Solutions for MATH 4604 (Advanced Calculus II)
Spring 2018

Homework 14: Due on Tuesday 1 May

14-1. Let W be a normed vector space, let $f : \mathbb{R} \dashrightarrow W$ and let $p \in \mathbb{R}$. Show: $(f_p^T)'(0) = f'(p)$.

Proof: Let $g := f_p^T$. Want: $g'(0) = f'(p)$. As $g'(0) = \lim\limits_0 SS_g^0$ and $f'(p) = \lim\limits_0 SS_f^p$, it suffices to show: $SS_g^0 = SS_f^p$. Want: $\forall h \in \mathbb{R}$, $(SS_g^0)(h) = (SS_f^p)(h)$. Given $h \in \mathbb{R}$. Want: $(SS_g^0)(h) = (SS_f^p)(h)$.

We have

$$\begin{aligned} [\ g(0+h)\ ] - [\ g(0)\ ] &= [\ g(h)\ ] - [\ g(0)\ ] \\ &= [\ (f_p^T)(h)\ ] - [\ (f_p^T)(0)\ ] \\ &= [\ [f(p+h)] - [f(p)]\ ] \\ &\qquad - [\ [f(p+0)] - [f(p)]\ ] \\ &= [\ f(p+h)\ ] - [\ f(p)\ ]. \end{aligned}$$

Then

$$\begin{aligned} (SS_g^0)(h) &= \frac{[\ g(0+h)\ ] - [\ g(0)\ ]}{h} \\ &= \frac{[\ f(p+h)\ ] - [\ f(p)\ ]}{h} = (SS_f^p)(h), \end{aligned}$$

as desired. QED

14-2. Let V, W be finite dimensional vector spaces, $\alpha, \beta : V \dashrightarrow W$, $p \geqslant 0$. Assume: $\alpha = \beta$ near 0_V and $\alpha \in \breve{\mathcal{O}}_p(V, W)$. Show: $\beta \in \breve{\mathcal{O}}_p(V, W)$.

Proof: By assumption, choose $U \in \mathcal{N}_V(0_V)$ s.t. $\alpha = \beta$ on U. Also,

(A) $\mathrm{dom}\,[\alpha] \in \mathcal{N}_V(0_V)$,
(B) $\alpha(0_V) = 0_W$ and
(C) α is continuous at 0_V.

Let $D := \mathrm{dom}\,[\alpha]$. By (A), we have $D \in \mathcal{N}_V(0_V)$. As $D, U \in \mathcal{N}_V(0_V)$, we get $D \cap U \in \mathcal{N}_V(0_V)$. We wish to show:

(1) $\mathrm{dom}\,[\beta] \in \mathcal{N}_V(0_V)$,
(2) $\beta(0_V) = 0_W$ and
(3) β is continuous at 0_V.

Proof of (1): Since we have $D \cap U \in \mathcal{N}_V(0_V)$, it suffices to show: $D \cap U \subseteq \text{dom}[\beta]$. We want: $\forall x \in D \cap U$, $x \in \text{dom}[\beta]$. Let $x \in D \cap U$ be given. We want to show: $x \in \text{dom}[\beta]$.

We have: $x \in D$ and $x \in U$. Since $x \in D$, we get $\alpha(x) \neq ☹$. Since $x \in U$, we get $\alpha(x) = \beta(x)$. Since $\beta(x) = \alpha(x) \neq ☹$, we get $x \in \text{dom}[\beta]$, as desired. *End of proof of (1).*

Proof of (2): Since $U \in \mathcal{N}_V(0_V)$, $0_V \in U$. So, since $\alpha = \beta$ on U, we get $\alpha(0_V) = \beta(0_V)$. By (B), we have $\alpha(0_V) = 0_W$.

Then $\beta(0_V) = \alpha(0_V) = 0_W$, as desired. *End of proof of (2).*

Proof of (3): Choose $|\bullet|_V \in \mathcal{N}(V)$ and $|\bullet|_W \in \mathcal{N}(W)$. We wish to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ such that, $\forall x \in \text{dom}[\beta]$,

$$[\ 0 < |x - 0_V|_V < \delta\ ] \quad \Rightarrow \quad [\ |[\beta(x)] - [\beta(0_V)]|_W < \varepsilon\ ].$$

Let $\varepsilon > 0$ be given. We wish to show: $\exists \delta > 0$ such that, $\forall x \in \text{dom}[\beta]$,

$$[\ 0 < |x - 0_V|_V < \delta\ ] \quad \Rightarrow \quad [\ |[\beta(x)] - [\beta(0_V)]|_W < \varepsilon\ ].$$

Since $D \cap U \in \mathcal{N}_V(0_V)$ and since $\mathcal{B}_V(0_V)$ is a neighborhood base at 0_V in V, choose $\rho > 0$ such that $B_V(0_V, \rho) \subseteq D \cap U$. By (C), choose $\tau > 0$ such that, $\forall x \in D$,

$$[\ 0 < |x - 0_V|_V < \tau\ ] \quad \Rightarrow \quad [\ |[\alpha(x)] - [\alpha(0_V)]|_W < \varepsilon\ ].$$

Let $\delta := \min\{\rho, \tau\}$. Then $\delta > 0$, $\delta \leqslant \rho$ and $\delta \leqslant \tau$. Want: $\forall x \in \text{dom}[\beta]$,

$$[\ 0 < |x - 0_V|_V < \delta\ ] \quad \Rightarrow \quad [\ |[\beta(x)] - [\beta(0_V)]|_W < \varepsilon\ ].$$

Let $x \in \text{dom}[\beta]$ be given. We wish to show:

$$[\ 0 < |x - 0_V|_V < \delta\ ] \quad \Rightarrow \quad [\ |[\beta(x)] - [\beta(0_V)]|_W < \varepsilon\ ].$$

Assume $0 < |x - 0_V|_V < \delta$. Want: $|[\beta(x)] - [\beta(0_V)]|_W$.

We have $0 < |x - 0_V| < \rho$ and $0 < |x - 0_V| < \tau$. Since $|x - 0_V|_V < \rho$, we get $x \in B_V(0_V, \rho)$. So, since $B_V(0_V, \rho) \subseteq D \cap U$, we see that $x \in D \cap U$. Then $x \in D$ and $x \in U$. Since $x \in U$, by choice of U, we have $\alpha(x) = \beta(x)$. By (2), we have $\beta(0_V) = 0_W$. So, by (B), we have $\beta(0_V) = \alpha(0_V)$. Since $x \in D$ and since $0 < |x - 0_V|_V < \tau$, by choice of τ, we conclude that $|[\alpha(x)] - [\alpha(0_V)]|_W < \varepsilon$. Then

$$|\,[\beta(x)] - [\beta(0_V)]\,|_W \quad = \quad |\,[\alpha(x)] - [\alpha(0_V)]\,|_W \quad < \quad \varepsilon,$$

as desired. *End of proof of (3).* QED

14-3. Let $\delta > 0$, let $I := (-\delta, \delta)$ and let $\alpha : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume: $\forall x \in I$, $\alpha(x) \in [0|x]$. Show that $\alpha \in \widehat{\mathcal{O}}_1(\mathbb{R}, \mathbb{R})$.

Proof: Let $|\bullet| \in \mathcal{N}(\mathbb{R})$ denote absolute value. We wish to show: $\alpha \in [|\bullet|] \cdot [\widehat{\mathcal{O}}(\mathbb{R},\mathbb{R})]$. Let $\beta := \mathrm{adj}_0^0(\alpha/|\bullet|)$. It suffices to show:

(1) $\beta \in \widehat{\mathcal{O}}(\mathbb{R},\mathbb{R})$ and

(2) $\alpha = [|\bullet|] \cdot \beta$.

Proof of (1): We wish to show: $\exists U \in \mathcal{N}_{\mathbb{R}}(0)$ such that

$$[\ U \subseteq \mathrm{dom}\,[\beta]\ ] \qquad \text{and} \qquad [\ \sup|\beta_*(U)| < \infty\ ].$$

Since $I = (-\delta, \delta)$, we see that I is open in $\mathbb{R}$. So since $0 \in I$, we get $I \in \mathcal{N}_{\mathbb{R}}(0)$. Let $U := I$. We wish to show:

(A) $U \subseteq \mathrm{dom}\,[\beta]$ and

(B) $\sup|\beta_*(U)| < \infty$.

Proof of (A): Since $U = I$, we wish to show $I \subseteq \mathrm{dom}\,[\beta]$. Since $\beta(0) = 0 \neq ☹$, we get $0 \in \mathrm{dom}\,[\beta]$. It remains to show: $I_0^\times \subseteq \mathrm{dom}\,[\beta]$.

We wish to show: $\forall x \in I_0^\times$, $x \in \mathrm{dom}\,[\beta]$. Let $x \in I_0^\times$ be given. We wish to show: $x \in \mathrm{dom}\,[\beta]$.

Since $x \in I_0^\times$, we get $x \in I$ and $x \neq 0$. Since $x \neq 0$, we get $|x| \neq 0$ and $\beta(x) = [|\alpha(x)|]/[|x|]$. Since $x \in I$, we get $\alpha(x) \in [0|x] \subseteq \mathbb{R}$. So, since $|x| \neq 0$, we get $[\alpha(x)]/[|x|] \in \mathbb{R}$. Then $\beta(x) = [|\alpha(x)|]/[|x|] \in \mathbb{R}$, so $\beta(x) \neq ☹$, and so $x \in \mathrm{dom}\,[\beta]$, as desired. *End of proof of (A).*

Proof of (B): It suffices to show $\sup|\beta_*(U)| \leqslant 1$. We wish to show: $|\beta_*(U)| \leqslant 1$. As $U = I$, we wish to show: $|\beta_*(I)| \leqslant 1$. Since

$$|\beta(0)| \quad = \quad |0| \quad = \quad 0 \quad \leqslant \quad 1,$$

it remains to show: $|\beta_*(I_0^\times)| \leqslant 1$.

We wish to show: $\forall x \in \mathrm{dom}\,[\beta]$,

$$[\ x \in I_0^\times\ ] \quad \Rightarrow \quad [\ |\beta(x)| \leqslant 1\ ].$$

Let $x \in \mathrm{dom}\,[\beta]$ be given. We wish to show:

$$[\ x \in I_0^\times\ ] \quad \Rightarrow \quad [\ |\beta(x)| \leqslant 1\ ].$$

Assume: $x \in I_0^\times$. Want: $|\beta(x)| \leqslant 1$.

Since $x \in I_0^\times$, we get $x \in I$ and $x \neq 0$. Let $y := |x|$. As $x \neq 0$, $y > 0$. Since $x \neq 0$, $\beta(x) = [\alpha(x)]/[|x|]$. That is, $\beta(x) = [\alpha(x)]/y$. Since $y > 0$, we get $|y| = y$. Then $|\beta(x)| = [|\alpha(x)|]/[|y|] = [|\alpha(x)|]/y$. Since $x \in I$, we get $\alpha(x) = [0|x]$. We have $-|x| \leqslant x \leqslant |x|$, *i.e.*, $-y \leqslant x \leqslant y$. Since $-y \leqslant 0$ and $-y \leqslant x$ we get $-y \leqslant \min\{0,x\}$. Since $0 \leqslant y$ and $x \leqslant y$, we get $\max\{0,x\} \leqslant y$. Then

$$[0|x] \quad = \quad [\,\min\{0,x\}\,,\,\max\{0,x\}\,] \quad \subseteq \quad [-y,y].$$

Then $\alpha(x) \in [0|x] = [-y, y]$, so $-y \leqslant \alpha(x) \leqslant y$, so $|\alpha(x)| \leqslant y$. Then $|\alpha(x)|/y \leqslant 1$. Then $|\beta(x)| = |\alpha(x)|/y \leqslant 1$, as desired. *End of proof of (B). End of proof of (1).*

Proof of (2): We have $0 \in (-\delta, \delta) = I$, so $\alpha(0) \in [0|0] = \{0\}$, so $\alpha(0) = 0$. Also $\beta(0) = 0$. Then

$$\alpha(0) \;=\; 0 \;=\; 0 \cdot 0 \;=\; [|0|] \cdot [\beta(0)|].$$

It remains to show: $\alpha = [| \bullet |] \cdot \beta$ on $\mathbb{R}_0^\times$. We wish to show: $\forall x \in \mathbb{R}_0^\times$, $\alpha(x) = [|x|] \cdot [\beta(x)]$. Let $x \in \mathbb{R}_0^\times$ be given. Want: $\alpha(x) = [|x|] \cdot [\beta(x)]$.

Since $x \neq 0$, it follows that $\beta(x) = [\alpha(x)]/[|x|]$. We conclude that $\alpha(x) = [|x|] \cdot [\beta(x)]$, as desired. *End of proof of (2).* QED

14-4. Let $k \in \mathbb{N}_0$ and let $f \in \breve{\mathcal{O}}_k(\mathbb{R}, \mathbb{R})$. Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume that $g' = f$ near 0. Assume that $g(0) = 0$. Show that $g \in \breve{\mathcal{O}}_{k+1}(\mathbb{R}, \mathbb{R})$.

Proof: Let $|\bullet| \in \mathcal{N}(\mathbb{R})$ denote absolute value. Let $\psi := \mathrm{adj}_0^0(g/[|\bullet|^{k+1}])$. So, since $\psi = g/[|\bullet|^{k+1}]$ on $\mathbb{R}_0^\times$, we get $g = |\bullet|^{k+1} \cdot \psi$ on $\mathbb{R}_0^\times$. So, since $g(0) = 0 = [|0|^{k+1}] \cdot [\psi(0)]$, we see that $g = |\bullet|^{k+1} \cdot \psi$ on $\mathbb{R}$. Then $g = |\bullet|^{k+1} \cdot \psi$. We wish to show: $g \in [|\bullet|^{k+1}] \cdot [\breve{\mathcal{O}}(\mathbb{R}, \mathbb{R})]$. It therefore suffices to show: $\psi \in \breve{\mathcal{O}}(\mathbb{R}, \mathbb{R})$. Since $\psi(0) = 0$, we need to show:

$$[\ \mathrm{dom}\,[\psi] \in \mathcal{N}_{\mathbb{R}}(0)\ ] \qquad \text{and} \qquad [\ \psi \text{ is continuous at } 0\ ].$$

We have $\breve{\mathcal{O}}_k(\mathbb{R}, \mathbb{R}) = [| \bullet |] \cdot [\breve{\mathcal{O}}(\mathbb{R}, \mathbb{R})]$, so, since $f \in \breve{\mathcal{O}}_k(\mathbb{R}, \mathbb{R})$, choose $\phi \in \breve{\mathcal{O}}(\mathbb{R}, \mathbb{R})$ s.t. $f = |\bullet| \cdot \phi$. Then $\mathrm{dom}\,[f] = (\mathrm{dom}\,[| \bullet |]) \cap (\mathrm{dom}\,[\phi])$, so, since $\mathrm{dom}\,[\phi] \subseteq \mathbb{R} = \mathrm{dom}\,[| \bullet |]$, we get $\mathrm{dom}\,[f] = \mathrm{dom}\,[\phi]$. Since $g = |\bullet|^{k+1} \cdot \psi$, we get $\mathrm{dom}\,[g] = (\mathrm{dom}\,[| \bullet |]) \cap (\mathrm{dom}\,[\psi])$, so, since $\mathrm{dom}\,[\psi] \subseteq \mathbb{R} = \mathrm{dom}\,[| \bullet |]$, we get $\mathrm{dom}\,[g] = \mathrm{dom}\,[\psi]$.

Since $g' = f$ near 0, choose $U \in \mathcal{N}_{\mathbb{R}}(0)$ s.t. $f = g'$ on U. Let $D := \mathrm{dom}\,[\phi]$. Since $\phi \in \breve{\mathcal{O}}(\mathbb{R}, \mathbb{R})$, we have $D \in \mathcal{N}_{\mathbb{R}}(0)$. So, since $U \in \mathcal{N}_{\mathbb{R}}(0)$, we get $D \cap U \in \mathcal{N}_{\mathbb{R}}(0)$. So, since $\mathcal{B}_{\mathbb{R}}(0)$ is a neighborhood base at 0 in $\mathbb{R}$, choose $\rho > 0$ s.t. $B_{\mathbb{R}}(0, \rho) \subseteq D \cap U$. Let $I := (-\rho, \rho)$. Then $I \in \mathcal{N}_{\mathbb{R}}(0)$. Also, $I = B_{\mathbb{R}}(0, \rho) \subseteq D \cap U$, so $I \subseteq D$ and $I \subseteq U$.

We have $I \subseteq D = \mathrm{dom}\,[\phi] = \mathrm{dom}\,[f]$. Since $f = g$ on U and since $I \subseteq U$, it follows that $f = g'$ on I. So, since $I \subseteq \mathrm{dom}\,[f]$, we get $I \subseteq \mathrm{dom}\,[g']$. Then $I \subseteq \mathrm{dom}\,[g'] \subseteq \mathrm{dom}\,[g] = \mathrm{dom}\,[\psi]$. So, since $I \in \mathcal{N}_{\mathbb{R}}(0)$, it follows that $\mathrm{dom}\,[\psi] \in \mathcal{N}_{\mathbb{R}}(0)$. It only remains to show that ψ is continuous at 0. We want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathrm{dom}\,[\psi]$,

$$[\ |x - 0| < \delta\ ] \quad \Rightarrow \quad [\ |[\psi(x)] - [\psi(0)]| < \varepsilon\ ].$$

Let $\varepsilon > 0$ be given. We wish to show: $\exists \delta > 0$ s.t., $\forall x \in \mathrm{dom}\,[\psi]$,

$$[\ 0 < |x - 0| < \delta\ ] \quad \Rightarrow \quad [\ |[\psi(x)] - [\psi(0)]| < \varepsilon\ ].$$

As $\phi \in \breve{\mathcal{O}}(\mathbb{R},\mathbb{R})$, ϕ is continuous at 0. Choose $\tau > 0$ s.t., $\forall \alpha \in D$,

$$[\ 0 < |\alpha - 0| < \tau\ ] \quad \Rightarrow \quad [\ |[\phi(\alpha)] - [\phi(0)]| < \varepsilon\ ].$$

Let $\delta := \min\{\rho, \tau\}$. Then $\delta \leqslant \rho$ and $\delta \leqslant \tau$. We want: $\forall x \in \mathrm{dom}\,[\psi]$,

$$[\ 0 < |x - 0| < \delta\ ] \quad \Rightarrow \quad [\ |[\psi(x)] - [\psi(0)]| < \varepsilon\ ].$$

Let $x \in \mathrm{dom}\,[\psi]$ be given. We wish to show:

$$[\ 0 < |x - 0| < \delta\ ] \quad \Rightarrow \quad [\ |[\psi(x)] - [\psi(0)]| < \varepsilon\ ].$$

Assume: $0 < |x - 0| < \delta$. We wish to show: $|[\psi(x)] - [\psi(0)]| < \varepsilon$. Since $\psi(0) = 0$, we wish to prove: $|\psi(x)| < \varepsilon$.

Since I is an interval and $0 \in (-\delta, \delta) = I$, we know: $[0|x] \subseteq I$. Then $[0|x] \subseteq I \subseteq \mathrm{dom}\,[g']$, so g is c/d on $[0|x]$. So, by the Mean Value Theorem, choose $\alpha \in (0|x)$ s.t. $g'(\alpha) = DQ_g(0,x)$. Since $g(0) = 0$, we get $DQ_g(0,x) = [g(x)]/x$. We have $\alpha \in (0|x) \subseteq [0|x] \subseteq I$. So, since $g' = f$ on I, we get $g'(\alpha) = f(\alpha)$. Recall that $f = |\bullet|^k \cdot \phi$ and that $g = |\bullet|^{k+1} \cdot \psi$. Then

$$\begin{aligned} |\alpha|^k \cdot [\phi(\alpha)] \quad &= \quad f(\alpha) \quad = \quad g'(\alpha) \quad = \quad DQ_g(0,x) \\ &= \quad [g(x)]\,/\,x \quad = \quad |x|^{k+1} \cdot [\psi(x)]\,/\,x. \end{aligned}$$

Let $y := |x|$. Taking absolute values gives $|\alpha|^k \cdot |\phi(\alpha)| = y^{k+1} \cdot |\psi(x)|/y$. Then $|\psi(x)| = [|\alpha|/y]^k \cdot |\phi(\alpha)|$. Want: $[|\alpha|/y]^k \cdot |\phi(\alpha)| < \varepsilon$.

We have $-|x| \leqslant x \leqslant |x|$, *i.e.*, $-y \leqslant x \leqslant y$. Since $-y \leqslant 0$ and $-y \leqslant x$, we see that $-y \leqslant \min\{0, x\}$. Since $0 \leqslant y$ and $x \leqslant y$, see that $\max\{0, x\} \leqslant y$. Then $[0|x] = [\min\{0,x\}, \max\{0,x\}] \subseteq [-y, y]$. Then $\alpha \in (0|x) \subseteq [0|x] = [-y, y]$, so $-y \leqslant \alpha \leqslant y$, so $|\alpha| \leqslant y$. Then $[|\alpha|/y]^k \leqslant 1$. It therefore suffices to show: $|\phi(\alpha)| < \varepsilon$.

Since $\alpha \in I = (-\delta, \delta)$, we see that $|\alpha| < \delta$. Since $\alpha \in (0|x)$, we see that $\alpha \neq 0$, so $|\alpha| > 0$. Then $0 < |\alpha| < \delta \leqslant \tau$. Then $0 < |\alpha - 0| < \tau$. So, since $\alpha \in I \subseteq D$, by choice of τ, we get $|[\phi(\alpha)] - [\phi(0)]| < \varepsilon$. Since $\phi \in \breve{\mathcal{O}}(\mathbb{R},\mathbb{R})$, we have $\phi(0) = 0$. Then $|\phi(\alpha)| < \varepsilon$, as desired. QED

14-5. Let V, W be finite dimensional vector spaces. Let $f, g : V \dashrightarrow W$. Let $p \in V$. Assume: $f = g$ near p. Show: $\mathrm{LINS}_p f \subseteq \mathrm{LINS}_p g$.

Proof: We wish to show: $\forall L \in \mathrm{LINS}_p f$, $L \in \mathrm{LINS}_p g$. Let $L \in \mathrm{LINS}_p f$ be given. We wish to show: $L \in \mathrm{LINS}_p g$.

As $L \in \mathrm{LINS}_p f$, we get $f_p^T - L \in \check{\mathcal{O}}_1(V,W)$. Want: $g_p^T - L \in \check{\mathcal{O}}_1(V,W)$. By HW#14-2, it suffices to show: $f_p^T - L = g_p^T - L$ near 0_V.

Since $f = g$ near p, choose $U \in \mathcal{N}_V(p)$ s.t. $f = g$ on U. Choose $|\bullet| \in \mathcal{N}(V)$. Since $\mathcal{B}_V(p)$ is a neighborhood base at p in V, choose $\delta > 0$ s.t. $B_V(p,\delta) \subseteq U$. Let $A := B_V(0_V,\delta)$. Then $A \in \mathcal{N}_V(0_V)$, so it suffices to show: $f_p^T - L = g_p^T - L$ on A. We wish to show: $\forall h \in A$, $(f_p^T - L)(h) = (g_p^T - L)(h)$. Let $h \in A$ be given. We wish to show: $(f_p^T - L)(h) = (g_p^T - L)(h)$. Want: $[f_p^T(h)] - [L(h)] = [g_p^T(h)] - [L(h)]$. It suffices to prove: $f_p^T(h) = g_p^T(h)$.

Since $h \in A = B_V(0_V,\delta)$, we get $|h - 0_V| < \delta$. Let $x := p + h$. Then $|x - p| = |h| = |h - 0_V| < \delta$, so $x \in B_V(p,\delta)$. So, since $B_V(p,\delta) \subseteq U$, we get $x \in U$. So, since $f = g$ on U, we get $f(x) = g(x)$. As $U \in \mathcal{N}_V(p)$, we get $p \in U$. So, since $f = g$ on U, we get $f(p) = g(p)$. Then

$$\begin{aligned}(f_p^T)(h) &= [f(p+h)] - [f(p)]\\ &= [f(x)] - [f(p)]\\ &= [g(x)] - [g(p)]\\ &= [g(p+h)] - [g(p)] \quad = \quad (g_p^T)(h),\end{aligned}$$

as desired. QED

Homework 13: Due on Tuesday 24 April

13-1. Let V, W be finite dimensional VSs. Show: $L(V,W) \subseteq \widehat{\mathcal{O}}_1(V,W)$.

Proof: Want: $\forall T \in L(V,W)$, $T \in \widehat{\mathcal{O}}_1(V,W)$. Let $T \in L(V,W)$ be given. Want: $T \in \widehat{\mathcal{O}}_1(V,W)$.

Since $T \in L(V,W)$, it follows that $T(0_V) = 0_W$. Let $|\bullet|_V \in \mathcal{N}(V)$, $|\bullet|_W \in \mathcal{N}(W)$. Let $\alpha := \mathrm{adj}_{0_V}^{0_W}(T/|\bullet|_V)$. Then $T = |\bullet|_V \cdot \alpha$ on $V_{0_V}^{\times}$. So, since $T(0_V) = 0_W = 0 \cdot 0_W = |0_V|_V \cdot [\alpha(0_V)]$, we conclude that $T = |\bullet|_V \cdot \alpha$. It suffices to show: $\alpha \in \widehat{\mathcal{O}}(V,W)$. We wish to show: $\exists U \in \mathcal{N}_V(0_V)$ such that: $[\ U \subseteq \mathrm{dom}\,[\alpha]\ ]$ and $[\ \sup|\alpha_*(U)|_W < \infty\ ]$.

We have $V \in \mathcal{N}_V(0_V)$. Let $U := V$. We wish to show:

$$[\ U \subseteq \mathrm{dom}\,[\alpha]\ ] \quad \text{and} \quad [\ \sup|\alpha_*(U)|_W < \infty\ ].$$

Since $U = V \subseteq V = \mathrm{dom}\,[\alpha]$, it remains to show: $\sup|\alpha_*(U)|_W < \infty$.

By (1) of Theorem 46.2 p. 331, $T : V \to W$ is bounded. We define $M := \widehat{T}_{VW}$. Then $0 \leqslant M < \infty$. Also, $T : V \to W$ is M-bounded. It suffices to show: $\sup|\alpha_*(U)|_W \leqslant M$. We wish to show: $|\alpha_*(U)|_W \leqslant M$.

We wish to show: $\forall x \in \operatorname{dom}[\alpha]$, $[\ x \in U\ ] \ \Rightarrow\ [\ |\alpha(x)|_W \leqslant M\ ]$. Let $x \in \operatorname{dom}[\alpha]$ be given. We want: $[\ x \in U\ ] \ \Rightarrow\ [\ |\alpha(x)|_W \leqslant M\ ]$. Assume: $x \in U$. We wish to show: $|\alpha(x)|_W \leqslant M$.

One of the following must be true:

(1) $x = 0_V$ or
(2) $x \neq 0_V$.

Case (1): As $\alpha(x) = \alpha(0_V) = 0_W$, $|\alpha(x)|_W = |0_W|_W = 0 \leqslant M$, as desired. *End of case (1).*

Case (2): We have $\alpha(x) = [T(x)]/[|x|_V]$. Since T is M-bounded, we have $|T(x)|_W \leqslant M \cdot |x|_V$. Then

$$|\alpha(x)|_W \quad = \quad \left|\frac{T(x)}{|x|_V}\right|_W \quad = \quad \frac{|T(x)|_W}{|x|_V} \quad \leqslant \quad M,$$

as desired. *End of case (2).* QED

13-2. Let $m, n \in \mathbb{N}$. Let $V := (\mathbb{R}^m, |\bullet|_{m,1})$ and let $W := (\mathbb{R}^n, |\bullet|_{n,1})$. Let Z be a normed vector space. Let $* \in B(V, W, Z)$. Show: $\exists K \geqslant 0$ such that, $\forall v \in V$, $\forall w \in W$, $\quad |v * w|_Z \leqslant K \cdot |v|_V \cdot |w|_W$.

Proof: Let $K := \max\{|e_i^m * e_j^n|_Z \mid i \in [1..m], j \in [1..n]\}$. We wish to show: $\forall v \in V$, $\forall w \in W$, $|v * w|_Z \leqslant K \cdot |v|_V \cdot |w|_W$. Let $v \in V$, $w \in W$ be given. We wish to show: $|v * w|_Z \leqslant K \cdot |v|_V \cdot |w|_W$.

Since $|\bullet|_V = |\bullet|_{m,1}$, we see that $|v|_V = \sum_{i=1}^{m} |v_i|$. Since $|\bullet|_W = |\bullet|_{n,1}$, we see that $|w|_W = \sum_{j=1}^{n} |w_j|$. Since $v = \sum_{i=1}^{m} v_i e_i^m$ and $w = \sum_{j=1}^{n} w_j e_j^n$, we get $v * w = \sum_{i=1}^{m} \sum_{j=1}^{n} v_i w_j (e_i^m * e_j^n)$. Then, by subadditivity of $|\bullet|_Z$, we have $|v * w|_Z \leqslant \sum_{i=1}^{m} \sum_{j=1}^{n} |v_i w_j (e_i^m * e_j^n)|_Z$. We have: $\forall i \in [1..m]$, $\forall j \in [1..n]$,

$$|v_i w_j (e_i^m * e_j^n)|_Z \quad = \quad |v_i| \cdot |w_j| \cdot |e_i^m * e_j^n|_Z \quad \leqslant \quad |v_i| \cdot |w_j| \cdot K.$$

Then

$$\begin{aligned} |v * w|_Z &\leqslant \sum_{i=1}^{m} \sum_{j=1}^{n} (\,|v_i| \cdot |w_j| \cdot K\,) \\ &= K \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} (\,|v_i| \cdot |w_j|\,) \\ &= K \cdot \left[\sum_{i=1}^{m} |v_i| \right] \cdot \left[\sum_{j=1}^{n} |w_j| \right] \\ &= K \cdot |v|_V \cdot |w|_W, \end{aligned}$$

as desired. QED

13-3. Let $\ell, m, n \in \mathbb{N}$. Let $U := (\mathbb{R}^\ell, |\bullet|_{\ell,1})$, let $V := (\mathbb{R}^m, |\bullet|_{m,1})$ and let $W := (\mathbb{R}^n, |\bullet|_{n,1})$. Let Z be a normed vector space. Let $F \in T(U, V, W, Z)$. Show: $\exists K \geqslant 0$ such that, $\forall u \in U$, $\forall v \in V$, $\forall w \in W$,

$$|F(u, v, w)|_Z \quad \leqslant \quad K \cdot |u|_U \cdot |v|_V \cdot |w|_W.$$

Proof: Let $K := \max\{|F(e_h^\ell, e_i^m, e_j^n)|_Z \,|\, h \in [1..\ell], i \in [1..m], j \in [1..n]\}$. We wish to show: $\forall u \in U$, $\forall v \in V$, $\forall w \in W$,

$$|F(u, v, w)|_Z \quad \leqslant \quad K \cdot |u|_U \cdot |v|_V \cdot |w|_W.$$

Given $u \in U$, $v \in V$, $w \in W$. Want: $|F(u, v, w)|_Z \leqslant K \cdot |u|_U \cdot |v|_V \cdot |w|_W$.

Since $|\bullet|_U = |\bullet|_{\ell,1}$, we see that $|u|_U = \sum_{h=1}^{\ell} |u_h|$. Since $|\bullet|_V = |\bullet|_{m,1}$, we see that $|v|_V = \sum_{i=1}^{m} |v_i|$. Since $|\bullet|_W = |\bullet|_{n,1}$, we see that $|w|_W = \sum_{j=1}^{n} |w_j|$. Since $u = \sum_{h=1}^{m} u_h e_h^\ell$ and $v = \sum_{i=1}^{m} v_i e_i^m$ and $w = \sum_{j=1}^{n} w_j e_j^n$, we get

$$F(u, v, w) \quad = \quad \sum_{h=1}^{\ell} \sum_{i=1}^{m} \sum_{j=1}^{n} u_h v_i w_j (F(e_h^\ell, e_i^m, e_j^n)).$$

Then, by subadditivity of $|\bullet|_Z$, we have

$$|F(u, v, w)|_Z \quad \leqslant \quad \sum_{h=1}^{\ell} \sum_{i=1}^{m} \sum_{j=1}^{n} |u_h v_i w_j (F(e_h^\ell, e_i^m, e_j^n))|_Z.$$

We have: $\forall h \in [1..\ell]$, $\forall i \in [1..m]$, $\forall j \in [1..n]$,

$$\begin{aligned} |u_h v_i w_j (F(e_h^\ell, e_i^m, e_j^n))|_Z &= |u_h| \cdot |v_i| \cdot |w_j| \cdot |F(e_h^\ell, e_i^m, e_j^n)|_Z \\ &\leqslant |u_h| \cdot |v_i| \cdot |w_j| \cdot K. \end{aligned}$$

Then

$$\begin{aligned} |F(u,v,w)|_Z &\leqslant \sum_{h=1}^{\ell} \sum_{i=1}^{m} \sum_{j=1}^{n} (\, |u_h| \cdot |v_i| \cdot |w_j| \cdot K \,) \\ &= K \cdot \sum_{h=1}^{\ell} \sum_{i=1}^{m} \sum_{j=1}^{n} (\, |u_h| \cdot |v_i| \cdot |w_j| \,) \\ &= K \cdot \left[\sum_{h=1}^{\ell} |u_h| \right] \cdot \left[\sum_{i=1}^{m} |v_i| \right] \cdot \left[\sum_{j=1}^{n} |w_j| \right] \\ &= K \cdot |u|_U \cdot |v|_V \cdot |w|_W, \end{aligned}$$

as desired. QED

13-4. Let U, V and W be finite dimensional normed vector spaces. Let Z be a normed vector space. Let $F \in T(U,V,W,Z)$. Show: $\exists K \geqslant 0$ such that, $\forall u \in U$, $\forall v \in V$, $\forall w \in W$,

$$|F(u,v,w)|_Z \quad \leqslant \quad K \cdot |u|_U \cdot |v|_V \cdot |w|_W.$$

Proof: Let $\ell := \dim U$, $m := \dim V$ and $n := \dim W$. Choose

$$A \in \mathrm{OB}(U), \qquad B \in \mathrm{OB}(V) \qquad \text{and} \qquad C \in \mathrm{OB}(W).$$

Let $U' := (\mathbb{R}^\ell, |\bullet|_{\ell,1})$, let $V' := (\mathbb{R}^m, |\bullet|_{m,1})$ and let $W' := (\mathbb{R}^n, |\bullet|_{n,1})$. Define $F' \in T(U',V',W',Z)$ by

$$F'\,(\,u\,,\,v\,,\,w\,) \quad = \quad F\,(\,L_A(u)\,,\,L_B(v)\,,\,L_C(w)\,).$$

By HW#13-3, choose $K' \geqslant 0$ s.t., $\forall u' \in U'$, $\forall v' \in V'$, $\forall w' \in W'$,

$$|F'(u',v',w')|_Z \quad \leqslant \quad K' \cdot |u'|_{U'} \cdot |v'|_{V'} \cdot |w'|_{W'}.$$

By (1) of Theorem 46.2 p. 331, the maps

$$L_A^{-1} : U \to U', \qquad L_B^{-1} : V \to V' \qquad \text{and} \qquad L_C^{-1} : W \to W'$$

are all bounded. Let

$$Q := (L_A^{-1})\widehat{_{U,U'}}, \quad R := (L_B^{-1})\widehat{_{V,V'}} \quad \text{and} \quad S := (L_C^{-1})\widehat{_{W,W'}}.$$

Then $L_A^{-1} : U \to U'$ is Q-bounded, $L_B^{-1} : V \to V'$ is R-bounded and $L_C^{-1} : W \to W'$ is S-bounded. Let $K := K' \cdot Q \cdot R \cdot S$. We wish to show: $\forall u \in U$, $\forall v \in V$, $\forall w \in W$,

$$|F(u,v,w)|_Z \quad \leqslant \quad K \cdot |u|_U \cdot |v|_V \cdot |w|_W.$$

Let $u \in U$, $v \in V$ and $w \in W$ be given. We wish to show:

$$|F(u,v,w)|_Z \quad \leqslant \quad K \cdot |u|_U \cdot |v|_V \cdot |w|_W.$$

Since $L_A^{-1} : U \to U'$ is Q-bounded, since $L_B^{-1} : V \to V'$ is R-bounded and since $L_C^{-1} : W \to W'$ is S-bounded, we get

$$\begin{aligned} |L_A^{-1}(u)|_{U'} &\leqslant Q \cdot |u|_U, \\ |L_B^{-1}(v)|_{V'} &\leqslant R \cdot |v|_V \qquad \text{and} \\ |L_C^{-1}(w)|_{W'} &\leqslant S \cdot |w|_W. \end{aligned}$$

Let $u' := (L_A^{-1})(u)$, $v' := (L_B^{-1})(v)$ and $w' := (L_C^{-1})(w)$. Then

$$|u'|_{U'} \leqslant Q \cdot |u|_U, \quad |v'|_{V'} \leqslant R \cdot |v|_V, \quad \text{and} \quad |w'|_{W'} \leqslant S \cdot |w|_W.$$

By choice of K', we have

$$|F'(u',v',w')|_Z \quad \leqslant \quad K' \cdot |u'|_{U'} \cdot |v'|_{V'} \cdot |w'|_{W'}.$$

Since $L_A(u') = u$, $L_B(v') = v$ and $L_C(w') = w$, by definition of F', we get $F'(u',v',w') = F(u,v,w)$. Then

$$\begin{aligned} |F(u,v,w)|_Z &\leqslant |F'(u',v',w')|_Z \\ &\leqslant K' \cdot |u'|_{U'} \cdot |v'|_{V'} \cdot |w'|_{W'} \\ &\leqslant K' \cdot Q \cdot |u|_U \cdot R \cdot |v|_V \cdot S \cdot |w|_W \\ &= K' \cdot Q \cdot R \cdot S \cdot |u|_U \cdot |v|_V \cdot |w|_W \\ &= K \cdot |u|_U \cdot |v|_V \cdot |w|_W, \end{aligned}$$

as desired. QED

13-5. Let S V, W and Z all be finite dimensional vector spaces, and let $* \in B(V,W,Z)$. Show: $[\widehat{\mathcal{O}}(S,V)] \underset{S}{*} [\widehat{\mathcal{O}}(S,W)] \subseteq \widehat{\mathcal{O}}(S,Z)$.

Proof: We wish to show: $\forall \alpha \in \widehat{\mathcal{O}}(S,V)$, $\forall \beta \in \widehat{\mathcal{O}}(S,W)$, we have: $\alpha \underset{S}{*} \beta \in \widehat{\mathcal{O}}(S,Z)$. Let $\alpha \in \widehat{\mathcal{O}}(S,V)$ and $\beta \in \widehat{\mathcal{O}}(S,W)$ be given. We wish to show: $\alpha \underset{S}{*} \beta \in \widehat{\mathcal{O}}(S,Z)$.

Choose $|\bullet|_S \in \mathcal{N}(S)$ and $|\bullet|_V \in \mathcal{N}(V)$ and $|\bullet|_W \in \mathcal{N}(W)$ and $|\bullet|_Z \in \mathcal{N}(Z)$. By Theorem 49.1 p. 359, choose $C \geqslant 0$ such that, $\forall v \in V$, $\forall w \in W$, $|v * w|_Z \leqslant C \cdot |v|_V \cdot |w|_W$.

Since $\alpha \in \widehat{\mathcal{O}}(S, V)$, choose $P \in \mathcal{N}_S(0_S)$ such that

$$P \subseteq \mathrm{dom}\,[\alpha] \qquad \text{and} \qquad \sup |\alpha_*(P)|_V < \infty.$$

Let $K := \sup |\alpha_*(P)|_V$. Then $0 \leqslant K < \infty$ and $|\alpha_*(P)|_V \leqslant K$.

Since $\beta \in \widehat{\mathcal{O}}(S, W)$, choose $Q \in \mathcal{N}_S(0_S)$ such that

$$Q \subseteq \mathrm{dom}\,[\beta] \qquad \text{and} \qquad \sup |\beta_*(Q)|_W < \infty.$$

Let $L := \sup |\beta_*(Q)|_W$. Then $0 \leqslant L < \infty$ and $|\beta_*(Q)|_W \leqslant L$.

Let $\gamma := \alpha \underset{S}{*} \beta$. We wish to show: $\gamma \in \widehat{\mathcal{O}}(S, Z)$. We wish to show: $\exists R \in \mathcal{N}_S(0_S)$ such that

$$R \subseteq \mathrm{dom}\,[\gamma] \qquad \text{and} \qquad \sup |\gamma_*(R)|_Z < \infty.$$

Since $P \in \mathcal{N}_S(0_S)$ and $Q \in \mathcal{N}_S(0_S)$, it follows that $P \cap Q \in \mathcal{N}_S(0_S)$. Let $R := P \cap Q$. We wish to show:

$$R \subseteq \mathrm{dom}\,[\gamma] \qquad \text{and} \qquad \sup |\gamma_*(R)|_Z < \infty.$$

Since $\gamma = \alpha \underset{S}{*} \beta$, it follows that $\mathrm{dom}\,[\gamma] = (\mathrm{dom}\,[\alpha]) \cap (\mathrm{dom}\,[\beta])$. Since $P \subseteq \mathrm{dom}\,[\alpha]$ and $Q \subseteq \mathrm{dom}\,[\beta]$, it follows that

$$P \cap Q \qquad \subseteq \qquad (\mathrm{dom}\,[\alpha]) \;\cap\; (\mathrm{dom}\,[\beta]).$$

Then $R = P \cap Q \subseteq (\mathrm{dom}\,[\alpha]) \cap (\mathrm{dom}\,[\beta]) = \mathrm{dom}\,[\gamma]$. It remains to show that $\sup |\gamma_*(R)|_Z < \infty$.

Let $M := CKL$. Then $0 \leqslant M < \infty$. It suffices to show that $\sup |\gamma_*(R)|_Z \leqslant M$. We wish to show: $\forall x \in \mathrm{dom}\,[\gamma]$,

$$[\ x \in R\ ] \quad \Rightarrow \quad [\ |\gamma(x)|_Z \leqslant M\ ].$$

Let $x \in \mathrm{dom}\,[\gamma]$ be given. We wish to show:

$$[\ x \in R\ ] \quad \Rightarrow \quad [\ |\gamma(x)|_Z \leqslant M\ ].$$

Assume: $x \in R$. Want: $|\gamma(x)|_Z \leqslant M$.

We have $x \in \mathrm{dom}\,[\gamma] \subseteq \mathrm{dom}\,[\alpha]$ and $x \in R \subseteq P$, so $\alpha(x) \in \alpha_*(P)$. Then $|\alpha(x)|_V \in |\alpha_*(P)|_V \leqslant K$. Let $v := \alpha(x)$. Then $|v|_V \leqslant K$. We have $x \in \mathrm{dom}\,[\gamma] \subseteq \mathrm{dom}\,[\beta]$ and $x \in R \subseteq Q$, so $\beta(x) \in \beta_*(Q)$. Then $|\beta(x)|_W \in |\beta_*(Q)|_W \leqslant L$. Let $w := \beta(x)$. Then $|w|_W \leqslant L$.

We have $\gamma(x) = (\alpha \underset{S}{*} \beta)(x) = [\alpha(x)] * [\beta(x)] = v * w$. By the choice of C, we have $|v * w|_Z \leqslant C \cdot |v|_V \cdot |w|_W$.

Then $|\gamma(x)|_Z = |v * w|_Z \leqslant C \cdot |v|_V \cdot |w|_W \leqslant CKL = M$. QED

12-1. Let V and W both be finite dimensional vector spaces, and let $|\bullet|, \|\bullet\| \in \mathcal{N}(W)$. Show: $\widehat{\mathcal{O}}(V,W,|\bullet|) \subseteq \widehat{\mathcal{O}}(V,W,\|\bullet\|)$.

Proof: Want: $\forall \alpha \in \widehat{\mathcal{O}}(V,W,|\bullet|)$, $\alpha \in \widehat{\mathcal{O}}(V,W,\|\bullet\|)$. Let $\alpha \in \widehat{\mathcal{O}}(V,W,|\bullet|)$ be given. Want: $\alpha \in \widehat{\mathcal{O}}(V,W,\|\bullet\|)$. Want: $\exists U \in \mathcal{N}_V(0_V)$ such that

$$[\ U \subseteq \mathrm{dom}\,[\alpha]\ ] \qquad \text{and} \qquad [\ \sup \|\alpha_*(U)\| < \infty\ ].$$

Since $\alpha \in \widehat{\mathcal{O}}(V,W,|\bullet|)$, choose $U \in \mathcal{N}_V(0_V)$ such that

$$[\ U \subseteq \mathrm{dom}\,[\alpha]\ ] \qquad \text{and} \qquad [\ \sup |\alpha_*(U)| < \infty\ ].$$

Want: $\sup \|\alpha_*(U)\| < \infty$.

Let $M := \sup |\alpha_*(U)|$. Then $|\alpha_*(U)| \leqslant M$. By Theorem 46.3 p. 332, $|\bullet| \approx \|\bullet\|$, so $\|\bullet\| << |\bullet|$, so choose $K > 0$ s.t. $\|\bullet\| \leqslant K \cdot |\bullet|$. It suffices to show: $\|\alpha_*(U)\| \leqslant KM$. We want: $\forall x \in \mathrm{dom}\,[\alpha]$,

$$[\ x \in U\ ] \quad \Rightarrow \quad [\ \|\alpha(x)\| \leqslant KM\ ].$$

Let $x \in \mathrm{dom}\,[\alpha]$ be given. We wish to prove:

$$[\ x \in U\ ] \quad \Rightarrow \quad [\ \|\alpha(x)\| \leqslant KM\ ].$$

Assume that $x \in U$. We wish to show: $\|\alpha(x)\| \leqslant KM$.

Since $x \in \mathrm{dom}\,[\alpha]$ and $x \in U$, we get $|\alpha(x)| \in |\alpha_*(U)|$. So, since $|\alpha_*(U)| \leqslant M$, we get $|\alpha(x)| \leqslant M$. Since $\|\bullet\| \leqslant K \cdot |\bullet|$, we get $\|\alpha(x)\| \leqslant K \cdot |\alpha(x)|$. Then $\|\alpha(x)\| \leqslant K \cdot |\alpha(x)| \leqslant KM$, as desired. QED

12-2. Let V and W both be finite dimensional vector spaces, and let $\alpha : V \dashrightarrow W$. Assume that $\mathrm{dom}\,[\alpha] \in \mathcal{N}_V(0_V)$. Assume that α is continuous at 0_V. Show: $\alpha \in \widehat{\mathcal{O}}(V,W)$.

Proof: Choose $|\bullet|_V \in \mathcal{N}(V)$ and $|\bullet|_W \in \mathcal{N}(W)$. We wish to show: $\exists U \in \mathcal{N}_V(0_V)$ such that

$$[\ U \subseteq \mathrm{dom}\,[\alpha]\ ] \quad \text{and} \quad [\ \sup |\alpha_*(U)|_W < \infty\ ].$$

Let $z := \alpha(0_V)$. Since α is continuous at 0_V, choose $\delta > 0$ such that, $\forall x \in \mathrm{dom}\,[\alpha]$, we have: $[\,|x|_V < \delta\,] \Rightarrow [\,|[\alpha(x)] - z|_W < 1\,]$. We define $U := (\mathrm{dom}\,[\alpha]) \cap (B_V(0_V,\delta))$. Since $\mathrm{dom}\,[\alpha] \in \mathcal{N}_V(0_V)$ and since $B_V(0_V,\delta) \in \mathcal{N}_V(0_V)$, it follows that $U \in \mathcal{N}_V(0_V)$. Moreover, we have both $U \subseteq \mathrm{dom}\,[\alpha]$ and $U \subseteq B_V(0_V,\delta)$. It remains to show: $\sup |\alpha_*(U)|_W < \infty$. Let $M := 1 + [\,|z|_W\,]$. It suffices to show: $|\alpha_*(U)|_W \leqslant M$. We wish to prove: $\forall x \in \mathrm{dom}\,[\alpha]$,

$$[\ x \in U\ ] \quad \Rightarrow \quad [\ |\alpha(x)|_W \leqslant M\ ].$$

Let $x \in \mathrm{dom}\,[\alpha]$ be given. We wish to prove:

$$[\ x \in U\ ] \quad \Rightarrow \quad [\ |\alpha(x)|_W \leqslant M\ ].$$

Assume that $x \in U$. We wish to prove: $|\alpha(x)|_W \leqslant M$.

Since $x \in U \subseteq B_V(0_V, \delta)$, we conclude that $|x|_V < \delta$. So, since $x \in \mathrm{dom}\,[\alpha]$, by choice of δ, we conclude that: $|[\alpha(x)] - z|_W < 1$. Let $y := \alpha(x)$. Then $|y - z|_W < 1$, and we wish to prove: $|y|_W \leqslant M$.

Since $y = (y - z) + z$, we get $|y|_W \leqslant [\,|y - z|_W\,] + [\,|z|_W\,]$. Then $|y|_W < 1 + [\,|z|_W\,] = M$, as desired. QED

12-3. Let V and W both be finite dimensional vector spaces, and let $|\bullet|, \|\bullet\| \in \mathcal{N}(V)$, and let $p > 0$. Show:

(1) $\breve{\mathcal{O}}_p(V, W, |\bullet|) \subseteq \breve{\mathcal{O}}_p(V, W, \|\bullet\|)$ and

(2) $\widehat{\mathcal{O}}_p(V, W, |\bullet|) \subseteq \widehat{\mathcal{O}}_p(V, W, \|\bullet\|)$.

Proof: Let $\alpha := \mathrm{adj}^1_{0_V}\,(\,[\,\|\bullet\|^p\,]/[\,|\bullet|^p\,]\,)$, $\beta := \mathrm{adj}^1_{0_V}\,(\,[\,|\bullet|^p\,]/[\,\|\bullet\|^p\,]\,)$. Then $\alpha, \beta : V \to \mathbb{R}$. Also, for all $x \in V\backslash\{0_V\}$, we have:

$$(\alpha\beta)(x) \quad = \quad \left[\frac{\|x\|^p}{|x|^p}\right] \cdot \left[\frac{|x|^p}{\|x\|^p}\right] \quad = \quad 1 \quad = \quad C^1_V(x).$$

So, since $(\alpha\beta)(0_V) = 1 \cdot 1 = 1 = C^1_V(0_V)$, we conclude that $\alpha\beta = C^1_V$.

By Theorem 46.3 p. 332, $|\bullet| \approx \|\bullet\|$, so $|\bullet| << \|\bullet\|$, so choose $C > 0$ such that $|\bullet| \leqslant C \cdot \|\bullet\|$. Then, for all $x \in V\backslash\{0_V\}$,

$$\beta(x) \quad = \quad \frac{|x|^p}{\|x\|^p} \quad = \quad \left[\frac{|x|}{\|x\|}\right]^p \quad \leqslant \quad \left[\frac{C \cdot \|x\|}{\|x\|}\right]^p \quad = \quad C^p.$$

Let $K := \max\{C^p, 1\}$. Then $K \geqslant 1$, so $K > 0$. Also, for all $x \in V$, we have $0 \leqslant \beta(x) \leqslant K$, and so $|\beta(x)| = \beta(x) \leqslant K$. Choose $|||\bullet||| \in \mathcal{N}(W)$.

Claim A: $\breve{o}(V, W) \subseteq \alpha \cdot [\,\breve{o}(V, W)\,]$. *Proof of Claim A:* We wish to show: $\forall \gamma \in \breve{o}(V, W)$, $\gamma \in \alpha \cdot [\,\breve{o}(V, W)\,]$. Let $\gamma \in \breve{o}(V, W)$ be given. Want: $\gamma \in \alpha \cdot [\,\breve{o}(V, W)\,]$. Since $\gamma = \gamma \cdot C^1_V = \gamma \cdot [\alpha\beta] = \alpha \cdot [\beta\gamma]$, it suffices to show that $\beta\gamma \in \breve{o}(V, W)$. We wish to show:

- $\mathrm{dom}\,[\beta\gamma] \in \mathcal{N}_V(0_V)$,
- $(\beta\gamma)(0_V) = 0_W$ and
- $\beta\gamma : V \dashrightarrow W$ is continuous at 0_V.

Since $\gamma \in \breve{o}(V, W)$, it follows that:

- $\operatorname{dom}[\gamma] \in \mathcal{N}_V(0_V)$,
- $\gamma(0_V) = 0_W$ and
- $\gamma : V \dashrightarrow W$ is continuous at 0_V.

Since $\operatorname{dom}[\beta] = V$ and since $\operatorname{dom}[\gamma] \subseteq V$, we get $\operatorname{dom}[\beta\gamma] = \operatorname{dom}[\gamma]$. Then $\operatorname{dom}[\beta\gamma] = \operatorname{dom}[\gamma] \in \mathcal{N}_V(0_V)$. Also, $(\beta\gamma)(0_V) = 1 \cdot 0_W = 0_W$. It remains to show: $\beta\gamma : V \dashrightarrow W$ is continuous at 0_V. We wish to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ such that, for all $x \in \operatorname{dom}[\beta\gamma]$,

$$[\ |x| < \delta\ ] \quad \Rightarrow \quad [\ |||(\beta\gamma)(x)||| < \varepsilon\ ].$$

Let $\varepsilon > 0$ be given. Want: $\exists \delta > 0$ such that, for all $x \in \operatorname{dom}[\beta\gamma]$,

$$[\ |x| < \delta\ ] \quad \Rightarrow \quad [\ |||(\beta\gamma)(x)||| < \varepsilon\ ].$$

As $\gamma : V \dashrightarrow W$ is continuous at 0_V, choose $\delta > 0$ s.t., $\forall x \in \operatorname{dom}[\gamma]$,

$$[\ |x| < \delta\ ] \quad \Rightarrow \quad [\ |||\gamma(x)||| < \varepsilon/K\ ].$$

We wish to show: $\forall x \in \operatorname{dom}[\beta\gamma]$,

$$[\ |x| < \delta\ ] \quad \Rightarrow \quad [\ |||(\beta\gamma)(x)||| < \varepsilon\ ].$$

Let $x \in \operatorname{dom}[\beta\gamma]$ be given. We wish to show:

$$[\ |x| < \delta\ ] \quad \Rightarrow \quad [\ |||(\beta\gamma)(x)||| < \varepsilon\ ].$$

Assume that $|x| < \delta$. We wish to show: $|||(\beta\gamma)(x)||| < \varepsilon$.

We have $x \in \operatorname{dom}[\beta\gamma] \subseteq \operatorname{dom}[\gamma]$. So, since $|x| < \delta$, by choice of δ, we get: $|||\gamma(x)||| < \varepsilon/K$.

We have $x \in \operatorname{dom}[\beta\gamma] \subseteq \operatorname{dom}[\gamma]$, so, as $x \in U$, $\gamma(x) \in \gamma_*(U)$. Then

$$|||\gamma(x)||| \quad \in \quad |||\gamma_*(U)||| \quad \leqslant \quad \sup |||\gamma_*(U)||| \quad = \quad M.$$

Recall: $|\beta(x)| \leqslant K$. Then

$$\begin{aligned} |||(\beta\gamma)(x)||| &= |||[\beta(x)] \cdot [\gamma(x)]||| \\ &= |\beta(x)| \cdot |||\gamma(x)||| \\ &< K \cdot (\varepsilon/K) \ = \ \varepsilon, \end{aligned}$$

as desired. *End of proof of Claim A.*

Claim B: $\widehat{\mathcal{O}}(V,W) \subseteq \alpha \cdot [\,\widehat{\mathcal{O}}(V,W)\,]$. *Proof of Claim B:* We wish to show: $\forall \gamma \in \widehat{\mathcal{O}}(V,W)$, $\gamma \in \alpha \cdot [\,\widehat{\mathcal{O}}(V,W)\,]$. Let $\gamma \in \widehat{\mathcal{O}}(V,W)$ be given. Want: $\gamma \in \alpha \cdot [\,\widehat{\mathcal{O}}(V,W)\,]$. Since $\gamma = \gamma \cdot C_V^1 = \gamma \cdot [\alpha\beta] = \alpha \cdot [\beta\gamma]$, it suffices to show that $\beta\gamma \in \widehat{\mathcal{O}}(V,W)$. Want: $\exists U \in \mathcal{N}_V(0_V)$ such that

$$U \subseteq \operatorname{dom}[\beta\gamma] \qquad \text{and} \qquad \sup |||(\beta\gamma)_*(U)||| < \infty.$$

Since $\gamma \in \widehat{\mathcal{O}}(V,W) = \widehat{\mathcal{O}}(V,W,|\bullet|)$, choose $U \in \mathcal{N}_V(0_V)$ such that

$$U \subseteq \mathrm{dom}\,[\gamma] \qquad \text{and} \qquad \sup|||\gamma_*(U)||| < \infty.$$

We wish to show:

$$U \subseteq \mathrm{dom}\,[\beta\gamma] \qquad \text{and} \qquad \sup|||(\beta\gamma)_*(U)||| < \infty.$$

As $\mathrm{dom}\,[\beta] = V$ and $\mathrm{dom}\,[\gamma] \subseteq V$, we get $\mathrm{dom}\,[\beta\gamma] = \mathrm{dom}\,[\gamma]$. Then $U \subseteq \mathrm{dom}\,[\gamma] = \mathrm{dom}\,[\beta\gamma]$. It remains to show: $\sup|||(\beta\gamma)_*(U)||| < \infty$.

Let $M := \sup|||\gamma_*(U)|||$. It suffices to show: $|||(\beta\gamma)_*(U)||| \leqslant KM$. We wish to show: $\forall x \in \mathrm{dom}\,[\beta\gamma]$,

$$[\ x \in U\ ] \quad \Rightarrow \quad [\ |||(\beta\gamma)(x)||| \leqslant KM\ ].$$

Let $x \in \mathrm{dom}\,[\beta\gamma]$ be given. We wish to show:

$$[\ x \in U\ ] \quad \Rightarrow \quad [\ |||(\beta\gamma)(x)|| \leqslant KM\ ].$$

Assume that $x \in U$. Want: $|||(\beta\gamma)(x)||| \leqslant KM$

We have $x \in \mathrm{dom}\,[\beta\gamma] \subseteq \mathrm{dom}\,[\gamma]$, so, as $x \in U$, $\gamma(x) \in \gamma_*(U)$. Then

$$|||\gamma(x)||| \quad \in \quad |||\gamma_*(U)||| \quad \leqslant \quad \sup|||\gamma_*(U)||| \quad = \quad M.$$

Recall: $|\beta(x)| \leqslant K$. Then

$$|||(\beta\gamma)(x)||| \;=\; |||[\beta(x)]\cdot[\gamma(x)]||| \;=\; |\beta(x)|\cdot|||\gamma(x)||| \;\leqslant\; KM,$$

as desired. *End of proof of Claim B.*

Claim C: $[\,|\bullet|^p\,]\cdot\alpha = \|\bullet\|^p$. *Proof of Claim C:* On $V\backslash\{0_V\}$, we have $[\,|\bullet|^p\,]\cdot\alpha = [\,|\bullet|^p\,]\cdot[\,\|\bullet\|^p/|\bullet|^p\,] = \|\bullet\|^p$, so it suffices to show that $(\,[\,|\bullet|^p\,]\cdot\alpha\,)(0_V) = (\,\|\bullet\|^p\,)(0_V)$.

We have $(\,[\,|\bullet|^p\,]\cdot\alpha\,)(0_V) = 0^p\cdot 1 = 0^p = (\,\|\bullet\|^p\,)(0_V)$, as desired. *End of proof of Claim C.*

Proof of (1): By Claim A, $[\,|\bullet|^p\,]\cdot[\,\breve{\mho}(V,W)\,] \subseteq [\,|\bullet|^p\,]\cdot\alpha\cdot[\,\breve{\mho}(V,W)\,]$. By Claim C, $[\,|\bullet|^p\,]\cdot\alpha\cdot[\,\breve{\mho}(V,W)\,] = \|\bullet\|^p\cdot[\,\breve{\mho}(V,W)\,]$. Then

$$\begin{aligned}
\breve{\mho}_p(V,W,|\bullet|) &= [\,|\bullet|^p\,]\cdot[\,\breve{\mho}(V,W)\,]\\
&\subseteq [\,|\bullet|^p\,]\cdot\alpha\cdot[\,\breve{\mho}(V,W)\,]\\
&= [\,\|\bullet\|^p\,]\cdot[\,\breve{\mho}(V,W)\,]\\
&= \breve{\mho}_p(V,W,\|\bullet\|),
\end{aligned}$$

as desired. *End of proof of (1).*

Proof of (2): By Claim B, $[\,|\bullet|^p\,]\cdot[\,\widehat{\mathcal{O}}(V,W)\,]\subseteq[\,|\bullet|^p\,]\cdot\alpha\cdot[\,\widehat{\mathcal{O}}(V,W)\,]$. By Claim C, $[\,|\bullet|^p\,]\cdot\alpha\cdot[\,\widehat{\mathcal{O}}(V,W)\,]=\|\bullet\|^p\cdot[\,\widehat{\mathcal{O}}(V,W)\,]$. Then

$$\begin{aligned}\widehat{\mathcal{O}}_p(V,W,|\bullet|) &= [\,|\bullet|^p\,]\cdot[\,\widehat{\mathcal{O}}(V,W)\,]\\ &\subseteq [\,|\bullet|^p\,]\cdot\alpha\cdot[\,\widehat{\mathcal{O}}(V,W)\,]\\ &= [\,\|\bullet\|^p\,]\cdot[\,\widehat{\mathcal{O}}(V,W)\,]\\ &= \widehat{\mathcal{O}}_p(V,W,\|\bullet\|),\end{aligned}$$

as desired. *End of proof of (2).* QED

12-4. Let V and W both be finite dimensional vector spaces, and let $p,q\geqslant 0$. Assume $p<q$. Show: $\breve{o}_p(V,W)\supseteq\widehat{\mathcal{O}}_q(V,W)$.

Proof: We want: $\forall\gamma\in\widehat{\mathcal{O}}_q(V,W)$, $\gamma\in\breve{o}_p(V,W)$. Let $\gamma\in\widehat{\mathcal{O}}_q(V,W)$ be given. We wish to show: $\gamma\in\breve{o}_p(V,W)$.

Chose $|\bullet|\in\mathcal{N}(V)$. Then $\widehat{\mathcal{O}}_q(V,W)=[|\bullet|^q]\cdot[\widehat{\mathcal{O}}(V,W)]$ and $\breve{o}_p(V,W)=[|\bullet|^p]\cdot[\breve{o}(V,W)]$. We have $\gamma\in[|\bullet|^q]\cdot[\widehat{\mathcal{O}}(V,W)]$. We wish to show: $\gamma\in[|\bullet|^p]\cdot\breve{o}(V,W)$.

Since $\gamma\in\widehat{\mathcal{O}}_q(V,W)=[|\bullet|^q]\cdot[\widehat{\mathcal{O}}(V,W)]$, choose $\alpha\in\widehat{\mathcal{O}}(V,W)$ such that $\gamma=[|\bullet|^q]\cdot\alpha$. Choose $\|\bullet\|\in\mathcal{N}(W)$. Since $\alpha\in\widehat{\mathcal{O}}(V,W)$, choose $U\in\mathcal{N}_V(0_V)$ such that

$$U\subseteq\operatorname{dom}[\alpha]\qquad\text{and}\qquad\sup\|\alpha_*(U)\|<\infty.$$

Since $\operatorname{dom}[\alpha]\supseteq U\in\mathcal{N}_V(0_V)$, we see that $\operatorname{dom}[\alpha]\in\mathcal{N}_V(0_V)$.

Let $\beta:=[|\bullet|^{q-p}]\cdot\alpha$. Then $[|\bullet|^p]\cdot\beta=[|\bullet|^q]\cdot\alpha$. Then

$$\gamma\quad=\quad[\,|\bullet|^q\,]\cdot\alpha\quad=\quad[\,|\bullet|^p\,]\cdot\beta,$$

and so it suffices to show: $\beta\in\breve{o}(V,W)$. We wish to show:

- $\operatorname{dom}[\beta]\in\mathcal{N}_V(0_V)$,
- $\beta(0_V)=0_W$ and
- $\beta:V\dashrightarrow W$ is continuous at 0_V.

Because $\operatorname{dom}[|\bullet|^{q-p}]=V$ and because $\operatorname{dom}[\alpha]\subseteq V$, we conclude that $\operatorname{dom}[\,[|\bullet|^{q-p}]\cdot\alpha\,]=\operatorname{dom}[\alpha]$. Then

$$\operatorname{dom}[\beta]\quad=\quad\operatorname{dom}[\,[|\bullet|^{q-p}]\cdot\alpha\,]\quad=\quad\operatorname{dom}[\alpha].$$

Then $\operatorname{dom}[\beta]=\operatorname{dom}[\alpha]\in\mathcal{N}_v(0_V)$. Also,

$$\beta(0_V)\quad=\quad([|\bullet|^{q-p}]\cdot\alpha)(0_V)\quad=\quad 0^{q-p}\cdot[\alpha(0_V)]\quad=\quad 0_W.$$

It remains to show: $\beta : V \dashrightarrow W$ is continuous at 0_V. We wish to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ such that, $\forall x \in \mathrm{dom}\,[\beta]$,

$$[\ |x| < \delta\ ] \quad \Rightarrow \quad [\ \|\beta(x)\| < \varepsilon\ ].$$

Let $\varepsilon > 0$ be given. We wish to show: $\exists \delta > 0$ such that, $\forall x \in \mathrm{dom}\,[\beta]$,

$$[\ |x| < \delta\ ] \quad \Rightarrow \quad [\ \|\beta(x)\| < \varepsilon\ ].$$

Since $U \in \mathcal{N}_V(0_V)$ and since $\mathcal{B}_V(0_V)$ is a neighborhood base at 0_V in V, choose $\rho > 0$ such that $B(0_V, \rho) \subseteq U$. Let $M := (\sup \|\alpha_*(U)\|)+1$. Then $M > 0$ and $\|\alpha_*(U)\| < M$. Let $\tau := (\varepsilon/M)^{1/(q-p)}$. We define $\delta := \min\{\rho, \tau\}$. We wish to show: $\forall x \in \mathrm{dom}\,[\beta]$,

$$[\ |x| < \delta\ ] \quad \Rightarrow \quad [\ \|\beta(x)\| < \varepsilon\ ].$$

Let $x \in \mathrm{dom}\,[\beta]$ be given. We wish to show:

$$[\ |x| < \delta\ ] \quad \Rightarrow \quad [\ \|\beta(x)\| < \varepsilon\ ].$$

Assume that $|x| < \delta$. We wish to show: $\|\beta(x)\| < \varepsilon$.

Since $|x| < \delta \leqslant \rho$, we get $x \in B_V(0_V, \rho)$. So, since $B_V(0_V, \rho) \subseteq U$, $x \in U$. So, since $x \in \mathrm{dom}\,[\beta] = \mathrm{dom}\,[\alpha]$, we have $\alpha(x) \in \alpha_*(U)$. Then $\|\alpha(x)\| \in \|\alpha_*(U)\| < M$. Since $\beta = [|\bullet|^{q-p}] \cdot \alpha$, we conclude that $\beta(x) = [|x|^{q-p}] \cdot [\alpha(x)]$. Since $|x| < \delta \leqslant \tau$, we get $|x|^{q-p} < \tau^{q-p}$. Then $\|\beta(x)\| = [|x|^{q-p}] \cdot \|\alpha(x)\| < [\tau^{q-p}] \cdot M = [\varepsilon/M] \cdot M = \varepsilon$. QED

12-5. Let $p, q \geqslant 0$. Show:

$$(\ [\ (\breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R})) \cdot (\breve{\mathcal{O}}_q(\mathbb{R},\mathbb{R})) \subseteq \breve{\mathcal{O}}_{p+q}(\mathbb{R},\mathbb{R})\ ] \qquad \text{and}$$
$$[\ (\breve{\mathcal{O}}_q(\mathbb{R},\mathbb{R})) \circ (\breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R})) \subseteq \breve{\mathcal{O}}_{qp}(\mathbb{R},\mathbb{R})\ ]\ ).$$

Proof: Let $\breve{\mathcal{O}} := \breve{\mathcal{O}}(\mathbb{R},\mathbb{R})$. By (1) of Fact 47.10 p. 342, we have $\breve{\mathcal{O}} \cdot \breve{\mathcal{O}} \subseteq \breve{\mathcal{O}}$. Then

$$\begin{aligned}
(\breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R})) \cdot (\breve{\mathcal{O}}_q(\mathbb{R},\mathbb{R})) &= |\bullet|^p \cdot \breve{\mathcal{O}} \cdot |\bullet|^q \cdot \breve{\mathcal{O}} \\
&= |\bullet|^p \cdot |\bullet|^q \cdot \breve{\mathcal{O}} \cdot \breve{\mathcal{O}} \\
&\subseteq |\bullet|^{p+q} \cdot \breve{\mathcal{O}} \\
&= \breve{\mathcal{O}}_{p+q}(\mathbb{R},\mathbb{R}).
\end{aligned}$$

It remains to show: $(\breve{\mathcal{O}}_q(\mathbb{R},\mathbb{R})) \circ (\breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R})) \subseteq \breve{\mathcal{O}}_{qp}(\mathbb{R},\mathbb{R})$. We want: $\forall f \in \breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R})$, $\forall g \in \breve{\mathcal{O}}_q(\mathbb{R},\mathbb{R})$, $g \circ f \in \breve{\mathcal{O}}_{qp}(\mathbb{R},\mathbb{R})$. Let $f \in \breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R})$ and $g \in \breve{\mathcal{O}}_q(\mathbb{R},\mathbb{R})$ be given. We wish to prove: $g \circ f \in \breve{\mathcal{O}}_{qp}(\mathbb{R},\mathbb{R})$.

Since $f \in \breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R}) = |\bullet|^p \cdot \breve{\mathcal{O}}$, choose $\delta \in \breve{\mathcal{O}}$ s.t. $f = |\bullet|^p \cdot \delta$. Since $g \in \breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R}) = |\bullet|^q \cdot \breve{\mathcal{O}}$, choose $\varepsilon \in \breve{\mathcal{O}}$ s.t. $g = |\bullet|^q \cdot \varepsilon$. For all $x \in \mathbb{R}$,

$$\begin{aligned} g(f(x)) &= (|\bullet|^q \cdot \varepsilon)(f(x)) = \big|f(x)\big|^q \cdot [\varepsilon(f(x))] \\ &= \big|\,(|\bullet|^p \cdot \delta)(x)\,\big|^q \cdot [\varepsilon(f(x))] \\ &= \big|\,|x|^p \cdot [\delta(x)]\,\big|^q \cdot [\varepsilon(f(x))] \\ &= |x|^{qp} \cdot |\delta(x)|^q \cdot [\varepsilon(f(x))]. \end{aligned}$$

Then $g \circ f = |\bullet|^{qp} \cdot |\delta|^q \cdot [\varepsilon \circ f]$.

By Fact 47.12 p. 342, (with V and W both replaced by $\mathbb{R}$, $\|\bullet\|$ by $|\bullet|$ and p by q), $|\breve{\mathcal{O}}|^q \subseteq \breve{\mathcal{O}}$. Then $|\delta|^q \in |\breve{\mathcal{O}}|^q \subseteq \breve{\mathcal{O}}$.

We have $f \in \breve{\mathcal{O}}_p(\mathbb{R},\mathbb{R}) \subseteq \breve{\mathcal{O}}_0(\mathbb{R},\mathbb{R}) = \breve{\mathcal{O}}$. By (1) of Fact 47.11 p. 342 (with V, W and X all replaced by $\mathbb{R}$), $\breve{\mathcal{O}} \circ \breve{\mathcal{O}} \subseteq \breve{\mathcal{O}}$. Then $\varepsilon \circ f \in \breve{\mathcal{O}} \circ \breve{\mathcal{O}} \subseteq \breve{\mathcal{O}}$.

Recall that $\breve{\mathcal{O}} \cdot \breve{\mathcal{O}} \subseteq \breve{\mathcal{O}}$. Then $|\delta|^q \cdot [\varepsilon \circ f] \in \breve{\mathcal{O}} \cdot \breve{\mathcal{O}} \subseteq \breve{\mathcal{O}}$. It follows that $g \circ f = |\bullet|^{qp} \cdot |\delta|^q \cdot [\varepsilon \circ f] \in |\bullet|^{qp} \cdot \breve{\mathcal{O}} = \breve{\mathcal{O}}_{qp}(\mathbb{R},\mathbb{R})$, as desired. QED

Homework 11: Due on Tuesday 10 April

11-1. Let $S \subseteq \mathbb{R}$. Assume that S has a minimum. (That is, assume: $\exists a \in S$ s.t. $a \leqslant S$.) Show that $\inf S = \min S \in S$.

Proof: Choose $a \in S$ s.t. $a \leqslant S$. Since $a \leqslant S$, we get $a \in \mathrm{LB}(S)$. Since $a \in S$ and $a \in \mathrm{LB}(S)$, we get $a \in S \cap [\mathrm{LB}(S)]$.

Claim: $S \cap [\mathrm{LB}(S)] = \{a\}$. *Proof of Claim:* Since $a \in S \cap [\mathrm{LB}(S)]$, we have $\{a\} \subseteq S \cap [\mathrm{LB}(S)]$. We wish to show: $S \cap [\mathrm{LB}(S)] \subseteq \{a\}$. We wish to show: $\forall z \in S \cap [\mathrm{LB}(S)]$, $z \in \{a\}$. Let $z \in S \cap [\mathrm{LB}(S)]$ be given. We wish to show: $z \in \{a\}$. We wish to show: $z = a$.

We have both $z \in S$ and $z \in \mathrm{LB}(S)$. Also, since $a \leqslant S$, it follows that $S \geqslant a$. Since $z \in S \geqslant a$, we get $z \geqslant a$. It remains to show: $z \leqslant a$.

Since $z \in \mathrm{LB}(S)$, it follows that $z \leqslant S$. So, since $a \in S$, we conclude that $z \leqslant a$, as desired. *End of proof of Claim.*

By the Claim, $\mathrm{ELT}(S \cap [\mathrm{LB}(S)]) = \mathrm{ELT}\{a\}$. Then

$$\min S = \mathrm{ELT}\,(\,S \cap [\mathrm{LB}(S)]\,) = \mathrm{ELT}\{a\} = a.$$

Since $a \in S$, we get $a \neq ☹$. By Fact 8.13 p. 69, $\inf S \;=^*\; \min S$. Then $\inf S \;=^*\; \min S = a \neq ☹$, so $\inf S = a$. Then $\inf S = a = \min S$. It remains to show: $\min S \in S$.

We have $\min S = a \in S$, as desired. QED

11-2. Let U be a vector space, let $|\bullet|, \|\bullet\| \in \mathcal{N}(V)$ and let $S \subseteq U$. Let $V := (U, |\bullet|)$ and let $W := (U, \|\bullet\|)$. Assume both that $|\bullet| << \|\bullet\|$, and that S is bounded in W. Show that S is bounded in V.

Proof: Since S is bounded in W, choose $A \in \mathcal{B}_W$ s.t. $S \subseteq A$. Since $A \in \mathcal{B}_W$, choose $p \in W$ and $r > 0$ s.t. $A = B_W(p, r)$. Since $|\bullet| << \|\bullet\|$, choose $K > 0$ s.t. $|\bullet| \leqslant K \cdot \|\bullet\|$. It suffices to show: $S \subseteq B_V(p, Kr)$. We wish to show: $\forall q \in S$, $q \in B_V(p, Kr)$. Let $q \in S$ be given. We wish to show: $q \in B_V(p, Kr)$. That is, we wish to show: $|q - p|_V < Kr$.

Since $q \in S \subseteq A = B_W(p, r)$, it follows that $|q - p|_W < r$. So, since $K > 0$, we get $K \cdot |q-p|_W < Kr$. We have $|\bullet|_V = |\bullet|$ and $|\bullet|_W = \|\bullet\|$. Then $|q - p|_V = |q - p|$ and $|q - p|_W = \|q - p\|$.

Since $|\bullet| \leqslant K \cdot \|\bullet\|$, we get $|q - p| \leqslant K \cdot \|q - p\|$. Then

$$|q-p|_V \;=\; |q-p| \;\leqslant\; K \cdot \|q-p\| \;=\; K \cdot |q-p|_W \;<\; Kr,$$

as desired. QED

11-3. Let V and W be normed vector spaces, and let $T \in L(V, W)$. Assume that $T : V \to W$ is bounded below. Show that T is 1-1.

Proof: By (3) of Fact 44.3 p. 319, it suffices to show: $\ker[T] = \{0_V\}$. By (1) of Fact 44.3 p. 319, $\ker[T]$ is a vector subspace of V, and so $\{0_V\} \subseteq \ker[T]$. We wish to show: $\ker[T] \subseteq \{0_V\}$. We wish to show: $\forall x \in \ker[T]$, $x \in \{0_V\}$. Given $x \in \ker[T]$. Want: $x \in \{0_V\}$.

Since $x \in \ker[T] = T^*(\{0_W\})$, we get $Tx \in \{0_W\}$. Then $Tx = 0_W$.

Since T is bounded below, we see that $\check{T} > 0$. Let $\varepsilon := \check{T}$. Then $\varepsilon > 0$ and T is ε-bounded below. Then, by HW#10-1, $|Tx|_W \geqslant \varepsilon |x|_V$. Then $\varepsilon |x|_V \leqslant |Tx|_W = |0_W|_W = 0$. Since $\varepsilon > 0$ and $\varepsilon |x|_V \leqslant 0$, we see that $|x|_V \leqslant 0$. So, since $|x|_V \geqslant 0$, we get $|x|_V = 0$. It follows that $x = 0_V$, so $x \in \{0_V\}$, as desired. QED

11-4. Let $T \in L(\mathbb{R}^4, \mathbb{R}^4)$ be defined by $Tx = (6x_1, 5x_2, 8x_3, 7x_4)$, and define $V := (\mathbb{R}^4, |\bullet|_{4,2})$. Show that $\check{T}_{VV} = 5$ and that $\hat{T}_{VV} = 8$.

Proof: We wish to show:

(1) $\check{T}_{VV} \leqslant 5$,
(2) $\check{T}_{VV} \geqslant 5$,
(3) $\hat{T}_{VV} \leqslant 8$ and
(4) $\hat{T}_{VV} \geqslant 8$.

Let $e_\bullet$ be the standard basis of $\mathbb{R}^4$. Then

$$e_1 = (1,0,0,0),\ \ e_2 = (0,1,0,0),\ \ e_3 = (0,0,1,0),\ \ e_4 = (0,0,0,1).$$

So, since $|\bullet|_V = |\bullet|_{4,2}$, we get

$$|e_1|_V \quad = \quad |e_2|_V \quad = \quad |e_3|_V \quad = \quad |e_4|_V \quad = \quad 1,$$

so $e_1, e_2, e_3, e_4 \in S_V$. So, since $e_1, e_2, e_3, e_4 \in V = \operatorname{dom}[T]$, we conclude that $T(e_1), T(e_2), T(e_3), T(e_4) \in T_*(S_V)$. Let $E := E\ell\ell_T$. Then

$$E = T_*(S_V), \qquad \check{T}_{VV} = \inf |E|_V, \qquad \hat{T}_{VV} = \sup |E|_V.$$

We have $T(e_1), T(e_2), T(e_3), T(e_4) \in T_*(S_V) = E$.

Proof of (1): We have $T(e_2) = 5e_2$. Also, $|5e_2|_V = 5\cdot|e_2|_V = 5\cdot 1 = 5$. Then $5e_2 = T(e_2) \in E$. Then $5 = |5e_2|_V \in |E|_V \geqslant \inf|E_V| = \check{T}_{VV}$. Then $\check{T}_{VV} \leqslant 5$, as desired. *End of proof of (1).*

Proof of (2): We want: $T : V \to V$ is 5-bounded below. We want: $\forall x \in V$, $|Tx|_V \geqslant 5 \cdot |x|_V$. Given $x \in V$. Want: $|Tx|_V \geqslant 5 \cdot |x|_V$.

We have

$$\begin{aligned}
|Tx|_V^2 \ &= \ |(6x_1, 5x_2, 8x_3, 7x_4)|_V^2 \ = \ |(6x_1, 5x_2, 8x_3, 7x_4)|_{4,2}^2 \\
&= \ (6x_1)^2 + (5x_2)^2 + (8x_3)^2 + (7x_4)^2 \\
&= \ 36x_1^2 + 25x_2^2 + 64x_3^2 + 49x_4^2 \\
&\geqslant \ 25x_1^2 + 25x_2^2 + 25x_3^2 + 25x_4^2 \\
&= \ 25(x_1^2 + x_2^2 + x_3^2 + x_4^2) \\
&= \ 25 \cdot |x|_{4,2}^2 \ = \ 25 \cdot |x|_V^2.
\end{aligned}$$

So, since $25 \cdot |x|_V^2 \geqslant 0$, we get $\sqrt{|Tx|_V^2} \geqslant \sqrt{25 \cdot |x|_V^2}$. Then

$$|Tx|_V \ = \ \sqrt{|Tx|_V^2} \ \geqslant \ \sqrt{25 \cdot |x|_V^2} \ = \ \sqrt{25} \cdot \sqrt{|x|_V^2} \ = \ 5 \cdot |x|_V,$$

as desired. *End of proof of (2).*

Proof of (3): We wish to show: $T : V \to V$ is 8-bounded. We want: $\forall x \in V$, $|Tx|_V \leqslant 8 \cdot |x|_V$. Given $x \in V$. Want: $|Tx|_V \leqslant 8 \cdot |x|_V$.

We have

$$\begin{aligned} |Tx|_V^2 &= |(6x_1, 5x_2, 8x_3, 7x_4)|_V^2 = |(6x_1, 5x_2, 8x_3, 7x_4)|_{4,2}^2 \\ &= (6x_1)^2 + (5x_2)^2 + (8x_3)^2 + (7x_4)^2 \\ &= 36x_1^2 + 25x_2^2 + 64x_3^2 + 49x_4^2 \\ &\leqslant 64x_1^2 + 64x_2^2 + 64x_3^2 + 64x_4^2 \\ &= 64(x_1^2 + x_2^2 + x_3^2 + x_4^2) \\ &= 64 \cdot |x|_{4,2}^2 = 64 \cdot |x|_V^2. \end{aligned}$$

So, since $0 \leqslant |Tx|_V^2$, we get $\sqrt{|Tx|_V^2} \leqslant \sqrt{64 \cdot |x|_V^2}$. Then

$$|Tx|_V = \sqrt{|Tx|_V^2} \leqslant \sqrt{64 \cdot |x|_V^2} = \sqrt{64} \cdot \sqrt{|x|_V^2} = 8 \cdot |x|_V,$$

as desired. *End of proof of (3).*

Proof of (4): We have $T(e_3) = 8e_3$. Also, $|8e_3|_V = 8 \cdot |e_3|_V = 8 \cdot 1 = 8$. Then $8e_3 = T(e_3) \in E$. Then $8 = |8e_3|_V \in |E|_V \leqslant \sup |E_V| = \widehat{T}_{VV}$. Then $\check{T}_{VV} \geqslant 8$, as desired. *End of proof of (4).* QED

11-5. Let V and W be normed vector spaces and let $T : V \hookrightarrow\!> W$ be a vector space isomorphism. Assume $T : V \to W$ is bounded below. Show that $T^{-1} : W \to V$ is bounded.

Proof: As $T : V \to W$ is bounded below, $\check{T}_{VW} > 0$. Let $\varepsilon := \check{T}_{VW}$. Then $\varepsilon > 0$ and $T : V \to W$ is ε-bounded below. Let $K := 1/\varepsilon$. Then $K > 0$, $K\varepsilon = 1$, and it suffices to show: $T^{-1} : W \to V$ is K-bounded. We wish to show: $\forall y \in W$, $|T^{-1}(y)|_V \leqslant K \cdot |y|_W$. Let $y \in W$ be given. We wish to show: $|T^{-1}(y)|_V \leqslant K \cdot |y|_W$.

Let $x := T^{-1}(y)$. Then $Tx = y$. Since $T : V \to W$ is ε-bounded below, we get $|Tx|_W \geqslant \varepsilon \cdot |x|_V$. So, since $K > 0$, $K \cdot |Tx|_W \geqslant (K\varepsilon) \cdot |x|_V$. So, since $K\varepsilon = 1$, we get $K \cdot |Tx|_W \geqslant |x|_V$. That is, $|x|_V \leqslant K \cdot |Tx|_W$. Then $|T^{-1}(y)|_V = |x|_V \leqslant K \cdot |Tx|_W = K \cdot |y|_W$, as desired. QED

Homework 10: Due on Tuesday 3 April

10-1. Let V and W be normed vector spaces, $T \in L(V, W)$ and $\varepsilon > 0$. Show: $[\, T$ is ε-bounded below$\,] \Leftrightarrow [\, \forall x \in V,\ |Tx|_W \geqslant \varepsilon |x|_V \,]$.

Proof: Proof of $\Rightarrow$: Assume that T is ε-bounded below. We want: $\forall x \in V$, $|Tx|_W \geqslant \varepsilon |x|_V$. Let $x \in V$ be given. Want: $|Tx|_W \geqslant \varepsilon |x|_V$.

By Fact 42.5 p. 306, choose $a \geqslant 0$ and $u \in S_V$ s.t. $x = au$. Since $a \geqslant 0$, we get $|a| = a$. Since $u \in S_V$, it follows that $|u|_V = 1$. Then $|x|_V = |au|_V = |a| \cdot |u|_V = a \cdot 1 = a$. Want: $|Tx|_W \geqslant \varepsilon a$.

We have $u \in S_V \subseteq V = \operatorname{dom}[T]$. So, as $u \in S_V$, we get $Tu \in T_*(S_V)$. So, since $T_*(S_V) = E\ell\ell_T$, we get $Tu \in E\ell\ell_T$. Since T is ε-bounded below, we have $\check{T} \geqslant \varepsilon$. Then

$$|Tu|_W \quad \in \quad |E\ell\ell_T|_W \quad \geqslant \quad \inf |E\ell\ell_T|_W \quad = \quad \check{T} \quad \geqslant \quad \varepsilon,$$

so, since $|a| = a$, we get $|a| \cdot |Tu|_W \geqslant \varepsilon a$. Then

$$|Tx|_W \;=\; |T(au)|_W \;=\; |a \cdot [Tu]|_W \;=\; |a| \cdot |Tu|_W \;\geqslant\; \varepsilon a,$$

as desired. *End of proof of* $\Rightarrow$.

Proof of $\Leftarrow$*:* Assume that $\forall x \in V$, $|Tx|_W \geqslant \varepsilon |x|_V$. We wish to show: T is ε-bounded below. Want: $\check{T} \geqslant \varepsilon$. So, since $\check{T} = \inf |E\ell\ell_T|_W$, it suffices to show: $|E\ell\ell_T|_W \geqslant \varepsilon$. We wish to show: $\forall s \in |E\ell\ell_T|_W$, $s \geqslant \varepsilon$. Let $s \in |E\ell\ell_T|_W$ be given. We wish to show: $s \geqslant \varepsilon$.

Since $s \in |E\ell\ell_T|_W$, choose $y \in E\ell\ell_T$ such that $s = |y|_W$. Since $y \in E\ell\ell_T = T_*(S_V)$, choose $u \in S_V$ such that $y = Tu$. By assumption, $|Tu|_W \geqslant \varepsilon |u|_V$. Since $u \in S_V$, we conclude that $|u|_V = 1$. Then $s = |y|_W = |Tu|_W \geqslant \varepsilon |u|_V = \varepsilon \cdot 1 = \varepsilon$. *End of proof of* $\Leftarrow$. QED

10-2. Let $m \in \mathbb{N}$. Show: $|\bullet|_{m,\infty} \leqslant |\bullet|_{m,2} \leqslant |\bullet|_{m,1} \leqslant m \cdot |\bullet|_{m,\infty}$.

Proof: We wish to show:

(1) $|\bullet|_{m,\infty} \leqslant |\bullet|_{m,2}$,
(2) $|\bullet|_{m,2} \leqslant |\bullet|_{m,1}$, and
(3) $|\bullet|_{m,1} \leqslant m \cdot |\bullet|_{m,\infty}$.

Proof of (1): Want: $\forall x \in \mathbb{R}^m$, $|x|_{m,\infty} \leqslant |x|_{m,2}$. Let $x \in \mathbb{R}^m$ be given. We wish to prove: $|x|_{m,\infty} \leqslant |x|_{m,2}$.

Let $a := |x|_{m,2}$. Want: $|x|_{m,\infty} \leqslant a$. As $|x|_{m,\infty} = \max\{|x_1|, \ldots, |x_m|\}$, it suffices to show $\{|x_1|, \ldots, |x_m|\} \leqslant a$. Want: $\forall t \in \{|x_1|, \ldots, |x_m|\}$, $t \leqslant a$. Let $t \in \{|x_1|, \ldots, |x_m|\}$ be given. Want: $t \leqslant a$.

Since $t \in \{|x_1|, \ldots, |x_m|\}$, choose $j \in [1..m]$ such that $t = |x_j|$. We have $0 \leqslant x_j^2 \leqslant x_1^2 + \cdot + x_m^2$. Then $\sqrt{x_j^2} \leqslant \sqrt{x_1^2 + \cdot + x_m^2}$. Then

$$t \quad = \quad |x_j| \quad = \quad \sqrt{x_j^2} \quad \leqslant \quad \sqrt{x_1^2 + \cdot + x_m^2} \quad = \quad |x|_2 \quad = \quad a,$$

as desired. *End of proof of (1).*

Proof of (2): Want: $\forall x \in \mathbb{R}^m$, $|x|_{m,2} \leqslant |x|_{m,1}$. Let $x \in \mathbb{R}^m$ be given. We wish to prove: $|x|_{m,2} \leqslant |x|_{m,1}$.

Let $I := [1..m]$. For all $j \in I$, let $a_j := |x_j|$. For all $j \in I$, we have $a_j^2 = |x_j|^2 = x_j^2$. Then $|x|_{m,2} = \sqrt{x_1^2 + \cdots + x_m^2} = \sqrt{a_1^2 + \cdots + a_m^2}$. Also, we have $|x|_{m,1} = |x_1| + \cdots + |x_m| = a_1 + \cdots + a_m$.

Let $K := \{r \in I^2 \,|\, r_1 \neq r_2\}$. For all $j \in I$, we have $a_j = |x_j| \geqslant 0$. Then, for all $r \in K$, we have $a_{r_1} a_{r_2} \geqslant 0$. It follows that $\sum_{r \in K} a_{r_1} a_{r_2} \geqslant 0$. Let $S := \sum_{r \in K} a_{r_1} a_{r_2}$. Then $S \geqslant 0$. We have

$$(a_1 + \cdots + a_m)^2 \quad = \quad (a_1^2 + \cdots + a_m^2) + S.$$

So, since $S \geqslant 0$, we get $(a_1 + \cdots + a_m)^2 \geqslant a_1^2 + \cdots + a_m^2$. Since $0 \leqslant a_1^2 + \cdots + a_m^2 \leqslant (a_1 + \cdots + a_m)^2$, we see that

$$\sqrt{a_1^2 + \cdots + a_m^2} \quad \leqslant \quad \sqrt{(a_1 + \cdots + a_m)^2}.$$

Recall: $\forall j \in I$, $a_j \geqslant 0$. It follows that $a_1 + \cdots + a_m \geqslant 0$. Then we have $|a_1 + \cdots + a_m| = a_1 + \cdots + a_m$. Then

$$\begin{aligned} |x|_{m,2} &= \sqrt{a_1^2 + \cdots + a_m^2} \\ &\leqslant \sqrt{(a_1 + \cdots + a_m)^2} \\ &\leqslant |a_1 + \cdots + a_m| \\ &= a_1 + \cdots + a_m \;=\; |x|_{m,1}, \end{aligned}$$

as desired. *End of proof of (2).*

Proof of (3): Want: $\forall x \in \mathbb{R}^m$, $|x|_{m,1} \leqslant m \cdot |x|_{m,\infty}$. Let $x \in \mathbb{R}^m$ be given. Want: $|x|_{m,1} \leqslant m \cdot |x|_{m,\infty}$. Let $a := |x|_{m,\infty}$. Want: $|x|_{m,1} \leqslant ma$.

For all $j \in [1..m]$, we have $|x_j| \leqslant \max\{|x_1|, \ldots, |x_m|\} = |x|_{m,\infty} = a$. Then $|x|_{m,1} = |x_1| + \cdots + |x_m| \leqslant ma$. *End of proof of (3).* QED

10-3. Let V be a normed vector space, let Z be a topological space, let $\mu : V \dashrightarrow Z$ and let $p \in V$. Let $\lambda := \mu(p + \bullet)$. Show: $\lim\limits_{0_V} \lambda = \lim\limits_{p} \mu$.

Proof: It suffices to show: $\mathop{\mathrm{LIMS}}\limits_{0_V} \lambda = \mathop{\mathrm{LIMS}}\limits_{p} \mu$.

Proof of $\subseteq$*:* Want: $\forall y \in \mathop{\mathrm{LIMS}}\limits_{0_V} \lambda$, $y \in \mathop{\mathrm{LIMS}}\limits_{p} \mu$. Given $y \in \mathop{\mathrm{LIMS}}\limits_{0_V} \lambda$. Want: $y \in \mathop{\mathrm{LIMS}}\limits_{p} \mu$. Know: $\lambda \to y$ near 0_V. Want: $\mu \to y$ near p. We wish to show: $\forall U \in \mathcal{N}_Z(y)$, $\exists \delta > 0$ s.t., $\forall x \in \mathrm{dom}\,[\mu]$,

$$[\; 0 < |x - p|_V < \delta \;] \quad \Rightarrow \quad [\; \mu(x) \in U \;].$$

Let $U \in \mathcal{N}_Z(y)$ be given. We wish to show: $\exists \delta > 0$ s.t., $\forall x \in \text{dom}\,[\mu]$,

$$[\ 0 < |x - p|_V < \delta\ ] \quad \Rightarrow \quad [\ \mu(x) \in U\ ].$$

Since $\lambda \to y$ near 0_V, choose $\delta > 0$ s.t., $\forall h \in \text{dom}\,[\lambda]$,

$$[\ 0 < |h|_V < \delta\ ] \quad \Rightarrow \quad [\ \lambda(h) \in U\ ].$$

We wish to show: $\forall x \in \text{dom}\,[\mu]$,

$$[\ 0 < |x - p|_V < \delta\ ] \quad \Rightarrow \quad [\ \mu(x) \in U\ ].$$

Let $x \in \text{dom}\,[\mu]$ be given. We wish to show:

$$[\ 0 < |x - p|_V < \delta\ ] \quad \Rightarrow \quad [\ \mu(x) \in U\ ].$$

Assume that $0 < |x - p|_V < \delta$. We wish to show: $\mu(x) \in V$.

Let $h := x - p$. Then $h \in (\text{dom}\,[\mu]) - p = \text{dom}\,[\lambda]$. Also, $0 < |h|_V < \delta$, so, by choice of δ, we have $\lambda(h) \in U$. By definition of λ, we have $\lambda(h) = \mu(p + h)$. So, since $p + h = x$, we get $\lambda(h) = \mu(x)$. Then $\mu(x) = \lambda(h) \in U$. *End of proof of* $\subseteq$.

Proof of $\supseteq$*:* Want: $\forall y \in \underset{p}{\text{LIMS}}\ \mu$, $y \in \underset{0_V}{\text{LIMS}}\ \lambda$. Given $y \in \underset{p}{\text{LIMS}}\ \mu$. Want: $y \in \underset{0_V}{\text{LIMS}}\ \lambda$. Know: $\mu \to y$ near p. Want: $\lambda \to y$ near 0_V. We wish to show: $\forall U \in \mathcal{N}_Z(y)$, $\exists \delta > 0$ s.t., $\forall h \in \text{dom}\,[\lambda]$,

$$[\ 0 < |h|_V < \delta\ ] \quad \Rightarrow \quad [\ \lambda(h) \in U\ ].$$

Let $U \in \mathcal{N}_Z(y)$ be given. We wish to show: $\exists \delta > 0$ s.t., $\forall h \in \text{dom}\,[\lambda]$,

$$[\ 0 < |h|_V < \delta\ ] \quad \Rightarrow \quad [\ \lambda(h) \in U\ ].$$

Since $\mu \to y$ near p, choose $\delta > 0$ s.t., $\forall x \in \text{dom}\,[\mu]$,

$$[\ 0 < |x - p|_V < \delta\ ] \quad \Rightarrow \quad [\ \mu(x) \in U\ ].$$

We wish to show: $\forall h \in \text{dom}\,[\lambda]$,

$$[\ 0 < |h|_V < \delta\ ] \quad \Rightarrow \quad [\ \lambda(h) \in U\ ].$$

Let $h \in \text{dom}\,[\lambda]$ be given. We wish to show:

$$[\ 0 < |h|_V < \delta\ ] \quad \Rightarrow \quad [\ \lambda(h) \in U\ ].$$

Assume that $0 < |h|_V < \delta$. We wish to show: $\lambda(h) \in U$.

Let $x := p + h$. Then $x \in (\text{dom}\,[\lambda]) + p = \text{dom}\,[\mu]$. Also, $h = x - p$ Then $0 < |x - p|_V < \delta$, so, by choice of δ, we have $\mu(x) \in U$. By definition of λ, we have $\lambda(h) = \mu(p + h)$. So, since $p + h = x$, we get $\lambda(h) = \mu(x)$. Then $\lambda(h) = \mu(x) \in U$. *End of proof of* $\supseteq$. QED

10-4. Let X be a metric space, let $D \subseteq X$ and let $p \in \mathrm{Int}_X D$. Show that there exists $B \in \mathcal{B}_X(p)$ such that $B \subseteq D$.

Proof: Since $p \in \mathrm{Int}_X D$, choose an open subset U of X such that $p \in U \subseteq D$. Since U is open in X, we have $U \in \mathcal{T}_X$. We have $\mathcal{T}_X = \langle \mathcal{B}_X \rangle_\cup$. Then $p \in U \in \langle \mathcal{B}_X \rangle_\cup$. Choose $C \in \mathcal{B}_X$ such that $p \in C \subseteq U$. By the Recentering Down Lemma (Lemma 14.2 p. 104), choose $B \in \mathcal{B}_X(p)$ such that $B \subseteq C$. We wish to show: $B \subseteq D$.

We have $B \subseteq C \subseteq U \subseteq D$, as desired. QED

10-5. Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$. Let $p, q \in \mathbb{R}$. Assume g has a local unique min at p in $\mathbb{R}$. Show: $g + C_{\mathbb{R}}^q$ has a local unique min at p in $\mathbb{R}$.

Proof: Let $h := g + C_{\mathbb{R}}^q$. We wish to show: $\exists V \in \mathcal{N}_X(p)$ s.t.

$$(\ V \subseteq \mathrm{dom}\,[h]\ ) \qquad \text{and} \qquad (\ h(p) < h_*(V_p^\times)\ ).$$

Since g has a local unique min at p in $\mathbb{R}$, choose $V \in \mathcal{N}_X(p)$ s.t.

$$(\ V \subseteq \mathrm{dom}\,[g]\ ) \qquad \text{and} \qquad (\ g(p) < g_*(V_p^\times)\ ).$$

We wish to show:

$$(\ V \subseteq \mathrm{dom}\,[h]\ ) \qquad \text{and} \qquad (\ h(p) < h_*(V_p^\times)\ ).$$

We have $\mathrm{dom}\,[h] = \mathrm{dom}\,[g + C_{\mathbb{R}}^q] = (\mathrm{dom}\,[g]) \cap (\mathrm{dom}\,[C_{\mathbb{R}}^q])$. So, since $\mathrm{dom}\,[g] \subseteq \mathbb{R} = \mathrm{dom}\,[C_{\mathbb{R}}^q]$, we get $\mathrm{dom}\,[h] = \mathrm{dom}\,[g]$. Then we have $V \subseteq \mathrm{dom}\,[g] = \mathrm{dom}\,[h]$. It remains to show: $h(p) < h_*(V_p^\times)$. We wish to show: $\forall y \in h_*(V_p^\times)$, $h(p) < y$. Let $y \in h_*(V_p^\times)$ be given. We wish to show: $h(p) < y$. Since $y \in h_*(V_p^\times)$, choose $x \in \mathrm{dom}\,[h]$ such that: both $x \in V_p^\times$ and $y = h(x)$. We wish to show $h(p) < h(x)$.

Since $x \in \mathrm{dom}\,[h] = \mathrm{dom}\,[g]$ and since $x \in V_p^\times$, it follows that $g(x) \in g_*(V_p^\times)$. So, since $g(p) < g_*(V_p^\times)$, we get $g(p) < g(x)$. Then $[g(p)] + q < [g(x)] + q$. So since

$$\begin{aligned} h(p) &= (g + C_{\mathbb{R}}^q)(p) = [g(p)] + (C_{\mathbb{R}}^q)(p)] = [g(p)] + q \\ \text{and}\quad h(x) &= (g + C_{\mathbb{R}}^q)(x) = [g(x)] + (C_{\mathbb{R}}^q)(x)] = [g(x)] + x, \end{aligned}$$

we conclude that $h(p) < h(x)$, as desired. QED

Homework 9: Due on Tuesday 27 March

9-1. Let V and W be normed vector spaces, $T \in L(V, W)$ and $K \geqslant 0$. Show: $[\ T \text{ is } K\text{-bounded}\ ] \Leftrightarrow [\ \forall x \in V,\ |Tx|_W \leqslant K \cdot |x|_V\ ]$.

Proof: Proof of $\Rightarrow$*:* Assume that T is K-bounded. We wish to show: $\forall x \in V$, $|T(x)|_W \leqslant K \cdot |x|_V$. Given $x \in V$. Want: $|T(x)|_W \leqslant K \cdot |x|_V$.

Since T is K-bounded, we have $\widehat{T} \leqslant K$. By definition of $\widehat{T}$, we have $|E\ell\ell_T|_W \leqslant \widehat{T}$. By the polar decomposition (Fact 42.5), choose $a \geqslant 0$ and $u \in S_V$ s.t. $x = au$. Since $u \in S_V$, we get $|u|_V = 1$. Then $K \cdot |x|_V = K \cdot |au|_V = K \cdot |a| \cdot |u|_V = K \cdot |a| \cdot 1 = |a| \cdot K$.

We have $|T(u)|_W \in |T_*(S_V)|_W = |E\ell\ell_T|_W \leqslant \widehat{T} \leqslant K$. Then

$$\begin{aligned} |T(x)|_W &= |T(au)|_W = |a \cdot [T(u)]|_W \\ &= |a| \cdot |T(u)|_W \leqslant |a| \cdot K = K \cdot |x|_V, \end{aligned}$$

as desired. *End of proof of* $\Rightarrow$*.*

Proof of $\Leftarrow$*:* Assume: $\forall x \in V$, $|T(x)|_W \leqslant K \cdot |x|_V$. We wish to show that T is K-bounded. Want: $\widehat{T} \leqslant K$. Want: $\sup |E\ell\ell_T|_W \leqslant K$. We wish to show: $|E\ell\ell_T|_W \leqslant K$. Want: $\forall r \in |E\ell\ell_T|_W, \quad r \leqslant K$. Let $r \in |E\ell\ell_T|_W$ be given. We want: $r \leqslant K$.

Choose $y \in E\ell\ell_T$ s.t. $r = |y|_W$. Since $y \in E\ell\ell_T = T_*(S_V)$, choose $u \in S_V$ s.t. $y = T(u)$. We have $|T(u)|_W \leqslant K \cdot |u|_V$. Since $u \in S_V$, we have $|u|_V = 1$. Then $r = |y|_W = |T(u)|_W \leqslant K \cdot |u|_V = K \cdot 1 = K$, as desired. *End of proof of* $\Leftarrow$*.* QED

9-2. Let V and W be normed vector spaces and let $K \geqslant 0$. Let $T \in L(V, W)$ be K-bounded. Show: T is K-Lipschitz. That is, show:

$$\forall x, y \in V, \qquad |\,[T(x)] - [T(y)]\,|_W \leqslant K \cdot |\,x - y\,|_V.$$

Proof: Let $x, y \in V$ be given. Want: $|\,[T(x)] - [T(y)]\,|_W \leqslant K \cdot |x - y|_V$.

By HW#9-1, we know: $\forall z \in V$, $|T(z)|_W \leqslant K \cdot |z|_V$. It follows that $|T(x - y)|_W \leqslant K \cdot |x - y|_V$. By linearity, $T(x - y) = [T(x)] - [T(y)]$. Then $|\,[T(x)] - [T(y)]\,|_W = |T(x - y)|_W \leqslant K \cdot |x - y|_V$. QED

9-3. Let Y and Z be metric spaces and let $K > 0$. Let $f : Y \dashrightarrow Z$ be a K-Lipschitz function. Let $x \in \mathrm{dom}\,[f]$ and let $r > 0$. Show that: $f_*(B_Y(x, r)) \subseteq B_Z(f(x), Kr)$.

Proof: We want to show: $\forall p \in \mathrm{dom}\,[f]$,

$$[\; p \in B_Y(x, r) \;] \quad \Rightarrow \quad [\; f(p) \in B_Z(f(x), Kr) \;].$$

Let $p \in \mathrm{dom}\,[f]$ be given. We want to show:

$$[\; p \in B_Y(x, r) \;] \quad \Rightarrow \quad [\; f(p) \in B_Z(f(x), Kr) \;].$$

Assume $p \in B_Y(x, r)$. We want to show: $f(p) \in B_Z(f(x), Kr)$.

Since $p \in B_Y(x, r)$, we have $d_Y(p, x) < r$. So, since f is K-Lipschitz, $d_Z(f(p), f(x)) < Kr$. Then $f(p) \in B_Z(f(x), Kr)$, as desired. QED

9-4. Let V and W be normed vector spaces and let $T \in L(V, W)$. Assume that T is continuous at 0_V. Show that T is bounded.

Proof: Since T is linear, $T(0_V) = 0_W$. So, since T is continuous at 0_V, choose $\delta > 0$ s.t., for all $x \in V$,

$$[\ |x|_V < \delta\ ] \quad \Rightarrow \quad [\ |Tx|_W < 1\ ].$$

Let $K := 2/\delta$. We want: T is K-bounded. By $\Leftarrow$ of Remark 43.1 p. 311, it suffices to show: $\forall u \in S_V$, $|Tu|_W \leqslant K$. Let $u \in V$ be given: We want to prove: $|Tu|_W \leqslant K$. It suffices to show: $|Tu|_W < K$.

Since $K = 2/\delta > 0$, we get $|K| = K$. Also, $|1/K| = |\delta/2| = \delta/2$. Since $u \in S_V$, we have $|u|_V = 1$. Let $x := (1/K)u$. Then we have $|x|_V = |1/K| \cdot |u|_V = (\delta/2) \cdot 1 = \delta/2 < \delta$. So, by choice of δ, we see that $|Tx|_W < 1$. Then $K \cdot |Tx|_W < K \cdot 1 = K$. Since $x = (1/K)u$, we get $u = Kx$, and so, by linearity of T, we have $Tu = K \cdot (Tx)$.

Then $|Tu|_W = |K \cdot (Tx)|_W = |K| \cdot |Tx|_W = K \cdot |Tx|_W < K$. QED

9-5. Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume: $\exists M \in \mathbb{R}$ s.t. $(M, \infty) \subseteq \mathrm{dom}\,[g'/f']$. Assume: $\lim\limits_{\infty} f = 0 = \lim\limits_{\infty} g$. Show: $\lim\limits_{\infty} (g/f) \ =^* \ \lim\limits_{\infty} (g'/f')$.

Proof: Define $\phi, \psi : (0, \infty) \dashrightarrow \mathbb{R}$ by $\phi(t) = f(1/t)$ and $\psi(t) = g(1/t)$. Then, for all $t \in (0, \infty)$, we have $(\psi/\phi)(t) = (g/f)(1/t)$. It follows, by (1) of Fact 42.19, that $\lim\limits_{\infty} (g/f) = \lim\limits_{0^+} (\psi/\phi)$.

Claim 1: $\forall t \in (0, \infty)$, $(\psi'/\phi')(t) = (g'/f')(1/t)$. *Proof of Claim 1:* Let $t \in (0, \infty)$ be given. We wish to show: $(\psi'/\phi')(t) = (g'/f')(1/t)$.

By (2) of Fact 42.19, we have both

$$\phi'(t) \ = \ -\frac{f'(1/t)}{t^2} \qquad \text{and} \qquad \psi'(t) \ = \ -\frac{g'(1/t)}{t^2}.$$

Then $[\psi'(t)]/[\phi'(t)] = [g'(t)]/[f'(t)]$, *i.e.*, $(\psi'/\phi')(t) = (g'/f')(1/t)$, as desired. *End of proof of Claim 1.*

By Claim 1, for all $t \in (0, \infty)$, we have $(\psi'/\phi')(t) = (g'/f')(1/t)$. It follows, by (1) of Fact 42.19, that $\lim\limits_{\infty} (g'/f') = \lim\limits_{0^+} (\psi'/\phi')$.

By assumption, choose $M \in \mathbb{R}$ such that $(M, \infty) \subseteq \mathrm{dom}\,[g'/f']$.

Claim 2: $(0,1/M) \in \mathrm{dom}\,[\psi'/\phi']$. *Proof of Claim 2:* We wish to show: $\forall t \in (0,1/M)$, $t \in \mathrm{dom}\,[\psi'/\phi']$. Let $t \in (0,1/M)$ be given. We wish to show: $t \in \mathrm{dom}\,[\psi'/\phi']$.

Since $t \in (0,1/M)$, $1/t \in (M,\infty)$. Then $1/t \in (M,\infty) \subseteq \mathrm{dom}\,[g'/f']$, so $(g'/f')(1/t) \neq$ ☹. Since $t \in (0,1/M) \subseteq (0,\infty)$, by Claim 1, we get $(\psi'/\phi')(t) = (g'/f')(1/t)$. Then $(\psi'/\phi')(t) = (g'/f')(1/t) \neq$ ☹, and so $t \in \mathrm{dom}\,[\psi'/\phi']$, as desired. *End of proof of Claim 2.*

Since $\lim\limits_{\infty} f = 0 = \lim\limits_{\infty} g$, it follows, from (1) of Fact 42.19, that $\lim\limits_{0^+} \phi = 0 = \lim\limits_{0^+} \psi$. By Claim 2, we have $(0,1/M) \in \mathrm{dom}\,[\psi'/\phi']$. Therefore, by Theorem 42.18 (with p replaced by 0, f by ϕ and g by ψ), we conclude that $\lim\limits_{0^+} (\phi/\psi) \;=^*\; \lim\limits_{0^+} (\phi'/\psi')$.

Then $\lim\limits_{\infty} (g/f) = \lim\limits_{0^+} (\phi/\psi) \;=^*\; \lim\limits_{0^+} (\phi'/\psi') = \lim\limits_{\infty} (g'/f')$. QED

Homework 8: Due on Tuesday 20 March

8-1. Let W be a vector space and let $u, v, v' \in W$. Assume that $\mathbb{R}v = \mathbb{R}v'$ and that $u \| v$. Show that $u \| v'$.

Proof: We wish to show: $u \in \mathbb{R}v'$ or $v' \in \mathbb{R}u$. Since $u \| v$, at least one of the following must be true:

(1) $u \in \mathbb{R}v$ or
(2) $v \in \mathbb{R}u$.

Case (1): We wish to show: $u \in \mathbb{R}v'$. Since $u \in \mathbb{R}v$, $\mathbb{R}u \subseteq \mathbb{R}v$. So, as $\mathbb{R}v = \mathbb{R}v'$, $\mathbb{R}u \subseteq \mathbb{R}v'$. Then $u \in \mathbb{R}u \subseteq \mathbb{R}v'$. *End of Case (1).*

Case (2): We wish to show: $v' \in \mathbb{R}u$. Since $v \in \mathbb{R}u$, $\mathbb{R}v \subseteq \mathbb{R}u$. So, as $\mathbb{R}v = \mathbb{R}v'$, $\mathbb{R}v' \subseteq \mathbb{R}u$. Then $v' \in \mathbb{R}v' \subseteq \mathbb{R}u$. *End of Case (2).* QED

8-2. Let $u, v \in \mathbb{R}^2$. Show:

$$[\ u \| v\ ] \quad \Leftrightarrow \quad [\ (u = 0_2) \text{ or } (v = 0_2) \text{ or } (\mathrm{sl}\, u = \mathrm{sl}\, v)\ ].$$

Proof: Proof of $\Rightarrow$*:* Assume that $u \| v$. We wish to prove:

$$(u = 0_2) \quad \text{or} \quad (v = 0_2) \quad \text{or} \quad (\mathrm{sl}\, u = \mathrm{sl}\, v).$$

Equivalently, we want: $[\ (u \neq 0_2) \text{ and } (v \neq 0_2)\ ] \Rightarrow [\ \mathrm{sl}\, u = \mathrm{sl}\, v\ ]$.

Assume $(u \neq 0_2)$ and $(v \neq 0_2)$. We wish to prove: $\mathrm{sl}\, u = \mathrm{sl}\, v$.

Since $u \| v$, at least one of the following is true:

(1) $u \in \mathbb{R}v$ or

(2) $v \in \mathbb{R}u$.

Case (1): Choose $a \in \mathbb{R}$ such that $u = av$. Since $av = u \neq 0_2 = 0 \cdot v$, we conclude that $a \neq 0$. Since $(u_1, u_2) = u = av = (av_1, av_2)$, we see that $u_1 = av_1$ and that $u_2 = av_2$. Then

$$\mathrm{sl}\, u \quad = \quad \frac{u_2}{u_1} \quad = \quad \frac{av_2}{av_1} \quad = \quad \frac{v_2}{v_1} \quad = \quad \mathrm{sl}\, v,$$

as desired. *End of Case (1).*

Case (2): Choose $a \in \mathbb{R}$ such that $v = au$. Since $au = v \neq 0_2 = 0 \cdot u$, we conclude that $a \neq 0$. Since $(v_1, v_2) = v = au = (au_1, au_2)$, we see that $v_1 = au_1$ and that $v_2 = au_2$. Then

$$\mathrm{sl}\, u \quad = \quad \frac{u_2}{u_1} \quad = \quad \frac{au_2}{au_1} \quad = \quad \frac{v_2}{v_1} \quad = \quad \mathrm{sl}\, v,$$

as desired. *End of Case (2). End of proof of* $\Rightarrow$.

Proof of $\Leftarrow$*:* Assume

$$(*) \qquad (u = 0_2) \quad \text{or} \quad (v = 0_2) \quad \text{or} \quad (\mathrm{sl}\, u = \mathrm{sl}\, v).$$

We wish to prove: $u \| v$. We want: $u \in \mathbb{R}v$ or $v \in \mathbb{R}u$.

At least one of the following must be true:

(1) $u_1 = 0 = v_1$ or
(2) $u_1 \neq 0 = v_1$ or
(3) $u_1 = 0 \neq v_1$ or
(4) $u_1 \neq 0 \neq v_1$.

Case (1): We wish to prove that: $[\ u \notin \mathbb{R}v\ ] \Rightarrow [\ v \in \mathbb{R}u\ ]$. Assume that: $u \notin \mathbb{R}v$. We wish to prove that: $v \in \mathbb{R}u$.

Since $u \notin \mathbb{R}v$ and $0_2 = 0 \cdot v \in \mathbb{R}v$, we get: $u \neq 0_2$. Then we have: $(0, u_2) = (u_1, u_2) = u \neq 0_2 = (0, 0)$, so $u_2 \neq 0$. Let $a := v_2/u_2$. Then $au_2 = v_2$. Then $au = (au_1, au_2) = (a \cdot 0, v_2) = (0, v_2) = (v_1, v_2) = v$. Then $v = au \in \mathbb{R}u$, as desired. *End of Case (1).*

Case (2): We wish to show: $v \in \mathbb{R}u$.

Since $u_1 \neq 0$, we have $(u_1, u_2) \neq (0, 0)$, *i.e.*, $u \neq 0_2$. Since $u_1 \neq 0$, we get $u_2/u_1 \neq ☹$. Since $v_1 = 0$, we get $v_2/v_1 = ☹$. Then

$$\mathrm{sl}\, u \quad = \quad u_2/u_1 \quad \neq \quad ☹ \quad = \quad v_2/v_1 \quad = \quad \mathrm{sl}\, v.$$

Since $u \neq 0_2$ and $\mathrm{sl}\, u \neq \mathrm{sl}\, v$, we see, by $(*)$, that $v = 0_2$. Then $v = 0_2 = 0 \cdot u \in \mathbb{R}u$, as desired. *End of Case (2).*

Case (3): We wish to show: $u \in \mathbb{R}v$.

Since $v_1 \neq 0$, we have $(v_1, v_2) \neq (0,0)$, *i.e.*, $v \neq 0_2$. Since $v_1 \neq 0$, we get $v_2/v_1 \neq ☹$. Since $u_1 = 0$, we get $u_2/u_1 = ☹$. Then

$$\operatorname{sl} u \quad = \quad u_2/u_1 \quad = \quad ☹ \quad \neq \quad v_2/v_1 \quad = \quad \operatorname{sl} v.$$

Since $v \neq 0_2$ and $\operatorname{sl} u \neq \operatorname{sl} v$, we see, by $(*)$, that $u = 0_2$. Then $u = 0_2 = 0 \cdot v \in \mathbb{R}v$, as desired. *End of Case (3).*

Case (4): We wish to show: $v \in \mathbb{R}u$.

Since $u_1 \neq 0$, we have $(u_1, u_2) \neq (0,0)$, *i.e.*, $u \neq 0_2$. Since $v_1 \neq 0$, we have $(v_1, v_2) \neq (0,0)$, *i.e.*, $v \neq 0_2$. Since $u \neq 0_2$ and $v \neq 0_2$, we see, by $(*)$, that $\operatorname{sl} u = \operatorname{sl} v$. That is, $u_2/u_1 = v_2/v_1$. Let $a := v_1/u_1$. Then $au_1 = v_1$ and $au_2 = (v_1/u_1)u_2 = (u_2/u_1)v_1 = (v_2/v_1)v_1 = v_2$. Then $au = (au_1, au_2) = (v_1, v_2) = v$. Then $v = au \in \mathbb{R}u$, as desired. *End of Case (4). End of proof of* $\Leftarrow$. QED

8-3. Let $u, v \in \mathbb{R}^2$. Show:

$$(\ u \| v\ ) \quad \Leftrightarrow \quad \left(\operatorname{Det} \begin{bmatrix} u \\ v \end{bmatrix} = 0 \right).$$

Proof: Proof of $\Rightarrow$*:* Assume: $u \| v$. Want: $\operatorname{Det} \begin{bmatrix} u \\ v \end{bmatrix} = 0$.

Since $u \| v$, at least one of the following is true:

(1) $u \in \mathbb{R}v$ or
(2) $v \in \mathbb{R}u$.

Case (1): Choose $a \in \mathbb{R}$ such that $u = av$. Then

$$\begin{aligned} \operatorname{Det} \begin{bmatrix} u \\ v \end{bmatrix} &= \operatorname{Det} \begin{bmatrix} av \\ v \end{bmatrix} \\ &= \operatorname{Det} \begin{bmatrix} (av_1, av_2) \\ (v_1, v_2) \end{bmatrix} \\ &= (av_1) \cdot v_2 - (av_2) \cdot v_1 \quad = \quad 0. \end{aligned}$$

End of Case (1).

Case (2): Choose $a \in \mathbb{R}$ such that $v = au$. Then

$$\begin{aligned} \operatorname{Det} \begin{bmatrix} u \\ v \end{bmatrix} &= \operatorname{Det} \begin{bmatrix} u \\ au \end{bmatrix} \\ &= \operatorname{Det} \begin{bmatrix} (u_1, u_2) \\ (au_1, au_2) \end{bmatrix} \\ &= u_1 \cdot (au_2) - u_2 \cdot (au_1) \quad = \quad 0. \end{aligned}$$

End of Case (2). End of proof of $\Rightarrow$.

Proof of $\Leftarrow$*:* Assume: $\text{Det}\begin{bmatrix} u \\ v \end{bmatrix} = 0$. We wish to prove: $u \| v$. We want to show: $u \in \mathbb{R}v$ or $v \in \mathbb{R}u$.

Since $0 = \text{Det}\begin{bmatrix} u \\ v \end{bmatrix} = u_1 v_2 - u_2 v_1$, we see that $u_1 v_2 = u_2 v_1$. At least one of the following is true:

(1) $u_1 \neq 0$ or
(2) $u_2 \neq 0$ or
(3) $u_1 = 0 = u_2$.

Case (1): Want: $v \in \mathbb{R}u$. Let $a := v_1/u_1$. Want: $v = au$.

We have both $au_1 = v$ and $au_2 = u_2 v_1/u_1 = u_1 v_2/u_1 = v_2$. Then $v = (v_1, v_2) = (au_1, au_2) = au$, as desired. *End of Case (1).*

Case (2): Want: $v \in \mathbb{R}u$. Let $a := v_2/u_2$. Want: $v = au$.

We have both $au_1 = u_1 v_2/u_2 = u_2 v_1/u_2 = v_1$ and $au_2 = v_2$. Then $v = (v_1, v_2) = (au_1, au_2) = au$, as desired. *End of Case (2).*

Case (3): Want: $u \in \mathbb{R}v$. Let $a := 0$. Want $u = av$.

Since $u_1 = 0 = u_2$, we get $u = 0_2$. Since $a = 0$, we get $av = 0_2$. Then $u = 0_2 = av$, as desired. *End of Case (3). End of proof of* $\Leftarrow$. QED

8-4. Let V and W be normed vector spaces. Let $x : \mathbb{R} \dashrightarrow V$ and $y : \mathbb{R} \dashrightarrow W$. Let $p \in \mathbb{R}$. Assume that x and y are both continuous at p. Show that (x, y) is continuous at p.

Proof: Let $z := (x, y)$ and let $U := V \times W$. Then $z : \mathbb{R} \dashrightarrow U$ and $\text{dom}\,[z] \subseteq \text{dom}\,[x]$ and $\text{dom}\,[z] \subseteq \text{dom}\,[y]$.

We want: z is continuous at p. We wish to show: $\forall U_0 \in \mathcal{N}_U(z(p))$, $\exists A \in \mathcal{N}_{\mathbb{R}}(p)$ s.t. $z_*(A) \subseteq U_0$. Let $U_0 \in \mathcal{N}_U(z(p))$ be given. We wish to prove: $\exists A \in \mathcal{N}_{\mathbb{R}}(p)$ s.t. $z_*(A) \subseteq U_0$.

Choose $V_0 \in \mathcal{N}_V(x)$ and $W_0 \in \mathcal{N}_W(y)$ s.t. $V_0 \times W_0 \subseteq U_0$. Since x is continuous at p, choose $B \in \mathcal{N}_{\mathbb{R}}(p)$ s.t. $x_*(B) \subseteq V_0$. Since y is continuous at p, choose $C \in \mathcal{N}_{\mathbb{R}}(p)$ s.t. $y_*(C) \subseteq W_0$. As $B, C \in \mathcal{N}_{\mathbb{R}}(p)$, we get $B \cap C \in \mathcal{N}_{\mathbb{R}}(p)$. Let $A := B \cap C$. We wish to show: $z_*(A) \subseteq U_0$. We wish to show: $\forall q \in \text{dom}\,[z]$,

$$[\ q \in A\ ] \quad \Rightarrow \quad [\ z(q) \in U_0\ ].$$

Let $q \in \text{dom}\,[z]$ be given. We wish to show:

$$[\ q \in A\ ] \quad \Rightarrow \quad [\ z(q) \in U_0\ ].$$

Assume: $q \in A$. We wish to show: $z(q) \in U_0$.

Since $q \in \mathrm{dom}\,[z] \subseteq \mathrm{dom}\,[x]$ and since $q \in A = B \cap C \subseteq B$, we have $x(q) \in x_*(B)$. So, by choice of B, we have $x(q) \in V_0$. Since $q \in \mathrm{dom}\,[z] \subseteq \mathrm{dom}\,[y]$ and since $q \in A = B \cap C \subseteq C$, we have $y(q) \in y_*(C)$. So, by choice of C, we have $y(q) \in W_0$.

Then $z(q) = (x(q), y(q)) \in V_0 \times W_0$. Then, by choice of V_0 and W_0, we conclude that $z(q) \in U_0$, as desired. QED

8-5. Let V and W be normed vector spaces. Let $x : \mathbb{R} \dashrightarrow V$ and $y : \mathbb{R} \dashrightarrow W$. Let $p \in \mathrm{LPD}_{\mathbb{R}}(x, y)$. Show: $(x, y)'(p) \;\; =^* \;\; (\, x'(p)\, ,\, y'(p)\,)$.

Proof: Let $\phi := SS_x^p$, $\psi := SS_y^p$. Then $x'(p) = \lim\limits_0 \phi$ and $y'(p) = \lim\limits_0 \psi$. Also, $(x, y)'(p) = \lim\limits_0 SS_{(x,y)}^p$. For all $h \in \mathbb{R}$, we have

$$\begin{aligned}
SS_{(x,y)}^p(h) &= \frac{[(x,y)(p+h)] \;-\; [(x,y)(p)]}{h} \\
&= \frac{(\, x(p+h)\, ,\, y(p+h)\,) \;-\; (\, x(p)\, ,\, y(p)\,)}{h} \\
&= \frac{(\;\; [x(p+h)] \;-\; [x(p)] \;\; , \;\; [y(p+h)] \;-\; [y(p)] \;\;)}{h} \\
&= \left(\frac{[x(p+h)] - [x(p)]}{h} \; , \; \frac{[y(p+h)] - [y(p)]}{h} \right) \\
&= (\; SS_x^p(h) \; , \; SS_y^p(h) \;) \\
&= (\; SS_x^p \; , \; SS_y^p \;)\, (h) \;\; = \;\; (\phi, \psi)(h).
\end{aligned}$$

Then $SS_{(x,y)}^p = (\phi, \psi)$. Since $p \in \mathrm{LPD}_{\mathbb{R}}(x, y)$, by (3) of Remark 26.7 p. 197, we get $0 \in \mathrm{LPD}_{\mathbb{R}}(SS_{(x,y)}^p)$. So, since $SS_{(x,y)}^p = (\phi, \psi)$, we conclude that $0 \in \mathrm{LPD}_{\mathbb{R}}(\phi, \psi)$. Then, by Remark 40.4 p. 295, we have $\lim\limits_0 (\phi, \psi) \;\; =^* \;\; (\lim\limits_0 \phi\, ,\, \lim\limits_0 \psi\,)$. Then

$$\begin{aligned}
(x, y)'(p) \;\; &= \;\; \lim_0 SS_{(x,y)}^p \;\; = \;\; \lim_0 (\phi, \psi) \\
&=^* \;\; (\, \lim_0 \phi\, ,\, \lim_0 \psi\,) \;\; = \;\; (\, x'(p)\, ,\, y'(p)\,),
\end{aligned}$$

as desired. QED

Homework 7: Due on Tuesday 6 March

7-1. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$ and $\delta > 0$. Assume

(1) $(p-\delta,p+\delta) \;\subseteq\; \text{dom}\,[f]$,
(2) f is strictly decreasing on $(p-\delta,p]$ and
(3) f is strictly increasing on $[p,p+\delta)$.

Show that f has a local unique minimum at p in $\mathbb{R}$.

Proof: We want: $\exists V \in \mathcal{N}_{\mathbb{R}}(p)$ s.t. $V \subseteq \text{dom}\,[f]$ and $f_*(V_p^\times) > f(p)$.

Since $(p-\delta,p+\delta)$ is open in $\mathbb{R}$ and since $p \in (p-\delta,p+\delta)$, it follows that $(p-\delta,p+\delta) \in \mathcal{N}_{\mathbb{R}}(p)$. Let $V := (p-\delta,p+\delta)$. We wish to show: $V \subseteq \text{dom}\,[f]$ and $f_*(V_p^\times) > f(p)$.

Since $V = (p-\delta,p+\delta)$, by (1), we have $V \subseteq \text{dom}\,[f]$. It remains to show: $f_*(V_p^\times) > f(p)$. We wish to show: $\forall y \in f_*(V_p^\times)$, $y > f(p)$. Let $y \in f_*(V_p^\times)$ be given. We wish to show: $y > f(p)$.

Since $y \in f_*(V_p^\times)$, choose $t \in V_p^\times$ such that $y = f(t)$. We have $t \in V_p^\times = (p-\delta,p) \cup (p,p+\delta)$, so one of the following must be true:

(A) $t \in (p-\delta,p)$ or
(B) $t \in (p,p+\delta)$.

Case (A): We have $t,p \in (p-\delta,p]$ and $t < p$. So, by (2), $f(t) > f(p)$. So, as $y = f(t)$, we get $y > f(p)$. *End of Case (A).*

Case (B): We have $t,p \in [p,p+\delta)$ and $t > p$. So, by (3), $f(t) > f(p)$. So, as $y = f(t)$, we get $y > f(p)$. *End of Case (B).* QED

7-2. Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$ and $p \in \text{IntD}_{\mathbb{R}}g$. Assume both that $g(p) = 0$ and that $g'(p) > 0$. Show: $\exists \delta > 0$ such that all three of the following hold:

(A) $(p-\delta,p+\delta) \;\subseteq\; \text{dom}\,[g]$,
(B) $g < 0$ on $(p-\delta,p)$ and
(C) $g > 0$ on $(p,p+\delta)$.

Proof: By Lemma 32.1 p. 240, choose $U \in \mathcal{N}_{\mathbb{R}}(0)$ s.t. $(SS_g^p)_*(U) > 0$. Then $U + p \in \mathcal{N}_{\mathbb{R}}(p)$. Let $D := \text{dom}\,[g]$. Since $p \in \text{IntD}_{\mathbb{R}}g = \text{Int}_{\mathbb{R}}D$, $D \in \mathcal{N}_{\mathbb{R}}(p)$. As $U + p, D \in \mathcal{N}_{\mathbb{R}}(p)$, we get $(U+p) \cap D \in \mathcal{N}_{\mathbb{R}}(p)$. So, since $\mathcal{B}_{\mathbb{R}}(p)$ is a neighborhood base at p in $\mathbb{R}$, choose $\delta > 0$ such that $B_{\mathbb{R}}(p,\delta) \subseteq (U+p) \cap D$. We wish to show: (A) and (B) and (C).

Proof of (A): We have

$$(p-\delta,p+\delta) \;=\; B_{\mathbb{R}}(p,\delta) \;\subseteq\; (U+p) \cap D \;\subseteq\; D \;=\; \text{dom}\,[g],$$

as desired. *End of proof of (A).*

Proof of (B): We wish to show: $\forall t \in (p-\delta,p)$, $g(t) < 0$. Let $t \in (p-\delta,p)$ be given. We wish to show: $g(t) < 0$.

We have $t \in (p - \delta, p) \subseteq B_{\mathbb{R}}(p, \delta) \subseteq (U + p) \cap D$, so $t \in U + p$ and $t \in D$. Since $t \in U + p$, we get $t - p \in U$. Let $h := t - p$. Then $h \in U$ and $p + h = t$. Also, since $t < p$, we get $h < 0$. In particular, $h \neq 0$. We have $\mathrm{dom}\,[SS_g^p] = [(\mathrm{dom}\,[g]) - p]_0^\times = [D - p]_0^\times$. So, since $h = t - p \in D - p$ and since $h \neq 0$, we get $h \in \mathrm{dom}\,[SS_g^p]$. So, since $h \in U$, we get $(SS_g^p)(h) \in (SS_g^p)_*(U)$. So, since $(SS_g^p)_*(U) > 0$, we get $(SS_g^p)(h) > 0$. So, since $h < 0$, we get $h \cdot [(SS_g^p)(h)] < 0$. So, since $h \cdot [(SS_g^p)(h)] = [g(p + h)] - [g(p)]$, we get $g(p + h) < g(p)$. So, since $p + h = t$, we get $g(t) < g(p)$. By assumption, $g(p) = 0$. Then $g(t) < 0$, as desired. *End of proof of (B).*

Proof of (C): We wish to show: $\forall t \in (p, p + \delta)$, $g(t) > 0$. Let $t \in (p, p + \delta)$ be given. We wish to show: $g(t) > 0$.

We have $t \in (p, p + \delta) \subseteq B_{\mathbb{R}}(p, \delta) \subseteq (U + p) \cap D$, so $t \in U + p$ and $t \in D$. Since $t \in U + p$, we get $t - p \in U$. Let $h := t - p$. Then $h \in U$ and $p + h = t$. Also, since $t > p$, we get $h > 0$. In particular, $h \neq 0$. We have $\mathrm{dom}\,[SS_g^p] = [(\mathrm{dom}\,[g]) - p]_0^\times = [D - p]_0^\times$. So, since $h = t - p \in D - p$ and since $h \neq 0$, we get $h \in \mathrm{dom}\,[SS_g^p]$. So, since $h \in U$, we get $(SS_g^p)(h) \in (SS_g^p)_*(U)$. So, since $(SS_g^p)_*(U) > 0$, we get $(SS_g^p)(h) > 0$. So, since $h > 0$, we get $h \cdot [(SS_g^p)(h)] > 0$. So, since $h \cdot [(SS_g^p)(h)] = [g(p + h)] - [g(p)]$, we get $g(p + h) > g(p)$. So, since $p + h = t$, we get $g(t) > g(p)$. By assumption, $g(p) = 0$. Then $g(t) > 0$, as desired. *End of proof of (C).* QED

7-3. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathrm{IntD}_{\mathbb{R}}(f')$. Assume both that $f'(p) = 0$ and that $f''(p) > 0$. Show: f has a local unique minimum at p in $\mathbb{R}$.

Proof: Let $g := f'$. Then $p \in \mathrm{IntD}_{\mathbb{R}} g$ and $g(p) = 0$ and $g'(p) > 0$. So, by HW#7-2, choose $\delta > 0$ such that all three of the following hold:

(A) $(p - \delta, p + \delta) \subseteq \mathrm{dom}\,[g]$,
(B) $g < 0$ on $(p - \delta, p)$ and
(C) $g > 0$ on $(p, p + \delta)$.

By HW#7-1, it suffices to show:

(1) $(p - \delta, p + \delta) \subseteq \mathrm{dom}\,[f]$,
(2) f is strictly decreasing on $(p - \delta, p]$ and
(3) f is strictly increasing on $[p, p + \delta)$.

Proof of (1): Since $\mathrm{dom}\,[g] = \mathrm{dom}\,[f'] \subseteq \mathrm{dom}\,[f]$, by (A), we see that $(p - \delta, p + \delta) \subseteq \mathrm{dom}\,[f]$, as desired. *End of proof of (1).*

Proof of (2): Let $I := (p - \delta, p]$. We want: $f|I$ is strictly decreasing.

Since $g = f'$, by (A), we see that f is differentiable on $(p-\delta, p+\delta)$. So, as $I \subseteq (p-\delta, p+\delta)$, f is differentiable on I. Then f is c/d on I. Since $g = f'$ and $\mathrm{Int}_{\mathbb{R}} I = (p-\delta, p)$, by (B), we get: $f' < 0$ on $\mathrm{Int}_{\mathbb{R}} I$. Then, by (5) of Corollary 34.9 p. 259, we see that $f|I$ is strictly decreasing, as desired. *End of proof of (2).*

Proof of (3): Let $I := [p, p+\delta)$. We want: $f|I$ is strictly increasing.

Since $g = f'$, by (A), we see that f is differentiable on $(p-\delta, p+\delta)$. So, as $I \subseteq (p-\delta, p+\delta)$, f is differentiable on I. Then f is c/d on I. Since $g = f'$ and $\mathrm{Int}_{\mathbb{R}} I = (p, p+\delta)$, by (B), we get: $f' > 0$ on $\mathrm{Int}_{\mathbb{R}} I$. Then, by (3) of Corollary 34.9 p. 259, we see that $f|I$ is strictly increasing, as desired. *End of proof of (3).* QED

7-4. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathrm{IntD}_{\mathbb{R}}(f')$. Assume both that $f'(p) = 0$ and that $f''(p) < 0$. Show: f has a local unique maximum at p in $\mathbb{R}$.

Proof: We wish to show $\exists V \in \mathcal{N}_{\mathbb{R}}(p)$ such that $V \subseteq \mathrm{dom}\,[f]$ and $f_*(V_p^\times) < f(p)$.

Let $f_0 := -f$. Then $\mathrm{dom}\,[f_0] = \mathrm{dom}\,[f]$. Also, $f_0' = -f'$. Then $\mathrm{dom}\,[f_0'] = \mathrm{dom}\,[f']$, so $\mathrm{IntD}_{\mathbb{R}}(f_0') = \mathrm{IntD}_{\mathbb{R}}(f')$. Also, $f_0'' = -f''$. Then

- $f_0 : \mathbb{R} \dashrightarrow \mathbb{R}$,
- $p \in \mathrm{IntD}_{\mathbb{R}}(f') = \mathrm{IntD}_{\mathbb{R}}(f_0')$,
- $f_0'(p) = (-f')(p) = -0 = 0$ and
- $f_0''(p) = (-f'')(p) = -(f''(p)) > 0$.

Then, by HW#7-3, f_0 has a local unique minimum at p in $\mathbb{R}$. So choose $V \in \mathcal{N}_{\mathbb{R}}(p)$ such that $V \subseteq \mathrm{dom}\,[f_0]$ and such that $(f_0)_*(V_p^\times) > f_0(p)$. As $V \subseteq \mathrm{dom}\,[f_0] = \mathrm{dom}\,[f]$, it remains only to show: $f_*(V_p^\times) < f(p)$. We wish to show: $\forall t \in \mathrm{dom}\,[f]$,

$$[\ t \in V_p^\times\ ] \quad \Rightarrow \quad [\ f(t) < f(p)\ ].$$

Let $t \in \mathrm{dom}\,[f]$ be given. We wish to show:

$$[\ t \in V_p^\times\ ] \quad \Rightarrow \quad [\ f(t) < f(p)\ ].$$

Assume: $t \in V_p^\times$. We wish to prove: $f(t) < f(p)$.

Since $t \in \mathrm{dom}\,[f] = \mathrm{dom}\,[f_0]$, and $t \in V_p^\times$, we get $f_0(t) \in (f_0)_*(V_p^\times)$. So, since $(f_0)_*(V_p^\times) > f_0(p)$, we get $f_0(t) > f_0(p)$. Then

$$-(f(t)) \;=\; (-f)(t) \;=\; f_0(t) \;>\; f_0(p) \;=\; (-f)(p) \;=\; -(f(p)).$$

Multiplying by -1, we get $f(t) < f(p)$, as desired. QED

7-5. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Asume both that $0 \in \mathrm{IntD}_{\mathbb{R}} f$ and that $f(0) = 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^4$. Assume that $f/g \to 1$ near 0. Show that f has a local unique minimum at 0 in $\mathbb{R}$.

Proof: Want: $\exists V \in \mathcal{N}_{\mathbb{R}}(0)$ s.t. $V \subseteq \mathrm{dom}\,[f]$ and $f(0) < f_*(V_0^\times)$.

Let $D := \mathrm{dom}\,[f]$. Then $0 \in \mathrm{IntD}_{\mathbb{R}} f = \mathrm{Int}_{\mathbb{R}} D$. Then $D \in \mathcal{N}_{\mathbb{R}}(0)$. So, since $\mathcal{B}_{\mathbb{R}}(0)$ is a neighborhood base at 0 in $\mathbb{R}$, choose $\alpha > 0$ s.t. $B_{\mathbb{R}}(0, \alpha) \subseteq D$. Since $f/g \to 1$ near 0, choose $\beta > 0$ s.t., $\forall t \in \mathrm{dom}\,[f/g]$,

$$[\ 0 < |t| < \beta\ ] \quad \Rightarrow \quad [\ |\,[(f/g)(t)] - 1\,| < 1/2\ ].$$

Let $\delta := \min\{\alpha, \beta\}$. Then $\delta \leqslant \alpha$, so $B(0, \delta) \subseteq B(0, \alpha)$. Also, $\delta \leqslant \beta$, so $B(0, \delta) \subseteq B(0, \beta)$. Since $B_{\mathbb{R}}(0, \delta)$ is open in $\mathbb{R}$ and since $0 \in B_{\mathbb{R}}(0, \delta)$, it follows that $B_{\mathbb{R}}(0, \delta) \in \mathcal{N}_{\mathbb{R}}(0)$. Let $V := B_{\mathbb{R}}(0, \delta)$. We wish to show: $V \subseteq \mathrm{dom}\,[f]$ and $f(0) < f_*(V_0^\times)$.

We have $V = B_{\mathbb{R}}(0, \delta) \subseteq B_{\mathbb{R}}(0, \alpha) \subseteq D = \mathrm{dom}\,[f]$. It remains to show: $f(0) < f_*(V_0^\times)$. We wish to show: $\forall y \in f_*(V_0^\times)$, $f(0) < y$. Let $y \in f_*(V_0^\times)$ be given. We wish to show: $f(0) < y$. By assumption, $f(0) = 0$. We wish to show: $0 < y$. Since $y \in f_*(V_0^\times)$, choose $t \in V_0^\times$ such that $y = f(t)$. We wish to show: $0 < f(t)$.

Since $t \in V_0^\times = V \backslash \{0\}$, we see that $t \neq 0$. Then $t^4 > 0$ and $0 < |t|$. We have $t \in V_0^\times \subseteq V \subseteq \mathrm{dom}\,[f]$. Also, $t \in \mathbb{R} = \mathrm{dom}\,[g]$. Also, $g(t) = t^4 > 0$, so, in particular, $g(t) \neq 0$. Then $t \in \mathrm{dom}\,[f/g]$. Also, we have $t \in V_0^\times \subseteq V = B_{\mathbb{R}}(0, \delta) \subseteq B_{\mathbb{R}}(0, \beta)$, so $|t| < \beta$. Since $t \in \mathrm{dom}\,[f/g]$ and since $0 < |t| < \beta$, by the choice of β, $|\,[(f/g)(t)] - 1\,| < 1/2$. Then $1 - (1/2) < (f/g)(t) < 1 + (1/2)$. Then $1/2 < (f/g)(t)$. Multiplying this inequality by $g(t)$, since $g(t) > 0$, we get $[1/2] \cdot [g(t)] < [(f/g)(t)] \cdot [g(t)]$. Since $g(t) > 0$, we conclude that $0 < [1/2] \cdot [g(t)]$. By definition of f/g, since $t \in \mathrm{dom}\,[f/g]$, we see that $[(f/g)(t)] \cdot [g(t)] = f(t)$. Then we have $0 < [1/2] \cdot [g(t)] < [(f/g)(t)] \cdot [g(t)] = f(t)$, as desired. QED

Homework 6: Due on Tuesday 27 February

6-1. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ be continuous, and let $I \subseteq \mathrm{dom}\,[f]$. Assume that I is an interval. Show: $f_*(I)$ is an interval.

Proof: Since I is an interval, $I \neq \varnothing$. So, since $I \subseteq \mathrm{dom}\,[f]$, we get $f_*(I) \neq \varnothing$. So, by $\Leftarrow$ of Fact 37.1 p. 276, we need only show: $\forall a, b \in f_*(I)$, $[a|b] \subseteq f_*(I)$. Given $a, b \in f_*(I)$. Want: $[a|b] \subseteq f_*(I)$.

Since $a, b \in f_*(I)$, choose $s, t \in I$ such that $a = f(s)$ and $b = f(t)$. Since $s, t \in I$ and I is an interval, we conclude, by $\Rightarrow$ of Fact 37.1 p. 276, that $[s|t] \subseteq I$. Then $f_*([s|t]) \subseteq f_*(I)$. Also, $[s|t] \subseteq I \subseteq \mathrm{dom}\,[f]$, so, as f is continuous, f is continuous on $[s|t]$. Then, by the Intermediate Value Theorem (Theorem 29.7 p. 218), $[f(s)|f(t)] \subseteq f_*([s|t])$. Then $[a|b] = [f(s)|f(t)] \subseteq f_*([s|t]) \subseteq f_*(I)$, as desired. QED

6-2. Let $X, Y \subseteq \mathbb{R}$. Let $f : X \hookrightarrow\!\!> Y$ be continuous. Let $X_0 := \mathrm{Int}_{\mathbb{R}} X$. Show that f^{-1} is continuous on $f_*(X_0)$.

Proof: We wish to show: $\forall q \in f_*(X_0)$, f^{-1} is continuous at q. Let $q \in f_*(X_0)$ be given. We wish to show: f^{-1} is continuous at q. Let $g := f^{-1}$. We wish to show: g is continuous at q. We wish to show: $\forall U \in \mathcal{N}_{\mathbb{R}}(g(q))$, $\exists V \in \mathcal{N}_{\mathbb{R}}(q)$ s.t. $g_*(V) \subseteq U$. Let $U \in \mathcal{N}_{\mathbb{R}}(g(q))$ be given. We wish to show: $\exists V \in \mathcal{N}_{\mathbb{R}}(q)$ s.t. $g_*(V) \subseteq U$.

Since $q \in f_*(X_0)$, choose $p \in X_0$ s.t. $q = f(p)$. Then $g(q) = p$. Then $U \in \mathcal{N}_{\mathbb{R}}(g(q)) = \mathcal{N}_{\mathbb{R}}(p)$. That is, U is a neighborhood of p in X. Choose an open U_0 in $\mathbb{R}$ s.t. $p \in U_0 \subseteq U$. Let $V := f_*(U_0 \cap X_0)$. We wish to show: both $V \in \mathcal{N}_{\mathbb{R}}(q)$ and $g_*(V) \subseteq U$.

Since $g = f^{-1}$ and since $U_0 \cap X_0 \subseteq X_0 \subseteq X = \mathrm{dom}\,[f]$, it follows that $g_*(f_*(U_0 \cap X_0)) = U_0 \cap X_0$. Then

$$g_*(V) \quad = \quad g_*(f_*(U_0 \cap X_0)) \quad = \quad U_0 \cap X_0 \quad \subseteq \quad U_0 \quad \subseteq \quad U.$$

It remains to show: $V \in \mathcal{N}_{\mathbb{R}}(q)$.

Since $X_0 = \mathrm{Int}_{\mathbb{R}} X$, we get: X_0 is open in $\mathbb{R}$. So, since U_0 is open in $\mathbb{R}$, we see that $U_0 \cap X_0$ is open in $\mathbb{R}$. Also, $U_0 \cap X_0 \subseteq X_0 \subseteq X = \mathrm{dom}\,[f]$. Also, $f : \mathbb{R} \dashrightarrow \mathbb{R}$ is 1-1 and continuous. Therefore, by Invariance of Domain (Theorem 30.3 p. 227), $f_*(U_0 \cap X_0)$ is open in $\mathbb{R}$. That is, V is open in $\mathbb{R}$. Since $p \in U_0$ and $p \in X_0$, we get $p \in U_0 \cap X_0$. So, since $p \in X_0 \subseteq X = \mathrm{dom}\,[f]$, we get $f(p) \in f_*(U_0 \cap X_0)$. Then $q = f(p) \in f_*(U_0 \cap X_0) = V$. By Remark 16.4 p. 117, any open set is a neighborhood of each of its points. So, since $q \in V$ and since V is open in $\mathbb{R}$, it follows that $V \in \mathcal{N}_{\mathbb{R}}(q)$, as desired. QED

6-3. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ be str. increasing. Show: f^{-1} is str. increasing.

Proof: Let $g := f^{-1}$. We wish to show: g is str. increasing. We wish to show: $\forall s, t \in \mathrm{dom}\,[g]$, $(\,[s < t] \Rightarrow [g(s) < g(t)]\,)$. Let $s, t \in \mathrm{dom}\,[g]$ be given. We wish to show: $(\,[s < t] \Rightarrow [g(s) < g(t)]\,)$. Assume that

$s < t$. We want: $g(s) < g(t)$. Let $a := g(s)$, $b := g(t)$. We want: $a < b$. Assume that $a \geqslant b$. We aim for a contradiction.

Since f is strictly increasing, it follows that f is semiincreasing. So, since $a \geqslant b$, we get $f(a) \geqslant f(b)$. Since $a = g(s)$, we get $f(a) = s$. Since $b = g(t)$, we get $f(b) = t$. Then $s = f(a) \geqslant f(b) = t$, so $t \leqslant s$. Then $t \leqslant s < t$, so $t < t$. Contradiction. QED

6-4. Let $X, Y \subseteq \mathbb{R}$. Let $f : X \hookrightarrow\!> Y$ be strictly increasing. Let $a \in \mathbb{R}$. Assume: $X = [a, \infty)$. Show: f^{-1} is continuous at $f(a)$.

Proof: Let $g := f^{-1}$ and let $q = f(a)$. We wish to show: g is continuous at q. We wish to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall t \in \mathrm{dom}\,[g]$,

$$[\ |t - q| < \delta\ ] \quad \Rightarrow \quad [\ |[g(t)] - [g(q)]| < \varepsilon\ ].$$

Let $\varepsilon > 0$ be given. We wish to show: $\exists \delta > 0$ s.t., $\forall t \in \mathrm{dom}\,[g]$,

$$[\ |t - q| < \delta\ ] \quad \Rightarrow \quad [\ |[g(t)] - [g(q)]| < \varepsilon\ ].$$

We have $a, a + \varepsilon \in [a, \infty) = X = \mathrm{dom}\,[f]$. So, since f is strictly increasing and $a < a + \varepsilon$, we get $f(a) < f(a + \varepsilon)$. We conclude that $0 < [f(a+\varepsilon)] - [f(a)]$. Let $\delta := [f(a+\varepsilon)] - [f(a)]$. Want: $\forall t \in \mathrm{dom}\,[g]$,

$$[\ |t - q| < \delta\ ] \quad \Rightarrow \quad [\ |[g(t)] - [g(q)]| < \varepsilon\ ].$$

Let $t \in \mathrm{dom}\,[g]$ be given. We wish to show:

$$[\ |t - q| < \delta\ ] \quad \Rightarrow \quad [\ |[g(t)] - [g(q)]| < \varepsilon\ ].$$

Assume that $|t - q| < \delta$. We wish to show: $|[g(t)] - [g(q)]| < \varepsilon$.

Let $s := g(t)$. Since $q = f(a)$, we get $g(q) = a$. We want: $|s - a| < \varepsilon$. Equivalently, we wish to show: $a - \varepsilon < s < a + \varepsilon$.

Since $t \in \mathrm{dom}\,[g]$, we get $g(t) \in \mathrm{im}\,[g]$. Since $g = f^{-1}$, we get $\mathrm{im}\,[g] = \mathrm{dom}\,[f]$. Then $s = g(t) \in \mathrm{im}\,[g] = \mathrm{dom}\,[f] = X = [a, \infty)$, so $s \geqslant a$. Then $a - \varepsilon < a \leqslant s$. It remains to show: $s < a + \varepsilon$.

Since $|t - q| < \delta$, we see that $q - \delta < t < q + \delta$. Then

$$t \ < \ q + \delta \ = \ [f(a)] + [f(a + \varepsilon)] - [f(a)] \ = \ f(a + \varepsilon).$$

Since $a + \varepsilon \in \mathrm{dom}\,[f]$, we get $f(a + \varepsilon) \in \mathrm{im}\,[f]$. Since $g = f^{-1}$, we get $\mathrm{dom}\,[g] = \mathrm{im}\,[f]$. Then $f(a + \varepsilon) \in \mathrm{im}\,[f] = \mathrm{dom}\,[g]$. By HW#6-3, we see that f^{-1} is strictly increasing. That is, g is strictly increasing. So, since $t < f(a + \varepsilon)$, since $t \in \mathrm{dom}\,[g]$ and since $f(a + \varepsilon) \in \mathrm{dom}\,[g]$, we get $g(t) < g(f(a + \varepsilon))$. Since $a + \varepsilon \in \mathrm{dom}\,[f]$ and $g = f^{-1}$, we get $g(f(a + \varepsilon)) = a + \varepsilon$. Then $s = g(t) < g(f(a + \varepsilon)) = a + \varepsilon$. QED

6-5. Let $X, Y \subseteq \mathbb{R}$. Let $f : X \hookrightarrow\!> Y$ be strictly increasing. Let $a \in \mathbb{R}$. Assume: $\exists b \in (a, \infty)$ s.t. $X = [a, b)$. Show: f^{-1} is continuous at $f(a)$.

Proof: Let $g := f^{-1}$ and let $q = f(a)$. We wish to show: g is continuous at q. We wish to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall t \in \operatorname{dom}[g]$,

$$[\ |t - q| < \delta\ ] \quad \Rightarrow \quad [\ |[g(t)] - [g(q)]| < \varepsilon\ ].$$

Let $\varepsilon > 0$ be given. We wish to show: $\exists \delta > 0$ s.t., $\forall t \in \operatorname{dom}[g]$,

$$[\ |t - q| < \delta\ ] \quad \Rightarrow \quad [\ |[g(t)] - [g(q)]| < \varepsilon\ ].$$

Choose $b \in (a, \infty)$ s.t. $X = [a, b)$. Let $c := (a+b)/2$. Since $b \in (a, \infty)$, it follows that $a < c < b$. Then $c-a > 0$. Let $\varepsilon_0 := \min\{\varepsilon, c-a\}$. Then $\varepsilon_0 > 0$, so $a < a+\varepsilon_0$. Also, $\varepsilon_0 \leqslant c-a$. Then $a+\varepsilon_0 \leqslant a+c-a = c < b$. Then $a < a + \varepsilon_0 < b$. Then $a + \varepsilon_0 \in (a, b) \subseteq [a, b)$.

We have $a, a + \varepsilon_0 \in [a, b) = X = \operatorname{dom}[f]$. So, since f is strictly increasing and since $a < a + \varepsilon_0$, it follows that $f(a) < f(a + \varepsilon_0)$. Therefore, we have $0 < [f(a+\varepsilon_0)]-[f(a)]$. Let $\delta := [f(a+\varepsilon_0)]-[f(a)]$. We wish to show: $\forall t \in \operatorname{dom}[g]$,

$$[\ |t - q| < \delta\ ] \quad \Rightarrow \quad [\ |[g(t)] - [g(q)]| < \varepsilon\ ].$$

Let $t \in \operatorname{dom}[g]$ be given. We wish to show:

$$[\ |t - q| < \delta\ ] \quad \Rightarrow \quad [\ |[g(t)] - [g(q)]| < \varepsilon\ ].$$

Assume that $|t - q| < \delta$. We wish to show: $|[g(t)] - [g(q)]| < \varepsilon$.

Let $s := g(t)$. Since $q = f(a)$, we get $g(q) = a$. We want: $|s-a| < \varepsilon$. Equivalently, we wish to show: $a - \varepsilon < s < a + \varepsilon$.

Since $t \in \operatorname{dom}[g]$, we get $g(t) \in \operatorname{im}[g]$. Since $g = f^{-1}$, we get $\operatorname{im}[g] = \operatorname{dom}[f]$. Then $s = g(t) \in \operatorname{im}[g] = \operatorname{dom}[f] = X = [a, b)$, so $s \geqslant a$. Then $a - \varepsilon < a \leqslant s$. It remains to show: $s < a + \varepsilon$.

Since $|t - q| < \delta$, we see that $q - \delta < t < q + \delta$. Then

$$t \ < \ q + \delta \ = \ [f(a)] + [f(a + \varepsilon_0)] - [f(a)] \ = \ f(a + \varepsilon_0).$$

Since $a+\varepsilon_0 \in \operatorname{dom}[f]$, we get $f(a+\varepsilon_0) \in \operatorname{im}[f]$. Since $g = f^{-1}$, we get $\operatorname{dom}[g] = \operatorname{im}[f]$. Then $f(a+\varepsilon_0) \in \operatorname{im}[f] = \operatorname{dom}[g]$. By HW#6-3, we see that f^{-1} is strictly increasing. That is, g is strictly increasing. So, since $t < f(a + \varepsilon_0)$, since $t \in \operatorname{dom}[g]$ and since $f(a + \varepsilon_0) \in \operatorname{dom}[g]$, we get $g(t) < g(f(a + \varepsilon_0))$. Since $a + \varepsilon_0 \in \operatorname{dom}[f]$ and $g = f^{-1}$, we get $g(f(a + \varepsilon_0)) = a + \varepsilon_0$. Then $s = g(t) < g(f(a + \varepsilon_0)) = a + \varepsilon_0$. Since $a + \varepsilon_0 \in \operatorname{dom}[f]$ and $g = f^{-1}$, we get $g(f(a + \varepsilon_0)) = a + \varepsilon_0$. Then

$s = g(t) < g(f(a+\varepsilon_0)) = a+\varepsilon_0$. We have $\varepsilon_0 = \min\{\varepsilon, c-a\} \leqslant \varepsilon$. Then $a+\varepsilon_0 \leqslant a+\varepsilon$. Then $s < a+\varepsilon_0 \leqslant a+\varepsilon$, as desired. QED

Homework 5: Due on Tuesday 20 February

5-1. Show: $\forall w, x \in \mathbb{R}, \quad \cos(w+x) = (\cos w)\cdot(\cos x) - (\sin w)\cdot(\sin x)$.

Proof: Let $w \in \mathbb{R}$ be given. We wish to show: $\forall x \in \mathbb{R}$,

$$\cos(w+x) \quad = \quad (\cos w)\cdot(\cos x) \;-\; (\sin w)\cdot(\sin x).$$

Define $\tau : \mathbb{R} \to \mathbb{R}$ by $\tau(x) = w + x$.

Claim 1: $\forall x \in \mathbb{R}$, $\tau'(x) = 1$. *Proof of Claim 1:* Let $x \in \mathbb{R}$ be given. We wish to show: $\tau'(x) = 1$. We wish to show: $\lim\limits_{0} SS_\tau^x = 1$.

For all $h \in \mathbb{R}_0^\times$, we have

$$\begin{aligned} (SS_\tau^x)(h) &= \frac{[\tau(x+h)] - [\tau(x)]}{h} \\ &= \frac{[w+x+h] - [w+x]}{h} \\ &= \frac{h}{h} \;=\; 1 \;=\; C_{\mathbb{R}}^1(h). \end{aligned}$$

Then $SS_\tau^x = C_{\mathbb{R}}^1$ on $\mathbb{R}_0^\times$, so $\lim\limits_{0} SS_\tau^x = \lim\limits_{0} C_{\mathbb{R}}^1$. So, since $\lim\limits_{0} C_{\mathbb{R}}^1 = 1$, we get $\lim\limits_{0} SS_\tau^x = 1$, as desired. *End of proof of Claim 1.*

Let $f := \cos\circ\tau$. Then, for all $x \in \mathbb{R}$, we have

$$f(x) \;=\; (\cos\circ\tau)(x) \;=\; \cos(\tau(x)) \;=\; \cos(w+x).$$

We therefore wish to show: $\forall x \in \mathbb{R}$,

$$f(x) \quad = \quad (\cos w)\cdot(\cos x) \;-\; (\sin w)\cdot(\sin x).$$

Claim 2: $\forall x \in \mathbb{R}$, $f'(x) = (-\sin)(w+x)$. *Proof of Claim 2:* Let $x \in \mathbb{R}$ be given. We wish to show: $f'(x) = -[\sin(w+x)]$.

We have $\mathrm{dom}\,[\cos\circ\tau] = \mathbb{R}$. Then

$$x \;\in\; \mathbb{R} \;=\; \mathrm{LP}_{\mathbb{R}}\mathbb{R} \;=\; \mathrm{LP}_{\mathbb{R}}(\mathrm{dom}\,[\exp\circ\tau]) \;=\; \mathrm{LPD}_{\mathbb{R}}(\cos\circ\tau).$$

So, since $f = \cos\circ\tau$, by the Chain Rule, $f'(x) \;=^*\; [\cos'(\tau(x))]\,[\tau'(x)]$. By Claim 1, $\tau'(x) = 1$. Then $f'(x) \;=^*\; \cos'(\tau(x))$.

We have $\cos' = -\sin$ and $\tau(x) = w + x$. Then

$$f'(x) \;=^*\; (-\sin)(w+x) \;\neq\; ☹.$$

Then $f'(x) = (-\sin)(w+x)$, as desired. *End of proof of Claim 2.*

Claim 3: $\forall x \in \mathbb{R}$, $f''(x) = (-\cos)(w+x)$. *Proof of Claim 3:* By Claim 2, $f' = (-\sin) \circ \tau$. Let $x \in \mathbb{R}$ be given. We wish to show: $f''(x) = -[\cos(w+x)]$.

We have $\mathrm{dom}\,[(-\sin) \circ \tau] = \mathbb{R}$. Then

$$x \in \mathbb{R} = \mathrm{LP}_{\mathbb{R}}\mathbb{R} = \mathrm{LPD}_{\mathbb{R}}((-\sin) \circ \tau).$$

So, since $f' = (-\sin) \circ \tau$, by the Chain Rule, we conclude that $f''(x) =^* [(-\sin)'(\tau(x))]\,[\tau'(x)]$. By Claim 1, $\tau'(x) = 1$. Then $f''(x) =^* (-\sin)'(\tau(x))$.

We have $(-\sin)' = -\cos$ and $\tau(x) = w + x$. Then

$$f'(x) =^* (-\cos)(w+x) \neq ☹.$$

Then $f'(x) = (-\cos)(w+x)$, as desired. *End of proof of Claim 3.*

Claim 4: $f'' = -f$. *Proof of Claim 4:* We wish to show: $\forall x \in \mathbb{R}$, $f''(x) = (-f)(x)$. Let $x \in \mathbb{R}$ be given. Want: $f''(x) = (-f)(x)$.

By Claim 3, $f''(x) = (-\cos)(w+x)$. Then $f''(x) = -[\cos(w+x)]$. By definition of f, we have $f(x) = (\cos \circ \tau)(x)$. Then

$$f(x) = \cos(\tau(x)) = \cos(w+x).$$

We conclude that $f''(x) = -[\cos(w+x)] = -[f(x)] = (-f)(x)$, as desired. *End of proof of Claim 4.*

Let $a := f(0)$ and let $b := f'(0)$. By Claim 4 and Theorem 35.2, $f = a \cdot \cos + b \cdot \sin$. By definition of f, $f(0) = \cos(w+0)$. Then $a = f(0) = \cos(w+0) = \cos w$. By Claim 2, $f'(0) = (-\sin)(w+0)$. Then $b = f'(0) = (-\sin)(w+0) = -(\sin w)$. Then: $\forall x \in \mathbb{R}$,

$$\begin{aligned} f(x) &= (a \cdot \cos + b \cdot \sin)(x) \\ &= a \cdot (\cos x) + b \cdot (\sin x) \\ &= (\cos w) \cdot (\cos x) - (\sin w) \cdot (\sin x), \end{aligned}$$

as desired. QED

5-2. Show: $\forall x \in \mathbb{R}$,

$$\begin{aligned} \sin(2x) &= 2 \cdot (\sin x) \cdot (\cos x) \qquad \text{and} \\ \cos(2x) &= (\cos^2 x) - (\sin^2 x). \end{aligned}$$

Proof: Let $x \in \mathbb{R}$ be given. We wish to show:

$$\sin(2x) \;=\; 2 \cdot (\sin x) \cdot (\cos x) \qquad \text{and}$$
$$\cos(2x) \;=\; (\cos^2 x) \,-\, (\sin^2 x).$$

Let $w := x$. Then $\sin w = \sin x$ and $\cos w = \cos x$. By Theorem 35.3, we have $\sin(w + x) \;=\; (\sin w) \cdot (\cos x) \,+\, (\cos w) \cdot (\sin x)$. Then

$$\begin{aligned} \sin(2x) \;&=\; \sin(x + x) \;=\; \sin(w + x) \\ &=\; (\sin w) \cdot (\cos x) \,+\, (\cos w) \cdot (\sin x) \\ &=\; (\sin x) \cdot (\cos x) \,+\, (\cos x) \cdot (\sin x) \\ &=\; 2 \cdot (\sin x) \cdot (\cos x). \end{aligned}$$

It remains to show: $\cos(2x) \;=\; (\cos^2 x) \,-\, (\sin^2 x)$.

By HW#5-1, $\cos(w + x) \;=\; (\cos w) \cdot (\cos x) \,-\, (\sin w) \cdot (\sin x)$. Then

$$\begin{aligned} \cos(2x) \;&=\; \cos(x + x) \;=\; \cos(w + x) \\ &=\; (\cos w) \cdot (\cos x) \,-\, (\sin w) \cdot (\sin x) \\ &=\; (\cos x) \cdot (\cos x) \,-\, (\sin x) \cdot (\sin x) \\ &=\; (\cos^2 x) \,-\, (\sin^2 x), \end{aligned}$$

as desired. QED

5-3. Let $f : \mathbb{R} \to \mathbb{R}$, $a := f(0)$. Assume $f' = f$. Show: $f = a \cdot \exp$.

Proof: Let $\phi := f - (a \cdot \exp)$. We wish to show: $\phi = C^0_{\mathbb{R}}$. We have

$$\phi(0) \;=\; [f(0)] \,-\, a \cdot [\exp(0)] \;=\; [a] \,-\, a \cdot [1] \;=\; 0.$$

So, by Theorem 35.14, it suffices to show: $\phi' = \phi$. We wish to show: $\forall x \in \mathbb{R}$, $\phi'(x) = \phi(x)$. Let $x \in \mathbb{R}$ be given. Want: $\phi'(x) = \phi(x)$.

We have $\mathrm{dom}\,[f] = \mathbb{R}$ and $\mathrm{dom}\,[\exp] = \mathbb{R}$, so $\mathrm{dom}\,[\phi] = \mathbb{R}$. Then $x \in \mathbb{R} = \mathrm{dom}\,[\phi] = \mathrm{dom}\,[f - (a \cdot \exp)$. So, by linearity of differentiation

$$(f - (a \cdot \exp))'(x) \;=^*\; [f'(x)] \,-\, a \cdot [\exp'(x)].$$

So, since $f - (a \cdot \exp) = \phi$, we get $\phi'(x) \;=^*\; [f'(x)] \,-\, a \cdot [\exp'(x)]$. By assumption, we have $f' = f$. Then, since $\exp' = \exp$, we see that $\phi'(x) \;=^*\; [f(x)] \,-\, a \cdot [\exp(x)]$. So, since $\phi(x) = [f(x)] \,-\, a \cdot [\exp(x)]$, we get $\phi'(x) \;=^*\; \phi(x)$. Since $x \in \mathbb{R} = \mathrm{dom}\,[\phi]$, we get $\phi(x) \neq$ ☹. Then $\phi'(x) \;=^*\; \phi(x) \;\neq$ ☹, and so $\phi'(x) = \phi(x)$, as desired. QED

5-4. Show: $\forall w, x \in \mathbb{R}, \quad \exp(w + x) \;=\; (\exp w) \cdot (\exp x)$.

Proof: Let $w \in \mathbb{R}$ be given. We wish to show: $\forall x \in \mathbb{R}$,

$$\exp(w+x) \quad = \quad (\exp w) \cdot (\exp x).$$

Define $\tau : \mathbb{R} \to \mathbb{R}$ by $\tau(x) = w + x$.

Claim 1: $\forall x \in \mathbb{R}$, $\tau'(x) = 1$. *Proof of Claim 1:* Let $x \in \mathbb{R}$ be given. We wish to show: $\tau'(x) = 1$. We wish to show: $\lim_0 SS_\tau^x = 1$.

For all $h \in \mathbb{R}_0^\times$, we have

$$\begin{aligned} (SS_\tau^x)(h) &= \frac{[\tau(x+h)] - [\tau(x)]}{h} \\ &= \frac{[w+x+h] - [w+x]}{h} \\ &= \frac{h}{h} \;=\; 1 \;=\; C_{\mathbb{R}}^1(h). \end{aligned}$$

Then $SS_\tau^x = C_{\mathbb{R}}^1$ on $\mathbb{R}_0^\times$, so $\lim_0 SS_\tau^x = \lim_0 C_{\mathbb{R}}^1$. So, since $\lim_0 C_{\mathbb{R}}^1 = 1$, we get $\lim_0 SS_\tau^x = 1$, as desired. *End of proof of Claim 1.*

Let $f := \exp \circ \tau$. Then, for all $x \in \mathbb{R}$, we have

$$f(x) \;=\; (\exp \circ \tau)(x) \;=\; \exp(\tau(x)) \;=\; \exp(w+x).$$

We therefore wish to show: $\forall x \in \mathbb{R}$,

$$f(x) \quad = \quad (\exp w) \cdot (\exp x).$$

Claim 2: $\forall x \in \mathbb{R}$, $f'(x) = \exp(w+x)$. *Proof of Claim 2:* Let $x \in \mathbb{R}$ be given. We wish to show: $f'(x) = \exp(w+x)$.

We have $\mathrm{dom}\,[\exp \circ \tau] = \mathbb{R}$. Then

$$x \;\in\; \mathbb{R} \;=\; \mathrm{LP}_{\mathbb{R}}\mathbb{R} \;=\; \mathrm{LP}_{\mathbb{R}}(\mathrm{dom}\,[\exp \circ \tau]) \;=\; \mathrm{LPD}_{\mathbb{R}}(\exp \circ \tau).$$

So, since $f = \exp \circ \tau$, by the Chain Rule, $f'(x) \;=^*\; [\exp'(\tau(x))]\,[\tau'(x)]$. By Claim 1, $\tau'(x) = 1$. Then $f'(x) \;=^*\; \exp'(\tau(x))$.

We have $\exp' = \exp$ and $\tau(x) = w + x$. Then

$$f'(x) \;=^*\; \exp(w+x) \;\neq\; ☹.$$

Then $f'(x) \;=\; \exp(w+x)$, as desired. *End of proof of Claim 2.*

Claim 3: $f' = f$. *Proof of Claim 3:* We wish to show: $\forall x \in \mathbb{R}$, $f'(x) = f(x)$. Let $x \in \mathbb{R}$ be given. Want: $f'(x) = f(x)$.

By Claim 2, we have $f'(x) = \exp(w+x)$. By definition of f, we have $f(x) = (\exp \circ \tau)(x)$. Then $f(x) = \exp(\tau(x)) = \exp(w+x)$. We conclude: $f'(x) = \exp(w+x) = f(x)$. *End of proof of Claim 3.*

Let $a := f(0)$. By Claim 3 and HW#5-3, $f = a \cdot \exp$. By definition of f, $f(0) = \exp(w + 0)$. Then $a = f(0) = \exp(w + 0) = \exp w$. Then: $\forall x \in \mathbb{R}$, $f(x) = (a \cdot \exp)(x) = a \cdot (\exp x) = (\exp w) \cdot (\exp x)$. QED

5-5. Let $f : \mathbb{R} \to \mathbb{R}$. Assume that $f' = f$. Let $g := f^2$. Show: $g' = 2g$.

Proof: Define $P : \mathbb{R} \to \mathbb{R}$ by $P(x) = x^2$. By Theorem 16.3, for all $x \in \mathbb{R}$, we have $P'(x) = 2x$. Also, for all $x \in \mathbb{R}$, we have

$$(P \circ f)(x) \;=\; P(f(x)) \;=\; [f(x)]^2 \;=\; f^2(x) \;=\; g(x).$$

Then $g = P \circ f$. We wish to show: $\forall x \in \mathbb{R}$, $g'(x) = (2g)(x)$. Let $x \in \mathbb{R}$ be given. We wish to show: $g'(x) = (2g)(x)$.

We have $\operatorname{dom}[P \circ f] = \mathbb{R}$. so $\mathrm{LPD}_{\mathbb{R}}(P \circ f) = \mathrm{LP}_{\mathbb{R}}\mathbb{R}$. Then we have $x \in \mathbb{R} = \mathrm{LP}_{\mathbb{R}}\mathbb{R} = \mathrm{LPD}_{\mathbb{R}}(P \circ f)$. So, by the Chain Rule, we see that $(P \circ f)'(x) =^* [P'(f(x))] \cdot [f'(x)]$. Then $g'(x) =^* [P'(f(x))] \cdot [f'(x)]$. We have $P'(f(x)) = 2 \cdot [f(x)]$. By assumption, $f' = f$, so $f'(x) = f(x)$. Then $g'(x) =^* 2 \cdot [f(x)]^2 \neq$ ☹, so $g'(x) = 2 \cdot [f(x)]^2$. We have $g(x) = f^2(x) = [f(x)]^2$. Then $g'(x) = 2 \cdot [g(x)] = (2g)(x)$. QED

Homework 4: Due on Tuesday 13 February

4-1. Let $m \in \mathbb{R}$. Define $\lambda : \mathbb{R} \to \mathbb{R}$ by $\lambda(x) = mx$. Show: $\lambda' = C_{\mathbb{R}}^m$.

Proof: We want: $\forall x \in \mathbb{R}$, $\lambda'(x) = C_{\mathbb{R}}^m(x)$. Let $x \in \mathbb{R}$ be given. We want: $\lambda'(x) = C_{\mathbb{R}}^m(x)$. We have $\mathbb{C}_{\mathbb{R}}^m(x) = m$. We want: $\lambda'(x) = m$.

Let $\phi := C_{\mathbb{R}_0^\times}^m$. Since $\phi = C_{\mathbb{R}}^m$ on $\mathbb{R}_0^\times$, it follows that $\lim\limits_0 \phi = \lim\limits_0 C_{\mathbb{R}}^m$. So, since $\lim\limits_0 C_{\mathbb{R}}^m = m$, we get $\lim\limits_0 \phi = m$. We want: $\lim\limits_0 SS_\lambda^x = m$. It therefore suffices to show: $SS_\lambda^x = \phi$.

As $\mathbb{R}_0^\times$ is a common superdomain of SS_λ^x and ϕ, it suffices to show: $\forall h \in \mathbb{R}_0^\times$, $SS_\lambda^x(h) = \phi(h)$. Let $h \in \mathbb{R}_0^\times$ be given. Want: $SS_\lambda^x(h) = \phi(h)$.

We have $\phi(h) = C_{\mathbb{R}_0^\times}^m(h) = m$. Then

$$\begin{aligned} SS_\lambda^x(h) \;&=\; \frac{[\lambda(x+h)] - [\lambda(x)]}{h} \;=\; \frac{[m(x+h)] - [mx]}{h} \\ &=\; \frac{mx + mh - mx}{h} \;=\; \frac{mh}{h} \;=\; m \;= \phi(h), \end{aligned}$$

as desired. QED

4-2. Let $S \subseteq \mathbb{R}$. Show: $-(\mathrm{LP}_{\mathbb{R}}S) = \mathrm{LP}_{\mathbb{R}}(-S)$.

Proof: Define $\lambda : \mathbb{R} \to \mathbb{R}$ by $\lambda(x) = -x$. Then $\lambda : \mathbb{R} \to \mathbb{R}$ is continuous and one-to-one. Also, $\lambda^{-1} = \lambda$. Then $\lambda^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous. Then $\lambda : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Then $\lambda_*(\mathrm{LP}_{\mathbb{R}} S) = \mathrm{LP}_{\mathbb{R}}(\lambda_*(S))$. That is, $-(\mathrm{LP}_{\mathbb{R}} S) = \mathrm{LP}_{\mathbb{R}}(-S)$, as desried. QED

4-3. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$. Define $f_0 : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f_0(x) = f(-x)$. Let $p_0 := -p$. Assume that f has a local maximum at p in $\mathbb{R}$. Show that f_0 has a local maximum at p_0 in $\mathbb{R}$.

Proof: Choose $V \in \mathcal{N}_{\mathbb{R}}(p)$ s.t. $V \subseteq \mathrm{dom}\,[f]$ and $f(p) \geqslant f_*(V)$. We wish to show: $\exists U_0 \in \mathcal{N}_{\mathbb{R}}(p_0)$ s.t. $U_0 \subseteq \mathrm{dom}\,[f_0]$ and $f_0(p_0) \geqslant (f_0)_*(U_0)$.

Since $\mathcal{B}_{\mathbb{R}}(p)$ is a neighborhood base at p in $\mathbb{R}$, choose $\delta > 0$ such that $B_{\mathbb{R}}(p, \delta) \subseteq V$. Let $U := B_{\mathbb{R}}(p, \delta)$. Then $U \subseteq V$. Let $U_0 := B_{\mathbb{R}}(p_0, \delta)$. Since $\mathcal{B}_{\mathbb{R}}(p_0)$ is a neighborhood base at p_0 in $\mathbb{R}$, we get $U_0 \in \mathcal{N}_{\mathbb{R}}(p_0)$. Want: $U_0 \subseteq \mathrm{dom}\,[f_0]$ and $f_0(p_0) \geqslant (f_0)_*(U_0)$. Since $U \subseteq V \subseteq \mathrm{dom}\,[f]$, it follows that $-U \subseteq -(\mathrm{dom}\,[f])$. We have

$$U_0 \;=\; B_{\mathbb{R}}(p_0, \delta) \;=\; B_{\mathbb{R}}(-p, \delta) \;=\; -[B_{\mathbb{R}}(p, \delta)] \;=\; -U.$$

Then $U_0 = -U \subseteq -(\mathrm{dom}\,[f]) = \mathrm{dom}\,[f_0]$. Want: $f_0(p_0) \geqslant (f_0)_*(U_0)$. Want: $\forall y_0 \in (f_0)_*(U_0)$, $f_0(p_0) \geqslant y_0$. Let $y_0 \in (f_0)_*(U_0)$ be given. We wish to show: $f_0(p_0) \geqslant y_0$.

Since $y_0 \in (f_0)_*(U_0)$, choose $x_0 \in U_0$ s.t. $y_0 = f_0(x_0)$. Let $x := -x_0$. We have $f_0(p_0) = f(-p_0) = f(p)$ and $y_0 = f_0(x_0) = f(-x_0) = f(x)$. We wish to show that $f(p) \geqslant f(x)$, or, equivalently, that $f(x) \leqslant f(p)$.

We have $x = -x_0 \in -U_0 = -(-U) = U \subseteq V \subseteq \mathrm{dom}\,[f]$. Then $x \in U$ and $x \in \mathrm{dom}\,[f]$, and so $f(x) \in f_*(U)$. Since $U \subseteq V$, it follows that $f_*(U) \subseteq f_*(V)$. By choice of V, we have $f_*(V) \leqslant f(p)$. We conclude that $f(x) \in f_*(U) \subseteq f_*(V) \leqslant f(p)$, as desired. QED

4-4. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$. Define $f_1 : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f_1(x) = -[f(x)]$. Assume that f has a local minimum at p in $\mathbb{R}$. Show that f_1 has a local maximum at p in $\mathbb{R}$.

Proof: Want: $\exists V \in \mathcal{N}_{\mathbb{R}}(p)$ s.t. $V \subseteq \mathrm{dom}\,[f_1]$ and $f(p) \leqslant (f_1)_*(V)$. Choose $V \in \mathcal{N}_{\mathbb{R}}(p)$ s.t. $V \subseteq \mathrm{dom}\,[f]$ and $f(p) \leqslant f_*(V)$. We want: $V \subseteq \mathrm{dom}\,[f_1]$ and $f_1(p) \geqslant (f_1)_*(V)$. As $V \subseteq \mathrm{dom}\,[f] = \mathrm{dom}\,[f_1]$, it remains to show: $f_1(p) \geqslant (f_1)_*(V)$. We wish to show: $\forall y_1 \in (f_1)_*(V)$, $f_1(p) \geqslant y_1$. Let $y_1 \in (f_1)_*(V)$ be given. We wish to show: $f_1(p) \geqslant y_1$.

Since $y_1 \in (f_1)_*(V)$, choose $x \in V$ such that $y_1 = f_1(x)$. Since $x \in V \subseteq \text{dom}[f]$, we get $f(x) \in f_*(V)$. Then, by the choice of V, $f(p) \leqslant f(x)$. Then $f_1(p) = -[f(p)] \geqslant -[f(x)] = f_1(x) = y$. QED

4-5. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \text{dom}[f']$. Assume that f has a local extremum at p. Show that $f'(p) = 0$.

Proof: At least one of the following must be true:

(1) f has a local maximum at p in $\mathbb{R}$ or
(2) f has a local minimum at p in $\mathbb{R}$.

Case (1): By Lemma 33.16, $f'(p) = 0$, as desired. *End of Case (1).*

Case (2): Define $f_1 : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f_1(x) = -[f(x)]$. By HW#4-4, we see that f_1 has a local maximum at p in $\mathbb{R}$. Define $\lambda : \mathbb{R} \to \mathbb{R}$ by $\lambda(x) = -x$. Then $f_1 = \lambda \circ f$. Since f_1 has a local maximum at p in $\mathbb{R}$, by (3) of Remark 33.9, we get $p \in \text{LPD}_{\mathbb{R}} f_1$. Then $p \in \text{LPD}_{\mathbb{R}}(\lambda \circ f)$, so, by the Chain Rule, we get $(\lambda \circ f)'(p) \;=^*\; [\lambda'(f(p))][f'(p)]$. Since $p \in \text{dom}[f'] \subseteq \text{dom}[f]$, we get $f(p) \in \text{im}[f]$. So, since $\text{im}[f] \subseteq \mathbb{R}$, we get $f(p) \in \mathbb{R}$. By HW#4-2, $\lambda'(f(p)) = -1$. Then

$$f_1'(p) \;=\; (\lambda \circ f)'(p) \;=^*\; [\lambda'(f(p))][f'(p)] \;=\; -[f'(p)],$$

so $f_1'(p) \;=^*\; -[f'(p)]$. Since $p \in \text{dom}[f']$, we get $f'(p) \in \text{im}[f]$. So, since $\text{im}[f] \subseteq \mathbb{R}$, we get $f'(p) \in \mathbb{R}$. Then $-[f'(p)] \neq ☹$.

Since $f_1'(p) \;=^*\; -[f'(p)] \neq ☹$, we conclude that $f_1'(p) = -[f'(p)]$. Then $f_1'(p) \neq ☹$, so $p \in \text{dom}[f_1']$. So, since f_1 has a local maximum at p in $\mathbb{R}$, it follows, from Lemma 33.16, that $f_1'(p) = 0$.

Then $0 = f_1'(p) = -[f'(p)]$. Then $f'(p) = 0$. *End of Case (2).* QED

Homework 3: Due on Tuesday 6 February

3-1. Let X be a topological space, let $S \subseteq X$ and let $p \in X$. Show:

$$[\ p \in \text{Int}_X S\ ] \quad \Leftrightarrow \quad [\ S \in \mathcal{N}_X(p)\ ].$$

Proof: Proof of $\Rightarrow$*:* Assume: $p \in \text{Int}_X S$. We wish to show: $S \in \mathcal{N}_X(p)$. We wish to show: $\exists$open U in X s.t. $p \in U \subseteq S$.

Let $U := \text{Int}_X S$. Then U is open in X, and we wish to show: $p \in U \subseteq S$. We have $p \in \text{Int}_X S = U$, and it remains to show: $U \subseteq S$.

We have $U = \text{Int}_X S \subseteq S$, as desired. *End of proof of* $\Rightarrow$*.*

Proof of $\Leftarrow$*:* Assume: $S \in \mathcal{N}_X(p)$. We wish to show: $p \in \text{Int}_X S$.

Since $S \in \mathcal{N}_X(p)$, choose an open U in X s.t. $p \in U \subseteq S$. Since U is open in X, we get $\mathrm{Int}_X U = U$. Since $U \subseteq S$, $\mathrm{Int}_X U \subseteq \mathrm{Int}_X S$. Then $p \in U = \mathrm{Int}_X U \subseteq \mathrm{Int}_X S$, as desired. *End of proof of* $\Leftarrow$. QED

3-2. Let $\phi : \mathbb{R} \dashrightarrow \mathbb{R}$, let $p \in \mathbb{R}$ and let $q > 0$. Assume: $\phi \to q$ near p. Show: $\exists U \in \mathcal{N}_{\mathbb{R}}^{\times}(p)$ s.t. $\phi_*(U) > 0$.

Proof: Let $V := (0, \infty)$. Then V is open in $\mathbb{R}$ and $q \in V$. Therefore, by Remark 16.4, we conclude that $V \in \mathcal{N}_{\mathbb{R}}(q)$. So, since $\phi \to q$ near p, choose $U \in \mathcal{N}_{\mathbb{R}}^{\times}(p)$ s.t. $\phi_*(U) \subseteq V$. We wish to show: $\phi_*(U) > 0$.

We have $\phi_*(U) \subseteq V = (0, \infty) > 0$, as desired. QED

3-3. Let $\alpha, \beta, \gamma : \mathbb{R} \dashrightarrow \mathbb{R}$. Let $S \subseteq \mathrm{dom}\,[(\alpha, \beta, \gamma)]$. Let $p \in \mathrm{Int}_{\mathbb{R}} S$ and let $q \in \mathbb{R}$. Assume that $\alpha \leqslant \beta \leqslant \gamma$ on S. Assume that $\alpha(p) = \gamma(p)$. Assume that $\alpha'(p) = q = \gamma'(p)$. Show: $\beta'(p) = q$.

Proof: We wish to show: $\lim\limits_{0} SS_{\beta}^{p} = q$.

Claim: $\forall h \in (S - p)_0^{\times}$, $(SS_{\beta}^{p})(h) \in [\,(SS_{\alpha}^{p})(h) \,|\, (SS_{\gamma}^{p})(h)\,]$. *Proof of Claim:* Given $h \in (S - p)_0^{\times}$. Want: $(SS_{\beta}^{p})(h) \in [\,(SS_{\alpha}^{p})(h) \,|\, (SS_{\gamma}^{p})(h)\,]$.

We have $h \in (S - p)_0^{\times} \subseteq S - p$. Then $p + h \in S$. Then

$$\alpha(p + h) \quad \leqslant \quad \beta(p + h) \quad \leqslant \quad \gamma(p + h).$$

We have $p \in \mathrm{Int}_R S \subseteq S$. It follows that $\alpha(p) \leqslant \beta(p) \leqslant \gamma(p)$. So, since $\alpha(p) = \gamma(p)$, we conclude that $\alpha(p) = \beta(p) = \gamma(p)$. So, since $\alpha(p + h) \leqslant \beta(p + h) \leqslant \gamma(p + h)$, we get:

$$[\alpha(p + h)] - [\alpha(p)] \leqslant [\beta(p + h)] - [\beta(p)] \leqslant [\gamma(p + h)] - [\gamma(p)].$$

Dividing this by h, we get:

$$\begin{array}{l} (\ h > 0\ ) \ \Rightarrow \ (\ (SS_{\alpha}^{p})(h) \leqslant (SS_{\beta}^{p})(h) \leqslant SS_{\gamma}^{p}(h)\ ) \quad \text{and} \\ (\ h < 0\ ) \ \Rightarrow \ (\ (SS_{\alpha}^{p})(h) \geqslant (SS_{\beta}^{p})(h) \geqslant SS_{\gamma}^{p}(h)\ ). \end{array}$$

Then

$$\begin{array}{l} (\ h > 0\ ) \ \Rightarrow \ (\ (SS_{\beta}^{p})(h) \in [\,(SS_{\alpha}^{p})(h) \,|\, (SS_{\gamma}^{p})(h)\,]\ ) \quad \text{and} \\ (\ h < 0\ ) \ \Rightarrow \ (\ (SS_{\beta}^{p})(h) \in [\,(SS_{\alpha}^{p})(h) \,|\, (SS_{\gamma}^{p})(h)\,]\ ). \end{array}$$

Since $h \in (S - p)_0^{\times}$, we get $h \neq 0$, so: either $h > 0$ or $h < 0$. Then $(SS_{\beta}^{p})(h) \in [\,(SS_{\alpha}^{p})(h) \,|\, (SS_{\gamma}^{p})(h)\,]$, as desired. *End of proof of Claim.*

Let $D := \mathrm{dom}\,[\beta]$. Then $\mathrm{Int}_{\mathbb{R}} D \subseteq \mathrm{LP}_{\mathbb{R}} D = \mathrm{LPD}_{\mathbb{R}} \beta$. We have $S \subseteq \mathrm{dom}\,[(\alpha, \beta, \gamma)] \subseteq D$. Then $p \in \mathrm{Int}_{\mathbb{R}} S \subseteq \mathrm{Int}_{\mathbb{R}} D \subseteq \mathrm{LPD}_{\mathbb{R}} \beta$. It therefore suffices to show: $SS^p_\beta \to q$ near 0. We wish to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall h \in \mathrm{dom}\,[SS^p_\beta]$,

$$[\ 0 < |h| < \delta\ ] \quad \Rightarrow \quad [\ |[(SS^p_\beta)(h)] - q| < \varepsilon\ ].$$

Let $\varepsilon > 0$ be given. We wish to show: $\exists \delta > 0$ s.t., $\forall h \in \mathrm{dom}\,[SS^p_\beta]$,

$$[\ 0 < |h| < \delta\ ] \quad \Rightarrow \quad [\ |[(SS^p_\beta)(h)] - q| < \varepsilon\ ].$$

Since $p \in \mathrm{Int}_{\mathbb{R}} S$, by HW#3-1, we get $S \in \mathcal{N}_{\mathbb{R}}(p)$. So, since $\mathcal{B}_{\mathbb{R}}(p)$ is a neighborhood base at p in $\mathbb{R}$, choose $\eta > 0$ s.t. $B_{\mathbb{R}}(p, \eta) \subseteq S$. Since $\alpha'(p) = q$, it follows that $SS^p_\alpha \to q$ near p. Choose $\lambda > 0$ s.t., $\forall h \in \mathrm{dom}\,[SS^p_\alpha]$,

$$[\ 0 < |h| < \lambda\ ] \quad \Rightarrow \quad [\ |[(SS^p_\alpha)(h)] - q| < \varepsilon\ ].$$

Since $\gamma'(p) = q$, it follows that $SS^p_\gamma \to q$ near p. Choose $\mu > 0$ s.t., $\forall h \in \mathrm{dom}\,[SS^p_\gamma]$,

$$[\ 0 < |h| < \mu\ ] \quad \Rightarrow \quad [\ |[(SS^p_\gamma)(h)] - q| < \varepsilon\ ].$$

Let $\delta := \min\{\eta, \lambda, \mu\}$. We wish to show: $\forall h \in \mathrm{dom}\,[SS^p_\beta]$,

$$[\ 0 < |h| < \delta\ ] \quad \Rightarrow \quad [\ |[(SS^p_\beta)(h)] - q| < \varepsilon\ ].$$

Let $h \in \mathrm{dom}\,[SS^p_\beta]$ be given. We wish to show:

$$[\ 0 < |h| < \delta\ ] \quad \Rightarrow \quad [\ |[(SS^p_\beta)(h)] - q| < \varepsilon\ ].$$

Assume: $0 < |h| < \delta$. We wish to prove: $|[(SS^p_\beta)(h)] - q| < \varepsilon$. Let $I := (q - \varepsilon, q + \varepsilon)$. We wish to prove $(SS^q_\beta)(h) \in I$

Since $0 < |h|$, it follows that $h \neq 0$. Since $|(p+h) - p| = |h| < \delta \leqslant \eta$, we see that $p + h \in B_{\mathbb{R}}(p, \eta)$. By the choice of η, we have $B_{\mathbb{R}}(p, \eta) \subseteq S$. Since $p + h \in B_{\mathbb{R}}(p, \eta) \subseteq S$, we get $h \in S - p$. So, since $h \neq 0$, we have $h \in (S - p)^\times_0$. Then, by the Claim, $(SS^p_\beta)(h) \in [\,(SS^p_\alpha)(h)\,|\,(SS^p_\gamma)(h)\,]$.

Since $0 < |h| < \delta \leqslant \lambda$, by choice of λ, we get: $|[(SS^p_\alpha)(h)] - q| < \varepsilon$. Then $(SS^p_\alpha)(h) \in I$. Since $0 < |h| < \delta \leqslant \mu$, by choice of μ, we get: $|[(SS^p_\gamma)(h)] - q| < \varepsilon$. Then $(SS^p_\gamma)(h) \in I$. Since I is an interval and $(SS^p_\alpha)(h) \in I$ and $(SS^p_\gamma)(h) \in I$, we get $[\,(SS^p_\alpha)(h)\,|\,(SS^p_\gamma)(h)\,] \subseteq I$.

Then $(SS^p_\beta)(h) \in [\,(SS^p_\alpha)(h)\,|\,(SS^p_\gamma)(h)\,] \subseteq I$, as desired. QED

3-4. Let W be a normed vector space. Let $f, g : \mathbb{R} \dashrightarrow W$. Let U be an open subset of $\mathbb{R}$. Assume: $U \subseteq \mathrm{dom}\,[(f, g)]$. Assume: $f = g$ on U. Show: $f' = g'$ on U.

Proof: We wish to show: $\forall p \in U$, $f'(p) = g'(p)$. Let $p \in U$ be given. We wish to show: $f'(p) = g'(p)$. We wish to show: $\lim\limits_0 SS_f^p = \lim\limits_0 SS_g^p$.

Since U is open in $\mathbb{R}$, it follows that $U - p$ is open in $\mathbb{R}$. So, since $0 \in U - p$, we get $U - p \in \mathcal{N}_{\mathbb{R}}(0)$. Let $V := (U - p)_0^\times$. Then $V \in \mathcal{N}_{\mathbb{R}}^\times(0)$.

Claim: $SS_f^p = SS_g^p$ on V. *Proof of Claim:* We want: $\forall h \in V$, $(SS_f^p)(h) = (SS_g^p)(h)$. Given $h \in V$. We want: $(SS_f^p)(h) = (SS_g^p)(h)$.

Since $h \in (U - p)_0^\times \subseteq U - p$, we get $p + h \in U$. Since $p, p + h \in U$ and since $f = g$ on U, we get $f(p) = g(p)$ and $f(p + h) = g(p + h)$. Then

$$(SS_f^p)(h) = \frac{[f(p+h)] - [f(p)]}{h} = \frac{[g(p+h)] - [g(p)]}{h} = (SS_g^p)(h),$$

as desired. *End of proof of Claim:*

Since $V \in \mathcal{N}_{\mathbb{R}}^\times(0)$, by the Claim and by Theorem 20.8, we have both $\operatorname{LIMS}\limits_0 (SS_f^p) \subseteq \operatorname{LIMS}\limits_0 (SS_g^p)$ and $\operatorname{LIMS}\limits_0 (SS_g^p) \subseteq \operatorname{LIMS}\limits_0 (SS_f^p)$. Then $\operatorname{LIMS}\limits_0 (SS_g^p) = \operatorname{LIMS}\limits_0 (SS_f^p)$. Then

$$\lim_0 SS_f^p = \operatorname{ELT}(\operatorname{LIMS}_0(SS_f^p)) = \operatorname{ELT}(\operatorname{LIMS}_0(SS_g^p)) = \lim_0 SS_g^p,$$

as desired. QED

3-5. Define $f : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f(x) = [x^2][\sin(x^{-3})]$. Let $\beta := \operatorname{adj}_0^0 f$. Show: $\beta'(0) = 0$.

Proof: Define $\alpha, \gamma : \mathbb{R} \to \mathbb{R}$ by $\alpha(x) = -x^2$ and $\gamma(x) = x^2$. For all $x \in \mathbb{R}_0^\times$, we have both $-1 \leqslant \sin(x^{-3}) \leqslant 1$ and $x^2 \geqslant 0$, and so $[x^2][-1] \leqslant [x^2][\sin(x^{-3})] \leqslant [x^2][1]$, and so $\alpha(x) \leqslant f(x) \leqslant \gamma(x)$. Then $\alpha \leqslant f \leqslant \gamma$ on $\mathbb{R}_0^\times$. So, since $f = \beta$ on $\mathbb{R}_0^\times$, we get: $\alpha \leqslant \beta \leqslant \gamma$ on $\mathbb{R}_0^\times$. So, since $\alpha(0) = \beta(0) = \gamma(0) = 0$, we get $\alpha \leqslant \beta \leqslant \gamma$ on $\mathbb{R}$.

We have $\alpha(0) = 0 = \gamma(0)$. By the Power Rule, for all $x \in \mathbb{R}$, we have $\gamma'(x) = 2x$. Then $\gamma'(0) = 0$. So, since $\alpha = -\gamma$, by linearity of differentiation, we have $\alpha'(0) =^* -[\gamma'(0)]$. Then

$$\alpha'(0) \ =^* \ -[\gamma'(0)] \ = \ -0 \ = \ 0 \ \neq \ ☹.$$

Then $\alpha'(0) = 0$. Then $\alpha'(0) = 0 = \gamma'(0)$. Let $p := 0$ and $q := 0$. Then $\alpha(p) = \gamma(p)$ and $\alpha'(p) = q = \gamma'(p)$. Then, by HW#3-3, we get $\beta'(p) = q$. That is, $\beta'(0) = 0$. QED

Homework 2: Due on Tuesday 30 January

2-1. Let X and Y be topological spaces. Let $X_0 \subseteq X$ and let $Y_0 \subseteq Y$. Let $f : X_0 \dashrightarrow Y_0$. Show:

$$[\ f \text{ is } (X,Y)\text{-continuous}\ ] \quad \Rightarrow \quad [\ f \text{ is } (X_0,Y_0)\text{-continuous}\ ].$$

Proof: Assume: f is (X,Y)-continuous. Want: f is (X_0,Y_0)-continuous. Want: $\forall p \in \operatorname{dom}[f]$, f is (X_0,Y_0)-continuous at p. Given $p \in \operatorname{dom}[f]$. Want: f is (X_0,Y_0)-continuous at p. Since f is (X,Y)-continuous and $p \in \operatorname{dom}[f]$, it follows that f is (X,Y)-continuous at p. Want: $\forall V_0 \in \mathcal{N}_{Y_0}(f(p))$, $\exists U_0 \in \mathcal{N}_{X_0}(p)$ s.t. $f_*(U_0) \subseteq V_0$. Let $V_0 \in \mathcal{N}_{Y_0}(f(p))$ be given. We wish to show: $\exists U_0 \in \mathcal{N}_{X_0}(p)$ s.t. $f_*(U_0) \subseteq V_0$.

By extension, choose $V \in \mathcal{N}_Y(f(p))$ s.t. $V \cap Y_0 = V_0$. Since f is (X,Y)-continuous at p, choose $U \in \mathcal{N}_X(p)$ s.t. $f_*(U) \subseteq V$. We define $U_0 := U \cap X_0$. By restriction, $U_0 \in \mathcal{N}_{X_0}(p)$. We wish to show: $f_*(U_0) \subseteq V_0$. We wish to show: $\forall x \in \operatorname{dom}[f]$,

$$[\ x \in U_0\ ] \quad \Rightarrow \quad [\ f(x) \in V_0\ ].$$

Let $x \in \operatorname{dom}[f]$ be given. We wish to show:

$$[\ x \in U_0\ ] \quad \Rightarrow \quad [\ f(x) \in V_0\ ].$$

Assume: $x \in U_0$. We wish to show: $f(x) \in V_0$.

We have $x \in U_0 = U \cap X_0 \subseteq U$. So, since $x \in \operatorname{dom}[f]$, we get $f(x) \in f_*(U)$. Then $f(x) \in f_*(U) \subseteq V$. Since $f : X_0 \dashrightarrow Y_0$, it follows that $\operatorname{im}[f] \subseteq Y_0$. Then $f(x) \in \operatorname{im}[f] \subseteq Y_0$. So, since $f(x) \in V$, we get $f(x) \in V \cap Y_0$. Then $f(x) \in V \cap Y_0 = V_0$, as desired. QED

2-2. Let X, Y and Z be topological spaces. Let $f : X \dashrightarrow Y$ and let $g : X \dashrightarrow Z$. Let $a \in X$, $b \in Y$ and $c \in Z$. Assume:

(1) $f \to b$ in Y near a in X and
(2) $g \to c$ in Z near a in X.

Show: $(f,g) \to (b,c)$ in $Y \times Z$ near a in X.

Proof: Want: $\forall W \in \mathcal{N}_{Y\times Z}((b,c))$, $\exists R \in \mathcal{N}_X^\times(a)$ s.t. $(f,g)_*(R) \subseteq W$. Let $W \in \mathcal{N}_{Y\times Z}((b,c))$ be given. Want: $\exists R \in \mathcal{N}_X^\times(a)$ s.t. $(f,g)_*(R) \subseteq W$.

Choose $U \in \mathcal{N}_Y(b)$ and $V \in \mathcal{N}_Z(c)$ s.t. $U \times V \subseteq W$. Since $f \to b$ in Y near a in X and since $U \in \mathcal{N}_Y(b)$, choose $P \in \mathcal{N}_X^\times(a)$ s.t. $f_*(P) \subseteq U$. Since $g \to c$ in Z near a in X and since $V \in \mathcal{N}_Z(c)$, choose $Q \in \mathcal{N}_X^\times(a)$ s.t. $f_*(Q) \subseteq V$. As $P, Q \in \mathcal{N}_X^\times(a)$, $P \cap Q \in \mathcal{N}_X^\times(a)$. Let $R := P \cap Q$.

Then $R \in \mathcal{N}_x^\times(a)$. Want: $(f,g)_*(R) \subseteq W$. Want: $\forall x \in \text{dom}\,[(f,g)]$,

$$[\ x \in R\ ] \quad \Rightarrow \quad [\ (f,g)(x) \in W\ ].$$

Let $x \in \text{dom}\,[(f,g)]$ be given. We wish to show:

$$[\ x \in R\ ] \quad \Rightarrow \quad [\ (f,g)(x) \in W\ ].$$

Assume: $x \in R$. We wish to show: $(f,g)(x) \in W$.

We have $x \in \text{dom}\,[(f,g)] = (\text{dom}\,[f]) \cap (\text{dom}\,[g])$, so $x \in \text{dom}\,[f]$ and $x \in \text{dom}\,[g]$. Since $x \in \text{dom}\,[f]$ and since $x \in R = P \cap Q \subseteq P$, we get $f(x) \in f_*(P)$. Since $x \in \text{dom}\,[f]$ and since $x \in R = P \cap Q \subseteq Q$, we get $g(x) \in g_*(P)$. Then $f(x) \in f_*(P) \subseteq U$ and $g(x) \in g_*(P) \subseteq V$. Then $(f,g)(x) = (f(x), g(x)) \in U \times V \subseteq W$, as desired. QED

2-3. Show: $\forall z \in \mathbb{R}$, $\exists! x \in \mathbb{R}$ s.t. $x^5 + x^3 = z$.

Proof: Let $z \in \mathbb{R}$ be given. We wish to show: $\exists! x \in \mathbb{R}$ s.t. $x^5 + x^3 = z$.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^5 + x^3$. Then f is continuous. We wish to show: $\exists! x \in \mathbb{R}$ s.t. $f(x) = z$. We wish to show:

(1) $\exists x \in \mathbb{R}$ s.t. $f(x) = z$.
(2) $\forall w, x \in \mathbb{R}$, $(\ [\ f(w) = z = f(x)\ ] \ \Rightarrow\ [\ w = x\ ]\ )$.

Proof of (1): As $z \in \mathbb{R}$, $-|z| \leqslant z \leqslant |z|$. Let $b := \max\{|z|/2, 1\}$. Since $b \geqslant 1$, we get $b^5 \geqslant b$ and $b^3 \geqslant b$. Then $b^5 \geqslant b \geqslant |z|/2$ and $b^3 \geqslant b \geqslant |z|/2$. Then $f(b) = b^5 + b^3 \geqslant (|z|/2) + (|z|/2) = |z|$. Then $-[f(b)] \leqslant -|z|$. We have $f(-b) = (-b)^5 + (-b)^3 = -(b^5 + b^3) = -[f(b)] \leqslant -|z|$. Then $f(-b) \leqslant -|z| \leqslant z \leqslant |z| \leqslant f(b)$. Then $z \in [f(-b), f(b)]$. Also, since $b \geqslant 1$, we see that $-b \leqslant -1$. Then $-b \leqslant -1 < 1 \leqslant b$. It suffices to show: $\exists x \in [-b, b]$ s.t. $f(x) = z$.

By the Intermediate Value Theorem, $[f(-b), f(b)] \subseteq f_*([-b, b])$. Then $z \in [f(-b), f(b)] \subseteq f_*([-b, b])$, so $\exists x \in [-b, b]$ s.t. $f(x) = z$, as desired. *End of proof of (1).*

Proof of (2): Given $w, x \in \mathbb{R}$. Want: $[f(w) = z = f(x)] \Rightarrow [w = x]$. Assume: $f(w) = z = f(x)$. We wish to show: $w = x$. Assume: $w \neq x$. We aim for a contradiction.

Let $p := \min\{w, x\}$ and let $q := \max\{w, x\}$. Since $w \neq x$, it follows that $p < q$. Also, $f(p) \in \{f(w), f(x)\} = \{z\}$, so $f(p) = z$. Also, $f(q) \in \{f(w), f(x)\} = \{z\}$, so $f(q) = z$. Since $p < q$, we get $p^5 < q^5$ and $p^3 < q^3$. Then $p^5 + p^3 < q^5 + q^3$. Then

$$f(p) \ = \ p^5 + p^3 \ < \ q^5 + q^3 \ = \ f(q) \ = \ z \ = \ f(p),$$

so $f(p) < f(p)$. Contradiction. *End of proof of (2).* QED

2-4. Let X and Y be topological spaces. Let $f : X \dashrightarrow Y$. Let $X_0 \subseteq \operatorname{dom}[f]$. Assume that f is (X,Y)-continuous on X_0. Show that $f|X_0$ is (X_0,Y)-continuous.

Proof: Let $g := f|X_0$. Then $\operatorname{dom}[g] = X_0$. We wish to show: g is (X_0,Y)-continuous. By HW#2-1 (with Y_0 replaced by Y), want: g is (X,Y)-continuous. Want: $\forall p \in X_0$, g is (X,Y)-continuous at p. Let $p \in X_0$ be given. We wish to show: g is (X,Y)-continuous at p. Want: $\forall V \in \mathcal{N}_Y(g(p))$, $\exists U \in \mathcal{N}_X(p)$ s.t. $g_*(U) \subseteq V$. Let $V \in \mathcal{N}_Y(g(p))$ be given. Want: $\exists U \in \mathcal{N}_X(p)$ s.t. $g_*(U) \subseteq V$.

Since $p \in X_0 \subseteq \operatorname{dom}[f]$, and since f is (X,Y)-continuous, it follows that f is (X,Y)-continuous at p. Since $p \in X_0 \subseteq \operatorname{dom}[f]$, we have $(f|X_0)(p) = f(p)$. Then $g(p) = (f|X_0)(p) = f(p)$. Since $V \in \mathcal{N}_Y(g(p)) = \mathcal{N}_Y(f(p))$ and since f is (X,Y)-continuous at p, choose $U \in \mathcal{N}_X(p)$ s.t. $f_*(U) \subseteq V$. We wish to show: $g_*(U) \subseteq V$.

Recall that $\operatorname{dom}[g] = W_0$. We wish to show: $\forall w \in X_0$,

$$[\ w \in U\ ] \quad \Rightarrow \quad [\ g(w) \in V\ ].$$

Let $w \in X_0$ be given. We wish to show:

$$[\ w \in U\ ] \quad \Rightarrow \quad [\ g(w) \in V\ ].$$

Assume: $w \in U$. We wish to show: $g(w) \in V$.

As $w \in X_0 \subseteq \operatorname{dom}[f]$ and as $w \in U$, we get $f(w) \in f_*(U)$. By choice of U, we have $f_*(U) \subseteq V$. Since $w \in X_0 \subseteq \operatorname{dom}[f]$, $(f|X_0)(w) = f(w)$. Then $g(w) = (f|X_0)(w) = f(w) \in f_*(U) \subseteq V$. QED

2-5. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Let $\alpha, \beta \in \operatorname{dom}[f]$. Assume that f is semimonotone on $[\alpha|\beta]$. Show: $f_*([\alpha|\beta]) \subseteq [f(\alpha)|f(\beta)]$.

Proof: Since $\alpha, \beta \in \operatorname{dom}[f]$, we conclude that $\{\alpha, \beta\} \subseteq \operatorname{dom}[f]$. Let $a := \min\{\alpha, \beta\}$ and $b := \max\{\alpha, \beta\}$. Then $a \leqslant b$. Also, $[\alpha|\beta] = [a, b]$. Then f is semimonotone on $[a, b]$. Also, $a, b \in \{\alpha, \beta\} \subseteq \operatorname{dom}[f]$.

Either ($(a = \alpha)$ and $(b = \beta)$) or ($(a = \beta)$ and $(b = \alpha)$). Then either $[f(\alpha)|f(\beta)] = [f(a)|f(b)]$ or $[f(\alpha)|f(\beta)] = [f(b)|f(a)]$. So, since $[f(b)|f(a)] = [f(a)|f(b)]$, we see that $[f(\alpha)|f(\beta)] = [f(a)|f(b)]$. We wish to show: $f_*([a, b]) \subseteq [f(a)|f(b)]$. We wish to show: $\forall x \in \operatorname{dom}[f]$,

$$(\ x \in [a, b]\ ) \quad \Rightarrow \quad (\ f(x) \in [f(a)|f(b)]\ ).$$

Let $x \in \text{dom}\,[f]$ be given. We wish to show:

$$(\ x \in [a,b]\ ) \quad \Rightarrow \quad (\ f(x) \in [f(a)|f(b)]\ ).$$

Assume: $x \in [a,b]$. We wish to show: $f(x) \in [f(a)|f(b)]$.

Because $x \in [a,b]$, it follows that $a \leqslant x \leqslant b$. Since f is semimonotone on $[a,b]$, at least one of the following must be true:

(1) f is semiincreasing on $[a,b]$ or
(2) f is semidecreasing on $[a,b]$.

Case (1): Since $a, x, b \in \text{dom}\,[f]$, since $a \leqslant x \leqslant b$ and since f is semiincreasing on $[a,b]$, it follows that $f(a) \leqslant f(x) \leqslant f(b)$. Then $f(x) \in [f(a), f(b)] = [f(a)|f(b)]$, as desired. *End of Case (1).*

Case (2): Since $a, x, b \in \text{dom}\,[f]$, since $a \leqslant x \leqslant b$ and since f is semidecreasing on $[a,b]$, it follows that $f(a) \geqslant f(x) \geqslant f(b)$. Then $f(x) \in [f(b), f(a)] = [f(a)|f(b)]$, as desired. *End of Case (2).* QED

Homework 1: Due on Tuesday 23 January

1-1. Let X and Y be metric spaces, and let $f : X \to Y$. Assume that f is not uniformly continuous. Show: $\exists \varepsilon > 0$, $\exists p, q \in X^{\mathbb{N}}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ d_X\,(\,p_j\,,\,q_j\,)\ <\ 1/j\ ] \quad \text{and} \quad [\ d_Y\,(\,f(p_j)\,,\,f(q_j)\,)\ \geqslant\ \varepsilon\ ].$$

Proof: As f is not u.c., choose $\varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists p, q \in X$ s.t.

$$[\ d_X(p,q) < \delta\ ] \quad \text{and} \quad [\ d_Y(f(p), f(q)) \geqslant \varepsilon\ ].$$

We wish to show: $\exists p, q \in X^{\mathbb{N}}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ d_X\,(\,p_j\,,\,q_j\,)\ <\ 1/j\ ] \quad \text{and} \quad [\ d_Y\,(\,f(p_j)\,,\,f(q_j)\,)\ \geqslant\ \varepsilon\ ].$$

Claim: $\forall j \in \mathbb{N}$, $\exists p, q \in X$ s.t.

$$[\ d_X(p,q) < 1/j\ ] \quad \text{and} \quad [\ d_Y(f(p), f(q)) \geqslant \varepsilon\ ].$$

Proof of claim: Let $j \in \mathbb{N}$ be given. By the choice of ε, $\exists p, q \in X$ s.t.

$$[\ d_X(p,q) < 1/j\ ] \quad \text{and} \quad [\ d_Y(f(p), f(q)) \geqslant \varepsilon\ ],$$

as desired. *End of proof of claim.*

By the claim and the Axiom of Choice, $\exists p, q \in X^{\mathbb{N}}$ s.t. $\forall j \in \mathbb{N}$,

$$[\ d_X(p_j, q_j) < 1/j\ ] \quad \text{and} \quad [\ d_Y(f(p_j), f(q_j)) \geqslant \varepsilon\ ],$$

as desired. QED

1-2. Let K be a sequentially compact metric space, let Y be a metric space, and let $f : K \to Y$. Assume that f is not uniformly continuous. Show: $\exists \varepsilon > 0$, $\exists s, t \in K^{\mathbb{N}}$, $\exists u \in K$ s.t.

$$(\quad [\ s_\bullet \to u \text{ in } K\ ] \quad \text{and} \quad [\ t_\bullet \to u \text{ in } K\ ] \quad \text{and} \\ [\ \forall j \in \mathbb{N}, \qquad d_Y\,(\,f(s_j)\,,\,f(t_j)\,)\ \geqslant\ \varepsilon\ ] \qquad).$$

Proof: By HW#1-1, choose $\varepsilon > 0$ and $p, q \in K^{\mathbb{N}}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ d_K\,(\,p_j\,,\,q_j\,)\ <\ 1/j\ ] \quad \text{and} \quad [\ d_Y\,(\,f(p_j)\,,\,f(q_j)\,)\ \geqslant\ \varepsilon\ ].$$

Since K is sequentially compact, $p_\bullet$ is subconvergent in K. Choose a subsequence s of $p_\bullet$ s.t. $s_\bullet$ is convergent in K. Choose $u \in K$ s.t. $s_\bullet \to u$ in K. Since $s_\bullet$ is a subsequence of $p_\bullet$, choose a strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $s_\bullet = (p \circ \ell)_\bullet$. Let $t := (q \circ \ell)_\bullet$. Then $s, t \in K^N$ and $u \in K$. Want:

$$(\quad [\ s_\bullet \to u \text{ in } K\ ] \quad \text{and} \quad [\ t_\bullet \to u \text{ in } K\ ] \quad \text{and} \\ [\ \forall j \in \mathbb{N}, \qquad d_Y\,(\,f(s_j)\,,\,f(t_j)\,)\ \geqslant\ \varepsilon\ ] \qquad).$$

By choice of u, we have $s_\bullet \to u$ in K, so it remains to show:

(1) $t \to u$ in K and

(2) $\forall j \in \mathbb{N}, \qquad d_Y\,(\,f(s_j)\,,\,f(t_j)\,)\ \geqslant\ \varepsilon$.

Proof of (1): We wish to show: $\forall \eta > 0$, $\exists I \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ j \geqslant I\ ] \quad \Rightarrow \quad [\ d_K(t_j, u) < \eta\ ].$$

Let $\eta > 0$ be given. We wish to show: $\exists I \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ j \geqslant I\ ] \quad \Rightarrow \quad [\ d_K(t_j, u) < \eta\ ].$$

Since $s_\bullet \to u$ in K, choose $R \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ j \geqslant R\ ] \quad \Rightarrow \quad [\ d_K(s_j, u) < \eta/2\ ].$$

By the Archimidean Principle, choose $S \in \mathbb{N}$ such that $S > 2/\eta$. Let $I := \max\{R, S\}$. We wish to show: $\forall j \in \mathbb{N}$,

$$[\ j \geqslant I\ ] \quad \Rightarrow \quad [\ d_K(t_j, u) < \eta\ ].$$

Let $j \in \mathbb{N}$ be given. We wish to show:

$$[\ j \geqslant I\ ] \quad \Rightarrow \quad [\ d_K(t_j, u) < \eta\ ].$$

Assume: $j \geqslant I$. We wish to show: $d_K(t_j, u) < \eta$.

Since $j \geqslant I \geqslant R$, by choice of R, we conclude that $d_K(s_j, u) < \eta/2$. Since $\ell \in \mathbb{N}^{\mathbb{N}}$ and since $\ell_\bullet$ is strictly increasing, by Remark 23.12, we get $\ell_j \geqslant j$. It follows that $1/\ell_j \leqslant 1/j$. Since $j \geqslant I \geqslant S > 2/\eta$, we get $1/j < \eta/2$. By the choice of $p_\bullet$ and $q_\bullet$, we have $d_K(p_{\ell_j}, q_{\ell_j}) < 1/\ell_j$. Then $d_K(t_j, s_j) = d_K(s_j, t_j) = d_K(p_{\ell_j}, q_{\ell_j}) < 1/\ell_j \leqslant 1/j < \eta/2$. Then

$$d_K(t_j, u) \leqslant [d_K(t_j, s_j)] + [d(s_j, u)] < (\eta/2) + (\eta/2) = \eta,$$

as desired. *End of proof of (1).*

Proof of (2): Let $j \in \mathbb{N}$ be given. Want: $d_Y(f(s_j), f(t_j)) \geqslant \varepsilon$.

By the choice of p and q, we have $d_Y(f(p_{\ell_j}), f(q_{\ell_j})) \geqslant \varepsilon$. Then $d_Y(f(s_j), f(t_j)) = d_Y(f(p_{\ell_j}), f(q_{\ell_j})) \geqslant \varepsilon$. *End of proof of (2).* QED

1-3. Let X be a metric space, and let $s \in X^{\mathbb{N}}$. Assume that $s_\bullet$ is convergent in X. Show that $s_\bullet$ is Cauchy in X.

Proof: We wish to show: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,

$$[\ i, j \geqslant K\ ] \quad \Rightarrow \quad [\ d_X(s_i, s_j) < \varepsilon\ ].$$

Let $\varepsilon > 0$ be given. We wish to show: $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,

$$[\ i, j \geqslant K\ ] \quad \Rightarrow \quad [\ d_X(s_i, s_j) < \varepsilon\ ].$$

Since $s_\bullet$ is convergent in X, choose $u \in X$ s.t. $s_\bullet \to u$ in X. Choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ j \geqslant K\ ] \quad \Rightarrow \quad [\ d_X(s_j, u) < \varepsilon/2\ ].$$

We wish to show: $\forall i, j \in \mathbb{N}$,

$$[\ i, j \geqslant K\ ] \quad \Rightarrow \quad [\ d_X(s_i, s_j) < \varepsilon\ ].$$

Let $i, j \in \mathbb{N}$ be given. We wish to show:

$$[\ i, j \geqslant K\ ] \quad \Rightarrow \quad [\ d_X(s_i, s_j) < \varepsilon\ ].$$

Assume $i, j \geqslant K$. We wish to show: $d_X(s_i, s_j) < \varepsilon$.

Since $i \geqslant K$, by choice of K, we have $d(s_i, u) < \varepsilon/2$. Since $j \geqslant K$, by choice of K, we have $d(s_j, u) < \varepsilon/2$.

Then $d_X(s_i, s_j) \leqslant [d_X(s_i, u)] + [d_X(u, s_j)] < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$. QED

1-4. Let X be a metric space, and let $s \in X^{\mathbb{N}}$. Assume that $s_\bullet$ is Cauchy and subconvergent in X. Show that $s_\bullet$ is convergent in X.

Proof: Since $s_\bullet$ is subconvergent in X, choose a subsequence t of $s_\bullet$ such that $t_\bullet$ is convergent in X. Choose $u \in X$ s.t. $t_\bullet \to u$ in X. We wish to show: $s_\bullet \to u$ in X. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ j \geqslant K\ ] \quad \Rightarrow \quad [\ d_X(s_j, u) < \varepsilon\ ].$$

Let $\varepsilon > 0$ be given. We wish to show: $\exists I \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ j \geqslant I\ ] \quad \Rightarrow \quad [\ d_X(s_j, u) < \varepsilon\ ].$$

Since $t_\bullet \to u$ in X, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[\ j \geqslant L\ ] \quad \Rightarrow \quad [\ d_X(t_j, u) < \varepsilon/2\ ].$$

Since $s_\bullet$ is Cauchy, choose $M \in \mathbb{N}$ s.t., $\forall j, k \in \mathbb{N}$,

$$[\ j, k \geqslant M\ ] \quad \Rightarrow \quad [\ d_X(s_j, s_k) < \varepsilon/2\ ].$$

Let $I := \max\{L, M\}$. We wish to show: $\forall j \in \mathbb{N}$,

$$[\ j \geqslant I\ ] \quad \Rightarrow \quad [\ d_X(s_j, u) < \varepsilon\ ].$$

Let $j \in \mathbb{N}$ be given. We wish to show:

$$[\ j \geqslant I\ ] \quad \Rightarrow \quad [\ d_X(s_j, u) < \varepsilon\ ].$$

Assume: $j \geqslant I$ We wish to show: $d_X(s_j, u) < \varepsilon$.

Since $j \geqslant I \geqslant L$, by choice of L, we conclude that $d_X(t_j, u) < \varepsilon/2$. By Corollary 23.13, choose $k \in [j..\infty]$ such that $t_j = s_k$. Then we have $k \geqslant j \geqslant I \geqslant M$. Then $j, k \geqslant M$, so, by choice of M, we get: $d_X(s_j, s_k) < \varepsilon/2$. Since $t_j = s_k$, we get $d(s_j, t_j) = d(s_j, s_k)$. Then

$$\begin{aligned} d(s_j, u) &\leqslant [d(s_j, t_j)] + [d(t_j, u)] \\ &= [d(s_j, s_k)] + [d(t_j, u)] \\ &< [\varepsilon/2] + [\varepsilon/2] = \varepsilon, \end{aligned}$$

as desired. QED

1-5. Let X be a metric space, and let $s \in X^{\mathbb{N}}$. Assume that $s_\bullet$ is Cauchy in X. Show that $s_\bullet$ is bounded in X.

Proof: We wish to show: $\mathrm{im}\,[s_\bullet]$ is bounded in X.

Since $s_\bullet$ is Cauchy in X, choose $K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,

$$[\ i, j \geqslant K\ ] \quad \Rightarrow \quad [\ d_X(s_i, s_j) < 1\ ].$$

Let $p := s_K$. Let $a := \max\{d_X(p, s_1), \ldots, d_X(p, s_{K-1})\}$. Let $R := 1 + a$. We wish to show: $\mathrm{im}\,[s_\bullet] \subseteq B_X(p, R)$. We wish to show: $\forall q \in \mathrm{im}\,[s_\bullet]$, $q \in B_X(p, R)$. Let $q \in \mathrm{im}\,[s_\bullet]$ be given. We wish to show: $q \in B_X(p, R)$. We wish to show $d_X(p, q) < R$. Since $q \in \mathrm{im}\,[s_\bullet]$, choose $j \in \mathbb{N}$ such that $q = s_j$. We wish to show: $d_X(p, s_j) < R$.

At least one of the follwing must be true:

(1) $j < K$ or

(2) $j \geqslant K$.

Case (1): As $j \in [1..K)$, $d_X(p, s_j) \in \{d_X(p, s_1), \ldots, d_X(p, s_{K-1})\}$. Then $d_X(p, s_j) \leqslant \max\{d_X(p, s_1), \ldots, d_X(p, s_{K-1})\} = a < 1 + a = R$, as desired. *End of Case (1).*

Case (2): Since $p = s_K$, we conclude that $d_X(p, s_j) = d_X(s_K, s_j)$. Also, since $K, j \geqslant K$, by choice of K, we see that $d_X(s_K, s_j) < 1$. By definition of a, we have $a \geqslant 0$. It follows that $1 \leqslant 1 + a$. Then $d_X(p, s_j) = d_X(s_K, s_j) < 1 \leqslant 1 + a = R$. *End of Case (2).* QED