. Then:

$$[\ T \text{ is } \delta\text{-bdd below}\ ] \Leftrightarrow [\ \forall u \in S_V,\ \ |T(x)|_W \geqslant \delta\ ].$$

(25) Another δ-bounded below quantified equivalence: Let V and W be normed vector spaces, $T \in L(V, W)$ and $\delta > 0$. Then:

$$[\ T \text{ is } \delta\text{-bdd below}\ ] \quad \Leftrightarrow \quad [\ \forall x \in V,\ \ |T(x)|_W \geqslant \delta \cdot |x|_V\ ].$$

Other useful facts

(A) For any set $\mathcal{S}$ of sets, for any $X \in \mathcal{S}$, we have: $X \subseteq \bigcup \mathcal{S}$.

(B) For any vector space V, for any $k \in \mathbb{N}$, for any $x_1, \ldots, x_k \in V$,

$$\langle \{x_1, \ldots, x_k\} \rangle_{\text{lin}} = \{ c_1 x_1 + \cdots + c_k x_k \mid c_1, \ldots, c_k \in \mathbb{R} \}.$$

(C) For any function f, for any set S, for any x, if [($x \in \text{dom}[f]$) and ($x \in S$)], then [$f(x) \in f_*(S)$].