CLASS NOTES

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Contents

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1. What is Mathematics?

What is mathematics? What distinguishes it from the many subjects that *use* mathematics? Answer: Mathematics is the study of absolute truth. Following Antis Pilate, we ask: What is truth? In the view of the pure mathematician, to distinguish truth from falsehood, we need to have a precise formal language and a set of formal rules for identifying which statements are mathematical statements. Then we need another set of formal rules for proceeding from a collection of assumed statements, called axioms, to a collection of statements, called theorems.

In this section, we will outline one way of formalizing the meaning of a mathematical statement. It is not necessarily the easiest formalism to use, but it is relatively easy to describe, and is tailored to the needs of a real analysis course like this one. In what follows, by a **logic purist**, I will mean someone who is ONLY willing to consider the highly restrictive formalism described in this section.

We will be using the following abbreviations:

$\forall$	forall (or, sometimes, for any)
$\exists$	there exists (or, sometimes, there exist)
$\exists 1$	there exists a unique
s.t.	such that
$\neg$	not
&	and
$\vee$	or
$\therefore$	therefore
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
iff	if and only if
$\in$	is an element of
☹	undefined

The symbols "$\forall$" and "$\exists$" are called **quantifiers**. The first one, "$\forall$", is called the **universal quantifier**, and the second one, "$\exists$", is called the **existential quantifier**. The symbol "$\in$" is *NOT* the Greek letter epsilon, written "ε". Our use of ☹ is unconventional. We will use it to indicate that some particular computation cannot be completed in a conventional manner. So, for example, in this course, $1/0 =$ ☹.

Now let's start with a mathematical sounding statement:

Every real number is not an integer.

This sentence is quite ambiguous. It may mean that there are real numbers that are not integers, or it may mean that no real number is ever an integer. Ambiguity is one of the many enemies of truth, and so we need to know which kinds of statements we even consider worth trying to prove, and which statements are so poorly worded that it's a waste of time to even try to understand them. The statement above would likely lose you some points, if you write it on homework or an exam. Written in a more precise way, it would come out as:

$$\neg\,(\,\forall\text{real } x, \quad x \text{ an integer}) \quad .$$

For our course, this statment will be considered close enough to a formal mathematical statement that it will be acceptable. However, once we give our exact definition of a formal statement, we will see that, to be perfectly correct, we need to say:

$$\neg\,(\,(\forall x)(\,(x \text{ is a real number}) \;\Rightarrow\; (x \text{ is an integer})\,)\,) \qquad .$$

To describe which streams of symbols are formal statements, we begin by deciding which symbols we will use to build such statements. In this course, our symbols will come in three types:

First, **alphabetic characters**:

lowercase Roman letters: a,b,c,. . . ,z ;
uppercase Roman letters: A,B,C,. . . ,Z ;
a blank space to separate words.

Later, we may add more (like period and comma) as needed.

Second, **variables**. We will use:

lowercase italic Roman letters: a,b,c,. . . ,z ;
uppercase italic Roman letters: A,B,C,. . . ,Z ;
uppercase script letters: $\mathcal{A}$,$\mathcal{B}$,$\mathcal{C}$,. . . ,$\mathcal{Z}$;
lowercase Greek letters: α,β,γ,. . . ,ω ;
some uppercase Greek letters: Γ,Δ,Θ,Λ,Ξ,Π,Σ,Φ,Ψ,Ω.

Some uppercase Greek letters look exactly like uppercase Roman letters, *e.g.*, a capital η is H. They are not exactly omitted from our list of variables; they are just not listed twice. This completes our list of variables. The Roman alphabet has 26 letters. The Greek alphabet has 24 letters. We compute $26+26+26+24+10=112$. We therefore have 112 variables to start. Later, we may add more, as needed.

Third, various **special characters**:

$$\forall \; , \; \exists \; , \; (\; , \;) \; , \; + \; , \; - \; , \; \cdot \; , \; \in \; , \; \neg \; , \; \& \; , \; \vee \; , \; \Rightarrow \; , \; < \; , \; = \; ,$$
$$\infty \; , \; ☹ \; , \; 0 \; , \; 1 \; , \; 2 \; , \; 3 \; , \; 4 \; , \; 5 \; , \; 6 \; , \; 7 \; , \; 8 \; , \; 9 \qquad .$$

Later, we may add more, as needed.

Using our characters, we form our list of constants:

$$\infty \; , \; -\infty \; , \; ☹ \; , \; 0 \; , \; 1 \; , \; 2 \; , \; 3 \; , \; 4 \; , \; 5 \; , \; 6 \; , \; 7 \; , \; 8 \; , \; 9 \qquad .$$

Note that most constants are single characters, but character streams, like "$-\infty$" are okay, too. Later, we may add more constants, as needed.

Next, we have our list of ten **starter statements**. Keep in mind that every formal mathematical statement has a list of free variables, abbreviated "FV". In each of our ten starter statements, every variable is free. We will get to more complicated statements later, and you will see how some variables can fail to be free. Here are our ten starter statements, together with the list free variables in each:

Statement	FV
x is a real number	x
j is an integer	j
S is a set	S
$a = b$	a, b
$a \in S$	a, S
$x < y$	x, y
$x + y = z$	x, y, z
$x \cdot y = z$	x, y, z
$\mathrm{UE}(S) = a$	S, a
$\mathrm{CH}(S) = a$	S, a

These ten character streams are simply declared to be formal statements. We may add more starter statements, as needed.

If we pick a starter statement, and then replace the variables by variables or constants, we end up with a **atomic statement**. Sometimes, more than one variable is replaced by the same variable, *e.g.*, "$x < y$" becomes "$z < z$", on replacing both "x" and "y" by "z". Sometimes, more than one variable is replaced by the same constant, *e.g.*, "$x < y$" becomes "$1 < 1$", on replacing both "x" and "y" by "1". All of the following are "atomic statements":

Statement	FV
a is an integer	a
b is a real number	b
∞ is a real number	
$q = q$	q
$A \in \mathcal{B}$	$A, \mathcal{B}$
$-\infty = \infty$	
$3 < 4$	
$4 < 3$	
$3 < z$	z
$x + y = x$	x, y
$p \cdot q = 1$	p, q
$p \cdot p = p$	p
$3 \cdot 4 = 5$	
$3 + 4 = 5$	

As you can see, some atomic statements are simply untrue. Since there are (currently) 112 variables and 13 constants, the starter statement "x is an integer" yields 125 atomic statements. Similarly, "$x + y = z$" yields $125^3 = 1,953,125$ atomic statements. Continuing, we see that there are only finitely many atomic statements. It would not be difficult to write computer code that would print them all out.

There are two general methods by which we can take known statements and develop them into new statements:

(1) Quantification development and
(2) Construction development .

First, **quantification development**. This kind of development comes in two types: $\exists$-quantification and $\forall$-quantification. To illustrate $\exists$-quantification, we will start with:

Statement	FV
$r = z$	r, z

We select one of its free variables, say "z". We then surround the statement by parentheses. We then place "$(\exists z)$" in front. We then remove "z" from the list of free variables, obtaining:

Statement	FV
$(\exists z)(r = z)$	r

Next, to illustrate ∀-quantification, we will select the only remaining free variable, "r", and then apply $(\forall r)$-quantification, obtaining:

Statement	FV
$(\forall r)((\exists z)(r = z))$	

Note that there are now no free variables remaining, so we cannot apply quantification to this statement.

Second, **construction devleopment**. This comes in four types:

¬-construction ,
&-construction ,
∨-construction and
⇒-construction.

To illustrate ¬-construction, we will start with the statement

Statement	FV
$a \in B$	a, B

We now surround the statement by parentheses. We then place the symbol "¬" in front. The list of free variables is unchanged:

Statement	FV
$\neg(a \in B)$	a, B

To illustrate &-construction, we start with two statements:

Statement	FV
$a \in B$	a, B
$s < t$	s, t

We now surround each by parentheses, and concatenate them, but with "&" in bewteen. The free variables are also concatenated:

Statement	FV
$(a \in B) \,\&\, (s < t)$	a, B, s, t

As you might expect, ∨-construction and ⇒-construction work similarly. Start again with the two statements:

Statement	FV
$a \in B$	a, B
$s < t$	s, t

If we apply ∨-construction and ⇒-construction, we obtain:

Statement	FV
$(a \in B) \;\vee\; (s < t)$	a, B, s, t
$(a \in B) \Rightarrow (s < t)$	a, B, s, t

By a **formal statement**, we mean a finite sequence of characters (from the character list) that can be developed from

the collection of all atomic statements

via repeated quantification and construction. Example:

$$((\forall p)(p < q)) \Rightarrow (q = \infty) \qquad .$$

To get this formal statement, start with the atomic statement "$p < q$", then use $(\forall p)$-quantification on it, to get "$(\forall p)(p < q)$". Then, using the $\Rightarrow$-construction, combine that with the atomic statement "$q = \infty$". We will generally leave it as work for the reader to think about how a particular formal statement is developed, and about what free variables it has. In "$((\forall p)(p < q)) \Rightarrow (q = \infty)$" the only free variable is "q". Applying $(\exists q)$-quantification, we get

$$(\exists q)(((\forall p)(p < q)) \Rightarrow (q = \infty)) \qquad ,$$

which is a formal statement that has no free variables.

Unassigned homework: Write code that would take a string as input and, after analyzing it, would ouput either

"the string is not a formal statement" or

"the string is a formal statement, with free variables:" ,

followed by a list of

all of the free variables in the formal statement .

This code implements an algorithm that we will call the **formal statement algorithm**. Once this code is written, a more precise definition of a formal sentence would simply be a string that, if input into the formal sentence algorithm, yields "the string is a formal statement, . . .".

By a **formal sentence**, we mean a formal statement that has no free variables. Example: "$(\exists q)(((\forall p)(p < q)) \Rightarrow (q = \infty))$". The logic purist would be aghast, but we sometimes replace some parentheses by brackets, for readability, *e.g.*: "$(\exists q)([(\forall p)(p < q)] \Rightarrow [q = \infty])$".

Unassigned homework: Write code that would take a string as input and, after analyzing it, would ouput either

"the string is not a formal sentence" or

"the string is a formal sentence" .

This code implements an algorithm that we will call the **formal sentence algorithm**. Once this code is written, a more precise definition of a formal sentence would simply be a string that, if input into the formal sentence algorithm, yields "the string is a formal sentence".

A **formal axiom** means: a formal sentence that is accepted as true without proof. In the first few weeks of this course, we will be describing various formal sentences as axioms. Unassigned homework: After those few weeks are over, write code that would take a string as input and, after analyzing it, would output either

"the string is not an axiom" or

"the string is an axiom" .

This code implements an algorithm that we will call the **axiom algorithm**. Once this code is written, a more precise definition of an axiom would simply be a string that, if input into the axiom algorithm, yields "the string is an axiom".

Let's say that two strings are **inference ready** if the first

is a finite sequence of formal sentences,

separated by commas ,

and the second

is a single formal sentence .

Unassigned homework: After studying truth tables and logical rules of inference, write code that would take as input two strings and, after analyzing them, would output

"the two strings are not infrence ready" or

"the single sentence follows from the sequence of sentences" or

"the single sentence does not follow

from the sequence of sentences" .

This code implements an algorithm that we will call the **inference algorithm**. Once this code is written, if $\mathbf{P}$ and $\mathbf{Q}$ are two strings, then, by $\mathbf{P} \models \mathbf{Q}$, we mean: if you input $\mathbf{P}$ and $\mathbf{Q}$ into the inference algorithm, the output will be "the single sentence follows from the sequence of sentences". If this code is written correctly, then, for example,

$$(2+2=5) \vee (2+2=4) \, , \, \neg(2+2=5) \quad \models \quad 2+2=4 \qquad .$$

In fact, part of the coding should enable the inference algorithm to know that, for any two formal sentences $\mathbf{S}$ and $\mathbf{T}$,

$$(\mathbf{S}) \vee (\mathbf{T}) \, , \, \neg(\mathbf{S}) \quad \models \quad \mathbf{T} \qquad .$$

Another part of the coding should enable the inference algorithm to know that, for any two formal sentences $\mathbf{S}$ and $\mathbf{T}$,

$$(\mathbf{S}) \Rightarrow (\mathbf{T}) \, , \, \mathbf{S} \quad \models \quad \mathbf{T} \qquad .$$

There are a few other logical inferences that will need to be coded into the inference algorithm. In this course, we will not take the time to write them all out. However, if you have ever learned to compute truth tables, then you know enough logic to write the code for the inference algorithm. Keep in mind that

$$(2+2=5) \vee (2+2=4)\ ,\ \neg(2+2=5) \quad \not\models \quad 1+1=2 \qquad .$$

The inference algorithm should not know how to add, and should not know any mathematics except for the basics of propositional logic (*i.e.*, the logic of truth tables).

A **formal proof** is a finite sequence of formal sentences, separated by commas, such that each one either

is an axiom or follows from earlier sentences according to the inference algorithm . In this course, we will not write down any completely formal proofs. We *WILL* develop standards of proof, but not be at the level required to please the logic purist. By a **formal theorem**, we mean the last formal sentence in a formal proof.

In this course, we relax standards from "formal" to "pidgin". That is, we usually use pidgin statements, pidgin sentences, pidgin axioms, pidgin theorems, pidgin proofs. By a **pidgin statement**, we mean

a finite sequence of characters

that can be rewritten as a formal statement. As the course goes on, you should come to understand, better and better, how this rewriting process is done. In pidgin statements, we will allow a few extra characters, like comma and period.

There are simliar meanings for **pidgin sentences**, **pidgin axioms**, **pidgin theorems** and **pidgin proofs**. Since we will not be developing formal proofs, we will not describe how to rewrite a pidgin proof into a formal proof. However, you should be aware that such rewriting is always possible, and it is this formalism that makes mathematics rigorous, even if it only operates in the background. For more information on formal proofs, read up on the foundations of mathematics.

Our first axiom expresses the idea that

Everything is equal to itself.

As written, such a statement is not sufficiently formalized to be acceptable in this course. If it were to appear in homework or exams, would result in loss of points. In pidgin form, it becomes acceptable:

AXIOM 1.1. $\forall x, \qquad x = x.$

The logic purist would say "tsk!" and insist on a formal statement:

$$(\forall x)(x = x) \qquad .$$

We will look at more and more pidgin statements, and rewrite them into formal statements. Our first theorem:

THEOREM 1.2. *$\forall x, y, \quad [\,(x = y) \;\Rightarrow\; (y = x)\,]$.*

Proof. See Theorem 29.1. □

Our main focus now is not on proofs, but on converting from pidgin to formal. To rewrite Theorem 1.2 as a formal sentence, we start with:

$$(\forall x)(\forall y)([x = y] \Rightarrow [y = x]) \qquad .$$

To be completely pure, we need to change brackets to parentheses. Also, following our rules for quantification development, we should surround "$(\forall y)((x = y) \Rightarrow (y = x))$" by parentheses, obtaining:

$$(\forall x)((\forall y)((x = y) \Rightarrow (y = x))) \qquad .$$

Our next theorem:

THEOREM 1.3. *$\forall x, y, z, \quad [\,(x = y = z) \;\Rightarrow\; (x = z)\,]$.*

Proof. See Theorem 29.2. □

To rewrite this as a formal sentence, we start with

$$(\forall x)(\forall y)(\forall z)((x = y = z) \;\Rightarrow\; (x = z)) \qquad .$$

We should change "$x = y = z$" to "$(x = y)\,\&\,(y = z)$". Also, because of the rules of quantification development, we need more parentheses:

$$(\forall x)((\forall y)((\forall z)(((x = y)\,\&\,(y = z)) \;\Rightarrow\; (x = z)))) \qquad .$$

Our first definition:

DEFINITION 1.4. *$\forall a, b$, by $a \neq b$ we mean: $\neg(a = b)$.*

In our formalism, the logic purist does not tolerate definitions. The logic purist would ask that we remove this definition, and that we

- extend our character list to by adding: "$\neq$" ,
- add a new starter statement: "$a \neq b$" and
- put in a new axiom: $(\forall a)((\forall b)([a \neq b] \Leftrightarrow [\neg(a = b)]))$.

This would be followed replacement of brackets by parentheses. Also, the "$\Leftrightarrow$" needs to be broken into two implications, yielding:

$$(\forall a)((\forall b)(((a \neq b) \Rightarrow (\neg(a = b)))\,\&\,((\neg(a = b)) \Rightarrow (a \neq b)))) \quad .$$

But who can read such dense code?

Finally, let's look at:

THEOREM 1.5. *$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\delta^2 + \delta \leqslant \varepsilon$.*

By convention, in this course, "$\forall\varepsilon > 0$" means "$\forall$real $\varepsilon > 0$". Also, "$\exists\delta > 0$" means "$\exists$real $\delta > 0$. The logic purist would prefer:

$$
\begin{array}{l}
(\forall\varepsilon)\,(\,[\,(\,\varepsilon \text{ is a real number}\,)\,\&\,(\,\varepsilon > 0\,)\,] \Rightarrow [\\
(\exists\delta)\,(\,[\,(\,\delta \text{ is a real number}\,)\,\&\,(\,\delta > 0\,)\,]\&\,[\\
\qquad\qquad \delta^2 + \delta \leqslant \varepsilon\\
\qquad\qquad\quad]\,)\,]\,) \qquad .
\end{array}
$$

The logic purist would replace the text "$\delta^2 + \delta \leqslant \varepsilon$" by something like:

$$
\begin{array}{l}
\exists\text{real } a, L \text{ is a real number s.t.}\\
\quad (\,(\,\delta\cdot\delta = a\,)\,\&\,(\,a+\delta = L\,)\,)\,\&\,(\,(\,L < \varepsilon\,) \vee (\,L = \varepsilon\,)\,) \qquad .
\end{array}
$$

After a bit more "tsk!"ing, we get to

$$
\begin{array}{l}
(\forall\varepsilon)\,(\,[\,(\,\varepsilon \text{ is a real number}\,)\,\&\,(\,\varepsilon > 0\,)\,] \Rightarrow [\\
(\exists\delta)\,(\,[\,(\,\delta \text{ is a real number}\,)\,\&\,(\,\delta > 0\,)\,]\&\,[\\
(\exists a)\,(\,[\,a \text{ is a real number}\,]\,\&\,[\\
(\exists L)\,(\,[\,L \text{ is a real number}\,]\,\&\,[\\
\quad (\,(\,\delta\cdot\delta = a\,)\,\&\,(\,a+\delta = L\,)\,)\,\&\,(\,(\,L < \varepsilon\,) \vee (\,L = \varepsilon\,)\,)\\
\qquad\qquad]\,)\,]\,)\,]\,)\,]\,) \qquad .
\end{array}
$$

Finally, change brackets to parentheses and write this all on one line.

Life ain't easy for the logic purist.

2. Some set theory

DEFINITION 2.1. $\forall S, T,$ $\quad S \subseteq T$ *means:*
$(\,S$ *and* T *are sets*$)$ $\quad$ *and* $\quad$ $(\forall x \in S,\ x \in T)$.

Logic purist: Introduce a new special character "$\subseteq$", then introduce a new starter statement "$S \subseteq T$", then introduce a new axiom:

$$
\begin{array}{l}
(\forall S)(\\
(\forall T)(\\
\quad [\,S \subseteq T\,] \quad \Leftrightarrow\\
\quad [\,(\,(\,S \text{ is a set}\,)\,\&\,(\,T \text{ is a set}\,)\,)\,\&\\
\qquad\qquad\qquad (\,(\forall x)([x \in S] \Rightarrow [x \in T])\,)\,]\\
\qquad\quad)\,) \qquad .
\end{array}
$$

The logic purist would ask us to change brackets to parentheses, break the "$\Leftrightarrow$" into two implications, and to put all of this on one line.

The text "$S \subseteq T$" is read "S is a **subset** of T".

DEFINITION 2.2. $\forall S, T,$ $\qquad T \supseteq S$ *means:* $\quad S \subseteq T$.

Logic purist: Introduce a new special character "$\supseteq$", then introduce a new starter statement "$T \supseteq S$", then introduce a new axiom:

$$\begin{array}{l} (\forall S)(\\ (\forall T)(\\ \qquad (T \supseteq S) \Leftrightarrow (S \subseteq T) \\ \qquad\qquad\qquad)\,) \qquad . \end{array}$$

The logic purist would ask us to break the "$\Leftrightarrow$" into two implications, and to put all of this on one line.

The text "$S \supseteq T$" is read "S is a **superset** of T".

The following is sometimes called the **Axiom of Extensionality**. It is a quantified equivalence for equality of sets.

AXIOM 2.3. *$\forall$sets S, T, $[\ (\,S = T\,)\ \Leftrightarrow\ (\,[\,S \subseteq T\,]\,\&\,[\,T \subseteq S\,]\,)\ ]$.*

Logic purist:

$$\begin{array}{l} (\forall S)([S \text{ is a set}] \Rightarrow [\\ (\forall T)([T \text{ is a set}] \Rightarrow [\\ \qquad (S = T) \Leftrightarrow ([S \subseteq T]\,\&\,[T \subseteq S]) \\ \qquad\qquad\qquad]\,)\,]\,) \qquad . \end{array}$$

The logic purist would ask us to change brackets to parentheses, break the "$\Leftrightarrow$" into two implications, and to put all of this on one line.

DEFINITION 2.4. *$\forall a$, $\forall$set S, by $a \notin S$, we mean $\neg(a \in S)$.*

Logic purist: Introduce a new special character "$\notin$", then introduce a new starter statement "$a \notin S$", then introduce a new axiom:

$$\begin{array}{l} (\forall a)(\\ (\forall S)([S \text{ is a set}] \Rightarrow [\\ \qquad (a \notin S) \Leftrightarrow (\neg(a \in S)) \\ \qquad\qquad\qquad]\,)\,) \qquad . \end{array}$$

The logic purist would ask us to change brackets to parentheses, break the "$\Leftrightarrow$" into two implications, and to put all of this on one line.

The preceding remarks about Definition 2.4 apply, *mutatis mutandis*, to the following two definitions.

DEFINITION 2.5. *$\forall$sets S, T, by $S \not\subseteq T$, we mean $\neg(S \subseteq T)$.*

DEFINITION 2.6. *$\forall$sets S, T, by $S \not\supseteq T$, we mean $\neg(S \supseteq T)$.*

We sometimes put a definition within an axiom or a theorem, *e.g.*:

AXIOM 2.7. *$\exists 1$set S, denoted $\varnothing$, s.t. $\forall x$, $x \notin S$.*

We will say a few words about "denoted $\varnothing$". A logic purist would insist that, instead of Axiom 2.7, we add

a new special character: $\varnothing$,
a new constant: $\varnothing$,
a new axiom: $\varnothing$ is a set and
a new axiom: $\forall$set S, $[\,(\,\forall x,\ x \notin S\,) \Leftrightarrow (\,S = \varnothing\,)\,]$.

Exercise: Formalize these two axioms. The set $\varnothing$ is called the **empty set**. The symbol "$\varnothing$" is *NOT* the Greek letter phi, written "ϕ".

AXIOM 2.8. *$\forall a, S$, $[\,(\,a \in S\,) \Rightarrow (\,S$ is a set$\,)\,]$.*

Logic purist:

$(\forall a)($
$(\forall S)($
$(\,a \in S\,) \Rightarrow (\,S$ is a set$\,)$
$)\,)$.

Can there be a set that is an element of itself? It would have to be a pretty weird set, and, in fact, we will not allow such a set to exist:

AXIOM 2.9. *$\forall a$, $a \notin a$.*

Logic purist:

$(\forall a)($
$a \notin a$
$)$.

The next axiom states that ☹ "lives outside of set theory":

AXIOM 2.10. *(☹ is not a set) & ($\forall$set S, ☹ $\notin S$).*

Logic purist: $[\,\neg($☹ is a set$)\,]\,\&\,[\,(\forall S)([S$ is a set$] \Rightarrow [$☹ $\notin S\,]\,)\,]$. The logic purist would then ask us to change brackets to parentheses.

It follows that ☹ has no elements:

THEOREM 2.11. *$\forall a$, $a \notin$ ☹.*

Proof. Given a. Want: $a \notin$ ☹.
Assume $a \in$ ☹. Want: Contradiction.
By Axiom 2.8, ☹ is a set.
By Axiom 2.10, ☹ is not a set.
Contradiction. □

The logic purist would prefer: $(\forall a)(a \notin$ ☹$)$.

3. Sets of up to nine objects

We will use "$\forall\smile a$, ..." to mean "$(\forall a)([\, a \neq ☹\,] \Rightarrow [\,......\,])$". Similar conventions are adopted for all the 112 variables, not just "a".

AXIOM 3.1. *$\forall\smile a$, $\exists 1$set S, denoted $\{a\}$, s.t.*

$$\forall x, \qquad [\,(\, x \in S\,) \Leftrightarrow (\, x = a\,)\,].$$

Logic purist: Introduce two new special characters "{" and }, then a new starter statement "$\{a\} = S$", then two new axioms:

$\forall\smile a$, $\exists$set S s.t. $\{a\} = S$ and
$\forall\smile a$, $\forall$set S, $[\,(\,\{a\} = S\,) \Leftrightarrow (\forall x, [\,(\, x \in S\,) \Leftrightarrow (\, x = a\,)\,]\,)\,]$.

Exercise: Formalize these two axioms.

We have similar axioms for $\{a, b\}$ and $\{a, b, c\}$:

AXIOM 3.2. *$\forall\smile a, \forall\smile b$, $\exists 1$set S, denoted $\{a, b\}$, s.t.*

$$\forall x, \qquad [\,(\, x \in S\,) \Leftrightarrow (\,[x = a] \vee [x = b]\,)\,].$$

AXIOM 3.3. *$\forall\smile a, \forall\smile b, \forall\smile c$, $\exists 1$set S, denoted $\{a, b, c\}$, s.t.*

$$\forall x, \qquad [\,(\, x \in S\,) \Leftrightarrow (\,[x = a] \vee [x = b] \vee [x = c]\,)\,].$$

There are more of these axioms, ending with:

AXIOM 3.4. *$\forall\smile a, \ldots, \forall\smile i$, $\exists 1$set S, denoted $\{a, \ldots, i\}$, s.t.*

$$\forall x, \qquad [\,(\, x \in S\,) \Leftrightarrow (\,[x = a] \vee \cdots \vee [x = i\,)\,].$$

AXIOM 3.5. $\{☹\} = ☹$.

AXIOM 3.6. *$\forall a$, $[\ (\,\{a, ☹\} = \{☹, a\} = ☹\,)\ ]$.*

AXIOM 3.7. *$\forall a, \forall b$,*
$[\ (\,\{a, b, ☹\} = \{a, ☹, b\} = \{☹, a, b\} = ☹\,)\ ]$.

This continues until:

AXIOM 3.8. *$\forall a, \ldots, \forall h$,*
$[\ (\,\{a, \ldots, h, ☹\} = \cdots = \{☹, a, \ldots, h\} = ☹\,)\ ]$.

In Axiom 3.4 and Axiom 3.8, the use of an ellipsis ("$\cdots$") causes the logic purist great pain, but we think you can fill in those blanks. Also, we leave it to you to fill and formalize the missing axioms between Axiom 3.3 and Axiom 3.4, as well as the missing axioms beween Axiom 3.7 and Axiom 3.8. We could continue with sets of ten elements, but nine should be enough.

Axiom 3.5 through Axiom 3.8 are part of a general understanding that ☹ is "infective". That is, if an expression has ☹ inside, then it equals ☹.

THEOREM 3.9. $\{1,2\} = \{2,1\}$.

More formally:
$\exists L$, $\exists R$ s.t.
$[\,(\,\{1,2\} = L\,]\;\&\;[\,\{2,1\} = R\,]\;\&\;[\,L = R\,]$.

A simpler way to formalize Theorem 3.9:
$(\exists S)(\,(\,\{1,2\} = S\,)\,\&\,(\,\{2,1\} = S\,)\,)$.

THEOREM 3.10. $\{3,3\} = \{3\}$.

More formally:
$\exists L$, $\exists R$ s.t.
$[\,\{3,3\} = L\,]\;\&\;[\,\{3\} = R\,]\;\&\;[\,L = R\,]$.

THEOREM 3.11. $\{\{3\},\{3,3\}\} = \{\{3\},\{3\}\} = \{\{3\}\}$.

More formally:
(
$\exists S$, $\exists T$, $\exists L$, $\exists R$ s.t.
$[\,\{3\} = S\,]\;\&\;[\,\{3,3\} = T\,]\;\&\;[\,\{S,T\} = L\,]\,\&$
$[\,\{S,S\} = R\,]\;\&\;[\,L = R\,]$
) & (
$\exists S$, $\exists L$, $\exists R$ s.t.
$[\,\{3\} = S\,]\;\&[\,\{S,S\} = L\,]\;\&[\,\{S\} = R\,]\;\&[\,L = R\,]$
) .

A simpler way to formalize Theorem 3.11:
$(\exists S)(\,(\exists T)(\;\;[\,(\,\{3,3\} = S\,)\,\&\,(\,\{3\} = S\,)\,]\;\;\&$
$[\,(\,\{S,S\} = T\,)\,\&\,(\,\{S\} = T\,)\,]\;\;)\,)$.

THEOREM 3.12. $3 \notin \{\{3\}\}$.

More formally:
$\exists L$, $\exists S$, $\exists R$ s.t.
$[\,3 = L\,]\;\&\;[\,\{3\} = S\,]\;\&\;[\,\{S\} = R\,]\;\&\;[\,L = R\,]$.

THEOREM 3.13. $3 \in \{3\} \in \{\{3\}\}$.

More formally:
(

$\exists L$, $\exists R$ s.t.
$[\,3 = L\,]\,[\,\{3\} = R\,]\,[\,L \in R\,]$
) & (
$\exists S$, $\exists L$, $\exists R$ s.t.
$[\,\{3\} = L\,]\,[\,\{3\} = S\,]\,[\,\{S\} = R\,]\,[\,L \in R\,]$
) .

A simpler way to formalize Theorem 3.13:

$$(\exists S)(\,(\exists T)(\quad [\,(\,\{3\} = S\,)\,\&\,(\,\{S\} = T\,)\,] \quad \& \quad [\,(\,3 \in S\,)\,\&\,(\,S \in T\,)\,] \quad)\,) \quad .$$

Unassigned homework: Formalize the next four theorems.

THEOREM 3.14. $(\,1 \notin \{\{1,2\}\}\,)\,\&\,(\,2 \notin \{\{1,2\}\}\,)$.

THEOREM 3.15. $1, 2 \in \{1,2\} \in \{\{1,2\}\}$.

THEOREM 3.16. $\{1, ☹, 3, 4, 5\} = ☹$.

THEOREM 3.17. $\{1, \{☹, 3, 4, 5\}\} = \{1, ☹\} = ☹$.

4. Picking an element from a set

We will use "$\exists\smile a, \ldots\ldots$" to mean "$(\exists a)(\,[\,a \neq ☹\,]\,\&\,[\,\ldots\ldots\,]\,)$". Similar conventions are adopted for all the 112 variables, not just "a".

DEFINITION 4.1. *$\forall S$, by S is a* **singleton**, *we mean:*

$$(\ S \textit{ is a set}\ )\ \&\ (\ \exists\smile a \textit{ s.t. } S = \{a\}\ ).$$

The logic purist would prefer to introduce a new starter statement, "S is a singleton", and to make Definition 4.1 into an axiom:

$$\forall S, [\,(\,S \text{ is a singleton}\,) \Leftrightarrow ((\,S \text{ is a set}\,)\,\&\,(\,\exists a \text{ s.t. } [\,a \neq ☹\,]\,\&\,[\,\{a\} = S\,]\,)\,] \,.$$

THEOREM 4.2. *$\varnothing$ is not a singleton, $\{1\}$ is a singleton, $\{1,2\}$ is not a singleton and $\{\,\{1,2\}\,\}$ is a singleton.*

Recall the starter statement:

Statement	FV
$\mathrm{UE}(S) = a$	S, a

AXIOM 4.3. *$\forall$singleton S, $\forall a$,* $[\,(\,\mathrm{UE}(S) = a\,) \Leftrightarrow (\,a \in S\,)\,]$.

So, for any singleton S, $\mathrm{UE}(S)$ is the unique element of S.
We are sometimes sloppy and leave off parentheses, writing $\mathrm{CH}\,S$.
We can write Axiom 4.3 in a more "pure" way:

$$
\begin{array}{l}
(\forall S)([S \text{ is a singleton}] \Rightarrow [\\
(\forall a)(\\
\qquad [\,(\,\mathrm{UE}(S) = a\,) \Leftrightarrow (\,a \in S\,)\,]\\
\qquad\quad)\,]\,) \qquad .
\end{array}
$$

We insist that, when S is not a singleton, then S has no unique element, and therefore we should have $\mathrm{UE}(S) = ☹$:

AXIOM 4.4. *$\forall S$, $[\,(\,S$ is not a singleton$\,) \Rightarrow (\,\mathrm{UE}(S) = ☹\,)\,]$.*

As always, there is lots to "tsk!" about. More formally:

$$
\begin{array}{l}
(\forall S)(\\
\qquad (\,\neg[\,S \text{ is a singleton}\,]\,) \Rightarrow (\,\mathrm{UE}(S) = ☹\,)\\
\qquad\quad) \qquad .
\end{array}
$$

If you are feeling energetic, change the brackets to parentheses, and then write this entire stream of symbols on one line.

THEOREM 4.5. $\mathrm{UE}\,\varnothing = ☹$.

Recall the starter statement:

Statement	FV
$\mathrm{CH}(S) = a$	S, a

We next state the **Axiom of Choice**:

AXIOM 4.6. *$\forall$nonempty set S, $\exists a \in S$ s.t. $\mathrm{CH}(S) = a$.*

So, for any nonempty set S, $\mathrm{CH}(S)$ is some element of S. So CH chooses, from every nonempty set, one of its elements.

We are sometimes sloppy and leave off parentheses, writing $\mathrm{CH}\,S$.

We can write Axiom 4.6 in a more formal way:

$$
\begin{array}{l}
(\forall S)(\,[\,(\,S \text{ is a set}\,)\,\&\,(\,S \neq \varnothing\,)\,] \Rightarrow [\\
(\exists a)(\,[\,a \in S\,]\,\&\,[\\
\qquad\qquad \mathrm{CH}(S) = a\\
\qquad\qquad\quad]\,)\,]\,) \qquad .
\end{array}
$$

If you are feeling energetic, change the brackets to parentheses, and then write this entire stream of symbols on one line.

Sad to say, the set $\varnothing$ has no element to choose:

AXIOM 4.7. $\mathrm{CH}\,\varnothing = ☹$.

THEOREM 4.8. $\mathrm{UE}\{1\} = 1$.

THEOREM 4.9. $\mathrm{UE}\{2\} = 2$.

THEOREM 4.10. $\mathrm{UE}\{\{1,2\}\} = \{1,2\}$.

THEOREM 4.11. $\mathrm{UE}\{1,2\} = ☹$.

THEOREM 4.12. $\mathrm{CH}\{1\} = 1$.

THEOREM 4.13. $\mathrm{CH}\{2\} = 2$.

THEOREM 4.14. $\mathrm{CH}\{\{1,2\}\} = \{1,2\}$.

THEOREM 4.15. $\mathrm{CH}\{1,2\} \in \{1,2\}$.

THEOREM 4.16. $\mathrm{CH}\{1,2\} \neq ☹ = \mathrm{UE}\{1,2\}$.

AXIOM 4.17. $\mathrm{UE}(☹) = ☹ = \mathrm{CH}(☹)$.

5. Formalism and intuition

Logic purity really takes it out of a fellow. The point is not that we *SHOULD* rewrite every pidgin statement as a formal statement, only that it *CAN* be done, if the need for extra precision should arise. There are many reasons why we do not want to obsess about formalism. For one thing, it requires a great deal of effort, and produces results that are very difficult to read. More importantly, if we focus on formalism to the complete exclusion of intuition, then we have lost a crucial aspect of the mathematical experience.

Intuition and formalism are yin and yang. At first blush, they may seem in competition, but, in fact, each reinforces the other, and each depends on the other. For example, intuition depends on formalism: Each person's intuition is based on their own experiences, so rigor and formal writing provides a basis for resolving differences of opinion. Conversely, I find that formal writing almost always starts as vague intuitive ideas, refined repeatedly to increasing levels of formality. I cannot imagine proving any complicated theorem without having some intuitive insight driving my thinking.

From the purist's point of view, certain streams of symbols are statements, and others are just the ramblings of someone who has learned to speak the English language, and accepts all the lack of clarity that comes with it. From this purist point of view, a proof is a sequence of formal statements each of which follows, by precise rules, from earlier statements, or from a list of axioms. So we could feed a proof into a computer, and the computer can check its validity.

We, however, are not computers. When we see

$$(\forall x)(x = x)$$

we want it to have some intuitive meaning; otherwise, mathematics becomes a subject fit only for code-monkeys. Reading

$$(\forall x)(x = x)$$

or the less formal version, "$\forall x,\ x = x$", one might interpret it to mean

$$\text{for any mathematical object } x, \text{ we have: } x = x \quad ,$$

but this begs the question: Which objects are mathematial objects?

The answer actually varies from subject to subject, from mathematician to mathematician. Logicians someimes refer to the collection of all mathematical objects as the "domain of discourse", and so, we are asking: What is the domain of discourse in this particular course, *i.e.*, what is our mathematical universe? For us, it consists of

real numbers , sets ,
∞ , $-\infty$ and ☹ .

These terms are intuitive. We will not, in this course, try to define a real number or set, or any of the other three objects. Also, at the moment, we have not given a name to any real number, so, while can talk about all of them at once, we cannot yet talk about any particular real number. So, for example, the statement

1 is a real number

is just a stream of symbols that, according to our rules, is a formal statement. We do not yet have the axiomatic framework to determine whether or not it is a theorem. However, we rely on your intuition and earlier learning to know that we should eventually set up our axioms in such a way that that statement *IS* a formal theorem.

Similarly, "∞" is not just a sideways "8". If you see "∞" used, you do not need to turn your head to understand it. We hope you have some intuitive sense of the infinite, and the formal theorems that we will prove later on should dovetail with that intuition.

6. A doubly quantified theorem

In §14, we will explain how to prove:

THEOREM 6.1. $\forall \varepsilon > 0,\ \exists \delta > 0$ *s.t.* $\delta^2 + \delta \leqslant \varepsilon$.

In this section, we only attempt to understand, *at an intuitive level*, why the Theorem 6.1 is true. It is very difficult to prove a theorem until you believe in your heart that it is valid.

Imagine the following game, which is based on Theorem 6.1:

> You move first: You choose a real $\varepsilon > 0$, and reveal it to me.
> My move: I choose a real $\delta > 0$, and reveal it to you.
> We check to see if $\delta^2 + \delta \leqslant \varepsilon$.
> If so, then I win.
> If not, then you win.

Let's play: Say you choose $\varepsilon = 100$. I will laugh at your poor play, and choose $\delta = 1$. We check that $1^2 + 1 \leqslant 100$ is true, so I win. We play again. You try $\varepsilon = 1000$. I laugh even harder, and choose $\delta = 1$ again. We check that $1^2 + 1 \leqslant 1000$ is true, so I win again. You begin to see that making ε large is not in your interest. However, by the rules, you cannot make it negative or zero. You try $\varepsilon = 0.001$. Now I have to concentrate. I choose $\delta = 0.00001$. We check that $0.00001^2 + 0.00001 \leqslant 0.001$ is true, so I win again. You begin to think the game is rigged. Saying that the game is rigged against you is the same as saying that you believe that Theorem 6.1 is true, and that is really the first step in proving it.

Theorem 6.1 is "doubly quantified"; it has one "$\forall$" and one "$\exists$", totaling to two quantifiers. Most of us do not spend a great deal of time considering the validity of doubly quantified assertions, *EXCEPT* when we play games. The chess player may say: "whatever move my opponent makes, I will be able to checkmate him/her on my next move". This is an example of a doubly quantified statement:

$$\forall \text{move of my opponent}, \quad \exists \text{move of mine} \quad \text{s.t.} \quad \text{checkmate} \quad .$$

Somehow we are hardwired to deal with highly quantified statements while playing certain games, and you can piggyback off that hardwiring by converting highly quantified theorems into games.

Now that we believe in Theorem 6.1, we need a specific strategy to win. It is not enough to say, "Well, just make sure the δ is very small". We need a specific method for choosing δ after we know ε.

Sometimes, it helps to focus first on the finish, in order to see what is needed in the δ-strategy. We wish to force

$$\delta^2 + \delta \leqslant \varepsilon \quad .$$

Some students may have practiced solving quadratic inequalities, which is one route to setting up a δ-strategy. However, there are more complicated problems leading to, *e.g.*, $\delta^5 + \delta^2 + \delta \leqslant \varepsilon$, and this kind of inequality is hard to solve. We favor a more robust approach, in which we break the problem down, term by term. That is, we work separately on the first term δ^2 and the second term δ. If we can force

$$\delta^2 \leqslant \varepsilon/2 \quad \text{and} \quad \delta \leqslant \varepsilon/2 \quad .$$

then we will win the game. It is therefore enough to force

$$0 < \delta \leqslant \sqrt{\varepsilon/2} \quad \text{and} \quad \delta \leqslant \varepsilon/2 \quad .$$

This leads us to the strategy:

Let $\delta := \min\{\varepsilon/2, \sqrt{\varepsilon/2}\}$.

We now need to take this strategy and turn it into a pidgin proof. Before we can do that, however, we will need to expose the basics of arithmetic and inequalities. In particular, we need to define

$$\min\{\ \ ,\ \ \} \qquad \text{and} \qquad \sqrt{} \qquad .$$

That will take a few sections, but, in §14, we will prove Theorem 6.1.

7. Arithmetic

Here are two of our starter statements:

Statement	FV
x is a real number	x
j is an integer	j

AXIOM 7.1. $\exists 1$*set* S, *denoted* $\mathbb{Z}$, *s.t.:*
$\forall j$, $[\,(\,j \in S\,) \Leftrightarrow (\,j$ *is an integer*$\,)\,]$.

The logic purist would drop the axiom above, then add "$\mathbb{Z}$" to the list of special characters and to the list of constants, and would then add two axioms:

$\mathbb{Z}$ is a set and
$\forall$set S, $[\,(\,\forall j, [\,(j \in S) \Leftrightarrow (j$ is an integer$)\,]\,) \Leftrightarrow (\,S = \mathbb{Z}\,)\,]$.

Exercise: Formalize the last of these axioms.

AXIOM 7.2. $\exists 1$*set* S, *denoted* $\mathbb{R}$, *s.t.:*
$\forall x$, $[\,(\,x \in S\,) \Leftrightarrow (\,x$ *is a real number*$\,)\,]$.

The logic purist would drop the axiom above, then add "$\mathbb{R}$" to the list of special characters and to the list of constants, and would then add two axioms:

$$\mathbb{R} \text{ is a set} \qquad \text{and}$$
$$\forall \text{set } S,\ [(\forall x,\ [(x \in S) \Leftrightarrow (x \text{ is a real number})]) \Leftrightarrow (\ S = \mathbb{R}\ )] \quad .$$

Exercise: Formalize the last of these axioms.

Our next axiom says, in set-theoretic language, that 1 is an integer, and that every integer is a real number:

AXIOM 7.3. *$0, 1 \in \mathbb{Z} \subseteq \mathbb{R}$.*

Our next axiom says that any two real numbers have a real sum:

AXIOM 7.4. *$\forall x, y \in \mathbb{R}$, $\exists z \in \mathbb{R}$ s.t. $x + y = z$.*

More formally:

$$\begin{array}{l} \forall x,\ (\ [x \in \mathbb{R}]\ \Rightarrow\ [\\ \forall y,\ (\ [y \in \mathbb{R}]\ \Rightarrow\ [\\ \exists z \text{ s.t. } (\ [z \in \mathbb{R}]\ \&\ [\\ \qquad x + y = z \\ \qquad\qquad]\,)\,]\,)\,]\,) \qquad . \end{array}$$

Finally, if you are feeling energetic, change the brackets to parentheses, and then write the entire stream of symbols on one line.

Our next axiom:

AXIOM 7.5. *$\forall x \in \mathbb{R}, \qquad x + 0 = x$.*

More formally:

$$\forall x, \qquad (x \in \mathbb{R})\ \Rightarrow\ (x + 0 = x).$$

From there, the remaining "tsk!"s are easily dealt with:

$$(\forall x)((x \in \mathbb{R})\ \Rightarrow\ (x + 0 = x)) \qquad .$$

AXIOM 7.6. *$\forall x, y \in \mathbb{R}, \quad x + y = y + x$.*

The equation "$x + y = y + x$" needs to be broken apart into several atomic statements, like:

$$x + y = L, \qquad y + x = R \qquad , \qquad L = R \qquad .$$

So we could partially formalize Axiom 7.6 as

$$\begin{array}{l} \forall x, y \in \mathbb{R},\ \exists L, R \in \mathbb{R} \text{ s.t.} \\ \qquad (x + y = L)\ \&\ (y + x = R)\ \&\ (L = R) \qquad . \end{array}$$

From there, we can work on "$\forall \cdots \in \mathbb{R}$" and "$\exists \cdots \in \mathbb{R}$":

$$\begin{aligned}
&(\forall x)([x \in \mathbb{R}] \Rightarrow [\\
&(\forall y)([y \in \mathbb{R}] \Rightarrow [\\
&(\exists L)([L \in \mathbb{R}] \,\&\, [\\
&(\exists R)([R \in \mathbb{R}] \,\&\, [\\
&\qquad (x + y = L) \,\&\, (y + x = R) \,\&\, (L = R)\\
&\qquad\qquad\qquad\qquad]\,)\,]\,)\,]\,)\,]\,) \qquad .
\end{aligned}$$

Finally, if you are feeling energetic, change the brackets to parentheses, and then write the entire stream of symbols on one line.

AXIOM 7.7. $\forall x, y, z \in \mathbb{R}, \quad x + (y + z) = (x + y) + z.$

We could formalize the above axiom as:

$$\begin{aligned}
&\forall x, y, z \in \mathbb{R},\ \exists a, b, L, R \in \mathbb{R} \text{ s.t.}\\
&\qquad (y + z = a) \,\&\, (x + a = L) \,\&\\
&\qquad (x + y = b) \,\&\, (b + z = R) \,\&\\
&\qquad (L = R) \qquad .
\end{aligned}$$

From there, we can work on "$\forall \cdots \in \mathbb{R}$" and "$\exists \cdots \in \mathbb{R}$":

$$\begin{aligned}
&(\forall x)([x \in \mathbb{R}] \Rightarrow [\\
&(\forall y)([y \in \mathbb{R}] \Rightarrow [\\
&(\forall z)([z \in \mathbb{R}] \Rightarrow [\\
&(\exists a)([a \in \mathbb{R}] \,\&\, [\\
&(\exists b)([b \in \mathbb{R}] \,\&\, [\\
&(\exists L)([L \in \mathbb{R}] \,\&\, [\\
&(\exists R)([R \in \mathbb{R}] \,\&\, [\\
&\qquad\qquad (y + z = a) \,\&\, (x + a = L) \,\&\\
&\qquad\qquad (x + y = b) \,\&\, (b + z = R) \,\&\\
&\qquad\qquad (L = R)\\
&\qquad\qquad\qquad\qquad]\,)\,]\,)\,]\,)\,]\,)\,]\,)\,]\,)\,]\,) \qquad .
\end{aligned}$$

Finally, if you are feeling energetic, change the brackets to parentheses, and then write the entire stream of symbols on one line. Uff da!

For lack of time, going forward, we will avoid formalizing most of our pidgin statements. However, if any question arises about how to convert a pidgin statement into a formal statement, be sure to ask.

Some more axioms:

AXIOM 7.8. $\forall x, y \in \mathbb{R}$, $\exists z \in \mathbb{R}$ *s.t.* $xy = z$.

AXIOM 7.9. $\forall x \in \mathbb{R}, \qquad x \cdot 1 = x.$

AXIOM 7.10. $\forall x, y \in \mathbb{R}, \qquad xy = yx.$

AXIOM 7.11. $\forall x, y, z \in \mathbb{R}, \qquad x(yz) = (xy)z.$

AXIOM 7.12. $\forall x, y, z \in \mathbb{R}, \qquad x(y + z) = xy + xz.$

To formalize "$x(y + z) = xy + xz$", we would write:

$$\begin{aligned}&\exists a, b, c, L, R \in \mathbb{R} \text{ s.t.}\\ &\qquad (y + z = a)\, \&\, (x \cdot a = L)\, \&\\ &\qquad (x \cdot y = b)\, \&\, (x \cdot z = c)\, \&\, (b + c = R)\, \&\\ &\qquad (L = R) \qquad .\end{aligned}$$

We leave the rest of this formalization as an exercise for the reader.

Next, we develop negation and subtraction.

AXIOM 7.13. $\forall x \in \mathbb{R}$, $\exists 1 y \in \mathbb{R}$, *denoted* $-x$, *s.t.* $x + y = 0$.

DEFINITION 7.14. $\forall a, b \in \mathbb{R}, \qquad b - a \;:=\; b + (-a).$

In high school, we teach students to solve simple equations in a single unknown. For example, solving $4 + x = 7$ leads to $x = 3$. It should be no surprise that, for any two real numbers a and b, we can solve $a + x = b$, and find a real solution x. More formally, we have:

THEOREM 7.15. $\forall a, b \in \mathbb{R}$, $\exists x \in \mathbb{R}$ *s.t.* $a + x = b$.

We are not yet writing proofs, but, in the proof of this theorem, at some point, we would write "Let $x := b - a$", see Theorem 29.3.

THEOREM 7.16. $\forall a, x, y \in \mathbb{R}, \qquad (\, a + x = a + y \,) \;\Rightarrow\; (\, x = y \,).$

THEOREM 7.17. $\forall x \in \mathbb{R}, \qquad x \cdot 0 = 0.$

Next, division. The next axiom is the multiplicative analogue of Theorem 7.15. It says: $\forall a, b \in \mathbb{R}$, we can solve $ax = b$, *PROVIDED* $a \neq 0$. That solution x is unique, and is denoted b/a. Formally:

AXIOM 7.18. $\forall a, b \in \mathbb{R}$,

$$(\, a \neq 0 \,) \;\Rightarrow\; (\, \exists 1 x \in \mathbb{R}, \;\; \textit{denoted}\ b/a, \;\; \textit{s.t.}\ ax = b \,).$$

Logic Purist: Add a new special character "/", then add a new starter statement "$b/a = x$", then add two new axioms:

$$\forall a, b \in \mathbb{R},\ (\, a \neq 0 \,) \;\Rightarrow\; (\, \exists x \in \mathbb{R} \text{ s.t. } b/a = x \,) \qquad \text{and}$$
$$\forall a, b \in \mathbb{R},\ (\, a \neq 0 \,) \;\Rightarrow\; (\, \forall \text{real } x,\ [\, (\, ax = b \,) \;\Leftrightarrow\; (\, b/a = x \,) \,] \,) \quad .$$

AXIOM 7.19. $\forall a$, $a/0 = ☹$.

The next axioms are part of a general understanding that ☹ is "infective". That is, if an expression has ☹ inside, then it equals ☹.

AXIOM 7.20. $-☹ = ☹$.

AXIOM 7.21. $\forall x, \quad x + ☹ = ☹ + x = ☹$.

AXIOM 7.22. $\forall x, \quad x \cdot ☹ = ☹ \cdot x = ☹$.

AXIOM 7.23. $\forall x, \quad x/☹ = ☹/x = ☹$.

8. Some real numbers of interest

Let's pin down how $1, \ldots, 9$ are related:

AXIOM 8.1. *All of the following are true:*

$$\begin{array}{llll} 1+1=2, & 2+1=3, & 3+1=4, & 4+1=5, \\ 5+1=6, & 6+1=7, & 7+1=8, & 8+1=9. \end{array}$$

The logic purist would ask either that we create eight separate axioms, or that we combine with parentheses and "&"s:

$$\begin{array}{l} (1+1=2)\,\&\,(2+1=3)\,\&\,(3+1=4)\,\&\,(4+1=5)\,\& \\ (5+1=6)\,\&\,(6+1=7)\,\&\,(7+1=8)\,\&\,(8+1=9) \qquad . \end{array}$$

The logic purist would have us put this all on one line.

DEFINITION 8.2. $10 := 9 + 1$.

Logic purist: Make

a new constant: "10" and
a new axiom: "9 + 1 = 10" .

DEFINITION 8.3. $100 := 10 \cdot 10$.

DEFINITION 8.4. $1000 := 10 \cdot 100$.

DEFINITION 8.5. $10000 := 10 \cdot 1000$.

DEFINITION 8.6. $100000 := 10 \cdot 10000$.

DEFINITION 8.7. $0.1 := 1/10$ *and* $0.01 := 1/100$ *and* $0.001 := 1/1000$ *and* $0.0001 := 1/10000$ *and* $0.00001 := 1/100000$.

DEFINITION 8.8. $11 := 10 + 1$, $12 := 10 + 2$, $13 := 10 + 3$, $14 := 10 + 4$, $15 := 10 + 5$, $16 := 10 + 6$, $17 := 10 + 7$, $18 := 10 + 8$, $19 := 10 + 9$.

THEOREM 8.9. $0, 1, 2, 3, \ldots, 19 \in \mathbb{R}$.
Also, $100, 1000, 10000, 100000 \in \mathbb{R}$.
Also, $0.1, 0.01, 0.001, 0.0001, 0.00001 \in \mathbb{R}$.

9. Sets of sets, unions and intersections

A "set of sets" is just a set all of whose elements are sets:

DEFINITION 9.1. *$\forall \mathcal{S}$, by $\mathcal{S}$ is a* **set of sets**, *we mean:*
$(\, \mathcal{S}$ *is a set* $)$ & $(\, \forall A \in \mathcal{S},\ A$ *is a set* $)$.

THEOREM 9.2. *Let $A := \{5, 6, 7, 8\}$ and let $B := \{7, 8, 9\}$.*
Then $\{A, B\}$ is a set of sets.

THEOREM 9.3. *$\varnothing$ is a set of sets.*

Proof. See Theorem 15.4. □

THEOREM 9.4. *$\{\varnothing\}$ is a set of sets.*

AXIOM 9.5. *$\forall$set $\mathcal{S}$ of sets, $\exists 1$set U, denoted $\bigcup \mathcal{S}$, s.t.:*
$\forall x, \quad [\, (\, x \in U\,) \Leftrightarrow (\, \exists A \in \mathcal{S}$ *s.t.* $x \in A\,)\,]$.

AXIOM 9.6. $\bigcup ☹ = ☹$.

THEOREM 9.7. *Let $A := \{5, 6, 7, 8\}$ and let $B := \{7, 8, 9\}$. Then*
$$\bigcup \{A, B\} = \{5, 6, 7, 8, 9\}.$$

THEOREM 9.8. $\bigcup \varnothing = \varnothing$.

Proof. See Theorem 15.5. □

THEOREM 9.9. $\bigcup \{\varnothing\} = \varnothing$.

AXIOM 9.10. *$\forall$nonempty set $\mathcal{S}$ of sets, $\exists 1$set V, denoted $\bigcap \mathcal{S}$, s.t.:*
$\forall x, \quad [\, (\, x \in V\,) \Leftrightarrow (\, \forall A \in \mathcal{S},\, x \in A\,)\,]$.

AXIOM 9.11. $\bigcap ☹ = \bigcap \varnothing = ☹$.

THEOREM 9.12. *Let $A := \{5, 6, 7, 8\}$ and let $B := \{7, 8, 9\}$. Then*
$$\bigcap \{A, B\} = \{7, 8\}.$$

THEOREM 9.13. $\bigcap \{\varnothing\} = \varnothing$.

AXIOM 9.14. *$\forall \mathcal{S}$, $(\, \mathcal{S}$ is not a set of sets$\,) \Rightarrow (\, \bigcup \mathcal{S} = ☹ = \bigcap \mathcal{S}\,)$.*

THEOREM 9.15. *$\forall$set A, $\bigcup\{A\} = A = \bigcap\{A\}$.*

DEFINITION 9.16. *$\forall$sets A, B,*

$$A \cup B \;:=\; \bigcup\{A, B\} \qquad \text{and} \qquad A \cap B \;:=\; \bigcap\{A, B\}.$$

THEOREM 9.17. *Let $A := \{5, 6, 7, 8\}$ and let $B := \{7, 8, 9\}$. Then*

$$A \cup B = \{5, 6, 7, 8, 9\} \qquad \text{and} \qquad A \cap B = \{7, 8\}.$$

DEFINITION 9.18. *$\forall$sets A, B, C,*

$$A \cup B \cup C := \bigcup\{A, B, C\} \qquad \text{and}$$
$$A \cap B \cap C := \bigcap\{A, B, C\}.$$

DEFINITION 9.19. *$\forall$sets A, B, C, D,*

$$A \cup B \cup C \cup D := \bigcup\{A, B, C, D\} \qquad \text{and}$$
$$A \cap B \cap C \cap D := \bigcap\{A, B, C, D\}.$$

We leave it to you to continue these definitions until we have unions and intersections of nine sets $A, \ldots, I$, finishing by writing out the following definition, without ellipses.

DEFINITION 9.20. *$\forall$sets $A, \ldots, I$,*

$$A \cup \cdots \cup I := \bigcup\{A, \ldots, I\} \qquad \text{and}$$
$$A \cap \cdots \cap I := \bigcap\{A, \ldots, I\}.$$

10. Extended reals and inequalities

DEFINITION 10.1. *$\forall x, y$, by $x \leqslant y$, we mean:*

$$(x < y) \vee (x = y).$$

DEFINITION 10.2. *$\forall x, y$, by $x > y$, we mean: $y < x$.*

DEFINITION 10.3. *$\forall x, y$, by $x \geqslant y$, we mean:*

$$(x > y) \vee (x = y).$$

AXIOM 10.4. *$\forall x, y, z$, $\quad (x < y < z) \Rightarrow (x < z)$.*

AXIOM 10.5. *$\forall x$, $\quad \neg(x < x)$.*

THEOREM 10.6. *$\forall x, y$, $\quad (x < y) \Rightarrow (\neg(x \geqslant y)$.*

AXIOM 10.7. $\quad \infty \neq ☹ \neq -\infty$.

By Axiom 10.7 and Axiom 3.2, there is a set denoted $\{-\infty, \infty\}$ whose elements are exactly $-\infty$ and ∞.

DEFINITION 10.8. $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$.

Elements of $\mathbb{R}^*$ are called **extended real numbers**.

We can now formulate our domain of discourse, as an axiom:

AXIOM 10.9. *$\forall\smile x$, $[\,(\,x \in \mathbb{R}^*\,) \vee (\,x$ is a set$\,)\,]$.*

AXIOM 10.10. *$\forall x \in \mathbb{R}^*$, x is not a set.*

AXIOM 10.11. *$\forall x \in \mathbb{R}$, $-\infty < x < \infty$*

THEOREM 10.12. *$-\infty < \infty$.*

THEOREM 10.13. *$\forall x \in \mathbb{R}$, $-\infty \neq x \neq \infty \neq -\infty$.*

AXIOM 10.14. *$0 < 1$.*

AXIOM 10.15. *$\forall a, x, y \in \mathbb{R}$, $(x < y) \Rightarrow (a + x < a + y)$.*

THEOREM 10.16. *$1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10$.*

AXIOM 10.17. *$\forall a, x, y \in \mathbb{R}$, $[\,(a > 0)\,\&\,(x < y)\,] \Rightarrow [\,ax < ay\,]$.*

DEFINITION 10.18. *$20 := 2\cdot 10$, $30 := 3\cdot 10$, $40 := 4\cdot 10$, $50 := 5\cdot 10$, $60 := 6\cdot 10$, $70 := 7\cdot 10$, $80 := 8\cdot 10$ and $90 := 9\cdot 10$.*

THEOREM 10.19. *$10 < 20 < 30 < \cdots < 90 < 100$.*

AXIOM 10.20. *$\forall x, y \in \mathbb{R}$, $(x < y) \Rightarrow (-x > -y)$.*

THEOREM 10.21. *$-10 < -9 < -8 < \cdots < -1 < 0$.*

AXIOM 10.22. *$\forall x, y \in \mathbb{R}^*$, $(x < y) \vee (x = y) \vee (x > y)$.*

Combining Axiom 10.22 and Theorem 10.6, we get:

THEOREM 10.23. *$\forall x, y \in \mathbb{R}^*$, $(x < y) \Leftrightarrow (\neg(x \geqslant y))$.*

DEFINITION 10.24. *Let $x \in \mathbb{R}$.*
By x is **positive**, *we mean $x > 0$.*
By x is **semi-positive**, *we mean $x \geqslant 0$.*
By x is **negative**, *we mean $x < 0$.*
By x is **semi-negative**, *we mean $x \leqslant 0$.*

Finally, ☹ is strictly incomparable with everything:

AXIOM 10.25. *$\forall x$, $[\ (\neg[\,☹ < x\,])\ \&\ (\neg[\,x < ☹\,])\ ]$.*

THEOREM 10.26. *$\forall\smile x$, $[\ (\neg[\,☹ \leqslant x\,])\ \&\ (\neg[\,x \leqslant ☹\,])\ ]$.*

However, do keep in mind that, by Axiom 1.1, ☹ = ☹. It follows that ☹ compares *NONstrictly* with itself:

THEOREM 10.27. *☹ $\leqslant$ ☹.*

11. Axioms of specification

By a **specification triple**, we mean three pieces of data:

(1) a formal statement with at least one free variable ,
(2) one of its free variables and
(3) a set .

Since we know how to formalize pidgin statements, we will relax, and allow a pidgin statement in (1), so long it has at least one free variable.

We will soon see, example by example, that each specification triple leads to an **Axiom of Specification**. There are infinitely many such triples, leading to infintely many axioms. We illustrate a few:

We begin with the specification triple

(1) $a < x < b$,
(2) x and
(3) $\mathbb{R}$.

To get the corresponding Axiom of Specification, identify all the free variables in (1), *except* for the variable in (2). This yields: "a" and "b". Then the axiom we seek begins "$\forall a, \forall b$". It reads:

AXIOM 11.1. *$\forall a$, $\forall b$, $\exists 1$set S, denoted $\{x \in \mathbb{R} \,|\, a < x < b\}$, s.t., $\forall x$, $(\,[\,x \in S\,] \;\Leftrightarrow\; [\,(\,x \in \mathbb{R}\,) \,\&\, (\,a < x < b\,)\,]\,)$.*

Here is another specification triple:

(1) $\exists t \in \mathbb{R}$ s.t. $a + q + t^2 = 1$,
(2) a and
(3) $\mathbb{Z}$.

To get the corresponding Axiom of Specification, identify all the free variables in (1), *except* for the variable in (2). This yields: "q". Then the axiom we seek begins "$\forall q$". It reads:

AXIOM 11.2. *$\forall q$, $\exists 1$set S, denoted $\{a \in \mathbb{Z} \,|\, \exists t \in \mathbb{R}$ s.t. $a + q + t^2 = 1\}$, s.t., $\forall a$, $(\,[\,a \in S\,] \;\Leftrightarrow\; [\,(\,a \in \mathbb{Z}\,) \,\&\, (\,\exists t \in \mathbb{R}$ s.t. $a + q + t^2 = 1\,)\,]\,)$.*

Here is another specification triple:

(1) $z \neq 0$,
(2) z and
(3) $\mathbb{R}$.

To get the corresponding Axiom of Specification, identify all the free variables in (1), *except* for the variable in (2). There are none. The axiom reads:

AXIOM 11.3. *$\exists 1$set S, denoted $\{z \in \mathbb{R} \,|\, z \neq 0\}$,*
s.t., $\forall z$, $(\ [\,z \in S\,] \ \Leftrightarrow\ [\,(\,z \in \mathbb{R}\,)\,\&\,(\,z \neq 0\,)\,]\ )$.

Here is another specification triple:

(1) $x^2 = a$,
(2) x and
(3) $\mathbb{R}$.

To get the corresponding Axiom of Specification, look at all the free variables, *except* the one in (2). This yields: "a". Then the axiom we seek begins "$\forall a$". It reads:

AXIOM 11.4. *$\forall a$, $\exists 1$set S, denoted $\{x \in \mathbb{R} \,|\, x^2 = a\}$,*
s.t., $\forall x$, $(\ [\,x \in S\,] \ \Leftrightarrow\ [\,(\,x \in \mathbb{R}\,)\,\&\,(\,x^2 = a\,)\,]\ )$.

Here is another specification triple:

(1) $x^2 = 4$,
(2) x and
(3) $\mathbb{R}$.

To get the corresponding Axiom of Specification, look at all the free variables, *except* the one in (2). There are none. The axiom reads:

AXIOM 11.5. *$\exists 1$set S, denoted $\{x \in \mathbb{R} \,|\, x^2 = 4\}$,*
s.t., $\forall x$, $(\ [\,x \in S\,] \ \Leftrightarrow\ [\,(\,x \in \mathbb{R}\,)\,\&\,(\,x^2 = 4\,)\,]\ )$.

In high school algebra one learns that the solutions of $x^2 = 4$ are -2 and 2. We express that result as a theorem:

THEOREM 11.6. $\{x \in \mathbb{R} \,|\, x^2 = 4\} \quad = \quad \{-2, 2\}$.

The focus on the variable x is somewhat arbitrary. We also have:

THEOREM 11.7. $\{z \in \mathbb{R} \,|\, z^2 = 4\} \quad = \quad \{-2, 2\}$.

Because this is a real analysis course, and not a complex analysis course, our formalism is focused on $\mathbb{R}$. You may have learned, in high school, that $(1 + i)^2 = 2i$, but, for us, this is not a theorem. Consequently, we do *NOT* have a theorem that says

$$\{z \in \mathbb{C} \,|\, z^2 = 2i\} \quad = \quad \{1 + i, -1 - i\}.$$

In fact, $\mathbb{C}$ is not a set in this course, and "$z^2 = 2i$" is not a formal statement. So we do not have a specification axiom that defines the set $\{z \in \mathbb{C} \,|\, z^2 = 2i\}$. The point is: There are many formal systems of mathematics. We are tailoring ours to this particular course.

Now that we have specification, we can define many useful sets.

DEFINITION 11.8. $\forall a, b \in \mathbb{R}^*$,

$$[a; b] := \{x \in \mathbb{R}^* \,|\, a \leqslant x \leqslant b\},$$
$$(a; b) := \{x \in \mathbb{R}^* \,|\, a < x < b\},$$
$$[a; b) := \{x \in \mathbb{R}^* \,|\, a \leqslant x < b\} \quad \textit{and}$$
$$(a; b] := \{x \in \mathbb{R}^* \,|\, a < x \leqslant b\}.$$

DEFINITION 11.9. $1.3 := 1 + (3/10)$ *and* $2.6 := 2 + (6/10)$.

THEOREM 11.10. $[1.3; 1.3] = \{1.3\}$, $1.3 \notin (1.3; 2.6]$, $2.6 \in (1.3; 2.6]$, $(2.6; 2.6) = [2.6; 1.3) = [2.6; 1.3] = \varnothing$.

DEFINITION 11.11. $\mathbb{Z}^* := \mathbb{Z} \cup \{-\infty, \infty\}$.

Elements of $\mathbb{Z}^*$ are called **extended integers**.

DEFINITION 11.12. $\forall a, b \in \mathbb{R}^*$,

$$[a..b] := \{x \in \mathbb{Z}^* \,|\, a \leqslant x \leqslant b\} \quad ,$$
$$(a..b) := \{x \in \mathbb{Z}^* \,|\, a < x < b\} \quad ,$$
$$[a..b) := \{x \in \mathbb{Z}^* \,|\, a \leqslant x < b\} \quad \textit{and}$$
$$(a..b] := \{x \in \mathbb{Z}^* \,|\, a < x \leqslant b\} \quad .$$

THEOREM 11.13. $[1..1] = \{1\}$ *and* $(1..2] = \{2\}$ *and* $(1.3..2.6) = \{2\}$ *and* $(2..2) = [2..2) = (2..2] = [2..1) = \varnothing$.

DEFINITION 11.14. $\mathbb{N} := [1..\infty)$, $\quad \mathbb{N}_0 := [0..\infty)$, $\mathbb{N}^* := [1..\infty]$, $\quad \mathbb{N}_0^* := [0..\infty]$.

DEFINITION 11.15. *Let A and B be sets. Then we define* $\quad A \backslash B \;:=\; \{x \in A \,|\, x \notin B\}$.

THEOREM 11.16. *Let $A := \{5, 6, 7, 8\}$ and let $B := \{7, 8, 9\}$. Then* $\quad A \backslash B \quad = \quad \{5, 6\}$.

DEFINITION 11.17. $\qquad \mathbb{Q} \;:=\; \{\; j/k \in \mathbb{R} \;|\; (\, j \in \mathbb{Z}\,) \;\&\; (\, k \in \mathbb{N}\,)\,\}$.

According to specification, between "{" and "|", we *should* have:

a single variable, then "∈" then a set .

The logic purist would therefore do some rewriting of Definition 11.17:

$$\mathbb{Q} \;:=\; \{x \in \mathbb{R} \,|\, \exists j \in \mathbb{Z}, \exists k \in \mathbb{N} \text{ s.t. } j/k = x\}.$$

Elements of $\mathbb{Q}$ are called **rational numbers**.

12. Upper bounds, lower bounds, max, min, sup and inf

DEFINITION 12.1. $\forall S \subseteq \mathbb{R}^*$, $\forall a \in \mathbb{R}^*$,

$$\begin{array}{lll}
S \leqslant a \text{ means: } & \forall x \in S,\ x \leqslant a & , \\
a \geqslant S \text{ means: } & \forall x \in S,\ a \geqslant x & , \\
a \leqslant S \text{ means: } & \forall x \in S,\ a \leqslant x & , \\
S \geqslant a \text{ means: } & \forall x \in S,\ x \geqslant a & , \\
S < a \text{ means: } & \forall x \in S,\ x < a & , \\
a > S \text{ means: } & \forall x \in S,\ a > x & , \\
a < S \text{ means: } & \forall x \in S,\ a < x & \text{and} \\
S > a \text{ means: } & \forall x \in S,\ x > a & .
\end{array}$$

DEFINITION 12.2. $\forall S \subseteq \mathbb{R}^*$,

$$\mathrm{UB}(S) := \{a \in \mathbb{R}^* \mid S \leqslant a\} \quad \text{and} \quad \mathrm{LB}(S) := \{a \in \mathbb{R}^* \mid a \leqslant S\}.$$

We also define UB(☹) := ☹.

DEFINITION 12.3. $\forall S \subseteq \mathbb{R}^*$,

$$\max(S) := \mathrm{UE}(S \cap [\mathrm{UB}(S)]) \text{ and } \min(S) := \mathrm{UE}(S \cap [\mathrm{LB}(S)]).$$

DEFINITION 12.4. $\forall S \subseteq \mathbb{R}^*$,

$$\sup(S) := \min(\mathrm{UB}(S)) \quad \text{and} \quad \inf(S) := \max(\mathrm{LB}(S)).$$

Here, "sup(S)" is read "the supremum of S". Sometimes "supremum" is abbreviated to "sup", which is read "soup". We sometimes change "min" to "least" and "UB" to "upper bound", and then "sup" becomes "least upper bound".

Also, "inf(S)" is read "the infimum of S". Sometimes "infimum" is abbreviated to "inf", which is read as written. We sometimes change "max" to "greatest" and "LB" to "lower bound", and then "inf" becomes "greatest lower bound". Some examples:

S	LB	UB	min	max	inf	sup
$\{5\}$	$[-\infty;5]$	$[5;\infty]$	5	5	5	5
$[0;1]$	$[\infty;0]$	$[1;\infty]$	0	1	0	1
$(0;1)$	$[\infty;0]$	$[1;\infty]$	☹	☹	0	1
$[0;1)$	$[\infty;0]$	$[1;\infty]$	0	☹	0	1
$(0;1]$	$[\infty;0]$	$[1;\infty]$	☹	1	0	1
$\{0,1\}$	$[\infty;0]$	$[1;\infty]$	0	1	0	1
$\mathbb{R}^*$	$\{-\infty\}$	$\{\infty\}$	$-\infty$	∞	$-\infty$	∞
$\mathbb{R}$	$\{-\infty\}$	$\{\infty\}$	☹	☹	$-\infty$	∞
$\varnothing$	$\mathbb{R}^*$	$\mathbb{R}^*$	☹	☹	∞	$-\infty$

Up to this point, all of our axioms about $\mathbb{R}$ would be equally true about $\mathbb{Q}$. There is, however, a significant problem with trying to do real analysis using only rational numbers: Let $S := \{x \in \mathbb{Q} \,|\, x^2 \leqslant 2\}$. It turns out that the supremum of S is *NOT* a rational number, so, working over $\mathbb{Q}$ has the disadvantage that not every subset of $\mathbb{Q}$ has its supremum in $\mathbb{Q}$. Since $\sup \mathbb{R} = \infty$, we are aslo forced into working in $\mathbb{R}^*$ if we want to guarantee infima and suprema. And we do! The next axiom is called **completeness of the extended reals**:

AXIOM 12.5. $\forall S \subseteq \mathbb{R}^*, \quad \inf(S) \neq ☹ \neq \sup(S)$.

THEOREM 12.6. $\forall a \in [0;\infty)$, $\exists 1 r \in [0;\infty)$ *s.t.* $r^2 = a$.

More formally, Theorem 12.6 would be written:

$$\begin{array}{l} (\forall a)\,(\,[\,a \in [0;\infty)\,]\ \Rightarrow\ [\\ \quad (\ (\exists r)\,(\,[r \in [0;\infty)\,]\,\&\,[\,r^2 = a\,]\,)\ )\ \&\\ \quad (\ (\forall r)\,(\,(\forall s)\,(\\ \qquad (\quad [\,r \in [0;\infty)\,]\,\&\,[\,s \in [0;\infty)\,]\,\&\\ \qquad\qquad [\,r^2 = a\,]\,\&\,[\,s^2 = a\,]\quad)\\ \qquad\quad \Rightarrow\quad (\quad r = s\quad)\\ \qquad\ )\,)\,)\\ \qquad\ ]\,)\quad . \end{array}$$

THEOREM 12.7. $\forall a \in (-\infty;0)$, $\nexists r \in \mathbb{R}$ *s.t.* $r^2 = a$.

More formally, Theorem 12.7 would be written:

$$\begin{array}{l} (\forall a)\,(\,[a \in (-\infty;0)\,]\ \Rightarrow\ [\\ \quad \neg(\,(\exists r)\,(\,[r \in \mathbb{R}]\,\&\,[r^2 = a]\,)\,)\\ \qquad\quad]\,). \end{array}$$

DEFINITION 12.8. $\forall a \in \mathbb{R}, \quad \sqrt{a} := \mathrm{UE}\{r \in [0;\infty) \,|\, r^2 = a\}$.

By Theorem 12.6, $\forall a \in [0;\infty)$, $\sqrt{a} \in [0;\infty)$. On the other hand, by Theorem 12.7, $\forall a \in (-\infty;0)$, $\sqrt{a} = ☹$.

In this course, when we write $a \mathbin{^\frown\!\!\in} S$, we mean: $(\,a = ☹\,) \vee (\,a \in S\,)$.

THEOREM 12.9. $\forall S \subseteq \mathbb{R}^*, \quad (\,\min S \mathbin{^\frown\!\!\in} S\,)\,\&\,(\,\max S \mathbin{^\frown\!\!\in} S\,)$.

In this course, when we write $a \mathbin{^\frown\!\!<} b$, we mean: $(\,a = ☹\,) \vee (\,a < b\,)$.
In this course, when we write $a \mathbin{^\frown\!\!>} b$, we mean: $(\,a = ☹\,) \vee (\,a > b\,)$.
In this course, when we write $a \mathbin{^\frown\!\!\leqslant} b$, we mean: $(\,a = ☹\,) \vee (\,a \leqslant b\,)$.
In this course, when we write $a \mathbin{^\frown\!\!\geqslant} b$, we mean: $(\,a = ☹\,) \vee (\,a \geqslant b\,)$.

In this course, when we write $a \mathrel{<^\frown} b$, we mean: $(\,a < b\,) \vee (\,b = ☹\,)$.
In this course, when we write $a \mathrel{>^\frown} b$, we mean: $(\,a > b\,) \vee (\,b = ☹\,)$.
In this course, when we write $a \mathrel{\leqslant^\frown} b$, we mean: $(\,a \leqslant b\,) \vee (\,b = ☹\,)$.
In this course, when we write $a \mathrel{\geqslant^\frown} b$, we mean: $(\,a \geqslant b\,) \vee (\,b = ☹\,)$.

THEOREM 12.10. $\forall S \subseteq \mathbb{R}^*, \quad (\min S \mathrel{^\frown\!\leqslant} S\,) \,\&\, (\max S \mathrel{^\frown\!\geqslant} S\,)$.

THEOREM 12.11. $\forall S \subseteq \mathbb{R}^*, \quad (\inf S \leqslant S\,) \,\&\, (\sup S \geqslant S\,)$.

AXIOM 12.12. $\sup \mathbb{N} = \infty$.

13. Unassigned homework

THEOREM 13.1. $\forall S \subseteq \mathbb{R}^*, \quad [\,(\,S > 0\,) \Rightarrow (\,\min S \mathrel{^\frown\!>} 0)\,)\,]$.

THEOREM 13.2. $\forall S \subseteq \mathbb{R}^*, \forall x \in S, \quad [\,(\,x \leqslant S\,) \Rightarrow (\,\min S = x\,)\,]$.

THEOREM 13.3. $\forall a, b \in \mathbb{R}^*, [\,(\min\{a, b\} = a) \vee (\min\{a, b\} = b)\,]$.

THEOREM 13.4. $\forall a > 0, \forall b > 0, \quad \min\{a, b\} > 0$.

THEOREM 13.5. $\forall a, b \in \mathbb{R}^*, [\,(\,\min\{a, b\} \leqslant a\,) \,\&\, (\,\min\{a, b\} \leqslant b\,)\,]$.

THEOREM 13.6. $\forall \varepsilon > 0, \forall a > 0, \quad \varepsilon/a > 0$.

THEOREM 13.7. $\forall x > 0, \quad \sqrt{x} > 0$.

THEOREM 13.8. $\forall a, b \in \mathbb{R}, \quad [\,(\,0 \leqslant a \leqslant b\,) \Rightarrow (\,a^2 \leqslant b^2\,)\,]$.

We cannot square the inequality $-2 < -1$; in fact, $(-2)^2 > (-1)^2$. So, in Theorem 13.8, the assumption that $0 \leqslant a$ is important.

THEOREM 13.9. $\forall a, b, c, d \in \mathbb{R}$,
$$(\,[\,a < b\,] \,\&\, [\,c < d\,]\,) \Rightarrow (\,a + c < b + d\,).$$

THEOREM 13.10. $\forall s > 0, \quad (\sqrt{s})^2 = s$.

THEOREM 13.11. $\forall \varepsilon \in \mathbb{R}, \quad (\varepsilon/2) + (\varepsilon/2) = \varepsilon$.

THEOREM 13.12. $1^2 + 1 < 100 < 1000$.

THEOREM 13.13. $0.00001^2 + 0.00001 < 0.001$.

14. A doubly quantified theorem, redux

We next discuss the art of proof-writing, with a focus on proving Theorem 6.1. In writing proofs, the most common mistake made by students in this course is failure to follow the **Cardinal Binding Rule**:

You must **bind** a variable before you use it.

In any proof, *ANY* time you use a variable, you *MUST* be able to tell me where you did the binding of that variable, and that binding must happen *before* the variable is used. Otherwise, you lose some credit. There is only one exception to this rule, see §24. Also tricky: Some bindings are temporary, and only last until the end of the clause in which they appear. For example, suppose, in a homework, I see

$$(\,\forall x \in S,\, x > 3\,)\,\&\,(\,x+5 \text{ is an integer}\,) \qquad .$$

Then the binding on x expires before "$x + 5$ is an integer", and the student will lose some credit. By contrast, if I see

$$\forall x \in S,\, [\,(\,x > 3\,)\,\&\,(\,x+5 \text{ is an integer}\,)\,] \qquad ,$$

then the binding continues to "]", so there is no problem.

For the logic purist, each time "$(\forall x)(\ldots\ldots)$" or "$(\exists x)(\ldots\ldots)$" appears, the binding of the variable x continues inside "$(\ldots\ldots)$". Immediately after ")", that binding expires. This is a straightforward rule. In less formal ("pidgin") mathematical writing, to follow the Cardinal Binding Rule, it helps to know how to formalize pidgin statements, to determine where their clauses begin and end.

The past participle of "to bind" is "bound"; it is NOT "bounded". After you bind a variable, it becomes **bound**, NOT bounded. Confusion arises because the verb "to bound" is also used frequently in mathematics, and the past participle of "to bound" is "bounded". After you bound a variable, it becomes bounded. Within this section, we will bound no variables; we only bind them. So, in this section, no variables become bounded; they become bound.

Free is the opposite of bound. To say that a variable is **free** is to say that it is not bound. Read everything in the Exposition Handout (**EH**) up to, but not including, (7) on pp. 1–3. This describes how to tell if a given variable is free or bound. Recall Theorem 6.1, which we restate:

THEOREM 14.1. *$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\delta^2 + \delta \leqslant \varepsilon$.*

Recall, from §6, that Theorem 14.1 is doubly quantified: There is one "$\forall$" quantifier, and one "$\exists$" quantifier.

In this section, we explain how to write a proof of Theorem 14.1. First, observe that, in Theorem 14.1,

the quantifier "$\forall$" binds the variable "ε" and
the quantifier "$\exists$" binds the variable "δ",

but both of these bindings are temporary, and they expire at the end of the sentence. So, as we begin our proof, there are NO bound variables. We therefore cannot use any variables, until some binding happens. Now read (7)–(12) on the EH, pp. 3–4.

At the start of our proof, we will implement Template (10) on p. 4 of the EH. Following it, we write:

Given $\varepsilon > 0$.
Want: $\exists \delta > 0$ s.t. $\delta^2 + \delta \leqslant \varepsilon$.

At this point, the variable "ε" is bound until the end of the proof, and that is the only bound variable. The variable "δ" was temporarily bound, but that binding expired at the end of its sentence.

We next implement Template (11) on p. 4 of the EH. Following it, we leave a blank space, and keep in mind that, somewhere in that blank space, a line must eventually appear that binds the variable δ, and, moreover, it is important that $\delta > 0$. We will refer to this blank space as our "δ-strategy". After this blank space, we write:

Want: $\delta^2 + \delta \leqslant \varepsilon$.

Then we leave a blank space for the remainder of the proof, followed by a small rectangular box. We will call this second blank space the "finish". At this point we have finished **structuring the proof**, and the proof has the following apperance:

Proof. Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t. $\delta^2 + \delta \leqslant \varepsilon$.
BLANK SPACE FOR δ-strategy.
Want: $\delta^2 + \delta \leqslant \varepsilon$.
BLANK SPACE FOR finish. □

For a proof of a doubly quantified theorem, if you can even structure the proof correctly, then you should receive substantial credit, typically about one third of the available points. The structuring of a proof is straightforward: You just untangle the quantifiers, carefully following templates (10)–(12) on p. 4 of the EH, leaving blank spaces as needed.

The hard part comes next: We must fill in the blanks, which typically requires that you both understand the proof as a game and know a winning strategy. It also requires that you *communicate* that strategy, following all of the rules in the Exposition Handout (EH).

In the case of Theorem 14.1, recall the strategy from §6:

Let $\delta := \min\{\varepsilon/2, \sqrt{\varepsilon/2}\}$.

Because $\varepsilon > 0$, by Theorem 13.6 and Theorem 13.7, it follows that both $\varepsilon/2$ and $\sqrt{\varepsilon/2}$ are positive. Then, by Theorem 13.4, $\delta > 0$. So our δ-stragtegy could be expressed as follows:

Let $\delta := \min\{\varepsilon/2, \sqrt{\varepsilon/2}\}$. Then $\delta > 0$.

All we have left is the finish.

Read (24) on p. 8. We cannot stop until we *KNOW* that $\delta^2 + \delta \leqslant \varepsilon$. Read (25) on p. 8. We *MUST* stop once we know that $\delta^2 + \delta \leqslant \varepsilon$.

Following Theorem 13.5, because $\delta = \min\{\varepsilon/2, \sqrt{\varepsilon/2}\}$, we consider the inequalities $\delta \leqslant \sqrt{\varepsilon/2}$ and $\delta \leqslant \varepsilon/2$ to be obvious. Then, by Theorem 13.10, $(\sqrt{\varepsilon/2})^2 = \varepsilon/2$. Finally, by Theorem 13.11, $(\varepsilon/2) + (\varepsilon/2) = \varepsilon$. So, knowing Theorem 13.8 and Theorem 13.9, the finish might read:

$0 \leqslant \delta \leqslant \sqrt{\varepsilon/2}$, so $\delta^2 \leqslant \varepsilon/2$.
$\delta \leqslant \varepsilon/2$ and $\delta^2 \leqslant \varepsilon/2$, so $\delta + \delta^2 \leqslant \varepsilon$.

By (24) and (25) of the EH, we must stop writing because what we know matches what we want. The full proof now reads:

Proof. Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t. $\delta^2 + \delta \leqslant \varepsilon$.
Let $\delta := \min\{\varepsilon/2, \sqrt{\varepsilon/2}\}$. Then $\delta > 0$.
Want: $\delta^2 + \delta \leqslant \varepsilon$.
$0 \leqslant \delta \leqslant \sqrt{\varepsilon/2}$, so $\delta^2 \leqslant \varepsilon/2$.
$\delta \leqslant \varepsilon/2$ and $\delta^2 \leqslant \varepsilon/2$, so $\delta + \delta^2 \leqslant \varepsilon$. □

Question for discussion: Suppose, in some proof, a student shows that $\delta > 0$, and, somewhere after that, writes

$\delta \leqslant \sqrt{\varepsilon/2}$, so $\delta^2 \leqslant \varepsilon/2$

instead of

$0 \leqslant \delta \leqslant \sqrt{\varepsilon/2}$, so $\delta^2 \leqslant \varepsilon/2$.

Is this bad style? Should it result in a loss of credit? I say yes. Recall: We cannot square the inequality $-2 < -1$; in fact, $(-2)^2 > (-1)^2$.

For some students, it may be tempting to replace

$0 \leqslant \delta \leqslant \sqrt{\varepsilon/2}$, so $\delta^2 \leqslant \varepsilon/2$

by

$$0 \leqslant \delta \leqslant \sqrt{\varepsilon/2} \quad \Rightarrow \quad \delta^2 \leqslant \varepsilon/2 \qquad .$$

However, this is bad style, and would lead to a loss of credit. The problem is that we *KNOW* that $0 \leqslant \delta \leqslant \sqrt{\varepsilon/2}$, so this statement does not belong on the left of $\Rightarrow$. When we say $A \Rightarrow B$, the understood meaning is "I am not sure if A is true, but if it should turn out to be true, then B must be true as well." Typically, someone who knows for sure that it is raining outside would not say: "If it is raining outside, then I will need my umbrella." Instead, they would say: "It is raining outside, so I will need my umbrella." For more explanation of this, read (26) on pp. 8–9 of the EH. I accept "$\because$" as an abbreviation for "therefore" or "so". Consequently, if you wish, you may replace

$$0 \leqslant \delta \leqslant \sqrt{\varepsilon/2}, \text{ so } \delta^2 \leqslant \varepsilon/2$$

by

$$0 \leqslant \delta \leqslant \sqrt{\varepsilon/2} \quad \therefore \quad \delta^2 \leqslant \varepsilon/2 \qquad .$$

15. Three subtleties in mathematical logic

First, we discuss **null true** statements. Let $\mathbf{P}$ and $\mathbf{Q}$ be formal statements, and suppose we are, for some reason, interested in proving that $\mathbf{P} \Rightarrow \mathbf{Q}$. The rules of inference are set up in such a way that, if we can prove $\neg\mathbf{P}$, then $\mathbf{P} \Rightarrow \mathbf{Q}$ follows. One sometimes expresses this by saying that $\mathbf{P} \Rightarrow \mathbf{Q}$ is "null true", because $\mathbf{P}$ is false. Example:

THEOREM 15.1. $(3 \neq 3) \;\Rightarrow\; (1 = 2)$.

Read (13), p. 7 of the Exposition Handout, on proof by contradiction.

Proof. Assume $3 \neq 3$. Want: $1 = 2$.
Assume $1 \neq 2$. Want: Contradiction.
$3 \neq 3$. By Axiom 1.1, $3 = 3$. Contradiction. □

Keep in mind:
any false statement implies every statement, true or false.
So be careful what you believe in!

Second, we apply null truth to the empty set $\varnothing$, obtaining **void true** statements. For example, the two statements

$$\forall u \in \varnothing, \qquad u = 9$$

and

$$\forall u \in \varnothing, \qquad u \neq 9$$

are *both* true. In fact, for any formal statement $\mathbf{P}$, if u is the only free variable in $\mathbf{P}$, then

$$\forall u \in \varnothing, \qquad \mathbf{P}$$

is a theorem. For example:

THEOREM 15.2. $\forall u \in \varnothing$, $u = 9$.

Proof. Given $u \in \varnothing$. Want: $u = 9$.
Assume $u \neq 9$. Want: Contradiction.
$u \in \varnothing$. By Axiom 2.7, $u \notin \varnothing$. Contradiction. □

THEOREM 15.3. $\forall$*set* A, $\varnothing \subseteq A$.

Proof. Given a set A. Want: $\varnothing \subseteq A$.
Want: $\forall x \in \varnothing$, $x \in A$.
Given $x \in \varnothing$. Want: $x \in A$.
Assume $x \notin A$. Want: Contradiction.
$x \in \varnothing$. By Axiom 2.7, $x \notin \varnothing$. Contradiction. □

The following is Theorem 9.3:

THEOREM 15.4. $\varnothing$ *is a set of sets.*

Proof. Know: $\varnothing$ is a set.
Want: $\forall A \in \varnothing$, A is a set.
Given $A \in \varnothing$. Want: A is a set.
Assume A is not a set. Want: Contradiction.
$A \in \varnothing$. By Axiom 2.7, $A \notin \varnothing$. Contradiction. □

The following is Theorem 9.8:

THEOREM 15.5. $\bigcup \varnothing = \varnothing$.

Proof. By Theorem 15.3, $\varnothing \subseteq \bigcup \varnothing$.
So, by the Axiom of Extensionality (Axiom 2.3),
Want: $\bigcup \varnothing \subseteq \varnothing$.
Want: $\forall x \in \bigcup \varnothing$, $x \in \varnothing$.
Given $x \in \bigcup \varnothing$. Want: $x \in \varnothing$.
Assume $x \notin \varnothing$. Want: Contradiction.
Since $x \in \bigcup \varnothing$, choose $A \in \varnothing$ s.t. $x \in A$.
Then $A \in \varnothing$. By Axiom 2.7, $A \notin \varnothing$. Contradiction. □

Third, the **inclusive or**. When Hamlet says, “To be or not to be”, it is understood that a choice must be made. Hamlet cannot decide both both to “be” and “not be” at the same time. However, this is *not* how “or” is used in mathematics. The rules of inference are set up in such a way that, for any two formal statements **P** and **Q**, if both **P** and **Q** are known, then $\mathbf{P} \vee \mathbf{Q}$ is known. So for eample, we have:

THEOREM 15.6. *$\forall x \in \mathbb{R}^*$,* $[\,(\,x \leqslant 0\,) \vee (\,x \geqslant 0\,)\,]$.

Read (16)–(17) on p. 6 of the Exposition Handout (EH).

Proof. Given $x \in \mathbb{R}^*$. Want: $(\,x \leqslant 0\,) \vee (\,x \geqslant 0\,)$.
By Axiom 10.22, one of the following is true:

(1) $x < 0$,
(2) $x = 0$ or
(3) $x > 0$.

Case (1):
Since $x < 0$, $x \leqslant 0$, so $[\,(\,x \leqslant 0\,) \vee (\,x \geqslant 0\,)\,]$.
End of Case (1).

Case (2):
Since $x = 0$, $x \leqslant 0$, so $[\,(\,x \leqslant 0\,) \vee (\,x \geqslant 0\,)\,]$.
End of Case (2).

Case (3):
Since $x > 0$, $x \geqslant 0$, so $[\,(\,x \leqslant 0\,) \vee (\,x \geqslant 0\,)\,]$.
End of Case (3). □

16. Unassigned homework

THEOREM 16.1. *$\forall x, y, z \in \mathbb{R}^*$,* $(x \leqslant y \leqslant z) \Rightarrow (x \leqslant z)$.

THEOREM 16.2. *$\forall a, b, c \in \mathbb{R}$,* $[\,(\,0 \leqslant a \leqslant b \leqslant c\,) \Rightarrow (\,a^2 \leqslant c^2\,)\,]$.

17. A triply quantified theorem with implication

In this section, we explain how to write a proof of:

THEOREM 17.1. *$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,*

$$[\ 0 \leqslant x \leqslant \delta\ ] \quad \Rightarrow \quad [\ x^2 + x \leqslant \varepsilon\ ].$$

There are two "$\forall$" quantifiers, one "$\exists$" quantifier and one "$\Rightarrow$", so we describe Theorem 17.1 as: triply quantified with implication.

We begin by structuring the proof, using

(10) on p. 4, for "$\forall \varepsilon > 0, \ldots\ldots$", then
(11) on p. 4, for "$\exists \delta > 0$ s.t. $\ldots\ldots$", then

(10) on p. 4, for "$\forall x \in \mathbb{R}, \ldots\ldots$", then
(12) on p. 4, for "$[\cdots\cdots] \Rightarrow [\cdots\cdots]$" .

This yields:

Proof. Given $\varepsilon > 0$.
Want: $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$, $(\,[\,0 \leqslant x \leqslant \delta\,] \Rightarrow [\,x^2 + x \leqslant \varepsilon\,]\,)$.
BLANK SPACE FOR δ-strategy.
Want: $\forall x \in \mathbb{R}$, $(\,[\,0 \leqslant x \leqslant \delta\,] \Rightarrow [\,x^2 + x \leqslant \varepsilon\,]\,)$.
Given $x \in \mathbb{R}$. Want: $[\,0 \leqslant x \leqslant \delta\,] \Rightarrow [\,x^2 + x \leqslant \varepsilon\,]$.
Assume: $0 \leqslant x \leqslant \delta$. Want: $x^2 + x \leqslant \varepsilon$.
BLANK SPACE FOR finish. □

The first blank area is for our "δ-strategy", within which δ must become bound, satisfying $\delta > 0$. The second blank area is for our "finish". In this second blank area, we must show that $x^2 + x \leqslant \varepsilon$. Also, once we have proven $x^2 + x \leqslant \varepsilon$, we *MUST* immediately STOP.

Theorem 17.1 is triply quantified with implication, and, for the structuring of a proof of that kind of statement, I would typically give half credit. This is a good deal, so learn the structuring process. In particular, learn p. 4 of the Exposition Handout.

To go further, it helps to turn Theorem 17.1 into a game:

> You move first: You choose a real $\varepsilon > 0$, and reveal it to me.
> My move: I choose $\delta > 0$, and reveal it to you.
> Your move: You choose $x \in \mathbb{R}$, and reveal it to me.
> We check to see if $[0 \leqslant x \leqslant \delta] \Rightarrow [x^2 + x \leqslant \varepsilon]$.
> If so, then I win.
> If not, then you win.

Remember that, if you choose x so that $\neg[0 \leqslant x \leqslant \delta]$, then the implication

$$[0 \leqslant x \leqslant \delta] \Rightarrow [x^2 + x \leqslant \varepsilon]$$

is "null true", and so I will win. So you are effectively forced to choose x satisfying $0 \leqslant x \leqslant \delta$. if you want to have any hope of winning. For that reason, it is common to revise the game, and make it part of the rules that your choice of j must satisfy $j \geqslant K$. This revised game reads:

> You move first: You choose a real $\varepsilon > 0$, and reveal it to me.
> My move: I choose $\delta > 0$, and reveal it to you.

You move: You choose a $x \in \mathbb{R}$ s.t. $0 \leqslant x \leqslant \delta$, and reveal it to me.

We check to see if $x^2 + x \leqslant \varepsilon$.

If so, then I win.

If not, then you win.

Let's play. Say you choose $\varepsilon = 100$. I will laugh at your poor play, and choose $\delta = 3$. You choose, say, $x = 1$. Since $1^2+1 \leqslant 100$, I win. We play again. You try $\varepsilon = 1000$. I laugh even harder, and choose $\delta = 3$ again. Maybe this time, you try $x = 2$. Since $2^2 + 2 \leqslant 100$, I win. You begin to see that making ε large is not in your interest. However, by the rules, you cannot make it negative or zero. You try $\varepsilon = 0.001$. Now I have to concentrate. I choose $\delta = 0.00001$. You begin to understand that your goal, in choosing x, is to make x as large as possible, so that $x^2 + x$ will be large. However, you face a constraint: You are required to choose x so that $0 \leqslant x \leqslant 0.00001$. So your best move is $x = 0.00001$. Since $0.00001^2 + 0.00001 \leqslant 0.001$, I win. You begin to think the game is rigged. Saying that the game is rigged against you is the same as saying that you believe that Theorem 17.1 is true. Belief is the first step in proof. Now that we believe in Theorem 17.1, we need a specific strategy to win. It is not enough to say, "Well, just make sure the δ is really small". We have to come up with a specific method for choosing δ after we know ε.

Sometimes, it helps to focus first on the finish, in order to see what is needed in the δ-strategy. We wish to force

$$x^2 + x \leqslant \varepsilon \quad .$$

We break the problem down term-by-term. That is, work separately on the first term x^2 and the second term x. If we can force

$$x^2 \leqslant \varepsilon/2 \text{ and } x \leqslant \varepsilon/2 \quad .$$

then we will win the game. It is therefore enough to force

$$0 \leqslant x \leqslant \sqrt{\varepsilon/2} \text{ and } x \leqslant \varepsilon/2 \quad .$$

So, since $0 \leqslant x \leqslant \delta$, we can win by forcing

$$\delta \leqslant \sqrt{\varepsilon/2} \text{ and } \delta \leqslant \varepsilon/2 \quad .$$

This leads us to the same δ-strategy as for Theorem 14.1:

Let $\delta := \min\{\varepsilon/2, \sqrt{\varepsilon/2}\}$. Then $\delta > 0$.

For the finish, by Theorem 16.1 and Theorem 16.2, we could write:

$x \leqslant \delta \leqslant \varepsilon/2$, so $x \leqslant \varepsilon/2$.

$0 \leqslant x \leqslant \delta \leqslant \sqrt{\varepsilon/2}$, so $x^2 \leqslant \varepsilon/2$.
$x^2 \leqslant \varepsilon/2$ and $x \leqslant \varepsilon/2$, so $x^2 + x \leqslant \varepsilon$.

Here, then, is the full proof:

Proof. Given $\varepsilon > 0$.
Want: $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$, $(\ [\,0 \leqslant x \leqslant \delta\,] \Rightarrow [\,x^2 + x \leqslant \varepsilon\,]\ )$.
Let $\delta := \min\{\varepsilon/2, \sqrt{\varepsilon/2}\}$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{R}$, $(\ [\,0 \leqslant x \leqslant \delta\,] \Rightarrow [\,x^2 + x \leqslant \varepsilon\,]\ )$.
Given $x \in \mathbb{R}$. Want: $[\ 0 \leqslant x \leqslant \delta\ ] \Rightarrow [\ x^2 + x \leqslant \varepsilon\ ]$.
Assume: $0 \leqslant x \leqslant \delta$. Want: $x^2 + x \leqslant \varepsilon$.
$x \leqslant \delta \leqslant \varepsilon/2$, so $x \leqslant \varepsilon/2$.
$0 \leqslant x \leqslant \delta \leqslant \sqrt{\varepsilon/2}$, so $x^2 \leqslant \varepsilon/2$.
$x^2 \leqslant \varepsilon/2$ and $x \leqslant \varepsilon/2$, so $x^2 + x \leqslant \varepsilon$. □

DEFINITION 17.2. *For all $s \in \mathbb{R}$, we define $|x| := \max\{x, -x\}$.*

Let $x \in \mathbb{R}$. Then $|x|$ is called the **absolute value** of x.

THEOREM 17.3. $|3| = 3$ *and* $|-6| = 6$.

THEOREM 17.4. $\forall x \in \mathbb{R}$, $|x| \geqslant 0$.

THEOREM 17.5. $|(-2) + 3| \neq |-2| + |3|$.

THEOREM 17.6. *All of the following are true:*

(1) $\forall x \in \mathbb{R}$, $|x| \geqslant 0$.
(2) $\forall x, y \in \mathbb{R}$, $|x \cdot y| = |x| \cdot |y|$.
(3) $\forall x \in \mathbb{R}$, $|x^2| = |x|^2$.
(4) $\forall x, y \in \mathbb{R}$, $|x + y| \leqslant |x| + |y|$.

THEOREM 17.7. $\forall \varepsilon > 0$, $\exists \delta > 0$ *s.t.*, $\forall x \in \mathbb{R}$,

$$[\ ||x < \delta\ ] \quad \Rightarrow \quad [\ |x^2 + x| < \varepsilon\ ].$$

Proof. Given $\varepsilon > 0$.
Want: $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$, $(\ [\,|x| < \delta\,] \Rightarrow [\,|x^2 + x| < \varepsilon\,]\ )$.
Let $\delta := \min\{\varepsilon/2, \sqrt{\varepsilon/2}\}$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{R}$, $(\ [\,|x| < \delta\,] \Rightarrow [\,|x^2 + x| < \varepsilon\,]\ )$.
Given $x \in \mathbb{R}$. Want: $[\ |x| < \delta\ ] \Rightarrow [\ |x^2 + x| < \varepsilon\ ]$.
Assume: $|x| < \delta$. Want: $|x^2 + x| < \varepsilon$.
$|x| < \delta \leqslant \varepsilon/2$, so $|x| < \varepsilon/2$.

$0 \leqslant |x| < \delta \leqslant \sqrt{\varepsilon/2}$, so $0 \leqslant |x| < \sqrt{\varepsilon/2}$, so $|x|^2 < \varepsilon/2$.
Then

$$\begin{aligned} |x^2 + x| &\leqslant |x^2| + |x| = |x|^2 + |x| \\ &< (\varepsilon/2) + (\varepsilon/2) = \varepsilon, \end{aligned}$$

as desired. □

18. Unassigned homework

THEOREM 18.1. *Let $S \subseteq \mathbb{R}^*$ and let $x \in \mathbb{R}^*$.*
Assume $x < \sup S$. Then $\neg(S \leqslant x)$.

THEOREM 18.2. *Let $S \subseteq \mathbb{R}^*$ and let $x \in \mathbb{R}^*$.*
Assume $\neg(S \leqslant x)$. Then $\exists y \in S$ s.t. $y > x$.

19. The Archimedean Principle

The next theorem is called the **Archimedean Principle**.

THEOREM 19.1. $\forall x \in \mathbb{R}$, $\exists k \in \mathbb{N}$ *s.t.* $x < k$.

Proof. Given $x \in \mathbb{R}$. Want: $\exists k \in \mathbb{N}$ s.t. $x < k$.
By Axiom 12.12, $\sup \mathbb{N} = \infty$.
Since $x \in \mathbb{R}$, by Axiom 10.11, $x < \infty$.
Then $x < \sup \mathbb{N}$.
Then $\neg(\mathbb{N} \leqslant x)$.
Choose $k \in \mathbb{N}$ s.t. $k > x$.
Want: $x < k$.
Since $k > x$, we conclude that $x < k$, as desired. □

20. Arithmetic of sets of real numbers

DEFINITION 20.1. $\forall S \subseteq \mathbb{R}$, $-S := \{-x \in \mathbb{R} \,|\, x \in S\}$.

DEFINITION 20.2. $\forall S \subseteq \mathbb{R}$, $\forall a \in \mathbb{R}$,
$S + a := \{x + a \in \mathbb{R} \,|\, x \in S\}$, $S - a := \{x - a \in \mathbb{R} \,|\, x \in S\}$,
$a + S := \{a + x \in \mathbb{R} \,|\, x \in S\}$, $a - S := \{a - x \in \mathbb{R} \,|\, x \in S\}$,
$a \cdot S := \{ax \in \mathbb{R} \,|\, x \in S\}$ *and* $S \cdot a := \{xa \in \mathbb{R} \,|\, x \in S\}$.
Also, $\forall S \subseteq \mathbb{R}\backslash\{0\}$, $\forall a \in \mathbb{R}$, $a/S := \{a/x \in \mathbb{R} \,|\, x \in S\}$.
Also, $\forall S \subseteq \mathbb{R}$, $\forall a \in \mathbb{R}\backslash\{0\}$, $S/a := \{x/a \in \mathbb{R} \,|\, x \in S\}$.

THEOREM 20.3. $\forall S \subseteq \mathbb{R}$, $\forall a \in \mathbb{R}$, $a + S = S + a$.

THEOREM 20.4. *Let* $S := (0;1]$. *Then*

$$-S = [-1;0),$$
$$S+3 = 3+S = (3;4],$$
$$S-4 = (-4;-3],$$
$$4-S = [3;4),$$
$$6S = S\cdot 6 = (0;6],$$
$$1/S = [1;\infty) \qquad \textit{and}$$
$$S/5 = (0;1/5].$$

21. Primitive ordered pairs, relations and functions

DEFINITION 21.1. $\forall x,y$, $\langle\langle x,y\rangle\rangle := \{\{x\},\{x,y\}\}$.

THEOREM 21.2. $\langle\langle 1,2\rangle\rangle = \{\{1\},\{1,2\}\}$.

THEOREM 21.3. $\langle\langle 2,1\rangle\rangle = \{\{2\},\{1,2\}\} \neq \{\{1\},\{1,2\}\} = \langle\langle 1,2\rangle\rangle$.

THEOREM 21.4. $\forall x,y, \qquad \{x,y\} = \{y,x\}$.

THEOREM 21.5. $\langle\langle 3,3\rangle\rangle = \{\{3\},\{3,3\}\} = \{\{3\},\{3\}\} = \{\{3\}\}$.

THEOREM 21.6. $\langle\langle 5,☹\rangle\rangle = \{\{5\},\{5,☹\}\} = \{\{5\},☹\} = ☹$.

THEOREM 21.7. $\forall a, \quad \langle\langle a,☹\rangle\rangle = ☹ = \langle\langle ☹,a\rangle\rangle$.

THEOREM 21.8. $\forall\smile a, \forall\smile b, \forall\smile c, \forall\smile d$,

$$(\ \langle\langle a,b\rangle\rangle = \langle\langle c,d\rangle\rangle\ ) \ \Leftrightarrow\ (\ [\,a=c\,]\,\&\,[\,b=d\,]\ ) \qquad .$$

DEFINITION 21.9. $\forall q$, *by* q *is a* **primitive ordered pair**, *we mean:* $\exists\smile x$, $\exists\smile y$ *s.t.* $q = \langle\langle x,y\rangle\rangle$.

AXIOM 21.10. $\forall$*sets* A,B, $\exists 1$*set* C, *denoted* $A\times\!\!\times B$, *s.t.*, $\forall z$, $\qquad [\ (\ z \in C\ ) \quad \Leftrightarrow \quad (\ \exists x\in A, \exists y \in B$ *s.t.* $z = \langle\langle x,y\rangle\rangle\ )\ ]$.

Well refer to the set $A\times\!\!\times B$ of Definition 21.10 as the **primitive product** of A and B. A set of primitive ordered pairs is called a relation:

DEFINITION 21.11. $\forall R$, *by* R *is a* **relation**, *we mean:*

R is a set $\qquad$ *and*
$\forall q \in R$, $\quad$ *q is a primitive ordered pair.*

THEOREM 21.12. *Let* $R := \{\,\langle\langle 5,2\rangle\rangle, \langle\langle 5,9\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle\,\}$.
Then R is a relation.

THEOREM 21.13. *$\varnothing$ is a relation.*

AXIOM 21.14. *Let R be a relation. Then $\exists 1$set A, denoted* $\mathrm{dom}[R]$, *s.t.,* $\forall x, [\,(\,x \in A\,) \Leftrightarrow (\,\exists y$ *s.t.* $\langle\langle x,y\rangle\rangle \in R\,)\,]$.

AXIOM 21.15. *Let R be a relation. Then $\exists 1$set B, denoted* $\mathrm{im}[R]$, *s.t.,* $\forall y, [\,(\,y \in B\,) \Leftrightarrow (\,\exists x$ *s.t.* $\langle\langle x,y\rangle\rangle \in R\,)\,]$.

THEOREM 21.16. *Let* $R := \{\langle\langle 5,2\rangle\rangle, \langle\langle 5,9\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle\}$. *Then* $\mathrm{dom}[R] = \{5,7,8\}$.

THEOREM 21.17. *Let* $R := \{\langle\langle 5,2\rangle\rangle, \langle\langle 5,9\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle\}$. *Then* $\mathrm{im}[R] = \{2,4,6,9\}$.

DEFINITION 21.18. *Let R be a relation,*

$$A := \mathrm{dom}[R],\ B := \mathrm{im}[R].$$

Then $R^{-1} \;:=\; \{\,\langle\langle y,x\rangle\rangle \in B \times\!\!\times A \mid \langle\langle x,y\rangle\rangle \in R\,\}$.

According to specification, between "{" and "|", we *should* have:

a single variable, then "$\in$" then a set .

The logic purist would therefore do some rewriting of Definition 21.18, and define R^{-1} to be

$$\{z \in B \times\!\!\times A \mid \exists x \in A, \exists y \in B \text{ s.t. } [(\langle\langle x,y\rangle\rangle \in R)\&(\langle\langle y,x\rangle\rangle = z)]\}.$$

THEOREM 21.19. *Let* $R := \{\langle\langle 5,2\rangle\rangle, \langle\langle 5,9\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle\}$. *Then* $R^{-1} = \{\langle\langle 2,5\rangle\rangle, \langle\langle 9,5\rangle\rangle, \langle\langle 6,7\rangle\rangle, \langle\langle 4,8\rangle\rangle\}$,

$\mathrm{dom}[R^{-1}] = \{2,4,6,9\} = \mathrm{im}[R]$,

$\mathrm{im}[R^{-1}] = \{5,7,8\} = \mathrm{dom}[R]$ *and*

$(R^{-1})^{-1} = \{\langle\langle 5,2\rangle\rangle, \langle\langle 5,9\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle\} = R$.

THEOREM 21.20. *Let R be a relation.*
Then: $R \subseteq (\mathrm{dom}[R]) \times\!\!\times (\mathrm{im}[R])$ *and*

$\forall x \in \mathrm{dom}[R], \exists y \in \mathrm{im}[R]$ *s.t.* $\langle\langle x,y\rangle\rangle \in R$ *and*

$\forall y \in \mathrm{im}[R], \exists x \in \mathrm{dom}[R]$ *s.t.* $\langle\langle x,y\rangle\rangle \in R$.

THEOREM 21.21. $\forall$*relation R,* $[\ (\,(R^{-1})^{-1} = R\,)\ \&\ (\,\mathrm{dom}[R^{-1}] = \mathrm{im}[R]\,)\ \&\ (\,\mathrm{im}[R^{-1}] = \mathrm{dom}[R]\,)\ ]$.

DEFINITION 21.22. *$\forall f$, by f is a* **function**, *we mean:*

(1) *f is a relation* *and*
(2) $\forall x \in \mathrm{dom}[f]$, $\forall y,z \in \mathrm{im}[f]$,

$$[\,\langle\langle x,y\rangle\rangle, \langle\langle x,z\rangle\rangle \in f\,] \;\Rightarrow\; [\,y = z\,].$$

Condition (2) in Definition 21.22 is called the **vertical line test**.

THEOREM 21.23. *Let* $R := \{ \langle\langle 5,2\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle, \langle\langle 5,9\rangle\rangle \}$.
Then R *is a not a function.*

THEOREM 21.24. *Let* $f := \{ \langle\langle 5,2\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle \}$.
Then: f *is a function,* $\mathrm{dom}[f] = \{5,7,8\}$ *and* $\mathrm{im}[f] = \{2,4,6\}$.

THEOREM 21.25. *Let* $f := \{ \langle\langle 5,2\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,6\rangle\rangle \}$.
Then: f *is a function,* $\mathrm{dom}[f] = \{5,7,8\}$ *and* $\mathrm{im}[f] = \{2,6\}$.

THEOREM 21.26. *Let* $f := \{\langle\langle x,y\rangle\rangle \in \mathbb{R} \times\!\!\times \mathbb{R} \,|\, y = x^2\}$.
Then: f *is a function,* $\mathrm{dom}[f] = \mathbb{R}$ *and* $\mathrm{im}[f] = [0;\infty)$.

THEOREM 21.27. *Let* $f := \{\langle\langle x,y\rangle\rangle \in \mathbb{R} \times\!\!\times \mathbb{R} \,|\, y = x^3\}$.
Then: f *is a function,* $\mathrm{dom}[f] = \mathbb{R}$ *and* $\mathrm{im}[f] = \mathbb{R}$.

THEOREM 21.28. *Let* $f := \varnothing$.
Then: f *is a function,* $\mathrm{dom}[f] = \varnothing$ *and* $\mathrm{im}[f] = \varnothing$.

DEFINITION 21.29. *Let* f *be a function. Then,* $\forall x$,
$$f(x) \quad := \quad \mathrm{UE}\ \{\ y \in \mathrm{im}[f] \mid \langle\langle x,y\rangle\rangle \in f\ \}.$$

We also often use f_x instead of $f(x)$:

DEFINITION 21.30. *Let* f *be a function. Then,* $\forall x$,
$$f_x \quad := \quad \mathrm{UE}\ \{\ y \in \mathrm{im}[f] \mid \langle\langle x,y\rangle\rangle \in f\ \}.$$

THEOREM 21.31. *Let* $f := \{ \langle\langle 5,2\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle \}$.
Then $f(7) = 6$, $f_8 = 4$ *and* $f(0) = ☹$.

THEOREM 21.32. *Let* $f := \{ \langle\langle 2,8\rangle\rangle, \langle\langle 3,8\rangle\rangle, \langle\langle 4,9\rangle\rangle \}$.
Then f *is a function,* $f(2) = f(3) = 8$, $f(4) = 9$ *and* $f(5) = ☹$.

THEOREM 21.33. *Let* $f := \{\langle\langle x,y\rangle\rangle \in \mathbb{R} \times\!\!\times [0;\infty) \,|\, y = x^2\}$. *Then*
$$f(3) = f(-3) = 9, \quad f_2 = 4, \quad f(0) = 0,$$
$$f(\infty) = f(-\infty) = ☹ \quad \textit{and} \quad f(☹) = ☹.$$

DEFINITION 21.34. $\forall f$,
$$[\ f \text{ not a function}\ ] \ \Rightarrow\ [\ \forall x,\ (\ (\ f(x) := ☹\ ) \,\&\, (\ f_x := ☹\ )\ )\ ].$$

THEOREM 21.35. *Let* $R := \{\langle\langle 5,2\rangle\rangle, \langle\langle 5,9\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle\}$.
Then $R_5 = R_7 = R_8 = R_1 = ☹$.
Also, $\forall x$, $R_x = ☹$.

THEOREM 21.36. *Let* $f := \{\langle\langle 2,8\rangle\rangle, \langle\langle 3,8\rangle\rangle, \langle\langle 4,9\rangle\rangle\}$*. Then*

$f^{-1} = \{\langle\langle 8,2\rangle\rangle, \langle\langle 8,3\rangle\rangle, \langle\langle 9,4\rangle\rangle\}$,

f^{-1} *is not a function,*

$f^{-1}(0) = ☹, \quad f^{-1}(8) = ☹, \quad f^{-1}(9) = ☹$ *and*

$\forall x,\ f^{-1}(x) = ☹$

Unhappiness is infective:

THEOREM 21.37. $\forall f, \qquad f(☹) = f_{☹} = ☹.$

DEFINITION 21.38. $\forall S$*, by* S *is* **set-valued***, we mean:*

$(\,S$ *is a function*$\,)$ *and* $(\,\forall j \in \mathrm{dom}[S], \quad S_j$ *is a set*$\,)$.

THEOREM 21.39. *Let* $S := \{\langle\langle 1,\{2,5\}\rangle\rangle, \langle\langle 7,\{0\}\rangle\rangle, \langle\langle 9,\varnothing\rangle\rangle\}$*. Then* S *is set-valued,* $S_1 = \{2,5\}$*,* $S_7 = \{0\}$*,* $S_9 = \varnothing$ *and* $S_2 = ☹$.

We will use the following notational convention: By $\begin{pmatrix} 5 \mapsto 2 \\ 7 \mapsto 6 \\ 8 \mapsto 4 \end{pmatrix}$, we mean the function $\{\langle\langle 5,2\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle\}$. Following this convention, then $\begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 4 \\ 3 \mapsto 0 \\ 4 \mapsto 6 \end{pmatrix}$ is the function $\{\langle\langle 1,7\rangle\rangle, \langle\langle 2,4\rangle\rangle, \langle\langle 3,0\rangle\rangle, \langle\langle 4,6\rangle\rangle\}$. Also, $\begin{pmatrix} 1 \mapsto \{2,5\} \\ 7 \mapsto \{0\} \\ 9 \mapsto \varnothing \end{pmatrix}$ is the function $\{\langle\langle 1,\{2,5\}\rangle\rangle, \langle\langle 7,\{0\}\rangle\rangle, \langle\langle 9,\varnothing\rangle\rangle\}$.

We will use the following notational convention: By $(7,4,0,6)$, we mean the function $\begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 4 \\ 3 \mapsto 0 \\ 4 \mapsto 6 \end{pmatrix}$. Following this convention, then $(3,7)$ is the function $\begin{pmatrix} 1 \mapsto 3 \\ 2 \mapsto 7 \end{pmatrix}$, which, in turn, is equal to $\{\langle\langle 1,3\rangle\rangle, \langle\langle 3,7\rangle\rangle\}$. Also, $(\{3,7\})$ is the function $\begin{pmatrix} 1 \mapsto \{3,7\} \end{pmatrix}$, which, in turn, is equal to $\{\langle\langle 1,\{3,7\}\rangle\rangle\}$. Finally, $()$ is the empty set, sometimes called the empty function. That is, $() = \varnothing$.

The logic purist has no patience with conventions, and would insist that every function be written out as a set of primitive ordered pairs.

DEFINITION 21.40. $\forall q$*, by* q *is an* **ordered pair***, we mean:*

$$\exists\smile a,\ \exists\smile b \text{ s.t. } q = (a,b).$$

DEFINITION 21.41. *$\forall q$, by q is an* **ordered triple**, *we mean:*
$\exists\smile a$, $\exists\smile b$, $\exists\smile c$ *s.t.* $q = (a, b, c)$.

An ordered pair is sometimes called an **ordered 2-tuple**. An ordered triple is sometimes called an **ordered 3-tuple**. There are similar definitions for **ordered quadruple**, *a.k.a.* **ordered 4-tuple**, and for **ordered pentatuple**, *a.k.a.* **ordered 5-tuple**. Let's not use "hexatuple", "septuple", "octuple", "nonuple", and instead, keep it simple, by using "6-tuple", "7-tuple", "8-tuple", "9-tuple".

Exercise: Continue with the definitions appearing above, until you get to ordered 9-tuples. In particular, fill in the ellipses in:

DEFINITION 21.42. *$\forall q$, by q is an* **ordered 9-tuple**, *we mean:*
phantomx $\exists\smile a, \ldots, \exists\smile i$, *s.t.* $q = (a, \ldots, i)$.

THEOREM 21.43. *Let* $A := \{5, 6, 7, 8\}$ *and let* $B := \{7, 8, 9\}$.
Then $(A, B) = \begin{pmatrix} 1 \mapsto A \\ 2 \mapsto B \end{pmatrix} = \{\,\langle\langle 1, \{5, 6, 7, 8\}\rangle\rangle, \langle\langle 2, \{7, 8, 9\}\rangle\rangle\,\}$.
Also, we have $\mathrm{dom}[(A, B)] = \{1, 2\}$ *and* $\mathrm{im}[(A, B)] = \{A, B\}$.
Also, (A, B) is set-valued.

THEOREM 21.44. *Let* $A := \{5, 6, 7, 8\}$ *and let* $B := \{7, 8, 9\}$.
Let $S := \begin{pmatrix} 0 \mapsto A \\ 3 \mapsto B \end{pmatrix}$. *Then* $S = \{\langle\langle 0, A\rangle\rangle, \langle\langle 1, B\rangle\rangle\}$.
Also, $\mathrm{dom}[S] = \{0, 3\}$ *and* $\mathrm{im}[S] = \{A, B\}$.
Also, $S_0 = A$ *and* $S_3 = B$ *and* $S_1 = \frownie$.
Also S is set-valued.

THEOREM 21.45. *$\forall$set-valued S,* $\mathrm{im}[S]$ *is a set of sets.*

DEFINITION 21.46. *$\forall$set-valued S,* $\bigcup S_\bullet := \bigcup \mathrm{im}[S]$.

DEFINITION 21.47. *$\forall$set-valued S,* $\bigcap S_\bullet := \bigcap \mathrm{im}[S]$.

THEOREM 21.48. $\bigcup()_\bullet = \bigcup \varnothing_\bullet = \bigcup \mathrm{im}[\varnothing] = \bigcup \varnothing = \varnothing$.

THEOREM 21.49. $\bigcap()_\bullet = \bigcap \varnothing_\bullet = \bigcap \mathrm{im}[\varnothing] = \bigcap \varnothing = \frownie$.

THEOREM 21.50. *Let* $A := \{5, 6, 7, 8\}$ *and let* $B := \{7, 8, 9\}$.
Then $\bigcup(A, B)_\bullet = \{5, 6, 7, 8, 9\}$ *and* $\bigcap(A, B)_\bullet = \{7, 8\}$.

THEOREM 21.51. *Let* $A := \{5, 6, 7, 8\}$ *and let* $B := \{7, 8, 9\}$.
Let $S := \begin{pmatrix} 0 \mapsto A \\ 3 \mapsto B \end{pmatrix}$. *Then* $\bigcup S_\bullet = \{5, 6, 7, 8, 9\}$ *and* $\bigcap S_\bullet = \{7, 8\}$.

22. Injectivity

DEFINITION 22.1. *$\forall$function f, by f is* **one-to-one**, *we mean:*

$$(*)\ \forall w, x \in \mathrm{dom}[f], \quad (\ [\ f(w) = f(x)\ ] \Rightarrow [\ w = x\ ]\ ).$$

Condition $(*)$ in Definition 22.1 is called the **horizontal line test**.

The word **injective** is synonomous with one-to-one. We typically write "one-to-one" as "1-1".

THEOREM 22.2. *Let* $f := \{\langle\langle 5,2\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,6\rangle\rangle\}$. *Then*

f *is a function,*
$f(7) = 6 = f(8)$, f *is not 1-1,*
$f^{-1} = \{\langle\langle 2,5\rangle\rangle, \langle\langle 6,7\rangle\rangle, \langle\langle 6,8\rangle\rangle\}$,
$\langle\langle 6,7\rangle\rangle, \langle\langle 6,8\rangle\rangle \in f^{-1}$, f^{-1} *is not a function,*
$f^{-1}(6) = ☹$, $f^{-1}(2) = ☹$ *and* $(\ \forall x,\ f^{-1}(x) = ☹\ )$.

THEOREM 22.3. *Let* $f := \{\langle\langle 5,2\rangle\rangle, \langle\langle 7,6\rangle\rangle, \langle\langle 8,4\rangle\rangle\}$. *Then*

f *is a 1-1 function,* $f(7) = 6$,
$f^{-1} = \{\langle\langle 2,5\rangle\rangle, \langle\langle 4,8\rangle\rangle, \langle\langle 6,7\rangle\rangle\}$,
f^{-1} *is a function* *and* $f^{-1}(6) = 7$.

THEOREM 22.4. *Let* $f := \{\langle\langle x,y\rangle\rangle \in \mathbb{R}\times\mathbb{R} \mid y = x^2\}$. *Then*

f *is a function,*
$f(3) = 9 = f(-3)$, f *is not 1-1,*
$f^{-1} = \{\langle\langle y,x\rangle\rangle \in \mathbb{R}\times\mathbb{R} \mid y = x^2\}$,
$\langle\langle 9,3\rangle\rangle, \langle\langle 9,-3\rangle\rangle \in f^{-1}$,
f^{-1} *is not a function* *and* $f^{-1}(9) = f^{-1}(0) = ☹$
and $(\ \forall x,\ f^{-1}(x) = ☹\ )$.

THEOREM 22.5. *Let* $f := \{\langle\langle x,y\rangle\rangle \in \mathbb{R}\times\mathbb{R} \mid y = x^3\}$. *Then*

f *is a 1-1 function,* $f(2) = 8$,
$f^{-1} = \{\langle\langle y,x\rangle\rangle \in \mathbb{R}\times\mathbb{R} \mid y = x^3\}$,
f^{-1} *is a function* *and* $f^{-1}(8) = 2$.

THEOREM 22.6. *$\forall$function f,*

$$(\quad f \text{ is 1-1} \quad) \quad \Leftrightarrow \quad (\quad f^{-1} \text{ is a function} \quad).$$

23. Arrow notation for functions

DEFINITION 23.1. *Let f be a function and let A be a set. By A is a* **superdomain** *of f, we mean:* $A \supseteq \mathrm{dom}[f]$.

The following is a quantified equivalence for equality of functions. Two functions are equal iff they agree on a common superdomain:

THEOREM 23.2. *Let f and g be functions and let A be a set. Assume that $A \supseteq \mathrm{dom}[f]$ and that $A \supseteq \mathrm{dom}[g]$.*
Then: $(f = g) \Leftrightarrow (\forall x \in A,\ f(x) = g(x))$

DEFINITION 23.3. *Let f be a function and let B be a set. By B is a* **superimage** *or* **target** *of f, we mean: $A \supseteq \mathrm{im}[f]$.*

Any function has many superdomains and many superimages, but only one domain and one image.

In this course, we will not use the term "range", since it has different meanings to different people: Some take it to mean image, while others take it to mean target.

DEFINITION 23.4. $\forall f, A, B,$ *by $f : A \dashrightarrow B$, we mean*

f is a function and A and B are sets and $\mathrm{dom}[f] \subseteq A$ and $\mathrm{im}[f] \subseteq B$.

DEFINITION 23.5. $\forall f, A, B,$ *by $f : A \to B$, we mean*

f is a function and A and B are sets and $\mathrm{dom}[f] = A$ and $\mathrm{im}[f] \subseteq B$.

DEFINITION 23.6. $\forall f, A, B,$ *by $f : A \to> B$, we mean*

f is a function and A and B are sets and $\mathrm{dom}[f] = A$ and $\mathrm{im}[f] = B$.

DEFINITION 23.7. $\forall f, A, B,$ *by $f : A \hookrightarrow B$, we mean*

$f : A \to B$ and f is 1-1.

DEFINITION 23.8. $\forall f, A, B,$ *by $f : A \hookrightarrow> B$, we mean*

$f : A \to> B$ and f is 1-1.

THEOREM 23.9. *Let $f : A \hookrightarrow> B$. Then $f^{-1} : B \hookrightarrow> A$.*

DEFINITION 23.10. $\forall$*sets A, B,*

$\exists A \hookrightarrow B$ means: $\exists f$ s.t. $f : A \hookrightarrow B$,
$\exists A \to> B$ means: $\exists f$ s.t. $f : A \to> B$ and
$\exists A \hookrightarrow> B$ means: $\exists f$ s.t. $f : A \hookrightarrow> B$.

THEOREM 23.11. $\forall$*sets A, B,* $[\,\exists A \hookrightarrow> B\,] \Leftrightarrow [\,\exists B \hookrightarrow> A\,]$.

24. How to define a function

Instead of

$$f \quad := \quad \{\langle\langle x,y\rangle\rangle \in \mathbb{R}\times\mathbb{R} \,|\, y = x^2\},$$

the logic purist would prefer

$$f \quad := \quad \{z \in \mathbb{R}\times\mathbb{R} \,|\, \exists x,y \in \mathbb{R} \text{ s.t. } y = x^2 \& \langle\langle x,y\rangle\rangle\}.$$

We are not logic purists, but, nevertheless, from here on out, in this course, we will treat

$$f \quad := \quad \{\langle\langle x,y\rangle\rangle \in \mathbb{R}\times\mathbb{R} \,|\, y = x^2\}$$

as an example of poor style (with a loss of credit). Instead, the preferred syntax will be

Define $f : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f(x) = x^2$.

The variable x is free, and it might be better to write Define $f : \mathbb{R} \dashrightarrow [0;\infty)$ by: $\forall x \in \mathrm{dom}[f]$, $f(x) = x^2$.

However, in practice, the "$\forall x \in \mathrm{dom}[f]$" is typically omitted. This is our ONLY exception to the Cardinal Binding Rule. We compute

$\mathrm{dom}[f] = \{x \in \mathbb{R} \,|\, x^2 \in \mathbb{R}\} = \mathbb{R}$, $\mathrm{im}[f] = \{x^2 \in \mathbb{R} \,|\, x \in \mathbb{R}\} = [0;\infty)$. We could therefore just as easily have written:

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

Or:

Define $f : \mathbb{R} \to (-3;\infty]$ by $f(x) = x^2$.

All that is important is that the superimage (or "target") contain the image of f, which is $[0;\infty)$. In this course, it is unacceptable to say "Let $f(x) = x^2$". You must always specify a superdomain and superimage.

Another example:

Let $g : \mathbb{R} \dashrightarrow [5;\infty)$ be defined by by $g(x) = 1/x$.

We only know that $\mathbb{R}$ is a superdomain of g, *i.e.*, that $\mathbb{R}$ is a superset of $\mathrm{dom}[g]$. In this situation, $g(0) = ☹ \notin [5;\infty)$, and so it is understood that 0 is not in the domain of g. In fact, by convention, if we write

Let $g : \mathbb{R} \dashrightarrow [5;\infty)$ be defined by by $g(x) = 1/x$,

then the domain of g is given by:

$$\mathrm{dom}[g] \quad = \quad \{\ x \in \mathbb{R} \ |\ 1/x \in [5;\infty)\ \} \quad = \quad (0;1/5] \quad .$$

DEFINITION 24.1. *Let A be a set.*
We define $\mathrm{id}_A : A \to A$ by $\mathrm{id}_A(x) = x$.
The function id_A is called the **identity function** *on A.*

THEOREM 24.2. *$\forall$set A,* $\quad \mathrm{id}_A : A \hookrightarrow\!> A$.

THEOREM 24.3. *$\forall$set A,* $\quad \exists A \hookrightarrow\!> A$.

25. Restriction, forward image and preimage

DEFINITION 25.1. *$\forall$function f, $\forall$set A,*
the function $f|A : A \cap (\mathrm{dom}[f]) \to \mathrm{im}[f]$
is defined by $(f|A)(x) = f(x)$.

THEOREM 25.2. *Let* $B := \{3,4,5\}$, $C := \{8,9\}$, $f := \begin{pmatrix} 3 \mapsto 9 \\ 4 \mapsto 9 \\ 5 \mapsto 8 \end{pmatrix}$.
Let $A := \{0,3,5\}$. *Then* $f : B \to C$ *and* $f|A = \begin{pmatrix} 3 \mapsto 9 \\ 5 \mapsto 8 \end{pmatrix}$.

THEOREM 25.3. *Let A, B and C be sets. Let* $f : B \to C$.
Then $f|A$ is a function $\quad$ *and* $\quad \mathrm{dom}[f|A] = (\mathrm{dom}[f]) \cap A$.

THEOREM 25.4. *Let* $B := \{3,4,5\}$, $C := \{8,9\}$, $f := \begin{pmatrix} 3 \mapsto 9 \\ 4 \mapsto 9 \\ 5 \mapsto 8 \end{pmatrix}$.
Let $A := \{3,5\}$. *Then* $A \subseteq B$ *and* $f : B \to C$ *and* $f|A = \begin{pmatrix} 3 \mapsto 9 \\ 5 \mapsto 8 \end{pmatrix}$.

THEOREM 25.5. *Let B and C be sets. Let $A \subseteq B$. Let $f : B \to C$.*
Then, $\forall t \in A$, $(f|A)(t) = f(t)$. *Also,* $\forall t \notin A$, $(f|A)(t) = ☹$.

THEOREM 25.6. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x^2$. *Let* $A := [0;\infty)$.
Then: $\quad$ *f is not 1-1* $\quad$ *and* $\quad$ *$f|A$ is 1-1* $\quad$ *and*
$f|A : A \hookrightarrow\!> A$ $\quad$ *and* $\quad \forall y \in A$, $(f|A)^{-1}(y) = \sqrt{y}$.

DEFINITION 25.7. *Let f be a function. Let S be a set. Then*
$f_*(S) \;:=\; \{\, f(x) \in \mathrm{im}[f] \mid x \in S \cap (\mathrm{dom}[f]) \,\}$ $\qquad$ *and*
$f^*(S) \;:=\; \{\, x \in \mathrm{dom}[f] \mid f(x) \in S \,\}$.

THEOREM 25.8. *Let* $f := \begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 7 \\ 3 \mapsto 6 \\ 4 \mapsto 9 \end{pmatrix}$. *Then*
$f_*(\{0,1,2,4\}) = \{7,9\}$ $\quad$ *and* $\quad f^*(\{6,7,8\}) = \{1,2,3\}$.

THEOREM 25.9. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x + 2$.
Then: $\quad f_*([0;\infty)) = [2;\infty)$ $\quad$ *and* $\quad f^*([0;\infty)) = [-2;\infty)$.

THEOREM 25.10. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x^2$. *Then*
$f^*([9;16)) = (-4;-3] \cup [3;4)$.

26. Composition

THEOREM 26.1. *Let* f *and* g *be functions. Then* $\exists 1$*function* h*, denoted* $g \circ f$*,*
s.t., $\forall x, \quad h(x) = g(f(x))$.

THEOREM 26.2. *Define* $f, g : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x + 2$, $g(x) = \sqrt{x}$.
Then, $\forall x \in \mathbb{R}$, $(g \circ f)(x) = \sqrt{x+2}$.
Also, $\mathrm{dom}[g \circ f] = [-2;\infty) = f^*(\mathrm{dom}[g])$.
Also, $\mathrm{im}[g \circ f] = [0;\infty) = g_*(\mathrm{im}[f])$.

THEOREM 26.3. *Define* $f, g : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = \sqrt{x}$, $g(x) = x + 2$.
Then, $\forall x \in \mathbb{R}$, $(g \circ f)(x) = \sqrt{x} + 2$.
Also, $\mathrm{dom}[g \circ f] = [0;\infty) = f^*(\mathrm{dom}[g])$.
Also, $\mathrm{im}[g \circ f] = [2;\infty) = g_*(\mathrm{im}[f])$.

THEOREM 26.4. *Let* f *and* g *be functions. Then*
(1) $\mathrm{dom}[g \circ f] = f^*(\mathrm{dom}[g])$ *and*
(2) $\mathrm{im}[g \circ f] = g_*(\mathrm{im}[f])$.

Composition of functions is associative:

THEOREM 26.5. $\forall$*functions* f, g, h*, we have:* $h \circ (g \circ f) = (h \circ g) \circ f$.

THEOREM 26.6. *Let* $f : A \hookrightarrow\!\!> B$ *and let* $g : B \hookrightarrow\!\!> C$. *Then* $g \circ f : A \hookrightarrow\!\!> C$.

THEOREM 26.7. *Let* A*,* B *and* C *be sets.*
Assume that $\exists A \hookrightarrow B$ *and that* $\exists B \hookrightarrow C$.
Then: $\exists A \hookrightarrow C$.

27. Power sets and sets of functions

AXIOM 27.1. $\forall$*set* S, $\exists 1$*set* $\mathcal{P}$*, denoted* 2^S*, s.t.*
$\forall A$, $(A \in \mathcal{P}) \Leftrightarrow (A \subseteq S)$.

For any set S, the set 2^S is called the **power set** of S. It is the set of all subsets of S.

THEOREM 27.2. $2^{\{7,8,9\}} =$
$\{ \varnothing, \{9\}, \{8\}, \{8,9\},$
$\{7\}, \{7,9\}, \{7,8\}, \{7,8,9\} \}$.

DEFINITION 27.3. *$\forall$sets A, B, $B^A := \{f \subseteq A \times\!\!\times B \,|\, f : A \to B\}$.*

According to specification, between "{" and "|", we *should* have:

a single variable, then "$\in$" then a set .

The logic purist would therefore do some rewriting of Definition 21.18, and define B^A to be $B^A := \{f \in 2^{A \times\!\times B} \,|\, f : A \to B\}$.

THEOREM 27.4. $\{0,1\}^{\{7,8,9\}} =$

$$\left\{ \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 0 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 0 \\ 9 \mapsto 1 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 1 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 1 \\ 9 \mapsto 1 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 0 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 0 \\ 9 \mapsto 1 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 1 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 1 \\ 9 \mapsto 1 \end{pmatrix} \right\}.$$

THEOREM 27.6. $\{7,8,9\}^{\{1,2\}} =$

$$\left\{ \begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 7 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 8 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 9 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 8 \\ 2 \mapsto 7 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 8 \\ 2 \mapsto 8 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 8 \\ 2 \mapsto 9 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 9 \\ 2 \mapsto 7 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 9 \\ 2 \mapsto 8 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 9 \\ 2 \mapsto 9 \end{pmatrix} \right\} =$$

$$\{\,(7,7)\,,\,(7,8)\,,\,(7,9)\,,\;(8,7)\,,\,(8,8)\,,\,(8,9)\,,\;(9,7)\,,\,(9,8)\,,\,(9,9)\,\}.$$

THEOREM 27.5. *$\forall$set A, $\exists\{0,1\}^A \hookrightarrow\!\!> 2^A$.*

28. Orbits

DEFINITION 28.1. *Let f be a function. Let $k \in \mathbb{N}_0$. Let $S := (\mathrm{dom}[f]) \cup (\mathrm{im}[f])$. Then, $\forall a$,*

$$\mathrm{ORB}_f^k(a) := \mathrm{UE}\{x \in S^{[0..k]} \,|\, (x_0 = a) \& (\forall j \in [1..k],\ f(x_{j-1}) = x_j)\}.$$

The function $\mathrm{ORB}_f^k(a)$ is called the ***k*-orbit** of a under f.

THEOREM 28.2. *Define $f : [1,8] \to \mathbb{R}$ by $f(x) = x + 2$.*

Then $\mathrm{ORB}_f^0(3) = (0 \mapsto 3)$, $\mathrm{ORB}_f^1(3) = \begin{pmatrix} 0 \mapsto 3 \\ 1 \mapsto 5 \end{pmatrix}$,

$$\mathrm{ORB}_f^2(3) = \begin{pmatrix} 0 \mapsto 3 \\ 1 \mapsto 5 \\ 2 \mapsto 7 \end{pmatrix}, \quad \mathrm{ORB}_f^3(3) = \begin{pmatrix} 0 \mapsto 3 \\ 1 \mapsto 5 \\ 2 \mapsto 7 \\ 3 \mapsto 9 \end{pmatrix} \quad \textit{and}$$

$$\mathrm{ORB}_f^4(3) = ☹.$$

Also, $(\mathrm{ORB}^0_f(3))_0 = 3$, $(\mathrm{ORB}^1_f(3))_1 = 5$,
$(\mathrm{ORB}^2_f(3))_2 = 7$, $(\mathrm{ORB}^3_f(3))_3 = 9$ *and*
$(\mathrm{ORB}^4_f(3))_4 = ☹$.

DEFINITION 28.3. *Let f be a function, $k \in \mathbb{N}_0$.*
Then, $\forall a$, $f^k_\circ(a) := (\mathrm{ORB}^k_f(a))_k$.

THEOREM 28.4. *Define $f : [1, 8] \to \mathbb{R}$ by $f(x) = x + 2$.*
Then $f^0_\circ(3) = 3$, $f^1_\circ(3) = 5$, $f^2_\circ(3) = 7$, $f^3_\circ(3) = 9$, $f^4_\circ(3) = ☹$.

THEOREM 28.5. *Define $f : [1, 8] \to \mathbb{R}$ by $f(x) = x + 2$.*
Then $f^0_\circ(3) = 3$, $f^1_\circ(3) = f(3)$, $f^2_\circ(3) = (f \circ f)(3)$,
$f^3_\circ(3) = (f \circ f \circ f)(3)$ *and* $f^4_\circ(3) = (f \circ f \circ f \circ f)(3)$.

THEOREM 28.6. *$\forall$function f, $\forall a$,*
$f^0_\circ(a) = a$, $f^1_\circ(a) = f(a)$, $f^2_\circ(a) = (f \circ f)(a)$,
$f^3_\circ(a) = (f \circ f \circ f)(a)$ *and* $f^4_\circ(a) = (f \circ f \circ f \circ f)(a)$.

THEOREM 28.7. *$\forall$function f, $\forall a$,* $f(f^j_\circ(a)) = f^{j+1}_\circ(a)$.

DEFINITION 28.8. *Let $a \in \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = ax$.*
Then, $\forall k \in \mathbb{N}_0$, $a^k := f^k_\circ(1)$.

DEFINITION 28.9. $\forall a \in \mathbb{R}$, $\forall k \in \mathbb{N}$, $a^{-k} := 1/(a^k)$.

THEOREM 28.10. $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$,
$2^{-1} = 1/2$, $2^{-2} = 1/4$, $2^{-3} = 1/8$.

THEOREM 28.11. $\forall j \in \mathbb{N}_0$, $2 \cdot 2^j = 2^{j+1}$.

THEOREM 28.12. $\forall a \in \mathbb{R}$, $a^0 = 1$.

THEOREM 28.13. $0^0 = 1$.

THEOREM 28.14. $\forall a \in \mathbb{R}$, $a^1 = a$ *and* $a^{-1} = 1/a$.

DEFINITION 28.15. *$\forall$set A, $\forall k \in \mathbb{N}$,* $A^k := A^{[1..k]}$.

Review Theorem 27.6.

THEOREM 28.16. $\{7, 8, 9\}^2 = \{7, 8, 9\}^{\{1,2\}} =$
$\{ (7, 7), (7, 8), (7, 9), (8, 7), (8, 8), (8, 9), (9, 7), (9, 8), (9, 9) \}$.

DEFINITION 28.17. *$\forall$sets A and B,*

$$A \times B := \{ (a, b) \in (A \cup B)^2 \mid (a \in A) \,\&\, (b \in B) \}.$$

We call $A \times B$ of Definition 28.17 the **product** of A and B.

THEOREM 28.18. $\{7\} \times \{8,9\} = \{(7,8), (7,9)\}$.

THEOREM 28.19. $\{1,2,3\} \times \{8,9\} =$
$$\{(1,8), (2,8), (3,8), (1,9), (2,9), (3,9)\}$$

DEFINITION 28.20. *$\forall$sets A and B and C,*
$$A \times B \times C := \{(a,b,c) \in (A \cup B \cup C)^3 \,|\, (a \in A)\&(b \in B)\&(c \in C\}.$$

We call $A \times B \times C$ of Definition 28.20 the **product** of A, B and C.

THEOREM 28.21. $\{1,2,3\} \times \{8,9\} \times \{0\} =$
$$\{(1,8,0), (2,8,0), (3,8,0), (1,9,0), (2,9,0), (3,9,0)\}$$

We leave it to you to continue these definitions up to nine sets. For the last definition, fill in the ellipses $(\cdots)$ in:

DEFINITION 28.22. *$\forall$sets $A, \ldots, I$,*
$$A \times \cdots \times I := \{(a, \ldots, i) \in (A \cup \cdots \cup I)^9 \,|\, (a \in A)\& \cdots \&(i \in I)\}.$$

THEOREM 28.23. *$\forall$set A, $A^2 = A \times A$ and $A^3 = A \times A \times A$ and $A^4 = A \times A \times A \times A$ and $A^5 = A \times A \times A \times A \times A$.*

We also have similar formulas for A^6, A^7, A^8 and A^9.

Let A, B and C be sets. Let $f : A \times B \to C$. Then, $\forall x \in A$, $\forall y \in B$, we have $(x,y) \in A \times B$ and $f((x,y)) \in C$, but it is common to eliminate one set of parentheses, and write $f(x,y)$ instead of $f((x,y))$.

The logic purist eschews ellipses $(\cdots)$. Consider the theorem
$$1 + \cdots + 4 = 10 .$$
The logic purist would prefer
$$1 + 2 + 3 + 4 = 10 .$$

On the other hand, consider the theorem:

THEOREM 28.24. *$\forall \ell \in \mathbb{N}$, $1 + \cdots + \ell = \ell(\ell+1)/2$.*

We now have a challenge in eliminating the ellipsis, because ℓ is a variable. We can use composition powers to deal with this challenge:

THEOREM 28.25. *Define $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{Z} \times \mathbb{R}$ by*
$$f(j,x) = (j+1, x+j).$$
Then $f(1,0) = (2,1)$ and $f(2,1) = (3,3)$ and $f(3,3) = (4,6)$ and $f(4,6) = (5,10)$.

$$\begin{aligned} \textit{Also, } (f_\circ^1(1,0))_2 &= ((2,1))_2 = 1, \\ (f_\circ^2(1,0))_2 &= ((3,3))_2 = 3 = 1+2, \\ (f_\circ^3(1,0))_2 &= ((4,6))_2 = 6 = 1+2+3 \quad \textit{and} \\ (f_\circ^4(1,0))_2 &= ((5,10))_2 = 10 = 1+2+3+4. \end{aligned}$$

Instead of Theorem 28.24, the logic purist would prefer:

THEOREM 28.26. *Define* $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{Z} \times \mathbb{R}$ *by*

$$f(j,x) = (j+1, x+j).$$

Then, $\forall \ell \in \mathbb{N}$, *we have* $(f_\circ^\ell(1,0))_2 = \ell(\ell+1)/2$.

However, as is often the case, purity comes at the cost of readability, and, in this course, we will often use ellipses. Theorem 28.24 is proved below, see Theorem 30.7. Next, we introduce the summation notation:

DEFINITION 28.27. *Let* α *be a function, let* $k, \ell \in \mathbb{N}$. *Assume* $k \leqslant \ell$, $[k..\ell] \subseteq \mathrm{dom}[\alpha]$ *and* $\mathrm{im}[\alpha] \subseteq \mathbb{R}$.

Then: $$\sum_k^\ell \alpha_\bullet \quad := \quad \alpha_k + \cdots + \alpha_\ell.$$

Assuming that j is a free variable, we can also use the notation $\displaystyle\sum_{j=k}^{\ell} \alpha_j$ to denote $\displaystyle\sum_k^\ell \alpha_\bullet$. In this case, the variable j becomes bound between "$\displaystyle\sum_{j=k}^{\ell}$" and "$\alpha_j$", and is then free again. If j is not free, but i is free, then we could use $\displaystyle\sum_{i=k}^{\ell} \alpha_i$, and, again i is temporarily bound. Any free variable is acceptable, not just i or j. For this reason, the variable is sometimes called a "dummy variable", meaning a variable that is easily replaced by another, as a dummy mannequin is easly replaced by another in a department store.

Definition 28.27 is not acceptable to a logic purist because of the ellipsis. The following, while difficult to read, is formally better.

DEFINITION 28.28. *Let* α *be a function and let* $k, \ell \in \mathbb{N}$. *Assume* $k \leqslant \ell$, $[k..\ell] \subseteq \mathrm{dom}[\alpha]$ *and* $\mathrm{im}[\alpha] \subseteq \mathbb{R}$. *Define* $f : [k..\ell] \times \mathbb{R} \to \mathbb{Z} \times \mathbb{R}$ *by* $f(j,x) = (j+1, x+\alpha_j)$.

Then: $$\sum_k^\ell \alpha_\bullet \quad := \quad (\, f_\circ^{\ell-k+1}(k,0)\,)_2.$$

The product notation is similar:

DEFINITION 28.29. *Let* α *be a function and let* $k, \ell \in \mathbb{N}$.
Assume $k \leqslant \ell$, $[k..\ell] \subseteq \mathrm{dom}[\alpha]$ *and* $\mathrm{im}[\alpha] \subseteq \mathbb{R}$.
Then $\prod\limits_{k}^{\ell} \alpha_\bullet \quad := \quad \alpha_k \cdot \cdots \cdot \alpha_\ell$.

Assuming that j is a free variable, we can also use the notation $\prod\limits_{j=k}^{\ell} \alpha_j$ to denote $\prod\limits_{k}^{\ell} \alpha_\bullet$. In this case, the variable j becomes bound between "$\prod\limits_{j=k}^{\ell}$" and "$\alpha_j$", and is then free again. If j is not free, but i is free, then we could use $\prod\limits_{i=k}^{\ell} \alpha_i$, and, again i is temporarily bound. This is another dummy variable; any free variable is okay, not just i or j.

More formally:

DEFINITION 28.30. *Let* α *be a function and let* $k, \ell \in \mathbb{N}$.
Assume $k \leqslant \ell$, $[k..\ell] \subseteq \mathrm{dom}[\alpha]$ *and* $\mathrm{im}[\alpha] \subseteq \mathbb{R}$.
Define $f : [k..\ell] \times \mathbb{R} \to \mathbb{Z} \times \mathbb{R}$ *by* $f(j, x) = (j + 1, x \cdot \alpha_j)$.
Then: $\qquad \prod\limits_{k}^{\ell} \alpha_\bullet \quad := \quad (\, f_\circ^{\ell - k + 1}(k, 1)\,)_2$.

Using summation notation, we can rewrite Theorem 28.24 in a way that is readable and avoids ellipses:

THEOREM 28.31. $\qquad \forall \ell \in \mathbb{N}, \quad \sum\limits_{j=1}^{\ell} j = \ell(\ell + 1)/2.$

THEOREM 28.32. $\qquad$ *Then* $\forall \ell \in \mathbb{N}, \quad \sum\limits_{1}^{\ell} (\mathrm{id}_\mathbb{R})_\bullet = \ell(\ell + 1)/2.$

29. Appendix 1

The following is Theorem 1.2:

THEOREM 29.1. $\forall x, y, \quad [\,(x = y) \;\Rightarrow\; (y = x)\,]$.

Proof. Given x, y.
Want: $(x = y) \Rightarrow (y = x)$.
Assume $x = y$.
Want: $y = x$.
By Axiom 1.1, $x = x$.
Since $x = y$, we may replace the first x in $x = x$ by y.
Then $y = x$, as desired. □

The following is Theorem 1.3.

THEOREM 29.2. $\forall x, y, z, \quad [\,(x = y = z) \Rightarrow (x = z)\,]$.

Proof. We have $x = y$ and $y = z$.
Since $x = y$, we may replace y in $y = z$ by x.
Then $x = z$, sa desired. □

The following is Theorem 7.15:

THEOREM 29.3. *$\forall a, b \in \mathbb{R}$, $\exists x \in \mathbb{R}$ s.t. $a + x = b$.*

Proof. Given $a, b \in \mathbb{R}$. Want: $\exists x \in \mathbb{R}$ s.t. $a + x = b$.
Let $x := b - a$.
Then

$$\begin{aligned} a + x &= a + (b - a) = a + b + (-a) \\ &= b + a + (-a) = b + 0 = b, \end{aligned}$$

as desired. □

30. Principle of Mathematical Induction

DEFINITION 30.1. *$\forall S \subseteq \mathbb{R}$, by S is* **successor closed**, *we mean:*
$$\forall x \in S, \quad x + 1 \in S.$$

AXIOM 30.2. *$\mathbb{N}$ is successor closed.*

Recall that $\mathbb{N} = [1..\infty)$. Then $1 \in \mathbb{N}$. Recall:

$$\begin{aligned} &(1 + 1 = 2) \,\&\, (2 + 1 = 3) \,\&\, (3 + 1 = 4) \,\&\, (4 + 1 = 5) \,\&\, \\ &(5 + 1 = 6) \,\&\, (6 + 1 = 7) \,\&\, (7 + 1 = 8) \,\&\, (8 + 1 = 9) \qquad , \end{aligned}$$

and $9 + 1 = 10$. So, as $\mathbb{N} := [1..\infty) \subseteq \mathbb{Z}$, using Axiom 30.2, we have:

THEOREM 30.3. *$1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \in \mathbb{N} \subseteq \mathbb{Z}$.*

AXIOM 30.4. *$\mathbb{Z} = (-\mathbb{N}) \cup \{0\} \cup \mathbb{N}$.*

THEOREM 30.5. $0, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10 \in \mathbb{Z}$.

The next axiom is the **Principle of Mathematical Induction**.

AXIOM 30.6. *Let $S \subseteq \mathbb{N}$.*
Assume that $1 \in S$ *and that* S *is successor closed.*
Then $S = \mathbb{N}$.

THEOREM 30.7. $\forall \ell \in \mathbb{N}, \quad 1 + \cdots + \ell = \ell(\ell + 1)/2.$

Proof. Let $S := \{\ell \in \mathbb{N} \,|\, 1 + \cdots + \ell = \ell(\ell + 1)/2\}$.
Want: $S = \mathbb{N}$.
Since $1 = 1 \cdot (1 + 1)/2$, it follows that $1 \in S$.
So, by the PMI, it suffices to show: S is successor closed.
Want: $\forall \ell \in S$, $\ell + 1 \in S$.
Given $\ell \in S$. Want: $\ell + 1 \in S$.
Know: $1 + \cdots + \ell = \ell(\ell + 1)/2$.
Want: $1 + \cdots + \ell + (\ell + 1) = (\ell + 1)((\ell + 1) + 1)/2$.
We have:

$$\begin{aligned} 1 + \cdots + \ell + (\ell + 1) &= (\ell(\ell + 1)/2) + (\ell + 1) \\ &= ((\ell^2 + \ell)/2) + ((2\ell + 2)/2) \\ &= (\ell^2 + 3\ell + 2)/2 \;=\; (\ell + 1)(\ell + 2)/2 \\ &= (\ell + 1)((\ell + 1) + 1)/2, \end{aligned}$$

as desired. □

THEOREM 30.8. $\forall j, k \in \mathbb{N}, \quad j + k \in \mathbb{N}.$

Proof. Let $S := \{k \in \mathbb{N} \,|\, \forall j \in \mathbb{N},\, j + k \in \mathbb{N}\}$.
Want: $S = \mathbb{N}$.
Since $\mathbb{N}$ is successor closed, we know: $\forall j \in \mathbb{N}$, $j + 1 \in \mathbb{N}$.
Then $1 \in S$.
By the PMI, it suffices to show: S is successor closed.
Want: $\forall k \in S$, $k + 1 \in S$.
Given $k \in S$. Want: $k + 1 \in S$.
Know: $\forall j \in \mathbb{N}$, $j + k \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $j + (k + 1) \in \mathbb{N}$.
Given $j \in \mathbb{N}$. Want: $j + (k + 1) \in \mathbb{N}$.
$j + k \in \mathbb{N}$ and $\mathbb{N}$ is successor closed.
Then $(j + k) + 1 \in \mathbb{N}$.
Then $j + (k + 1) = (j + k) + 1 \in \mathbb{N}$, as desired □

We leave it as unassigned homework to show that successor closed is "translation invariant". That is:

$\forall S \subseteq \mathbb{R}$, $\forall a \in \mathbb{R}$,

$[\,(\,S$ is successor closed$\,) \Rightarrow (\,S + a$ is successor closed$\,)\,]$.

We can generalize the PMI:

THEOREM 30.9. *Let $k \in \mathbb{Z}$. Let $S \subseteq [k..\infty)$. Assume that $k \in S$ and that S is successor closed.*

Then: $S = [k..\infty)$.

Proof. Let $a := 1 - k$. Then $a + k = 1$.
Since S is successor closed, $S + a$ is successor closed.
Since $k \in S$, $k + a \in S + a$.
Since $S \subseteq [k..\infty)$, $S + a \subseteq [k..\infty) + a$.
Then $S + a$ is successor closed
and $1 = k + a \in S + a$
and $S + a \subseteq [k..\infty) + a = [1..\infty)$,
so, by Axiom 30.6, $S + a = \mathbb{N}$.
Then $S = \mathbb{N} - a = [1..\infty) - a = [k..\infty)$, as desired. □

THEOREM 30.10. $\forall j \in \mathbb{N}_0,\ 2^j \geqslant j + 1$.

Proof. Let $S := \{j \in \mathbb{N}_0 \,|\, 2^j \geqslant j + 1\}$.
Then $S \subseteq \mathbb{N}_0 = [0..\infty)$. Want: $S = \mathbb{N}_0$. Want: $S = [0..\infty)$.
Since $2^0 = 1 \geqslant 0 + 1$, we see that $0 \in S$.
Then, by Theorem 30.9, it suffices to show: S is successor closed.
Want: $\forall j \in S,\ j + 1 \in S$.
Given $j \in S$. Want: $j + 1 \in S$.
Know: $2^j \geqslant j + 1$. Want: $2^{j+1} \geqslant (j + 1) + 1$.
Since $2^j \geqslant j + 1$, we get $2 \cdot 2^j \geqslant 2 \cdot (j + 1)$.
Since $j \in S \subseteq \mathbb{N}_0 = [0..\infty) \geqslant 0$, we get $j \geqslant 0$.
Then $j + (j + 2) \geqslant 0 + (j + 2)$, so $2j + 2 \geqslant j + 2$.
Then $2^{j+1} = 2 \cdot 2^j \geqslant 2 \cdot (j + 1) = 2j + 2 \geqslant j + 2 = (j + 1) + 1$. □

THEOREM 30.11. $\forall j \in \mathbb{N}_0,\ 2^j > j$.

Proof. Given $j \in \mathbb{N}_0$. Want: $2^j > j$.
By Theorem 30.10, $2^j \geqslant j + 1$.
Then $2^j \geqslant j + 1 > j$, as desired. □

31. Well-ordered sets

DEFINITION 31.1. *Let $S \subseteq \mathbb{R}^*$. By S is* **well-ordered**, *we mean:*

$$\forall \text{nonempty } A \subseteq S, \quad \min A \neq ☹.$$

THEOREM 31.2. *Let $A \subseteq \mathbb{N}^*$ and let $j \in \mathbb{N}_0$. Let $k := j + 1$. Assume $[1..j] \cap A = \varnothing \neq [1..k] \cap A$. Then $\min A = k$.*

Proof. Unassigned homework. □

THEOREM 31.3. *$\mathbb{N}^*$ is well-ordered.*

Proof. Want: $\forall$nonempty $A \subseteq \mathbb{N}^*$, $\min A \neq ☹$.
Given nonempty $A \subseteq \mathbb{N}^*$.
Want: $\min A \neq ☹$.
Assume $\min A = ☹$.
Want: Contradiction.

Claim 1: $\forall j \in \mathbb{N}_0$, $[1..j] \cap A = \varnothing$.
Proof of Claim 1:
Let $S := \{j \in \mathbb{N}_0 \,|\, [1..j] \cap A = \varnothing\}$.
Want: $S = \mathbb{N}_0$. Want: $S = [0..\infty)$.
Since $[1..0] \cap A = \varnothing \cap A = \varnothing$, we see that $0 \in S$.
Then, by the GPMI,
Want: S is successor closed.
Want: $\forall j \in S$, $j + 1 \in S$.
Given $j \in S$. Want: $j + 1 \in S$.
Let $k := j + 1$. Want: $k \in S$.
Since $j \in S$, we have $[1..j] \cap A = \varnothing$.
So, since $k \neq ☹ = \min A$, by Theorem 31.2, we see that $[1..k] \cap A = \varnothing$.
Then $k \in A$, as desired. *End of proof of Claim 1.*

Claim 2: $A \subseteq \{\infty\}$.
Proof of Claim 2:
We have $\mathbb{N}^* \backslash \mathbb{N} = \{\infty\}$. Want: $A \subseteq \mathbb{N}^* \backslash \mathbb{N}$.
We have $A \subseteq \mathbb{N}^*$. Want: $\forall j \in \mathbb{N}$, $j \notin A$.
Given $j \in \mathbb{N}$. Want: $j \notin A$.
Since $j \in \mathbb{N}$, we conclude that $j \in [1..j]$.
By Claim 1, $[1..j] \cap A = \varnothing$.
Then $j \notin [1..j] \cap A$.
So, since $j \in [1..j]$, we see that $j \notin A$, as desired.

End of proof of Claim 2.

Since $\varnothing \neq A \subseteq \{\infty\}$, we see that $A = \{\infty\}$, so $\min A = \infty$.
Then $\min A = \infty \neq ☹$ and $\min A = ☹$.
Contradiction. □

THEOREM 31.4. *Let $T \subseteq \mathbb{R}^*$ and let $S \subseteq T$.*
Assume that T is well-ordered. Then S is well-ordered.

THEOREM 31.5. *$\mathbb{N}$ is well-ordered.*

THEOREM 31.6. *Let $S \subseteq \mathbb{R}^*$ and let $t \in \mathbb{R}$.*
Assume that S is well-ordered. Then $S + t$ is well-ordered.

THEOREM 31.7. *$\forall k \in \mathbb{Z}$, $[k..\infty]$ is well-ordered.*

THEOREM 31.8. *$\forall k \in \mathbb{Z}$, $[k..\infty)$ is well-ordered.*

DEFINITION 31.9. *Let $S \subseteq \mathbb{R}$.*
By S is **bounded below** *in $\mathbb{R}$, we mean: $\exists u \in \mathbb{R}$ s.t. $u \leqslant S$.*
By S is **bounded above** *in $\mathbb{R}$, we mean: $\exists u \in \mathbb{R}$ s.t. $S \leqslant u$.*

THEOREM 31.10. *The following are all true:*
$[1;\infty)$ is bounded below in $\mathbb{R}$, but not bounded above in $\mathbb{R}$,
$\mathbb{N}$ is bounded below in $\mathbb{R}$, but not bounded above in $\mathbb{R}$,
$(-\infty;5)$ is bounded above in $\mathbb{R}$, but not bounded below in $\mathbb{R}$,
$\mathbb{Z}$ is neither bounded above nor bounded below in $\mathbb{R}$ and
$(2;5]$ is both bounded above and bounded below in $\mathbb{R}$.
$\varnothing$ is both bounded above and bounded below in $\mathbb{R}$.

The following will be called the **Reverse Archimedean Principle**:

THEOREM 31.11. *$\forall u \in \mathbb{R}$, $\exists k \in -\mathbb{N}$ s.t. $k < u$.*

Proof. Given $u \in \mathbb{R}$. Want: $\exists k \in -\mathbb{N}$ s.t. $k < u$. By the Archimedean Principle (Theorem 19.1),
choose $j \in \mathbb{N}$ s.t. $j > -u$.
Let $k := -j$.
Want: $k < u$.
Since $j > -u$, we see that $-j < -(-u)$.
Then $k = -j < -(-u) = u$, as desired. □

THEOREM 31.12. *Let $S \subseteq \mathbb{Z}$ be nonempty.*
Assume that S is bounded below in $\mathbb{R}$. Then $\min S \neq ☹$.

Proof. Since S is bounded below in $\mathbb{R}$, choose $u \in \mathbb{R}$ s.t. $u \leqslant S$.
By the Reverse Archimedean Principle (Theorem 31.11),
choose $k \in -\mathbb{N}$ s.t. $k < u$.
Then $S \subseteq (k;\infty) \cap \mathbb{Z} = (k..\infty) \subseteq [k..\infty]$.
By Theorem 31.8, $[k..\infty]$ is well-ordered.
So, since $\varnothing \neq S \subseteq [k..\infty]$, we get $\min S \neq ☹$, as desired. □

THEOREM 31.13. *Let $S \subseteq \mathbb{R}$.*
Assume that S is bounded above in $\mathbb{R}$.
Then $-S$ is bounded below in $\mathbb{R}$.

Proof. Unassigned HW. □

THEOREM 31.14. $\forall S \subseteq \mathbb{R}$, $\quad \min(-S) = -(\max S)$.

Proof. Unassigned HW. □

THEOREM 31.15. *Let $S \subseteq \mathbb{Z}$ be nonempty.*
Assume that S is bounded above in $\mathbb{R}$. Then $\max S \neq ☹$.

Proof. Since $S \subseteq \mathbb{Z}$, we see that $-S \subseteq \mathbb{Z}$.
Since $S \neq \varnothing$, we see that $-S \neq \varnothing$.
Since S is bounded above in $\mathbb{R}$, by Theorem 31.13,
we see that $-S$ is bounded below in $\mathbb{R}$.
Then, by Theorem 31.12 (with S replaced by $-S$),
we see that $\min(-S) \neq ☹$.
By Theorem 31.14, $\quad \min(-S) = -(\max S)$.
Then $-(\max S) \neq ☹$. $\quad$ Then $\max S \neq ☹$, as desired. □

32. Constants, punctures, fills and adjustments

DEFINITION 32.1. $\forall$*set* A, $\forall\smile y$,
we define $C_A^y : A \to \{y\}$ *by* $\quad C_A^y(x) = y$.

The function C_A^y of Definition 32.1 is called the **constant function** on A with value y. For example, the graph of $C_{\mathbb{R}}^1$ is the horizontal line through the point $(0,1)$. Another example:

THEOREM 32.2. $C_{\{2,5,9\}}^6 \;=\; \begin{pmatrix} 2 \mapsto 6 \\ 5 \mapsto 6 \\ 9 \mapsto 6 \end{pmatrix}$.

In class, we graphed $C_{\mathbb{R}}^1$.

DEFINITION 32.3. *$\forall$set S,* $\quad 0_S := C_S^0$.

DEFINITION 32.4. *Let A be a set. Then, $\forall\smile b$,*

$$A_b^\times \ := \ A\backslash\{b\} \qquad \textit{and} \qquad A_b^+ \ := \ A \cup \{b\}.$$

Also, $\forall b$, the set $A_b^\times$ is called A **punctured** *at b.*
Also, $\forall b$, the set A_b^+ is called A **filled** *by b, or A* **adjoin** *b.*

THEOREM 32.5. *Let $A := \{5,6,7,8\}$. Then*

$$\begin{array}{ll} A_5^\times = \{6,7,8\}, & A_9^\times = \{5,6,7,8\} \quad , \\ A_5^+ = \{5,6,7,8\}, & A_9^+ = \{5,6,7,8,9\} \quad . \end{array}$$

Let $X := (0;2)$.
In class, we graphed X and then $X_1^\times$ and then X_1^+ on a number line. We then graphed $X_3^\times$ and then X_3^+.

THEOREM 32.6. *Let f be a function.*
Then, $\forall\smile p, \forall\smile q$, $\exists 1$function g, denoted $\operatorname{adj}_p^q f$, s.t.

$$[\ \forall x, \quad (\, x \neq p\,) \ \Rightarrow \ (\, g(x) = f(x)\,)\ ] \qquad \textit{and} \qquad [\ g(p) = q\ ].$$

The function $\operatorname{adj}_p^q f$ is called the **adjustment** *of f sending p to q.*

THEOREM 32.7. *Let* $f \ := \ \begin{pmatrix} 1 \mapsto 4 \\ 2 \mapsto -1 \\ 5 \mapsto 6 \end{pmatrix}, \quad \phi \ := \ \begin{pmatrix} 0 \mapsto 3 \\ 1 \mapsto 4 \\ 2 \mapsto -1 \\ 5 \mapsto 6 \end{pmatrix}.$

$$\textit{Then} \quad \operatorname{adj}_0^7 f \ = \ \operatorname{adj}_0^7 \phi \ = \ \begin{pmatrix} 0 \mapsto 7 \\ 1 \mapsto 4 \\ 2 \mapsto -1 \\ 5 \mapsto 6 \end{pmatrix}.$$

THEOREM 32.8. *Define $f : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f(x) = x/x$.*
Then $\operatorname{adj}_0^1 f = C_{\mathbb{R}}^1$.

THEOREM 32.9. *Define $f : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f(x) = (x^2+x-2)/(x-1)$.*
Let $g := \operatorname{adj}_1^3 f$. Then, $\forall x \in \mathbb{R}_1^\times$, $f(x) = x+2$. Also, $f(1) = ☹$.
Also, $\forall x \in \mathbb{R}$, $g(x) = x+2$.

THEOREM 32.10. *$\forall$function f, $\forall\smile p$, $\forall\smile q$,*

$$\operatorname{dom}[(\operatorname{adj}_p^q f)] = (\operatorname{dom}[f])_p^+ \quad \textit{and} \quad \operatorname{im}[(\operatorname{adj}_p^q f)] \subseteq (\operatorname{im}[f])_q^+.$$

33. Finite and infinite sets

THEOREM 33.1. *Let S be a set. Let $A := \{j \in \mathbb{N}_0 \,|\, \exists S \hookrightarrow [1..j]\}$. Let $B := A^+_\infty$. Then $\varnothing \neq B \subseteq \mathbb{N}^*_0$.*

DEFINITION 33.2. *Let S be a set. Let $A := \{j \in \mathbb{N}_0 \,|\, \exists S \hookrightarrow [1..j]\}$. Let $B := A^+_\infty$. Then $\#S := \min B$.*

THEOREM 33.3. $\#\{2,7,9\} = 3$ *and* $\#\mathbb{Z} = \infty = \#\mathbb{R}$ *and* $\#\varnothing = 0$.

THEOREM 33.4. $\forall$*set S,* $\#S \in \mathbb{N}^*_0$.

DEFINITION 33.5. $\forall$*set S, by S is* **finite**, *we mean $\#S < \infty$.*

DEFINITION 33.6. $\forall$*set S, by S is* **infinite**, *we mean $\#S = \infty$.*

THEOREM 33.7. *Let S be a set, $k \in \mathbb{N}_0$.*

$$\textit{Then:}\quad (\,\#S = k\,) \Leftrightarrow (\,\exists[1..k] \hookrightarrow\!> S\,).$$

THEOREM 33.8. $\forall$*finite sets A, B,*

$$\textit{Then:}\quad (\,\#A = \#B\,) \Leftrightarrow (\,\exists A \hookrightarrow\!> B\,).$$

THEOREM 33.9. $\forall$*sets A, B,* $[\,(\,\exists A \hookrightarrow B\,) \vee (\,\exists B \hookrightarrow A\,)\,]$.

The next theorem is the **Schroeder-Bernstein Theorem**:

THEOREM 33.10. $\forall$*sets A, B,*

$$[\quad (\,\exists A \hookrightarrow B\,) \quad \& \quad (\,\exists B \hookrightarrow A\,) \quad] \quad \Rightarrow \quad [\quad \exists A \hookrightarrow\!> B \quad].$$

We described the "World of Sets", as a big blob on the board, with no top. Inside, sets that are at the same horizontal level are bijective. If one set is above another then there's an injection from the lower one to the upper one, but not the other way around. Inside, starting at the bottom, we showed the empty set, then singletons, then unordered pairs, etc., and then a dividing line between finite and infinite.

THEOREM 33.11. *Let S be a set.*

$$\textit{Then:}\quad (\,\#S = \infty\,) \Leftrightarrow (\,\exists \mathbb{N} \hookrightarrow S\,).$$

In the World of Sets, we showed $\mathbb{N}$ at the bottom of the infinite sets. Sets at or below $\mathbb{N}$ are said to be "countable". Sets above $\mathbb{N}$ are "uncountable". Sets at the same horizontal level as $\mathbb{N}$ are "countably infite". That is:

DEFINITION 33.12. *Let S be a set.*
Then S is **countable** *means: $\exists S \hookrightarrow \mathbb{N}$.*
Also, S is **countably infinite** *means: $\exists S \hookrightarrow\!> \mathbb{N}$.*
Also, S is **uncountable** *means: $\nexists S \hookrightarrow \mathbb{N}$.*

THEOREM 33.13. *Let S be a set. Then:*
$[\,S \text{ is countably infnite}\,] \Leftrightarrow [\,(\,S \text{ is countable}\,)\,\&\,(\,S \text{ is infnite}\,)\,]$.

THEOREM 33.14. *$\forall$countable set C, $\forall A \subseteq C$, A is countable.*

THEOREM 33.15. *N_0 and $\mathbb{Z}$ and $\mathbb{Q}$ are all countably infinite.*

We put $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ all at the same level as $\mathbb{N}$.

THEOREM 33.16. *$\forall$ sets A, $[\ (\,\exists A \hookrightarrow 2^A\,)\,\&\,(\,\nexists 2^A \hookrightarrow A\,)\ ]$.*

We put in $2^{\mathbb{N}}$ and $2^{2^{\mathbb{N}}}$, and explained that there is no top.

THEOREM 33.17. $\exists 2^{\mathbb{N}} \hookrightarrow\!\!> \mathbb{R}$.

We put in $\mathbb{R}$ at the same level as $2^{\mathbb{N}}$. Sets at that level are said to have "continuum cardinality":

DEFINITION 33.18. *Let S be a set.*
By S has **continuum cardinality**, *we mean: $\exists S \hookrightarrow\!\!> \mathbb{R}$.*

Any Euclidean space had continuum cardinality:

THEOREM 33.19. *$\forall k \in \mathbb{N}$, $\mathbb{R}^k$ has continuum cardinality.*

We put $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$ at the same level as $\mathbb{R}$.
Any nondegenerate interval has continuum cardinality:

THEOREM 33.20. *Let $a, b \in \mathbb{R}^*$. Assume $a < b$. Then:*
$[a; b]$, $[a; b)$, $(a; b]$ and $(a; b)$ all have continuum cardinality.

We put $[0; 1]$ and $(0; 1)$ at the same level as $\mathbb{R}$.

Within our axiom system, there is no way to determine if there are any sets strictly between $\mathbb{N}$ and $2^{\mathbb{N}}$. The assertion

$\nexists$set S s.t. $(\,(\exists \mathbb{N} \hookrightarrow S)\,\&\,(\exists S \hookrightarrow 2^{\mathbb{N}})\,\&\,(\nexists S \hookrightarrow \mathbb{N})\,\&\,(\nexists 2^{\mathbb{N}} \hookrightarrow S)\,)$

is called the **Continuum Hypothesis** or **CH**. The axiom system of this course is equivalent to a standard axiomatic system called ZFC. Within ZFC, it is impossible to prove CH, but it is also impossible to prove ¬CH. To convey this, one says: "CH is independent of ZFC".

Within our axiom system, $\forall$infinite set A, there is no way to determine if there are any sets strictly between A and 2^A. The **Generalized Continuum Hypothesis** or **GCH** is the assertion: $\forall$infinite set A,

$\nexists$set S s.t. $(\,(\exists A \hookrightarrow S)\,\&\,(\exists S \hookrightarrow 2^A)\,\&\,(\nexists S \hookrightarrow A)\,\&\,(\nexists 2^A \hookrightarrow S)\,)$

Within ZFC, it is impossible to prove GCH, but it is also impossible to prove ¬GCH. That is, GCH is independent of ZFC.

Here are a few important sets:

Let $c_0 := \varnothing$.
Let $c_1 := \{c_0\}$.
Let $c_2 := \{c_0, c_1\}$.
Let $c_3 := \{c_0, c_1, c_2\}$.
$\vdots$
Let $\aleph_0 := \{c_0, c_1, c_2, c_3, \ldots\}$.

Then c_0 is called the 0th cardinal number, and it is the only set at the bottom level of the World of Sets. The first cardinal number is c_1, and we will position it as the leftmost set at the level of singleton sets. The second cardinal number is c_2, and we will position it as the leftmost set at the level of unordered pairs. The third cardinal number is c_3, and we will position it as the leftmost set at the level of sets with three elements. The countably infinite cardinal number is $\aleph_0$, and we will position it as the leftmost set at the level of countably infinite sets.

We will not go into more detail here, but there is a system for producing exactly one cardinal number at each horizontal level in the World of Sets, and I like to position these sets on the left. The "cardinality" of a set is the unique cardinal number that is bijective with that set. Then two sets are bijective iff they have the same cardinality.

THEOREM 33.21. *$\forall$finite, nonempty $A \subseteq \mathbb{R}^*$,* min $A \neq$ ☹ $\neq$ max A.

Recall Theorem 27.4:

THEOREM 33.22. $\{0,1\}^{\{7,8,9\}} =$

$$\left\{ \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 0 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 0 \\ 9 \mapsto 1 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 1 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 1 \\ 9 \mapsto 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 0 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 0 \\ 9 \mapsto 1 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 1 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 1 \\ 9 \mapsto 1 \end{pmatrix} \right\}.$$

Recall Theorem 27.6:

THEOREM 33.23. $\{7,8,9\}^{\{1,2\}} =$

$$\left\{ \begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 7 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 8 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 9 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 1 \mapsto 8 \\ 2 \mapsto 7 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 8 \\ 2 \mapsto 8 \end{pmatrix}, \begin{pmatrix} 1 \mapsto 8 \\ 2 \mapsto 9 \end{pmatrix},$$

$$\left(\begin{array}{c}1\mapsto 9\\ 2\mapsto 7\end{array}\right),\left(\begin{array}{c}1\mapsto 9\\ 2\mapsto 8\end{array}\right),\left(\begin{array}{c}1\mapsto 9\\ 2\mapsto 9\end{array}\right)\right\} =$$

$$\{\,(7,7)\,,\,(7,8)\,,\,(7,9)\,,\;\,(8,7)\,,\,(8,8)\,,\,(8,9)\,,\;\,(9,7)\,,\,(9,8)\,,\,(9,9)\,\}.$$

THEOREM 33.24. *$\forall$finite sets A, B,* $\quad \#(B^A) = (\#B)^{\#A}$.

Recall: Theorem 27.2:

THEOREM 33.25. $2^{\{7,8,9\}} =$

$$\{\ \varnothing,\ \{9\},\ \{8\},\ \{8,9\},\ \{7\},\ \{7,9\},\ \{7,8\},\ \{7,8,9\}\ \}.$$

Recall Theorem 27.5:

THEOREM 33.26. *$\forall$set A,* $\exists\{0,1\}^A \hookrightarrow\!> 2^A$.

THEOREM 33.27. *$\forall$finite set A,* $\quad \#(2^A) = 2^{\#A}$.

34. Arithmetic of functionals

DEFINITION 34.1. *$\forall f$, by f is a* **functional**, *we mean:*
$$(\,f \text{ is a function}\,)\ \&\ (\,\mathrm{im}[f] \subseteq \mathbb{R}\,).$$

DEFINITION 34.2. *Let $a \in \mathbb{R}$ and let f be a functional.*
Then $a \cdot f$, a/f and f/a are the functionals defined by: $\forall x$,

$$(a \cdot f)(x) = a \cdot [f(x)],$$
$$(a/f)(x) = a/[f(x)] \qquad \text{and}$$
$$(f/a)(x) = [f(x)]/a.$$

We often write af instead of $a \cdot f$.

DEFINITION 34.3. *For any functional f, we define $-f := (-1) \cdot f$.*

DEFINITION 34.4. *Let f and g be functionals.*
Assume that $\mathrm{im}[f] \subseteq \mathbb{R}$ and that $\mathrm{im}[g] \subseteq \mathbb{R}$.
Then $f + g$, $f - g$, $f \cdot g$ and f/g are the functionals defined by: $\forall x$,

$$(f + g)(x) = [f(x)] + [g(x)],$$
$$(f - g)(x) = [f(x)] - [g(x)],$$
$$(f \cdot g)(x) = [f(x)] \cdot [g(x)] \qquad \text{and}$$
$$(f/g)(x) = [f(x)]/[g(x)].$$

We often write fg instead of $f \cdot g$.

THEOREM 34.5. $(1, 2, 3) + (4, 0, -3) = (5, 2, 0)$ *and*
$$(1, 2, 3) - (4, 0, -3) = (-3, 2, 6).$$

THEOREM 34.6. $3 \cdot (2,0,-3,1) = (6,0,-9,3)$.

THEOREM 34.7. $(6,0,-9,3)/3 = (2,0,-3,1)$.

35. Absolute value and dot product

THEOREM 35.1. $\forall x \in \mathbb{R}, \quad x^2 \geqslant 0$.

THEOREM 35.2. $\forall x \in \mathbb{R}, \quad (\, x^2 = 0 \,) \Leftrightarrow (\, x = 0 \,)$.

THEOREM 35.3. $\forall x \geqslant 0, \quad |x| = x$

THEOREM 35.4. $\forall x \leqslant 0, \quad |x| = -x$

THEOREM 35.5. $\forall x \in \mathbb{R}, \quad |x| = \sqrt{x^2}$.

THEOREM 35.6. *All of the following are true:*

(1) $\forall x \in \mathbb{R}, \quad [\, (\, x = 0 \,) \Leftrightarrow (\, |x| = 0 \,) \,]$.
(2) $\forall a \in \mathbb{R}, \forall x \in \mathbb{R}, \quad |ax| = |a| \cdot |x|$.
(3) $\forall x, y \in \mathbb{R}, \quad |x + y| \leqslant |x| + |y|$.

In Theorem 35.6,

(1) says that "$|\bullet|$ separates zero" ,
(2) says that "$|\bullet|$ is absolute homogeneous" and
(3) says that "$|\bullet|$ is subadditive" .

The three properties together say "$|\bullet|$ is a norm".

DEFINITION 35.7. $\forall k \in \mathbb{N}, \forall v, w \in \mathbb{R}^k$,

$$v \bullet w := v_1 w_1 + \cdots + v_k w_k.$$

Logic purist: Replace "$v_1 w_1 + \cdots + v_k w_k$" by "$\sum_1^k (vw)_\bullet$".

DEFINITION 35.8. $44 := 4 \cdot 10 + 4$.

THEOREM 35.9. $(1,3,5) \bullet (2,4,6) = 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 = 44$.

THEOREM 35.10. $\forall k \in \mathbb{N}, \forall v, w \in \mathbb{R}^k, \quad v \bullet w = w \bullet v$.

THEOREM 35.11. $\forall k \in \mathbb{N}, \forall u, v, w \in \mathbb{R}^k$,

$$u \bullet (v + w) = (u \bullet v) + (u \bullet w).$$

THEOREM 35.12. $\forall k \in \mathbb{N}, \forall a \in \mathbb{R}, \forall v, w \in \mathbb{R}^k$,

$$(av) \bullet w = a \cdot (v \bullet w).$$

THEOREM 35.13. $\forall k \in \mathbb{N}, \forall v \in \mathbb{R}^k, \quad v \bullet v \geqslant 0$.

36. Standard norms on Euclidean spaces

FOR NEXT YEAR: Define $|v|_p := (|v_1|^p + \cdots + |v_k|^p)^{1/p}$ and note that $|v|_2 = \sqrt{v \bullet v}$. Don't use $|v|_k$; instead, use $|v|_2$. LATER, don't use $|v|_{k,p}$; instead, use $|v|_p$.

DEFINITION 36.1. *$\forall k \in \mathbb{N}$, $\forall v \in \mathbb{R}^k$, we define $|v|_k := \sqrt{v \bullet v}$.*

THEOREM 36.2. *$\forall k \in \mathbb{N}$, $\forall v \in \mathbb{R}^k$, $\quad v \bullet v = |v|_k^2$.*

THEOREM 36.3. $|(3,4)|_2 = \sqrt{(3,4) \bullet (3,4)} = \sqrt{3^2 + 4^2} = 5$.

DEFINITION 36.4. $194 := 1 \cdot 100 + 9 \cdot 10 + 4$.

THEOREM 36.5. $|(7,8,9)|_3 = \sqrt{7^2 + 8^2 + 9^2} = \sqrt{194}$.

Recall (Definition 32.3) that, $\forall$set S, we defined $0_S := C_S^0$.

DEFINITION 36.6. *$\forall k \in \mathbb{N}$, $\qquad 0_k := 0_{[1..k]}$.*

Then $0_2 = (0,0)$ and $0_3 = (0,0,0)$ and $0_4 = (0,0,0,0)$, *etc.*

THEOREM 36.7. *Let $k \in \mathbb{N}$. Then all of the following are true:*

(1) $\forall v \in \mathbb{R}^k$, $\qquad [\,(\,v = 0_k\,) \Leftrightarrow (\,|v|_k = 0\,)\,]$.
(2) $\forall a \in \mathbb{R}$, $\forall v \in \mathbb{R}^k$, $\qquad |av|_k = |a| \cdot |v|_k$.
(3) $\forall v, w \in \mathbb{R}^k$, $\qquad |v + w|_k \leqslant |v|_k + |w|_k$.

Let $k \in \mathbb{N}$. In Theorem 36.7,

(1) says that "$|\bullet|_k$ separates zero" ,
(2) says that "$|\bullet|_k$ is absolute homogeneous" and
(3) says that "$|\bullet|_k$ is subadditive" .

The three properties together say "$|\bullet|_k$ is a norm".

We sometimes refer to the absolute value function, $|\bullet| : \mathbb{R} \to [0;\infty)$, as the **standard norm on** $\mathbb{R}$. For all $k \in \mathbb{N}$, the **standard norm on** $\mathbb{R}^k$ is $|\bullet|_k : \mathbb{R}^k \to [0;\infty)$. Some use "Euclidean norm" instead of standard norm.

THEOREM 36.8. *Let $k \in \mathbb{N}$, $v \in \mathbb{R}^k$. Let $a := |v|_k$.*
Then $\exists u \in \mathbb{R}^k$ s.t.: $\qquad (\,|u|_k = 1\,)$ & $(\,v = au\,)$.

Proof. One of the following is true:

(1) $v = 0_k$ or
(2) $v \neq 0_k$.

Case (1):
Let $u := (1, 0, \ldots, 0)$.
Want: $(\,|u|_k = 1\,)\ \&\ (\,v = au\,)$.
We have $|u|_k = \sqrt{1^2 + 0^2 + \cdots + 0^2} = 1$.
Want: $v = au$.
We have $v = 0_k = 0 \cdot u = au$.
End of Case (1).

Case (2):
Since $v \neq 0_k$, we see that $|v|_k \neq 0$.
Then $a = |v|_k \neq 0$.
So, since $a \in [0; \infty)$, we conclude that $a \in \mathbb{R}_0^\times$.
Let $u := v/a$.
Want: $(\,|u|_k = 1\,)\ \&\ (\,v = au\,)$.
We have $|u|_k = |v/a|_k = (|v|_k)/a = a/a = 1$.
Want: $v = au$.
We have $v = a \cdot (v/a) = au$.
End of Case (2). ∎

The following theorem is the **Cauchy-Schwarz inequality**:

THEOREM 36.9. *$\forall k \in \mathbb{N}$, $\forall v, w \in \mathbb{R}^k$, $\quad |v \bullet w| \ \leqslant\ |v|_k \cdot |w|_k$.*

Proof. Let $\quad a := |v|_k \quad$ and $\quad b := |w|_k$.
By Theorem 36.8, choose $t \in \mathbb{R}^k$ s.t. $|t|_k = 1$ and $v = at$.
By Theorem 36.8, choose $u \in \mathbb{R}^k$ s.t. $|u|_k = 1$ and $w = bu$.
Then $t \bullet t = |t|_k^2 = 1^2 = 1$. Also, $u \bullet u = |u|_k^2 = 1^2 = 1$.
We have $(t - u) \bullet (t - u) \geqslant 0$.
Expanding this, we get $1 - 2 \cdot (t \bullet u) + 1 \geqslant 0$, so $2 - 2 \cdot (t \bullet u) \geqslant 0$.
Then $2 \geqslant 2 \cdot (t \bullet u)$, so $1 \geqslant t \bullet u$.
Then $t \bullet u \leqslant 1$, so $(ab) \cdot (t \bullet u) \leqslant ab$.
Then $v \bullet w = (at) \bullet (bu) = (ab) \cdot (t \bullet u) \leqslant ab = |v|_k \cdot |w|_k$. ∎

37. Unassigned homework

THEOREM 37.1. *$\forall a, z \in \mathbb{R}$, $[\,|a| \leqslant z\,] \Leftrightarrow [\,(\,a \leqslant z\,) \,\&\, (\,-a \leqslant z\,)\,]$.*

38. Metric spaces

FOR NEXT YEAR: Put nonemptyness as part of the definition of a metric space.

FOR NEXT YEAR: Maybe we should define "extended metric" as a function $d : S \times S \to [0;\infty]$ with the same properties as a metric, but with the target including ∞. So, some points could be at an infinite distance from others. This would allow for a standard *extended* metric on $\mathbb{R}^*$. We'd need to define: $\forall a \in \mathbb{R}^+_\infty$, $a + \infty = \infty$ to make the triangle inequality make sense.

DEFINITION 38.1. *Let S be a set, and let $d : S \times S \to [0;\infty)$. By d is a* **metric** *on S, we mean:*

(1) $\forall x, y \in S$, $\quad ([x = y] \Leftrightarrow [d(x, y) = 0])$,
(2) $\forall x, y \in S$, $\quad d(x, y) = d(y, x)$ *and*
(3) $\forall x, y, z \in S$, $\quad d(x, z) \leqslant [d(x, y)] + [d(y, z)]$.

In Definition 38.1,

(1) says that "d separates points" ,
(2) says that "d is symmetric" and
(3) says that "d satisfies the triangle inequality" .

DEFINITION 38.2. *For any set S,*

$$\mathcal{M}(S) \quad := \quad \{d : S \times S \to [0;\infty) \,|\, d \text{ is a metric on } S\}.$$

The logic purist would object because, according to our Axioms of Specification, in Definition 38.2, we should write "$\{d \in \ldots | \ldots\}$". To fix this, we could write

$$\mathcal{M}(S) \quad := \quad \{d \in [0;\infty)^{S\times S} \,|\, d \text{ is a metric on S}\}.$$

THEOREM 38.3. *$\exists 1 d \in \mathcal{M}(\mathbb{R})$, denoted d_0,*

$$\text{s.t.}, \ \forall x, y \in \mathbb{R}, \quad d(x, y) = |y - x|.$$

We call d_0 the **standard metric on** $\mathbb{R}$*.*

THEOREM 38.4. *Let $k \in \mathbb{N}$. Then $\exists 1 d \in \mathcal{M}(\mathbb{R}^k)$, denoted d_k,*

$$\text{s.t.}, \ \forall v, w \in \mathbb{R}^k, \quad d(v, w) = |w - v|_k.$$

We call d_k the **standard metric on** $\mathbb{R}^k$*.*

DEFINITION 38.5. *A* **metric space** *is an ordered pair (S, d) s.t.*

$$S \text{ is a set} \qquad \text{and} \qquad d \in \mathcal{M}(S).$$

DEFINITION 38.6. *Let X be a metric space.*
Then $X_{\text{set}} := X_1$ and $d_X := X_2$.
Also, X_{set} is called the **underlying set** *of X.*
Also, d_X is called the **metric** *on X.*

We sometimes omit the subscript "X" from "d_X".

We almost always omit the subscript "set" from "X_{set}", so, by sloppiness, the underlying set X_{set} of X is often denoted X. This means that X has two different meanings, and, in each usage, you have to figure out, by context, which X is intended. For example, if you see "d_X", then X is a metric space. On the other hand, if you see "$a : \mathbb{R} \to X$" or "$b \in X$", then X is a set.

For your confusion, $(\mathbb{R}, d_0)$ is denoted $\mathbb{R}$. Then $d_0 = d_{\mathbb{R}}$.

Let $k \in \mathbb{N}$. For confusion, $(\mathbb{R}^k, d_k)$ is denoted $\mathbb{R}^k$. Then $d_k = d_{\mathbb{R}^k}$.

THEOREM 38.7. $d_{\mathbb{R}}(5,7) = 2$ *and* $d_{\mathbb{R}}(9,3) = 6$ *and* $d_2((1,7),(4,3)) = \sqrt{(1-4)^2 - (7-3)^2} = 5$.

A basic property of $|\bullet|$ is that it is "distance semi-decreasing":

THEOREM 38.8. *$\forall x, y \in \mathbb{R}$, $d_{\mathbb{R}}(|x|,|y|) \leqslant d_{\mathbb{R}}(x,y)$.*

Proof. Unassigned HW. □

According to Theorem 38.8, $\forall x, y \in \mathbb{R}$,

$$|\ |x| - |y|\ | \leqslant |\ x - y\ |.$$

For each $k \in \mathbb{N}$, $|\bullet|_k$ is also "distance semi-decreasing":

THEOREM 38.9. *$\forall k \in \mathbb{N}$, $\forall v, w \in \mathbb{R}^k$, $d_{\mathbb{R}}(|v|_k, |w|_k) \leqslant d_k(v,w)$.*

Proof. Given $k \in \mathbb{N}$ and $v, w \in \mathbb{R}^k$.
Want: $d_{\mathbb{R}}(|v|_k, |w|_k) \leqslant d_k(v,w)$.
Let $a := |v|_k - |w|_k$ and let $z := |v - w|_k$.
Then $d_{\mathbb{R}}(|v|_k, |w|_k) = |a|$ and $d_k(v,w) = z$.
Want: $|a| \leqslant z$. Want: $a \leqslant z$ and $-a \leqslant z$.
We have $|v|_k = |w + (v - w)|_k \leqslant |w|_k + |v - w|_k = |w|_k + z$.
Subtracting $|w|_k$ from both sides, we see that $a \leqslant z$.
Want: $-a \leqslant z$.
We have $|w - v|_k = |v - w|_k = z$ and $-a = |w|_k - |v|_k$.
Then $|w|_k = |v + (w - v)|_k \leqslant |v|_k + |w - v|_k = |v|_k + z$.
Subtracting $|v|_k$ from both sides, we see that $-a \leqslant z$, as desired. □

According to Theorem 38.9, $\forall k \in \mathbb{N}$, $\forall v, w \in \mathbb{R}^k$,

$$|\ |v|_k - |w|_k\ | \leqslant |\ v - w\ |_k.$$

DEFINITION 38.10. *Let X be a metric space, $z \in X$, $r > 0$. Then $B_X(z,r) := \{q \in X \mid d_X(z,q) < r\}$.*

We sometimes omit the subscript "X" from "$B_X(z,r)$".

The set $B_X(z,r)$ of Definition 38.10 is called: the **open ball** about z of radius r. We sometimes omit "open" and simply say "**ball** about z of radius r".

THEOREM 38.11. *$\forall a \in \mathbb{R}$, $\forall \delta > 0$, $\quad B_{\mathbb{R}}(a,\delta) = (a-\delta; a+\delta)$.*

The next result says that any two points in a metric space can be separated by balls of equal radii.

THEOREM 38.12. *Let X be a metric space, and let $y, z \in X$. Assume $y \neq z$. Then $\exists r > 0$ s.t. $[B_X(y,r)] \cap [B_X(z,r)] = \varnothing$.*

Proof. This is HW#6-2. □

Theorem 38.12, above, is the **Hausdorff property** of metric spaces.

DEFINITION 38.13. *Let X be a metric space and let $z \in X$. We define $\mathcal{B}_X(z) := \{B(z,r) \mid r > 0\}$.*

We sometimes omit the subscript "X" from "$\mathcal{B}_X(z)$".

THEOREM 38.14. *Let X be a metric space and let $p, q \in X$. Assume that $p \neq q$. Then $\exists A \in \mathcal{B}_X(p)$, $\exists B \in \mathcal{B}_X(q)$ s.t. $A \cap B = \varnothing$.*

Proof. By Theorem 38.12,
choose $r > 0$ s.t. $[B(y,r)] \cap [B(z,r)] = \varnothing$. Let $A := B(y,r)$ and $B := B(z,r)$.
Want: $A \cap B = \varnothing$.
We have: $A \cap B = [B(y,r)] \cap [B(z,r)] = \varnothing$, as desired. □

DEFINITION 38.15. *Let X be a metric space.*

Then $\quad \mathcal{B}_X \; := \; \{\, B(z,r) \subseteq X \mid z \in X,\; r > 0 \,\}$.

THEOREM 38.16. *Let X be a metric space, $B \in \mathcal{B}_X$ and $p \in B$. Then $\exists A \in \mathcal{B}(p)$ s.t. $A \subseteq B$.*

Proof. Since $B \in \mathcal{B}_X$, choose $q \in X$ and $t > 0$ s.t. $B = B(q,t)$.
Since $p \in B = B(q,t)$, we get $d(p,q) < t$.
Let $s := d(p,q)$. Then $s < t$.
Let $r := t - s$. Then $r > 0$ and $r + s = t$.
Let $A := B(p,r)$. Then $A \in \mathcal{B}(p)$.
Want: $A \subseteq B$.
Want: $\forall z \in A$, $z \in B$.

Given $z \in A$. Want: $z \in B$.
Since $z \in A = B(p,r)$, we get $d(z,p) < r$.
Since $d(z,p) < r$ and $d(p,q) = s$,
we get $[d(z,p)] + [d(p,q)] < r + s$.
Then $d(z,q) \leqslant [d(z,p)] + [d(p,q)] < r + s = t$, so $z \in B(q,t)$.
Then $z \in B(q,t) = B$, as desired. □

THEOREM 38.17. *Let X be a metric space, $A \in \mathcal{B}_X$ and $q \in X$. Then $\exists B \in \mathcal{B}(q)$ s.t. $B \supseteq A$.*

Proof. This is HW#6-3. □

Theorem 38.16 is called the **Subset Recentering Lemma**.
Theorem 38.17 is called the **Superset Recentering Lemma**.

DEFINITION 38.18. *Let X be a metric space and let $S \subseteq X$. By S is **bounded** in X, we mean: $\exists B \in \mathcal{B}_X$ s.t. $S \subseteq B$.*

The following is the same as Theorem 31.10:

THEOREM 38.19. *The following are all true:*
$[1;\infty)$ is bounded below in $\mathbb{R}$, but not bounded above in $\mathbb{R}$,
$\mathbb{N}$ is bounded below in $\mathbb{R}$, but not bounded above in $\mathbb{R}$,
$(-\infty;5)$ is bounded above in $\mathbb{R}$, but not bounded below in $\mathbb{R}$,
$\mathbb{Z}$ is neither bounded above nor bounded below in $\mathbb{R}$ and
$(2;5]$ is both bounded above and bounded below in $\mathbb{R}$.
$\varnothing$ is both bounded above and bounded below in $\mathbb{R}$.

THEOREM 38.20. *The following are all true:*
$[1;\infty)$ is not bounded in $\mathbb{R}$,
$\mathbb{N}$ is not bounded in $\mathbb{R}$,
$(-\infty;5)$ is not bounded in $\mathbb{R}$,
$\mathbb{Z}$ is not bounded in $\mathbb{R}$ and
$(2;5]$ bounded in $\mathbb{R}$.
$\varnothing$ bounded in $\mathbb{R}$.

THEOREM 38.21. *Let $S \subseteq \mathbb{R}$. Then:*
$$[\ (S \text{ is bounded in } \mathbb{R}) \quad \Leftrightarrow$$
$$(\,(\,S \text{ is bounded below in } \mathbb{R}\,)\,\&\,(\,S \text{ is bounded above in } \mathbb{R}\,)\,)\ ].$$

39. Sequences

DEFINITION 39.1. *$\forall a$, by a is a **sequence**, we mean: a is a function and $\mathrm{dom}[a] = \mathbb{N}$.*

DEFINITION 39.2. *$\forall a$, $\forall X$, by a is a* **sequence** *in X, we mean:*
$a \in X^{\mathbb{N}}$.

DEFINITION 39.3. *Let f be a function and let X be a set.*
*By f is X-***valued***, we mean* $\mathrm{im}[f] \subseteq X$.

DEFINITION 39.4. *Let X be a metric space and let f be an X-valued function.*
By f is **bounded into** *X, we mean:*
$\mathrm{im}[f]$ *is bounded in X.*

We sometimes say "bounded in X" instead of "bounded into X".

FOR NEXT YEAR, let's just write "$a \to z$ in X", not "$a_\bullet \to z$ in X". Also, use s instead of a; think of a is indicating a sequence of reals, and s as a more general sequence.

DEFINITION 39.5. *Let X be a metric space, $a \in X^{\mathbb{N}}$ and $z \in X$.*
By $a_\bullet \to z$ in X, we mean: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ *s.t.*, $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d(a_j, z) < \varepsilon\ ).$$

We sometimes omit "in X" in "$a_\bullet \to z$ in X". For any sequence a, we sometimes denote a by $(a_1, a_2, a_3, \ldots)$. Then, for example, the text
Define $a \in \mathbb{R}^{\mathbb{N}}$ by $a_j = 1/j$
might be replaced by
Let $a := (1, 1/2, 1/3, \ldots)$.
This is very irksome to the logic purist who does not like ellipses.

THEOREM 39.6. $(1, 1/2, 1/3, \ldots)_\bullet \to 0$ *in* $\mathbb{R}$.

The purist would prefer:

THEOREM 39.7. *Define $a \in \mathbb{R}^{\mathbb{N}}$ by $a_j = 1/j$. Then $a_\bullet \to 0$ in $\mathbb{R}$.*

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d(a_j, 0) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d(a_j, 0) < \varepsilon\ ).$$
By the Archimedean Principle (Theorem 19.1),
choose $K \in \mathbb{N}$ s.t. $K > 1/\varepsilon$.
Want: $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ d(a_j, 0) < \varepsilon\ )\ ]$.
Given $j \in \mathbb{N}$. Want: $(\ j \geqslant K\ ) \Rightarrow (\ d(a_j, 0) < \varepsilon\ )$.
Assume $j \geqslant K$. Want: $d(a_j, 0) < \varepsilon$.
Since $j \geqslant K > 1/\varepsilon$, we get $j > 1/\varepsilon$.

Since $j > 1/\varepsilon > 0$, we get $1/j < \varepsilon$.
Since $a_j = 1/j > 0$, we get $|a_j| = a_j$.
Then $d(a_j, 0) = |a_j - 0| = |a_j| = a_j = 1/j < \varepsilon$, as desired. □

DEFINITION 39.8. *Let X be a metric space and let $a \in X^{\mathbb{N}}$. Then a is* **convergent** *in X means:*
$\exists z \in X$ s.t. $a_\bullet \to z$ in X.

Sometimes "in X" is omitted from "convergent in X".
From Definition 38.18 and Definition 39.4, we have:

THEOREM 39.9. *$\forall$metric space X, $\forall a \in X^{\mathbb{N}}$,*
$$(a \text{ is bounded in } X) \Leftrightarrow (\text{im}[a] \text{ is bounded in } X) \Leftrightarrow (\exists S \in \mathcal{B}_X \text{ s.t. } \text{im}[a] \subseteq S).$$

A bounded sequence is not necessarily convergent:

THEOREM 39.10. *Define $a \in \mathbb{R}^N$ by $a_j = (-1)^j$. Then: $a = (-1, 1, -1, 1-1, 1-1, 1-1, 1-1, 1, \ldots)$ and*
a is bounded in $\mathbb{R}$ and
a is not convergent in $\mathbb{R}$.

THEOREM 39.11. *Let X be a metric space, $a \in X^{\mathbb{N}}$ and $z \in X$. Assume that $a_\bullet \to z$ in X. Then: $\forall B \in \mathcal{B}_X(z)$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,*
$$(\ j \geqslant K\ ) \Rightarrow (\ a_j \in B\ ).$$

Proof. Given $B \in \mathcal{B}_X(z)$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \Rightarrow (\ a_j \in B\ ).$$
Since $B \in \mathcal{B}_X(z)$, choose $\varepsilon > 0$ s.t. $B = B_X(z, \varepsilon)$.
Since $a_\bullet \to z$ in X, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \Rightarrow (\ d_X(a_j, z) < \varepsilon\ ).$$
Want: $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ a_j \in B\ )\ ]$.
Given $j \in \mathbb{N}$. Want: $(\ j \geqslant K\ ) \Rightarrow (\ a_j \in B\ )$.
Assume $j \geqslant K$. Want: $a_j \in B$.
Since $j \geqslant K$, by choice of K,
we have $d_X(a_j, z) < \varepsilon$, and so $a_j \in B_X(z, \varepsilon)$.
Then $a_j \in B_X(z, \varepsilon) = B$, as desired. □

THEOREM 39.12. *Let X be a metric space, $p, q \in X$ and $s \in X^{\mathbb{N}}$. Assume: $(\ s_\bullet \to p$ in $X\ )$ & $(\ s_\bullet \to q$ in $X\ )$. Then $p = q$.*

Proof. Assume that $p \neq q$. Want: Contradiction.
By Theorem 38.14, choose $A \in \mathcal{B}_X(p)$ and $B \in \mathcal{B}_X(q)$ s.t. $A \cap B = \varnothing$.
Since $s_\bullet \to p$ in X, by Theorem 39.11, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ s_j \in A\ ).$$
Since $s_\bullet \to q$ in X, by Theorem 39.11, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant L\ ) \ \Rightarrow\ (\ s_j \in B\ ).$$
Let $j := \max\{K, L\}$. Then $j \in \mathbb{N}$.
Since $j \geqslant K$, by choice of K, we have $s_j \in A$.
Since $j \geqslant L$, by choice of L, we have $s_j \in B$.
Then $s_j \in A \cap B$, so $A \cap B \neq \varnothing$.
By choice of A and B, we have: $A \cap B = \varnothing$. Contradiction. □

40. Properties of limits

THEOREM 40.1. *Let $s, t \in \mathbb{R}^{\mathbb{N}}$ and let $x, y \in \mathbb{R}$.*
Assume: ($s_\bullet \to x$ in $\mathbb{R}$) & ($t_\bullet \to y$ in $\mathbb{R}$).
Then $(s + t)_\bullet \to x + y$ in $\mathbb{R}$.

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_{\mathbb{R}}((s+t)_j, x+y) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_{\mathbb{R}}((s+t)_j, x+y) < \varepsilon\ ).$$
Since $s_j \to x$ in $\mathbb{R}$, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant L\ ) \quad \Rightarrow \quad (\ d_{\mathbb{R}}(s_j, x) < \varepsilon/2\ ).$$
Since $t_j \to y$ in $\mathbb{R}$, choose $M \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant M\ ) \quad \Rightarrow \quad (\ d_{\mathbb{R}}(t_j, y) < \varepsilon/2\ ).$$
Let $K := \max\{L, M\}$. Then $K \in \mathbb{N}$ and $K \geqslant L$ and $K \geqslant M$.
Want: $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ d_{\mathbb{R}}((s+t)_j, x+y) < \varepsilon\ )\ ]$.
Given $j \in \mathbb{N}$. Want: $(\ j \geqslant K\ ) \Rightarrow (\ d_{\mathbb{R}}((s+t)_j, x+y) < \varepsilon\ )$.
Assume $j \geqslant K$. Want: $d_{\mathbb{R}}((s+t)_j, x+y) < \varepsilon$.
Since $j \geqslant K \geqslant L$, by choice of L, we have $d_{\mathbb{R}}(s_j, x) < \varepsilon/2$.
Since $j \geqslant K \geqslant M$, by choice of M, we have $d_{\mathbb{R}}(t_j, y) < \varepsilon/2$.
Then
$$\begin{aligned} d_{\mathbb{R}}((s+t)_j, x+y) &= d_{\mathbb{R}}(s_j + t_j, x+y) \\ &= |\,(s_j + t_j) - (x + y)\,| = |\,(s_j - x) + (t_j - y)\,| \\ &\leqslant |\,s_j - x\,| + |\,t_j - y\,| = [\,d_{\mathbb{R}}(s_j, x)\,] + [\,d_{\mathbb{R}}(t_j, y)\,] \\ &< [\,\varepsilon/2\,] + [\,\varepsilon/2)\,] = \varepsilon, \text{ as desired.} \end{aligned}$$
□

THEOREM 40.2. *Let $s \in \mathbb{R}^{\mathbb{N}}$ and let $a, y \in \mathbb{R}$.*
Assume that $s_\bullet \to x$ in $\mathbb{R}$. Then $(as)_\bullet \to ax$ in $\mathbb{R}$.

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$(\; j \geqslant K \;) \quad \Rightarrow \quad (\; d_{\mathbb{R}}((as)_j, ax) < \varepsilon \;)$.

Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$(\; j \geqslant K \;) \quad \Rightarrow \quad (\; d_{\mathbb{R}}((as)_j, ax) < \varepsilon \;)$.

Let $b := |a| + 1$. Then $b > 0$ and $|a|/b < 1$.

Since $s_j \to x$ in $\mathbb{R}$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$(\; j \geqslant K \;) \quad \Rightarrow \quad (\; d_{\mathbb{R}}(s_j, x) < \varepsilon/b \;)$.

Want: $\forall j \in \mathbb{N}$, $[\; (\; j \geqslant K \;) \; \Rightarrow \; (\; d_{\mathbb{R}}((as)_j, ax) < \varepsilon \;) \;]$.

Given $j \in \mathbb{N}$. Want: $(\; j \geqslant K \;) \; \Rightarrow \; (\; d_{\mathbb{R}}((as)_j, ax) < \varepsilon \;)$.

Assume $j \geqslant K$. Want: $d_{\mathbb{R}}((as)_j, ax) < \varepsilon$.

Since $j \geqslant K$, by choice of K, we have $d_{\mathbb{R}}(s_j, x) < \varepsilon/b$.

So, since $|a| \geqslant 0$, we get $|a| \cdot [d_{\mathbb{R}}(s_j, x)] \leqslant |a| \cdot [\varepsilon/b]$.

Since $|a|/b < 1$ and $\varepsilon > 0$, we get $[|a|/b] \cdot \varepsilon < \varepsilon$.

Then $d_{\mathbb{R}}((as)_j, ax) \; = \; d_{\mathbb{R}}(a \cdot s_j, a \cdot x) \; = \; | \, a \cdot s_j \; - \; a \cdot x \, |$

$= \; | \, a \cdot (s_j - x) \, | \; = \; |a| \cdot |s_j - x| \; = \; |a| \cdot [d_{\mathbb{R}}(s_j, x)]$

$\leqslant \; |a| \cdot [\varepsilon/b] \; = \; [|a|/b] \cdot \varepsilon \; < \; \varepsilon$, as desired. □

THEOREM 40.3. *Let X be a metric space and let $z \in X$. Then $C_{\mathbb{N}}^y \to y$ in X.*

Proof. Unassigned HW. □

THEOREM 40.4. *Let $s, t \in \mathbb{R}^{\mathbb{N}}$ and let $x, y \in \mathbb{R}$. Assume: ($s_\bullet \to x$ in $\mathbb{R}$) & ($t_\bullet \to y$ in $\mathbb{R}$). Then $(s - t)_\bullet \to x - y$ in $\mathbb{R}$.*

Proof. Unassigned HW. □

THEOREM 40.5. *Let $s, t \in \mathbb{R}^{\mathbb{N}}$ and $x, y \in \mathbb{R}$. Assume that $s_\bullet \to x$ in $\mathbb{R}$ and that $t_\bullet \to y$ in $\mathbb{R}$. Then $(st)_\bullet \to xy$ in $\mathbb{R}$.*

Proof. Since $s_\bullet$ is convergent in $\mathbb{R}$, by HW#6-4, $s_\bullet$ is bounded in $\mathbb{R}$.

Let $c := C_{\mathbb{N}}^y$. By Theorem 40.3, $c_\bullet \to y$ in $\mathbb{R}$.

So, since $t_\bullet \to y$ in $\mathbb{R}$ and since $y - y = 0$, by Theorem 40.4,

we see that $(t - c)_\bullet \to 0$ in $\mathbb{R}$.

So, since $s_\bullet$ is bounded in $\mathbb{R}$, by HW#6-5,

we see that $(s \cdot (t - c))_\bullet \to 0$ in $\mathbb{R}$.

So, since $s \cdot (t - c) = st - sc$, we get $(st - sc)_\bullet \to 0$ in $\mathbb{R}$.

Since $s_\bullet \to x$ in $\mathbb{R}$, by Theorem 40.2, we get $ys \to yx$ in $\mathbb{R}$.

So, since $ys = cs = sc$ and since $yx = xy$, we get $sc \to xy$ in $\mathbb{R}$.

So, since $(st - sc)_\bullet \to 0$ in $\mathbb{R}$, by Theorem 40.1,
we see that $(sc + st - sc)_\bullet \to xy + 0$ in $\mathbb{R}$.
So, since $sc + st - sc = st$ and since $xy + 0 = xy$,
we see that $(st)_\bullet \to xy$ in $\mathbb{R}$, as desired. □

Recall (Theorem 38.8): $|\bullet|$ is distance semi-decreasing.
That is, $\forall x, y \in \mathbb{R}$, we have: $d(|x|, |y|) \leqslant d(x, y)$.

THEOREM 40.6. *Let $b \in (\mathbb{R}_0^\times)^\mathbb{N}$ and $z \in \mathbb{R}_0^\times$.
Assume that $b_\bullet \to z$ in $\mathbb{R}$.
Then $(1/b)_\bullet \to 1/z$ in $\mathbb{R}$.*

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_\mathbb{R}((1/b)_j, 1/z) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_\mathbb{R}((1/b)_j, 1/z) < \varepsilon\ ).$$
Let $\eta := \min\{\,|z|/2\,,\ \varepsilon z^2/2\,\}$.
Then $\eta \leqslant |z|/2$ and $\eta < \varepsilon z^2/2$.
Since $z \in \mathbb{R}_0^\times$, we get $|z| > 0$ and $z^2 > 0$.
So, since $\varepsilon > 0$, we get $\eta > 0$.
So, since $b_j \to z$ in $\mathbb{R}$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_\mathbb{R}(b_j, z) < \eta\ ).$$
Want: $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ d_\mathbb{R}((1/b)_j, 1/z) < \varepsilon\ )\ ]$.
Given $j \in \mathbb{N}$. Want: $(\ j \geqslant K\ ) \Rightarrow (\ d_\mathbb{R}((1/b)_j, 1/z) < \varepsilon\ )$.
Assume $j \geqslant K$. Want: $d_\mathbb{R}((1/b)_j, 1/z) < \varepsilon$.
Since $j \geqslant K$, it follows, from the choice of K, that $d_\mathbb{R}(b_j, z) < \eta$.
Then $|z - b_j| = d_\mathbb{R}(z, b_j) = d_\mathbb{R}(b_j, z) < \eta$.
By Theorem 38.8, $d_\mathbb{R}(|b_j|, |z|) \leqslant d_\mathbb{R}(b_j, z)$.
Since $d_\mathbb{R}(|b_j|, |z|) \leqslant d_\mathbb{R}(b_j, z) < \eta$, we get $|z| - \eta \leqslant |b_j| \leqslant |z| + \eta$.
Since $\eta \leqslant |z|/2$, we get $|z| - \eta \geqslant |z| - (|z|/2) = |z|/2$.
Then $|b_j| > |z| - \eta \geqslant |z| - (|z|/2) = |z|/2$.
So, since $|z - b_j| < \eta$, we get: $\dfrac{|z - b_j|}{|b_j| \cdot |z|} < \dfrac{\eta}{(|z|/2) \cdot |z|}$.
Then
$$d_\mathbb{R}((1/b)_j, 1/z) = d_\mathbb{R}\left(\frac{1}{b_j}, \frac{1}{z}\right) = \left|\frac{1}{b_j} - \frac{1}{z}\right| = \left|\frac{z - b_j}{b_j z}\right|$$
$$= \frac{|z - b_j|}{|b_j| \cdot |z|} < \frac{\eta}{(|z|/2) \cdot |z|} = \frac{2 \cdot \eta}{|z|^2} = \frac{2 \cdot \eta}{z^2} \leqslant \frac{2 \cdot (\varepsilon z^2/2))}{z^2} = \varepsilon.$$ □

THEOREM 40.7. *Let $a \in \mathbb{R}^\mathbb{N}$, $y \in \mathbb{R}$, $b \in (\mathbb{R}_0^\times)T\mathbb{N}$, $z \in \mathbb{R}_0^\times$. Assume that $a_\bullet \to y$ in $\mathbb{R}$ and that $b_\bullet \to z$ in $\mathbb{R}$. Then $(a/b)_\bullet \to y/z$ in $\mathbb{R}$.*

Proof. By Theorem 40.6, $(1/b)_\bullet \to 1/z$ in $\mathbb{R}$.
So, since $a_\bullet \to y$ in $\mathbb{R}$, by Theorem 40.5, $((1/b) \cdot a)_\bullet \to (1/z) \cdot y$ in $\mathbb{R}$.
So, since $(1/z) \cdot y = y/z$, we get: $((1/b) \cdot a)_\bullet \to y/z$ in $\mathbb{R}$.
It therefore suffices to show: $a/b = (1/b) \cdot a$.
Want: $\forall j \in \mathbb{N}$, $(a/b)_j = ((1/b) \cdot a)_j$.
Given $j \in \mathbb{N}$. Want: $(a/b)_j = ((1/b) \cdot a)_j$.
We have $(a/b)_j = a_j/b_j = (1/b_j) \cdot a_j = ((1/b) \cdot a)_j$, as desired. □

41. Diamond and square norms

DEFINITION 41.1. *$\forall k \in \mathbb{N}$, $\forall v \in \mathbb{R}^k$, we define:*
$|v|_k^D := |v_1| + \cdots + |v_k|$ *and*
$|v|_k^S := \max\{|v_1|, \ldots, |v_k|\}$.

Let $k \in \mathbb{N}$. We leave it as an unassigned exercise to show that $|\bullet|_k^D$ separates 0_k, is symmetric and satisfies the triangle inequality. Thus $|\bullet|_k^D$ is a norm, called the **diamond norm** in $\mathbb{R}^k$. Since $|\bullet|_k^D$ is a norm, its unit level set

$$\{v \in \mathbb{R}^k \text{ s.t. } |v|_k^D = 1\}$$

is called its "unit sphere" and is denoted $\{|\bullet|_k^D = 1\}$. Since $|\bullet|_k^D$ is a norm, its open unit sublevel set

$$\{v \in \mathbb{R}^k \text{ s.t. } |v|_k^D < 1\}$$

is called its "unit ball" and is denoted $\{|\bullet|_k^D < 1\}$.

We graphed $\{|\bullet|_2^D = 1\}$, and observed that it is a diamond.

Let $k \in \mathbb{N}$. We leave it as an unassigned exercise to show that $|\bullet|_k^S$ separates 0_k, is symmetric and satisfies the triangle inequality. Thus $|\bullet|_k^S$ is a norm, called the **square norm** in $\mathbb{R}^k$. Since $|\bullet|_k^S$ is a norm, its unit level set

$$\{v \in \mathbb{R}^k \text{ s.t. } |v|_k^S = 1\}$$

is called its "unit sphere" and is denoted $\{|\bullet|_k^S = 1\}$. Since $|\bullet|_k^S$ is a norm, its open unit sublevel set

$$\{v \in \mathbb{R}^k k \text{ s.t. } |v|_k^S < 1\}$$

is called its "unit ball" and is denoted $\{|\bullet|_k^S < 1\}$.

We graphed $\{|\bullet|_2^S = 1\}$, and observed that it is a square.

We explained how to reccover any absolutely homgeneous function from its unit level set. The graph of that unit level set contains, in geometric form, all of the information of the function, and it is a skill to look at that graph, and, from it, to "see" properties of the function. For example, for any absolutely homogeneous function, that function is a norm iff

its unit level set is symmetric through the origin, and
its unit sublevel set is convex.

Let $k \in \mathbb{N}$. Every positive multiple of a norm is a norm. So, for example $2 \cdot |\bullet|_k^S$ is a norm. We observed that

$$\{2 \cdot |\bullet|_k^S < 1\} \subseteq \{|\bullet|_k^D < 1\} \subseteq \{|\bullet|_k < 1\} \subseteq \{|\bullet|_k^S < 1\}.$$

According to the "**Compensation Principle**", big norms have small balls and small norms have big balls. Thus, we expect that:

$$\forall v \in \mathbb{R}^k,\ 2 \cdot |v|_k^S \geqslant \cdot|v|_k^D \geqslant \cdot|v|_k \geqslant \cdot|v|_k^S,$$

and we will leave it as homework to verify these inequalities via symbolic proofs. (See HW#7-1, HW#7-2 and HW#7-3.)

42. Pairing together functions

DEFINITION 42.1. *$\forall$functions f, g, by $(f, g)_{\text{fn}}$, we mean the function defined by:* $\forall x,\ (f, g)_{\text{fn}}(x) = (f(x), g(x))$.

Keep in mind that, in Definition 42.1, (f, g) would refer to an ordered pair, and $(f, g) = \begin{pmatrix} 1 \mapsto f \\ 2 \mapsto g \end{pmatrix}$. Unfortunately, the subscript "fn" is almost always omitted from the notation "$(f, g)_{\text{fn}}$", and so "(f, g)" might mean $(f, g)_{\text{fn}}$ or it might mean $\begin{pmatrix} 1 \mapsto f \\ 2 \mapsto g \end{pmatrix}$. It is up to the reader to figure out, from context, which is meant.

THEOREM 42.2. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ be defined by $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$. Then*

$(f, g) : \mathbb{R} \dashrightarrow \mathbb{R}^2$,
$\forall x \in \mathbb{R},\ (f, g)(x) = (\sqrt{x}, \sqrt{1-x})$ *and*
$\text{dom}[(f, g)] = [0; \infty) \cap (-\infty; 1] = (\text{dom}[f]) \cap (\text{dom}[g])$.

The domain of the pairing is always the intersection of the domains, for any two functions, not just for the particular two functions f and g that were specified in Theorem 42.2:

THEOREM 42.3. *$\forall$functions f and g, we have:*

$$\text{dom}[(f, g)] = (\text{dom}[f]) \cap (\text{dom}[g]).$$

Since a sequence is just a function with domain $\mathbb{N}$, we see, from Theorem 42.3, that a pairing of two sequences is again a sequence:

THEOREM 42.4. *$\forall$sequences a and b, (a, b) is a sequence.*

Moreover, any evaulation of a paired sequence is done by evaluating the first and second parts of the pair:

THEOREM 42.5. *$\forall$sequences a and b, $\forall j \in \mathbb{N}$, $(a,b)_j = (a_j, b_j)$.*

Targets of paired functions also behave predictably:

THEOREM 42.6. *$\forall$sets S and T, $\forall$functions f and g,*
$$[\ (\,\mathrm{im}[f] \subseteq S\,)\ \&\ (\,\mathrm{im}[g] \subseteq T\,)\ ]\ \Rightarrow\ [\ \mathrm{im}[(f,g)] \subseteq S \times T\ ].$$

Targets of paired sequences also behave predictably:

THEOREM 42.7. *$\forall$sets S and T, $\forall a \in S^{\mathbb{N}}$, $\forall b \in T^{\mathbb{N}}$,*
$$(a,b)\ \in\ (S \times T)^{\mathbb{N}}.$$

Projection to the x-axis is distance semi-decreasing:

THEOREM 42.8. *$\forall v, w \in \mathbb{R}^2$, $d_{\mathbb{R}}(v_1, w_1) \leqslant d_2(v, w)$.*

Proof. Given $v, w \in \mathbb{R}^2$. Want: $d_{\mathbb{R}}(v_1, w_1) \leqslant d_2(v, w)$.
Let $x := v - w$. Then $|x|_2 = d_2(v, w)$.
Want: $d_{\mathbb{R}}(v_1, w_1) \leqslant |x|_2$.
Since $x_1 = v_1 - w_1$, we get $|x_1| = d_{\mathbb{R}}(v_1, w_1)$.
Want: $|x_1| \leqslant |x|_2$.
Since $0 \leqslant x_1^2$ and $0 \leqslant x_2^2$, we get $0 \leqslant x_1^2 \leqslant x_1^2 + x_2^2$.
It follows that $\sqrt{x_1^2} \leqslant \sqrt{x_1^2 + x_2^2}$.
Then $|x_1| = \sqrt{x_1^2} \leqslant \sqrt{x_1^2 + x_2^2} = |x|_2$, as desired. □

THEOREM 42.9. *Let $a, b \in \mathbb{R}^{\mathbb{N}}$ and let $p, q \in \mathbb{R}$.*
Assume that $a_\bullet \to p$ in $\mathbb{R}$ and that $b_\bullet \to q$ in $\mathbb{R}$.
Then $(a,b)_\bullet \to (p,q)$ in $\mathbb{R}^2$.

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ )\quad \Rightarrow\quad (\ d_2((a,b)_j, (p,q)) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ )\quad \Rightarrow\quad (\ d_2((a,b)_j, (p,q)) < \varepsilon\ ).$$
Let $\eta := \varepsilon/2$.
Then $\eta > 0$ and $2\eta = \varepsilon$.
Since $a_\bullet \to p$ in $\mathbb{R}$, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant L\ )\quad \Rightarrow\quad (\ d_{\mathbb{R}}(a_j, p) < \eta\ ).$$
Since $b_\bullet \to q$ in $\mathbb{R}$, choose $M \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant M\ )\quad \Rightarrow\quad (\ d_{\mathbb{R}}(b_j, q) < \eta\ ).$$
Let $K := \max\{L, M\}$. Then $K \geqslant L$ and $K \geqslant M$ and $K \in \mathbb{N}$.

Want: $\forall j \in \mathbb{N}$, $[\; (\; j \geqslant K \;) \; \Rightarrow \; (\; d_2((a,b)_j,(p,q)) < \varepsilon \;) \;]$.
Given $j \in \mathbb{N}$. Want: $(\; j \geqslant K \;) \; \Rightarrow \; (\; d_2((a,b)_j,(p,q)) < \varepsilon \;)$.
Assume $j \geqslant K$. Want: $d_2((a,b)_j,(p,q)) < \varepsilon$.
Since $j \geqslant K \geqslant L$, by choice of L, we have: $d_{\mathbb{R}}(a_j,p) < \eta$.
Since $j \geqslant K \geqslant M$, by choice of M, we have: $d_{\mathbb{R}}(b_j,q) < \eta$.
Let $v := (a_j - p, b_j - q)$.
Then $v = (a_j,b_j) - (p,q)$ and $|v|_2^D = |a_j - p| + |b_j - q|$.
By HW 7-1, we have: $|v|_2^D \geqslant |v|_2$.
Then $d_2((a,b)_j,(p,q)) = d_2((a_j,b_j),(p,q)) = |(a_j,b_j) - (p,q)|_2$
$= |v|_2 \leqslant |v|_2^D = |a_j - p| + |b_j - q|$
$= [d_{\mathbb{R}}(a_j,p)] + [d_{\mathbb{R}}(b_j,q)] < \eta + \eta = 2\eta = \varepsilon$, as desired. □

43. Product metrics and relative metrics

DEFINITION 43.1. *Let X and Y be metric spaces.*
Define $d \in \mathcal{M}(X \times Y)$ by

$$d(v,w) = \sqrt{[d_X(v_1,w_1)]^2 + [d_Y(v_2,w_2)]^2}.$$

Then d is called the **product metric** *on $X \times Y$ from X and Y.*

We leave it as an unassigned exercise to show that the function

$$d : (X \times Y) \times (X \times Y) \to [0;\infty)$$

of Definition 43.1 is, in fact, a metric on $X \times Y$.

In Definition 43.1, the phrase "from X and Y" is often omitted. For any metric spaces X and Y, the standard metric on $X \times Y$ is the product metric.

We can generalize Definition 43.1 to products $X \times Y \times Z$ of three metric spaces X and Y and Z. Or to four, or to five, *etc.*

THEOREM 43.2. *Let X and Y be metric spaces.*
Let $a \in X^{\mathbb{N}}$, $p \in X$, $b \in Y^{\mathbb{N}}$ and $q \in Y$.
Then: $(\; [\; (a,b)_\bullet \to (p,q) \text{ in } X \times Y \;] \; \Leftrightarrow$
$[\; (\; a_\bullet \to p \text{ in } X \;) \; \& \; (\; b_\bullet \to q \text{ in } Y \;) \;] \;)$.

DEFINITION 43.3. *Let X be a metric space and let $S \subseteq X$.*
Then $d_X|(S \times S)$ is called the **relative metric** *on S inherited from X.*

We leave it as an unassigned exercise to show that the function

$$d|(S \times S) : S \times S \to [0;\infty)$$

of Definition 43.3 is, in fact, a metric on S.

In Definition 43.3, the phrase "inherited from X" is often omitted. For any metric space X, for any $S \subseteq X$, the standard metric on S is the relative metric.

THEOREM 43.4. *Let* $C := \{v \in \mathbb{R}^2 \,|\, v_1^2 + v_2^2 = 1\}$
be the unit circle about the origin in $\mathbb{R}^2$.
Let d *be the product metric on* $\mathbb{R} \times \mathbb{R}$ *from* $(\mathbb{R}, d_{\mathbb{R}})$ *and* $(\mathbb{R}, d_{\mathbb{R}})$.
Let δ *be the relative metric on* C *inherited from* $(\mathbb{R} \times \mathbb{R}, d)$.
Let $v := (1, 0)$ *and* $w := (0, 1)$.
Then $d = d_2$ *and* $\delta(v, w) = \sqrt{2}$.

44. Continuity

DEFINITION 44.1. *Let* X *and* Y *be metric spaces.*
Let $f : X \to Y$ *and* $p \in X$.
By f *is* **continuous at** p *from* X *to* Y*, we mean:*
$\forall a \in X^{\mathbb{N}}$, ($a_\bullet \to p$ *in* X) $\Rightarrow$ ($(f \circ a)_\bullet \to f(p)$ *in* Y).

DEFINITION 44.2. *Let* X *and* Y *be metric spaces,* $f : X \to Y$.
By f *is* **continuous** *from* X *to* Y*, we mean:*
$\forall p \in X$, f *is continuous at* p *from* X *to* Y.
Also, $\forall S \subseteq X$*, by* f *is* **continuous on** S *from* X *to* Y*, we mean:*
$\forall p \in S$, f *is continuous at* p *from* X *to* Y.

In Definition 44.1 and in Definition 44.2, sometimes, the text "from X to Y" is omitted, provide the domain and target of f are clear.

DEFINITION 44.3. *Let* ϕ *be a functional and let* $k \in \mathbb{N}$.
Define $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x^k$.
Then $\phi^k := f \circ \phi$.

THEOREM 44.4. $\forall$*functional* ϕ,
$\phi^0 = C^1_{\mathrm{dom}[\phi]}$ *and* $\phi^1 = \phi$ *and* $\phi^2 = \phi \cdot \phi$ *and* $\phi^3 = \phi \cdot \phi \cdot \phi$.

THEOREM 44.5. $\forall a \in \mathbb{R}^{\mathbb{N}}$,
$a^0 = C^1_{\mathbb{N}}$ *and* $a^1 = a$ *and* $a^2 = a \cdot a$ *and* $a^3 = a \cdot a \cdot !a$.

THEOREM 44.6. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = x^2$.
Then f *is continuous from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. Want: $\forall x \in \mathbb{R}$, f is continuous at x from $\mathbb{R}$ to $\mathbb{R}$.
Given $x \in \mathbb{R}$. Want: f is continuous at x from $\mathbb{R}$ to $\mathbb{R}$.
Want: $\forall a \in \mathbb{R}^{\mathbb{N}}$, $[\ (\ a_\bullet \to x \text{ in } \mathbb{R}\ ) \Rightarrow (\ (f \circ a)_\bullet \to f(x) \text{ in } \mathbb{R}\ )\ ]$.
Given $a \in \mathbb{R}^{\mathbb{N}}$. Want: $(\ a_\bullet \to x \text{ in } \mathbb{R}\ ) \Rightarrow (\ (f \circ a)_\bullet \to f(x) \text{ in } \mathbb{R}\ )$.
Assume that $a_\bullet \to x$ in $\mathbb{R}$. Want: $(f \circ a)_\bullet \to f(x)$ in $\mathbb{R}$.
Since $a_\bullet \to X$ in $\mathbb{R}$ and since $a_\bullet \to x$ in $\mathbb{R}$,
by Theorem 40.5, we get $(a \cdot a)_\bullet \to x \cdot x$ in $\mathbb{R}$.
So, since $a \cdot a = a^2 = f \circ a$ and since $x \cdot x = x^2 = f(x)$,
we get $(f \circ a)_\bullet \to f(x)$ in $\mathbb{R}$, as desired. □

THEOREM 44.7. *Let $A : \mathbb{R}^2 \to \mathbb{R}$ be defined by $A(x, y) = x + y$. Then A is continuous from $\mathbb{R}^2$ to $\mathbb{R}$.*

Proof. Want: $\forall z \in \mathbb{R}^2$, A is continuous at z from $\mathbb{R}^2$ to $\mathbb{R}$.
Given $z \in \mathbb{R}^2$. Want: A is continuous at z from $\mathbb{R}^2$ to $\mathbb{R}$.
Want: $\forall v \in (\mathbb{R}^2)^{\mathbb{N}}$, $[\ (\ v_\bullet \to z \text{ in } \mathbb{R}^2\ ) \Rightarrow (\ (A \circ v)_\bullet \to A(z) \text{ in } \mathbb{R}\ )\ ]$.
Given $v \in (\mathbb{R}^2)^{\mathbb{N}}$. Want: $(\ v_\bullet \to z \text{ in } \mathbb{R}^2\ ) \Rightarrow (\ (A \circ v)_\bullet \to A(z) \text{ in } \mathbb{R}\ )$.
Assume that $v_\bullet \to z$ in $\mathbb{R}^2$. Want: $(A \circ v)_\bullet \to A(z)$ in $\mathbb{R}$.
Let $x := z_1$ and $y := z_2$. Then $z = (z_1, z_2) = (x, y)$.
Define $s, t \in \mathbb{R}^{\mathbb{N}}$ by $s_j = (v_j)_1$ and $t_j = (v_j)_2$.
Then, $\forall j \in \mathbb{N}$, $v_j = ((v_j)_1, (v_j)_2) = (s_j, t_j) = (s, t)_j$.
Then $v = (s, t)$.
Since $v_\bullet \to z$ in $\mathbb{R}^2$, since $v = (s, t)$ and since $z = (x, y)$,
we see that $(s, t)_\bullet \to (x, y)$ in $\mathbb{R}^2$.
Then, by Theorem 43.2,
we see that $s_\bullet \to x$ in $\mathbb{R}$ and $t_\bullet \to y$ in $\mathbb{R}$. Then, by Theorem 40.1,
we see that $(s + t)_\bullet \to x + y$ in $\mathbb{R}$.
So, since $A(x, y) = x + y$, we see that $(s + t)_\bullet \to A(x, y)$ in $\mathbb{R}$.
Recall that we want: $(A \circ v)_\bullet \to A(z)$ in $\mathbb{R}$.
It therefore suffices to show that $A \circ v = s + t$.
Want: $\forall j \in \mathbb{N}$, $(A \circ v)_j = (s + t)_j$.
Given $j \in \mathbb{N}$. Want: $(A \circ v)_j = (s + t)_j$.
We have $(A \circ v)_j = A(v_j) = A(s_j, t_j) = s_j + t_j = (s + t)_j$, as desired. □

The next two theorems are proved similarly.

THEOREM 44.8. *Let $S : \mathbb{R}^2 \to \mathbb{R}$ be defined by $S(x, y) = x - y$. Then S is continuous from $\mathbb{R}^2$ to $\mathbb{R}$.*

THEOREM 44.9. *Let $M : \mathbb{R}^2 \to \mathbb{R}$ be defined by $M(x, y) = xy$. Then M is continuous from $\mathbb{R}^2$ to $\mathbb{R}$.*

In Theorem 44.10, below, the metric on $\mathbb{R} \times \mathbb{R}_0^\times$ is the relative metric inherited from $(\mathbb{R}^2, d_2)$. In Theorem 44.10, below, the metric on $\mathbb{R}$ is the standard metric $d_{\mathbb{R}}$.

THEOREM 44.10. *Let $D : \mathbb{R} \times \mathbb{R}_0^\times \to \mathbb{R}$ be defined by $D(x, y) = x/y$. Then D is continuous from $\mathbb{R} \times \mathbb{R}_0^\times$ to $\mathbb{R}$.*

THEOREM 44.11. *Let X be a metric space.*
Let $S \subseteq X$, $a \in S^{\mathbb{N}}$ and $p \in S$.
Assume that $a_\bullet \to p$ in S. Then $a_\bullet \to p$ in X.

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_X(a_j, p) < \varepsilon\ )$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_X(a_j, p) < \varepsilon\ )$.
Since $a_\bullet \to p$ in S, choose $K \in \mathbb{N}$ s.t.,, $\forall j \in \mathbb{N}$,
$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_S(a_j, p) < \varepsilon\ )$.
Want: $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \ \Rightarrow\ (\ d_X(a_j, p) < \varepsilon\ )\ ]$.
Given $j \in \mathbb{N}$. Want: $[\ (\ j \geqslant K\ ) \ \Rightarrow\ (\ d_X(a_j, p) < \varepsilon\ )\ ]$.
Assume $j \geqslant K$. Want: $d_X(a_j, p) < \varepsilon$.
Since $j \geqslant K$, by choice of K, we get $d_S(a_j, p) < \varepsilon$.
Since $a_j, p \in S \subseteq X$, we have $d_S(a_j, p) = d_X(a_j, p)$.
Then $d_X(a_j, p) = d_S(a_j, p) < \varepsilon$, as desired. ☐

The converse of Theorem 44.11 holds, with similar proof:

THEOREM 44.12. *Let X be a metric space.*
Let $S \subseteq X$, $a \in S^{\mathbb{N}}$ and $p \in S$.
Assume that $a_\bullet \to p$ in X. Then $a_\bullet \to p$ in S.

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_S(a_j, p) < \varepsilon\ )$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_S(a_j, p) < \varepsilon\ )$.
Since $a_\bullet \to p$ in X, choose $K \in \mathbb{N}$ s.t.,, $\forall j \in \mathbb{N}$,
$(\ j \geqslant K\ ) \quad \Rightarrow \quad (\ d_X(a_j, p) < \varepsilon\ )$.
Want: $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \ \Rightarrow\ (\ d_S(a_j, p) < \varepsilon\ )\ ]$.
Given $j \in \mathbb{N}$. Want: $[\ (\ j \geqslant K\ ) \ \Rightarrow\ (\ d_X(a_j, p) < \varepsilon\ )\ ]$.
Assume $j \geqslant K$. Want: $d_X(a_j, p) < \varepsilon$.
Since $j \geqslant K$, by choice of K, we get $d_S(a_j, p) < \varepsilon$.
Since $a_j, p \in S \subseteq X$, we have $d_S(a_j, p) = d_X(a_j, p)$.
Then $d_S(a_j, p) = d_X(a_j, p) < \varepsilon$, as desired. ☐

The buzz phrase for Theorem 44.13, below, is "restriction maintains continuity". In Theorem 44.13, below, the metric on S is the relative metric, inherited from X.

THEOREM 44.13. *Let X and Y be metric spaces.*
Let $f : X \to Y$, $S \subseteq X$ and $p \in S$.
Assume that f is continuous at p from X to Y.
Then $f|S$ is continuous at p from S to Y.

Proof. Want: $\forall a \in S^{\mathbb{N}}$,
$(a_\bullet \to p \text{ in } S) \quad \Rightarrow \quad (((f|S) \circ a)_\bullet \to (f|S)(p) \text{ in } Y)$.
Given $a \in S^{\mathbb{N}}$.
Want: $(a_\bullet \to p \text{ in } S) \Rightarrow (((f|S) \circ a)_\bullet \to (f|S)(p) \text{ in } Y)$.
Assume $a_\bullet \to p$ in S. Want: $((f|S) \circ a)_\bullet \to (f|S)(p)$ in Y.
Since $a_\bullet \to p$ in S, it follows, by Theorem 44.11,
that $a_\bullet \to p$ in X.
Then, by continuity of f at p from X to Y,
we see that $(f \circ a)_\bullet \to f(p)$ in Y.
Since $p \in S$, we have $(f|S)(p) = f(p)$.
Then $(f \circ a)_\bullet \to (f|S)(p)$ in Y.
Recall that we want: $((f|S) \circ a)_\bullet \to (f|S)(p)$ in Y.
It therefore suffices to show that $f \circ a = (f|S) \circ a$.
Want: $\forall j \in \mathbb{N}$, $(f \circ a)_j = ((f|S) \circ a)_j$.
Given $j \in \mathbb{N}$. Want: $(f \circ a)_j = ((f|S) \circ a)_j$.
Since $a \in S^{\mathbb{N}}$, we get $a_j \in S$, and so $(f|S)(a_j) = f(a_j)$.
Then $(f \circ a)_j = f(a_j) = (f|S)(a_j) = ((f|S) \circ a)_j$, as desired. □

The buzz phrase for Theorem 44.14, below, is "decrease of target maintains continuity". The buzz phrase for Theorem 44.15, below, is "increase of target maintains continuity". In both Theorem 44.14 and Theorem 44.15, below, the metric on Y_0 is the relative metric, inherited from Y.

THEOREM 44.14. *Let X and Y be metric spaces.*
Let $p \in X$, $Y_0 \subseteq Y$ and $\phi : X \to Y_0$.
Assume that ϕ is continuous at p from X to Y.
Then ϕ is continuous at p from X to Y_0.

Proof. We have both $\phi : X \to Y_0$ and $\phi : X \to Y$.
Want: $\forall a \in X^{\mathbb{N}}$, $[(a_\bullet \to p \text{ in } X) \Rightarrow ((\phi \circ a)_\bullet \to \phi(p) \text{ in } Y_0)]$.
Given $a \in X^{\mathbb{N}}$. Want: $(a_\bullet \to p \text{ in } X) \Rightarrow ((\phi \circ a)_\bullet \to \phi(p) \text{ in } Y_0)$.

Assume $a_\bullet \to p$ in X. Want: $(\phi \circ a)_\bullet \to \phi(p)$ in Y_0.
Since $a_\bullet \to p$ in X, by continuity of ϕ at p from X to Y,
we see that $(\phi \circ a)_\bullet \to \phi(p)$ in Y.
Since $a \in X^{\mathbb{N}}$ and $\phi : X \to Y_0$, it follows that $\phi \circ a \in Y_0^{\mathbb{N}}$.
Since $p \in X$ and $\phi : X \to Y_0$, it follows that $\phi(p) \in Y_0$.
Then, by Theorem 44.12 (with b replaced by $\phi \circ a$, and q by $\phi(p)$),
we see that $(\phi \circ a)_\bullet \to \phi(p)$ in Y_0, as desired. □

The converse of Theorem 44.14 holds, with similar proof:

THEOREM 44.15. *Let X and Y be metric spaces.*
Let $p \in X$, $Y_0 \subseteq Y$ and $\phi : X \to Y_0$.
Assume that ϕ is continuous at p from X to Y_0.
Then ϕ is continuous at p from X to Y.

Proof. We have both $\phi : X \to Y_0$ and $\phi : X \to Y$.
Want: $\forall a \in X^{\mathbb{N}}$, $[\ (\ a_\bullet \to p \text{ in } X\ ) \Rightarrow (\ (\phi \circ a)_\bullet \to \phi(p) \text{ in } Y\ )\ ]$.
Given $a \in X^{\mathbb{N}}$. Want: $(\ a_\bullet \to p \text{ in } X\ ) \Rightarrow (\ (\phi \circ a)_\bullet \to \phi(p) \text{ in } Y\ )$.
Assume $a_\bullet \to p$ in X. Want: $(\phi \circ a)_\bullet \to \phi(p)$ in Y.
Since $a_\bullet \to p$ in X, by continuity of ϕ at p from X to Y_0,
we see that $(\phi \circ a)_\bullet \to \phi(p)$ in Y_0.
Since $a \in X^{\mathbb{N}}$ and $\phi : X \to Y_0$, it follows that $\phi \circ a \in Y_0^{\mathbb{N}}$.
Since $p \in X$ and $\phi : X \to Y_0$, it follows that $\phi(p) \in Y_0$.
Then, by Theorem 44.11 (with b replaced by $\phi \circ a$, and q by $\phi(p)$),
we see that $(\phi \circ a)_\bullet \to \phi(p)$ in Y, as desired. □

The next theorem, Theorem 44.16 below, is **transitivity of inherited metrics**; it follows from HW 8-2.

THEOREM 44.16. *Let X be a metric space, let $T \subseteq X$ and let $S \subseteq T$.*
Then $(d_X|(T \times T))|(S \times S) = d_X|(S \times S)$

In Theorem 44.17 below, the point is that, to prove that $f(p) \in f_*(S)$, it is not sufficient that $p \in S$; one also needs $p \in \mathrm{dom}[f]$. Otherwise, we get: $f(p)☹ \notin f_*(S)$.

THEOREM 44.17. *$\forall$function f, $\forall$set S, $\forall p$,*
$$[\ (\ p \in S\ )\ \&\ (\ p \in \mathrm{dom}[f]\ )\ ] \Rightarrow [\ f(p) \in f_*(S)\ ].$$

THEOREM 44.18. *Let S, T, U and V be sets, let $a : S \to T$ and let $f : U \to V$.*
Assume that $T \subseteq U$. Then $f \circ a : S \to f_*(T)$.

Proof. Unassigned HW. ☐

THEOREM 44.19. *$\forall$functions f and g, $\forall$set S, we have:*

(1) $f^*(S) \subseteq \mathrm{dom}[f]$ *and*
(2) $f_*(S) \in \mathrm{im}[f]$.

Proof. Unassigned homework. ☐

THEOREM 44.20. *Let X, Y and Z be metric spaces.*
Let $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ and $p \in X$.
Assume that f is continuous at p from dom$[f]$ *to Y.*
Assume that g is continuous at $f(p)$ from dom$[g]$ *to Z.*
Then $g \circ f$ is continuous at p from dom$[g \circ f]$ *to Z.*

Proof. Want: $\forall a \in (\mathrm{dom}[g \circ f])^{\mathbb{N}}$,
$(a_\bullet \to p \text{ in } \mathrm{dom}[g \circ f]) \Rightarrow (((g \circ f) \circ a)_\bullet \to (g \circ f)(p) \text{ in } Z)$.
Given $a \in (\mathrm{dom}[g \circ f])^{\mathbb{N}}$.
Want: $(a_\bullet \to p \text{ in } \mathrm{dom}[g \circ f]) \Rightarrow (((g \circ f) \circ a)_\bullet \to (g \circ f)(p) \text{ in } Z)$.
Assume that $a_\bullet \to p$ in dom$[g \circ f]$.
Want: $((g \circ f) \circ a)_\bullet \to (g \circ f)(p)$ in Z.
By (1) of Theorem 44.19, we have: $f^*(\mathrm{dom}[g]) \subseteq \mathrm{dom}[f]$.
Since $a \in (\mathrm{dom}[g \circ f])^{\mathbb{N}}$, we get $a : \mathbb{N} \to \mathrm{dom}[g \circ f]$.
Since $f : X \dashrightarrow Y$, we get $f : \mathrm{dom}[f] \to Y$.
By (1) of Theorem 26.4, we have $\mathrm{dom}[g \circ f] = f^*(\mathrm{dom}[g])$.
So, since $f^*(\mathrm{dom}[g]) \subseteq \mathrm{dom}[f]$, we conclude that $\mathrm{dom}[g \circ f] \subseteq \mathrm{dom}[f]$.
Since $a : \mathbb{N} \to \mathrm{dom}[g \circ f]$ and $\mathrm{dom}[g \circ f] \subseteq \mathrm{dom}[f]$ and $f : \mathrm{dom}[f] \to Y$,
it follows, from Theorem 44.18, that $f \circ a : \mathbb{N} \to f_*(\mathrm{dom}[g \circ f])$.
Recall that $\mathrm{dom}[g \circ f] = f^*(\mathrm{dom}[g])$.
By HW#8-1, we have $f_*(f^*(\mathrm{dom}[g])) \subseteq \mathrm{dom}[g]$.
Then $f_*(\mathrm{dom}[g \circ f]) = f_*(f^*(\mathrm{dom}[g])) \subseteq \mathrm{dom}[g]$.
So, since $f \circ a : \mathbb{N} \to f_*(\mathrm{dom}[g \circ f])$, we get $f \circ a : \mathbb{N} \to \mathrm{dom}[g]$.
Then $f \circ a \in (\mathrm{dom}[g])^{\mathbb{N}}$.
Since f is continuous at p, it follows that $p \in \mathrm{dom}[f]$.
Since g is continuous at $f(p)$, it follows that $f(p) \in \mathrm{dom}[g]$.
Since $(a_\bullet \to p \text{ in } \mathrm{dom}[g \circ f])$ and since $(\mathrm{dom}[g \circ f] \subseteq \mathrm{dom}[g])$,
we conclude, from Theorem 44.11, that $a_\bullet \to p$ in dom$[f]$.
So, since f is continuous at p from dom$[f]$ to Y,
we conclude that $(f \circ a)_\bullet \to f(p)$ in Y.

So, since $f \circ a \in (\mathrm{dom}[g])^{\mathbb{N}}$ and $f(p) \in \mathrm{dom}[g]$,
we conclude, from Theorem 44.12, that $(f \circ a)_\bullet \to f(p)$ in $\mathrm{dom}[g]$.
So, since g is continuous at p from $\mathrm{dom}[g]$ to Z,
we conclude that $(g \circ (f \circ a))_\bullet \to g(f(p))$ in Z.
So, since $g \circ (f \circ a) = (g \circ f) \circ a$ and since $g(f(p)) = (g \circ f)(p)$,
we conclude that $((g \circ f) \circ a)_\bullet \to (g \circ f)(p))$ in Z, as desired. ☐

THEOREM 44.21. *Let X, Y and Z be metric spaces.*
Let $f : X \dashrightarrow Y$ and $g : X \dashrightarrow Z$.
Assume that f is continuous at p.
Assume that g is continuous at p.
Then (f, g) is continuous at p.

Proof. Let $S := \mathrm{dom}[f]$ and $T := \mathrm{dom}[g]$. Then $S \cap T = \mathrm{dom}[(f, g)]$.
Also, $f : S \to Y$ and $g : T \to Z$ and $(f, g) : S \cap T \to Y \times Z$.
Also, f is continuous at p from S to Y.
Also, g is continuous at p from T to Z.
Want: (f, g) is continuous at p from $S \cap T$ to $Y \times Z$.
Want: $\forall a \in (S \cap T)^{\mathbb{N}}$,
($a_\bullet \to p$ in $S \cap T$) $\Rightarrow$ ($((f, g) \circ a)_\bullet \to (f, g)(p)$ in $Y \times Z$).
Given $a \in (S \cap T)^{\mathbb{N}}$.
Want: ($a_\bullet \to p$ in $S \cap T$) $\Rightarrow$ ($((f, g) \circ a)_\bullet \to (f, g)(p)$ in $Y \times Z$).
Assume $a_\bullet \to p$ in $S \cap T$. Want: $((f, g) \circ a)_\bullet \to (f, g)(p)$ in $Y \times Z$.
Since $a_\bullet \to p$ in $S \cap T$, by Theorem 44.11,
we conclude that $a_\bullet \to p$ in S.
So, since f is continuous at p from S to Y,
we conclude that $(f \circ a)_\bullet \to f(p)$ in Y.
Since $a_\bullet \to p$ in $S \cap T$, by Theorem 44.11,
we conclude that $a_\bullet \to p$ in T.
So, since g is continuous at p from T to Z,
we conclude that $(g \circ a)_\bullet \to g(p)$ in Z.
Since $(f \circ a)_\bullet \to f(p)$ in Y and $(g \circ a)_\bullet \to g(p)$ in Z,
by $\Leftarrow$ of Theorem 43.2, we get $(f \circ a, g \circ a)_\bullet \to (f(p), g(p))$ in Z.
So, since $(f, g)(p) = (f(p), g(p))$, we get $(f \circ a, g \circ a)_\bullet \to (f, g)(p)$ in Z.
Recall that we want: $((f, g) \circ a)_\bullet \to (f, g)(p)$ in $Y \times Z$.
It therefore suffices to show: $(f \circ a, g \circ a) = (f, g) \circ a$.
Want: $\forall j \in \mathbb{N}$, $(f \circ a, g \circ a)_j = ((f, g) \circ a)_j$.
Given $j \in \mathbb{N}$. Want: $(f \circ a, g \circ a)_j = ((f, g) \circ a)_j$.

We have $(f \circ a, g \circ a)_j = ((f \circ a)_j, (g \circ a)_j) = (f(a_j), g(a_j))$
$= (f, g)(a_j) = ((f, g) \circ a)_j$, as desired. □

THEOREM 44.22. *Let X be a metric space.*
Let $f, g : X \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.
Assume that f is continuous at p from $\mathrm{dom}[f]$ *to* $\mathbb{R}$.
Assume that g is continuous at p from $\mathrm{dom}[g]$ *to* $\mathbb{R}$.
Then $f + g$ is continuous at p from $\mathrm{dom}[f + g]$ *to* $\mathbb{R}$.

Proof. We have $(f, g) : X \dashrightarrow \mathbb{R}^2$.
Let $A : \mathbb{R}^2 \to \mathbb{R}$ be defined by $A(x, y) = x + y$.
Since (f, g) is continuous at p and A is continuous at $(f, g)(p)$,
it follows, from Theorem 44.20, that
$A \circ (f, g)$ is continuous at p from $\mathrm{dom}[A \circ (f, g)]$ to $\mathbb{R}$.
Recall that we want: $f + g$ is continuous at p from $\mathrm{dom}[f + g]$ to $\mathbb{R}$.
It therefore suffices to show: $A \circ (f, g) = f + g$.
Want: $\forall z \in X$, $(A \circ (f, g))(z) = (f + g)(z)$.
Given $z \in \mathbb{R}$. Want: $(A \circ (f, g))(z) = (f + g)(z)$.
We have $(A \circ (f, g))(z) = A((f, g)(z)) = A(f(z), g(z))$
$= [f(z)] + [g(z)] = (f + g)(z)$, as desired. □

THEOREM 44.23. *Let X be a metric space.*
Let $f, g : X \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.
Assume that f is continuous at p from $\mathrm{dom}[f]$ *to* $\mathbb{R}$.
Assume that g is continuous at p from $\mathrm{dom}[g]$ *to* $\mathbb{R}$.
Then fg is continuous at p from $\mathrm{dom}[f + g]$ *to* $\mathbb{R}$.

Proof. We have $(f, g) : X \dashrightarrow \mathbb{R}^2$.
Let $M : \mathbb{R}^2 \to \mathbb{R}$ be defined by $M(x, y) = xy$.
Since (f, g) is continuous at p and M is continuous at $(f, g)(p)$,
it follows, from Theorem 44.20, that
$M \circ (f, g)$ is continuous at p from $\mathrm{dom}[M \circ (f, g)]$ to $\mathbb{R}$.
Recall that we want: fg is continuous at p from $\mathrm{dom}[fg]$ to $\mathbb{R}$.
It therefore suffices to show: $M \circ (f, g) = fg$.
Want: $\forall z \in X$, $(M \circ (f, g))(z) = (fg)(z)$.
Given $z \in \mathbb{R}$. Want: $(M \circ (f, g))(z) = (fg)(z)$.
We have $(M \circ (f, g))(z) = M((f, g)(z)) = M(f(z), g(z))$
$= [f(z)] \cdot [g(z)] = (fg)(z)$, as desired. □

45. A squeeze theorem

THEOREM 45.1. *Let $u \in \mathbb{R}^{\mathbb{N}}$ and let $x \in \mathbb{R}$.*
Assume: $\forall j \in \mathbb{N}$, $x - (1/j) \leqslant u_j \leqslant x$.
Then $u_\bullet \to x$ in $\mathbb{R}$.

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(j \geqslant K) \Rightarrow (d_{\mathbb{R}}(u_j, x) < \varepsilon)$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(j \geqslant K) \Rightarrow (d_{\mathbb{R}}(u_j, x) < \varepsilon)$.
By the Archimedean Principle, choose $K \in \mathbb{N}$ s.t. $K > 1/\varepsilon$.
Want: $\forall j \in \mathbb{N}$, $[(j \geqslant K) \Rightarrow (d_{\mathbb{R}}(u_j, x) < \varepsilon)]$.
Given $j \in \mathbb{N}$. Want: $(j \geqslant K) \Rightarrow (d_{\mathbb{R}}(u_j, x) < \varepsilon)$.
Assume that $j \geqslant K$. Want: $d_{\mathbb{R}}(u_j, x) < \varepsilon$.
Want: $|u_j - x| < \varepsilon$. Want: $x - \varepsilon < u_j < x + \varepsilon$.
By assumption $u_j \leqslant x$. Since $\varepsilon > 0$, we get $x < x + \varepsilon$.
Then $u_j \leqslant x < x + \varepsilon$. Want: $x - \varepsilon < u_j$.
By assumption, $x - (1/j) \leqslant u_j$.
Want: $x - \varepsilon < x - (1/j)$. Want: $1/j < \varepsilon$.
Since $j \geqslant K > 1/\varepsilon$, we get $j > 1/\varepsilon$.
Since $\varepsilon > 0$, it follows that $1/\varepsilon > 0$.
Since $j > 1/\varepsilon > 0$, we conclude that $1/j < \varepsilon$, as desired. □

46. The supremum is a limit

THEOREM 46.1. *Let $S \subseteq \mathbb{R}$ and let $x := \sup S$.*
Assume that $S \neq \emptyset$ and that S is bounded above in $\mathbb{R}$.
Then $\exists u \in S^{\mathbb{N}}$ s.t. $u_\bullet \to x$ in $\mathbb{R}$.

Proof. We have $S \leqslant \sup S = x$, so $S \leqslant x$.
Also, $\forall w < x$, we have $\not(S \leqslant w)$.

Claim: $\forall j \in \mathbb{N}$, $[x - (1/j); x] \cap S \neq \emptyset$.

Proof of Claim:
Given $j \in \mathbb{N}$. Want: $[x - (1/j); x] \cap S \neq \emptyset$.
Since $(\forall w < x, \not(S \leqslant w))$ and since $x - (1/j) < x$,
we conclude: $\not(S \leqslant x - (1/j))$.
Then choose $t \in S$ s.t. $t > x - (1/j)$.
We have $t \in S \leqslant x$, so $t \leqslant x$.

Also, $x - (1/j) < t \leqslant x$, so $x - (1/j) < t$.
Then $x - (1/j) < t \leqslant x$, so $t \in (x - (1/j); x]$.
Since $t \in (x - (1/j); x] \subseteq [x - (1/j); x]$ and since $t \in S$,
we conclude: $t \in [x - (1/j); x] \cap S$.
Then $[x - (1/j); x] \cap S \neq \varnothing$, as desired.
End of proof of Claim.

By the Claim, and by Axiom 4.6, we have:
$\forall j \in \mathbb{N}, \quad \mathrm{CH}([x - (1/j); x] \cap S) \in [x - (1/j); x] \cap S.$
Define $u \in S^{\mathbb{N}}$ by $u_j = \mathrm{CH}([x - (1/j); x] \cap S)$.
Want $u_\bullet \to x$ in $\mathbb{R}$.
By Theorem 45.1, it suffices to show: $\forall j \in \mathbb{N}$, $x - (1/j) \leqslant u_j \leqslant x$.
Given $j \in \mathbb{N}$. Want: $x - (1/j) \leqslant u_j \leqslant x$.
We have $u_j = \mathrm{CH}([x-(1/j); x] \cap S) \in [x-(1/j); x] \cap S \subseteq [x-(1/j); x]$.
Then $x - (1/j) \leqslant u_j \leqslant x$, as desired.

□

47. Limit preserves nonstrict inequalities

THEOREM 47.1. *Let $w \in \mathbb{R}^{\mathbb{N}}$ and let $y, z \in \mathbb{R}$.*
Assume that $w_\bullet \to z$ in $\mathbb{R}$.
Assume: $\forall j \in \mathbb{N}$, $w_j \leqslant y$.
Then $z \leqslant y$.

Proof. HW#8-2. □

THEOREM 47.2. *Let $w \in \mathbb{R}^{\mathbb{N}}$ and let $y, z \in \mathbb{R}$.*
Assume that $w_\bullet \to z$ in $\mathbb{R}$.
Assume: $\forall j \in \mathbb{N}$, $w_j \geqslant y$.
Then $z \geqslant y$.

Proof. Unassigned HW. □

48. Increasing and decreasing functions $\mathbb{R} \dashrightarrow \mathbb{R}$

DEFINITION 48.1. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathrm{dom}[f]$.*
By f is **strictly increasing** *on S, we mean:*
$\forall t, u \in S, \; [\; (\; t < u \;) \;\Rightarrow\; (\; f(t) < f(u) \;) \;].$
By f is **strictly decreasing** *on S, we mean:*
$\forall t, u \in S, \; [\; (\; t < u \;) \;\Rightarrow\; (\; f(t) > f(u) \;) \;].$
By f is **semi-increasing** *on S, we mean:*

$$\forall t, u \in S, [\ (\ t \leqslant u\ )\ \Rightarrow\ (\ f(t) \leqslant f(u)\ )\ ].$$

By f is **semi-decreasing** *on S, we mean:*

$$\forall t, u \in S, [\ (\ t \leqslant u\ )\ \Rightarrow\ (\ f(t) \geqslant f(u)\ )\ ].$$

THEOREM 48.2. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathrm{dom}[f]$. Then:*
(1) (f is strictly increasing on S) $\Rightarrow$
(f is semi-increasing on S)
and (2) (f is strictly decreasing on S) $\Rightarrow$
(f is semi-dencreasing on S).

THEOREM 48.3. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathrm{dom}[f]$. Then:*
and (1) (f is strictly increasing on S) $\Leftrightarrow$
$(\forall t, u \in S, [\,(\,t > u\,) \Rightarrow (\,f(t) > f(u)\,)\,]\,)$
and (2) (f is strictly decreasing on S) $\Leftrightarrow$
$(\forall t, u \in S, [\,(\,t > u\,) \Rightarrow (\,f(t) < f(u)\,)\,]\,)$
and (3) (f is semi-increasing on S) $\Leftrightarrow$
$(\forall t, u \in S, [\,(\,t \geqslant u\,) \Rightarrow (\,f(t) \geqslant f(u)\,)\,]\,)$
and (4) (f is semi-decreasing on S) $\Leftrightarrow$
$(\forall t, u \in S, [\,(\,t \geqslant u\,) \Rightarrow (\,f(t) \leqslant f(u)\,)\,]\,)$.

DEFINITION 48.4. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
By f is **strictly increasing***, we mean:*
f is strictly increasing on $\mathrm{dom}[f]$.
By f is **strictly decreasing***, we mean:*
f is strictly decreasing on $\mathrm{dom}[f]$.
By f is **semi-increasing***, we mean:*
f is semi-increasing on $\mathrm{dom}[f]$.
By f is **semi-decreasing***, we mean:*
f is semi-decreasing on $\mathrm{dom}[f]$.

DEFINITION 48.5. $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall a, b$,

$$(\mathrm{DQ}_f)(a, b) := \frac{[f(b)] - [f(a)]}{b - a}.$$

In Definition 48.5, "DQ" stands for "**Difference Quotient**".

We drew a graph of a function f and demonstrated how $(\mathrm{DQ}_f)(a, b)$ is the slope of a secant line.

For Theorem 48.6 below, we showed the graphs of $\mathrm{id}_{\mathbb{R}}$ and $(\mathrm{id}_{\mathbb{R}})^2$ and $(\mathrm{id}_{\mathbb{R}})^3$ and $C^3_{\mathbb{R}}$. We discussed slopes of secant lines for these graphs and used various variants of HW#9-1.

THEOREM 48.6. *All of the following are true:*

(1) ($\mathrm{id}_{\mathbb{R}}$ *is strictly increasing*)
and (2) ($(\mathrm{id}_{\mathbb{R}})^3$ *is strictly increasing*)
and (3) ($(\mathrm{id}_{\mathbb{R}})^2$ *is neither strictly decreasing nor strictly increasing*)
and (4) ($(\mathrm{id}_{\mathbb{R}})^2$ *is strictly decreasing on* $(-\infty; 0]$)
and (5) ($(\mathrm{id}_{\mathbb{R}})^2$ *is strictly increasing on* $[0; \infty)$)
and (6) ($C_{\mathbb{R}}^3$ *is neither strictly decreasing nor strictly increasing*)
and (7) ($C_{\mathbb{R}}^3$ *is both semi-decreasing and semi-increasing*).

THEOREM 48.7. $\forall \ell \in \mathbb{Z}$, $[\,(\,\ell > 0\,) \Rightarrow (\,\ell \geqslant 1\,)\,]$.

Proof. Unassigned HW. □

Theorem 48.8, below, is of use in HW#9-2. It follows easily from Theorem 48.7, above.

THEOREM 48.8. $\forall j, k \in \mathbb{Z}$, $[\,(\,j < k\,) \Rightarrow (\,j + 1 \leqslant k\,)\,]$.

Proof. Unassigned HW. □

DEFINITION 48.9. *Let f be a functional.*
By f is **bounded above** *into $\mathbb{R}$, we mean:*
$\mathrm{im}[f]$ *is bounded above in* $\mathbb{R}$.
By f is **bounded below** *into $\mathbb{R}$, we mean:*
$\mathrm{im}[f]$ *is bounded below in* $\mathbb{R}$.

49. Cauchy sequences

DEFINITION 49.1. *Let X be a metric space, $S \subseteq X$ and $\varepsilon > 0$.*
*By S is ε-***small** *in X, we mean: $\forall y, z \in S$, $d_X(y, z) < \varepsilon$.*

DEFINITION 49.2. *Let X be a metric space and $a \in X^{\mathbb{N}}$.*
By a is **Cauchy** *in X, we mean: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,*
$$(\ i, j \geqslant K\ ) \Rightarrow (\ d_X(a_i, a_j) < \varepsilon\ ).$$

A buzz phrase for Definition 49.2 is:
"A sequence is Cauchy iff,
$\forall \varepsilon > 0$, the sequence has an ε-small tail."
More precisely, $\forall \varepsilon > 0$, there is a tail with ε-small image.

50. Intermediate Value Theorems (IVTs)

The following is HW#8-4:

THEOREM 50.1. *Let $w \in \mathbb{R}^{\mathbb{N}}$ and let $y, z \in \mathbb{R}$.*
Assume that $w_\bullet \to y$ in $\mathbb{R}$. Assume: $\forall j \in \mathbb{N}$, $w_j \leqslant z$.
Show: $y \leqslant z$.

The following is an unassigned exercise:

THEOREM 50.2. *Let $w \in \mathbb{R}^{\mathbb{N}}$ and let $y, z \in \mathbb{R}$.*
Assume that $w_\bullet \to y$ in $\mathbb{R}$. Assume: $\forall j \in \mathbb{N}$, $w_j \geqslant z$.
Show: $y \geqslant z$.

THEOREM 50.3. *Let f be a function and let S be a set. Then, $\forall p$,*
$[\,(\,p \in S\,) \,\&\, (\,p \in \mathrm{dom}[f]\,)\,] \quad \Rightarrow \quad [\,f(p) \in f_*(S)\,]$.

THEOREM 50.4. *Let $b, x \in \mathbb{R}$. Assume that $x < b$.*
Define $v \in \mathbb{R}^{\mathbb{N}}$ by $v_j = x + \dfrac{b - x}{j}$.
Then $(\,v_\bullet \to x$ in $\mathbb{R}\,)$ and $(\,\forall j \in \mathbb{N}, x < v_j \leqslant b\,)$.

THEOREM 50.5. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Let $a, b, y \in \mathbb{R}$. Assume $a \leqslant b$.
Assume f is continuous on $[a; b]$ from $\mathrm{dom}[f]$ to $\mathbb{R}$.
Assume $f(a) \leqslant y \leqslant f(b)$. Then $\exists x \in [a; b]$ s.t. $f(x) = y$.

Proof. Since f is continuous on $[a; b]$, we see that $[a; b] \subseteq \mathrm{dom}[f]$.
Let $S := \{t \in [a; b] \mid f(t) \leqslant y\}$.
Then $S \subseteq [a; b] \subseteq \mathrm{dom}[f]$. Then $S^{\mathbb{N}} \subseteq (\mathrm{dom}[f])^{\mathbb{N}}$.

Claim 1: $f_*(S) \leqslant y$.
Proof of Claim 1:
Want: $\forall q \in f_*(S)$, $q \leqslant y$.
Given $q \in f_*(S)$. Want: $q \leqslant y$.
Since $q \in f_*(S)$, choose $t \in S \cap (\mathrm{dom}[f])$ s.t. $f(t) = q$.
Since $t \in S$, it follows, from the definition of S, that $f(t) \leqslant y$.
Then $q = f(t) \leqslant y$, as desired.
End of proof of Claim 1.

By assumption, $f(a) \leqslant y$.
Since $a \in [a; b]$ and $f(a) \leqslant y$, we conclude,
from the definition of S, that $a \in S$.
Since $a \in S$, it follows that $S \neq \varnothing$.
We have $a \in S \leqslant \sup S$, so $a \leqslant \sup S$.
Since $S \subseteq [a; b] \leqslant b$, we get $S \leqslant b$.

Then $\sup S \leqslant b$. Then $a \leqslant \sup S \leqslant b$.
Let $x := \sup S$. Then $a \leqslant x \leqslant b$, so $x \in [a; b]$.
Want: $f(x) = y$.
Since $x \in [a; b]$ and since f is continuous on $[a; b]$ from $\mathrm{dom}[f]$ to $\mathbb{R}$,
we conclude that f is continuous at x from $\mathrm{dom}[f]$ to $\mathbb{R}$.
Since $S \leqslant b$, we conclude that S is bounded above in $\mathbb{R}$.
So, since $S \neq \varnothing$, by Theorem 46.1, choose $u \in S^{\mathbb{N}}$ s.t. $u_\bullet \to x$ in $\mathbb{R}$.
We have $u \in S^{\mathbb{N}} \subseteq (\mathrm{dom}[f])^{\mathbb{N}}$. Also, $x \in [a; b] \subseteq \mathrm{dom}[f]$.
Then, by Theorem 44.12, $u_\bullet \to x$ in $\mathrm{dom}[f]$.
So, since f is continuous at x from $\mathrm{dom}[f]$ to $\mathbb{R}$,
we see that $(f \circ u)_\bullet \to f(x)$ in $\mathbb{R}$.

Claim 2: $\forall j \in \mathbb{N}$, $(f \circ u)_j \leqslant y$.
Proof of Claim 2:
Given $j \in \mathbb{N}$. Want: $(f \circ u)_j \leqslant y$.
Since $u \in S^{\mathbb{N}}$, we get $u_j \in S$.
Then $u_j \in S \subseteq \mathrm{dom}[f]$.
By Claim 1, $f_*(S) \leqslant y$.
Since $u_j \in S$ and $u_j \in \mathrm{dom}[f]$, we get: $f(u_j) \in f_*(S)$.
Then $(f \circ u)_j = f(u_j) \in f_*(S) \leqslant y$, as desired.
End of proof of Claim 2.

Since $(f \circ u)_\bullet \to f(x)$ in $\mathbb{R}$, by Theorem 50.1,
it follows, from Claim 2, that $f(x) \leqslant y$.
It remains to show: $f(x) \geqslant y$.
Since $x \in [a; b]$, we conclude that one of the following is true:
(α) $x = b$ or (β) $x \in [a; b)$.

Case (α):
By assumption $y \leqslant f(b)$.
Then $f(x) = f(b) \geqslant y$, as desired.
End of Case α.

Case (β):
Define $v \in \mathbb{R}^{\mathbb{N}}$ by $v_j = x + \dfrac{b - x}{j}$.
By Theorem 50.4, we know both of the following:
(A) $v_\bullet \to x$ in $\mathbb{R}$

and (B) $\forall j \in \mathbb{N}$, $x < v_j \leqslant b$.

Claim 3: $v \in [a;b]^{\mathbb{N}}$.
Proof of Claim 3:
Since $v \in \mathbb{R}^{\mathbb{N}}$, we see that $\mathrm{dom}[v] = \mathbb{N}$.
Want: $\mathrm{im}[v] \subseteq [a;b]$.
Want: $\forall z \in \mathrm{im}[v]$, $z \in [a;b]$.
Given $z \in \mathrm{im}[v]$. Want $z \in [a;b]$.
Since $z \in \mathrm{im}[v]$ and $v : \mathbb{N} \to \mathbb{R}$, choose $j \in \mathbb{N}$ s.t. $z = v_j$.
By (B), $x < v_j \leqslant b$.
Since $x \in [a;b]$, we get $a \leqslant x$.
Then $a \leqslant x < v_j$, so $a < v_j$. Then $a \leqslant v_j$.
Since $a \leqslant v_j \leqslant b$, we get $v_j \in [a;b]$.
Then $z = v_j \in [a;b]$.
End of proof of Claim 3.

By (A), we have: $v_\bullet \to x$ in $\mathbb{R}$.
Recall that $x \in \mathrm{dom}[f]$.
Since $[a;b] \subseteq \mathrm{dom}[f]$, it follows that $[a;b]^{\mathbb{N}} \subseteq (\mathrm{dom}[f])^{\mathbb{N}}$.
Then, by Claim 3, we see that $v \in (\mathrm{dom}[f])^{\mathbb{N}}$.
It follows, from Theorem 44.12, that $v_\bullet \to x$ in $\mathrm{dom}[f]$.
So, since f is continuous at x from $\mathrm{dom}[f]$ to $\mathbb{R}$,
we see that $(f \circ v)_\bullet \to f(x)$ in $\mathbb{R}$.
Recall that we want to show: $f(x) \geqslant y$.
Then, by Theorem 50.2, it suffices to prove: $\forall j \in \mathbb{N}$, $(f \circ v)_j \geqslant y$.
Given $j \in \mathbb{N}$. Want: $(f \circ v)_j \geqslant y$.
By (B), $x < v_j$.
Since $S \leqslant \sup S = x < v_j$, we get $S < v_j$, so $v_j > S$, so $v_j \notin S$.
By Claim 3, $\mathrm{im}[v] \subseteq [a;b]$.
Then $v_j \in \mathrm{im}[v] \subseteq [a;b]$, so $v_j \in [a;b]$.
So, since $v_j \notin S$, by definition of S, we see that $\neg(f(v_j) \leqslant y)$.
Then $f(v_j) > y$. Then $f(v_j) \geqslant y$.
So, since $(f \circ v)_j = f(v_j)$, we get: $(f \circ v)_j \geqslant y$, as desired.
End of Case (β). □

THEOREM 50.6. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Let $a, b, y \in \mathbb{R}$. *Assume* $a \leqslant b$.

Assume f is continuous on $[a;b]$ from $\mathrm{dom}[f]$ *to* $\mathbb{R}$.
Assume $f(a) \geqslant y \geqslant f(b)$. Then $\exists x \in [a;b]$ s.t. $f(x) = y$.

Proof. Let $g := -f$ and let $z := -y$.
We have $g(a) = -(f(a))$ and $z = -y$ and $g(b) = -(f(b))$.
Since $f(a) \geqslant y \geqslant f(b)$, we get $-(f(a)) \leqslant -y \leqslant -(f(b))$.
Then $g(a) \leqslant z \leqslant g(b)$.
By HW#8-3, g is continuous on $[a;b]$.
Then, by Theorem 50.5 (with f replaced by g and y by z),
choose $x \in [a;b]$ s.t. $g(x) = z$.
Want: $f(x) = y$.
Since $g = -f$, we get $g(x) = -(f(x))$, and so $f(x) = -(g(x))$.
We have $z = -y$, so $-z = y$.
Then $f(x) = -(g(x)) = -z = y$, as desired. □

DEFINITION 50.7. $\forall a, b \in \mathbb{R}^*$, $[a|b] := [a;b] \cup [b;a]$.

THEOREM 50.8. $[1|3] = [1;3] = [3|1]$.

THEOREM 50.9. $\forall a, b \in \mathbb{R}^*$, $[\,a\,|\,b\,] = [\,\min\{a,b\}\,;\,\max\{a,b\}\,]$.

The following is the **Intermediate Value Theorem**.

THEOREM 50.10. *Let $f : \mathbb{R} \to \mathbb{R}$, $a, b \in \mathbb{R}$.*
Assume f is continuous on $[a|b]$. Then $[f(a)|f(b)] \subseteq f_*([a|b])$.

Proof. Want: $\forall y \in [f(a)|f(b)]$, $y \in f_*([a|b])$.
Given $y \in [f(a)|f(b)]$. Want: $y \in f_*([a|b])$.
Want: $\exists x \in [a|b] \cap (\mathrm{dom}[f])$ s.t. $f(x) = y$.
Since f is continuous on $[a|b]$,
it follows that $[a|b] \subseteq \mathrm{dom}[f]$,
so $[a|b] \cap (\mathrm{dom}[f]) = [a|b]$.
Want: $\exists x \in [a|b]$ s.t. $f(x) = y$.
Let $\alpha := \min\{a,b\}$ and $\beta := \max\{a,b\}$.
Then $[a|b] = [\alpha;\beta]$, so, by assumption, f is continuous on $[\alpha;\beta]$.
Also, $[f(a)|f(b)] = [f(\alpha)|f(\beta)]$, so $y \in [f(\alpha)|f(\beta)]$.
Want: $\exists x \in [\alpha;\beta]$ s.t. $f(x) = y$.
At least one of the following is true:
(1) $f(\alpha) \leqslant f(\beta)$ or (2) $f(\alpha) \geqslant f(\beta)$.

Case 1:
We have $[f(\alpha)|f(\beta)] = [f(\alpha);f(\beta)]$, so $y \in [f(\alpha);f(\beta)]$,

so $f(\alpha) \leqslant y \leqslant f(\beta)$.
Then, by Theorem 50.6 (with a replaced by α and b by β),
we see that $\exists x \in [\alpha; \beta]$ s.t. $f(x) = y$, as desired.
End of Case 1.

Case 2:
We have $[f(\alpha)|f(\beta)] = [f(\beta); f(\alpha)]$, so $y \in [f(\beta); f(\alpha)]$,
so $f(\alpha) \geqslant y \geqslant f(\beta)$.
Then, by Theorem 50.5 (with a replaced by α and b by β),
we see that $\exists x \in [\alpha; \beta]$ s.t. $f(x) = y$, as desired.
End of Case 2. □

51. Isometries and homeomorphisms

DEFINITION 51.1. *Let X and Y be metric spaces.*
Then, $\forall f$, by f is an **isometry** *from X to Y, we mean:*
$(\quad f : X \hookrightarrow\!> Y \quad) \quad \&$
$(\quad \forall p, q \in X, \quad d_Y\,(\,f(p)\,,\,f(q)\,) \,=\, d_X\,(\,p\,,\,q\,) \quad)$.
Also, by X and Y are **isometric***, we mean:*
$\exists f$ *s.t. f is an isometry from X to Y.*
Also, $\forall f$, by f is a **homeomorphism** *from X to Y, we mean:*
$(\quad f : X \hookrightarrow\!> Y \quad) \quad \&$
$(\quad f$ *is continuous from X to Y* $\quad) \quad \&$
$(\quad f^{-1}$ *is continuous from Y to X* $\quad)$.
Also, by X and Y are **homeomorphic***, we mean:*
$\exists f$ *s.t. f is a homeomorphism from X to Y.*

We sometimes omit "from X to Y". We sometimes say "X is isometric to Y" or "Y is isometric to X" instead of "X and Y are isometric". We sometimes say "X is homeomorphic to Y" or "Y is homeomorphic to X" instead of "X and Y are homeomorphic".

We drew pictures indicating that two circles of the same radius are isometric. We indicated that any circle is homeomorphic to any ellipse. We drew a wandring simple closed curve and indicated that it is homeomorphic to a circle.

THEOREM 51.2. $(-1; 1)$ *is homeomorphic to* $\mathbb{R}$.

Proof. Define $f : (-1;1) \to \mathbb{R}$ by $f(x) = x/\sqrt{1-x^2}$.
Unassigned HW: f is a homeomorphism from $(-1;1)$ to $\mathbb{R}$.
Then $(-1;1)$ is homeomorphic to $\mathbb{R}$. □

Theorem 51.2 shows that it is possible for a bounded subset of $\mathbb{R}$ like $(-1;1)$ to be homeomorphic to an unbounded one, like $\mathbb{R}$ itself.

DEFINITION 51.3. *Let X be a metric space, $p \in X$, $r > 0$. Then:*
$\overline{B}_X(p,r) := \{q \in X \,|\, d_X(p,q) \leqslant r\}$ *and*
$S_X(p,r) := \{q \in X \,|\, d_X(p,q) = r\}$.

We sometimes omit the subscript X from "$\overline{B}_X(p,r)$" and "$S_X(p,r)$".
The set $\overline{B}_X(p,r)$ is called the **closed ball** in X about p of radius r. When $X = \mathbb{R}$, $\overline{B}_X(p,r)$ is a closed interval. When $X = \mathbb{R}^2$, $\overline{B}_X(p,r)$ is a closed disk. The set $S_X(p,r)$ is called the **sphere** in X about p of radius r. When $X = \mathbb{R}$, $\overline{B}_X(p,r)$ is a set of two real numbers. When $X = \mathbb{R}^2$, $\overline{B}_X(p,r)$ is a circle.

THEOREM 51.4. *Let $C := S_{\mathbb{R}^2}(0_2,1)$ and let $p := (0,1)$.*
Then $C_p^\times$ is homeomorphic to $\mathbb{R}$.

Proof. Define $f : C_p^\times \to \mathbb{R}$ by $f(x,y) = x/(1-y)$.
Unassigned HW: f is a homeomorphism from $C_p^\times$ to $\mathbb{R}$.
Then $C_p^\times$ is homeomorphic to $\mathbb{R}$. □

Theorem 51.4 shows that it is possible for a bounded subset of $\mathbb{R}^2$ like $C_p^\times$ to be homeomorphic to an unbounded subset of $\mathbb{R}$, like $\mathbb{R}$ itself.

DEFINITION 51.5. *Let X be a metric space.*
By X is **geometrically bounded***, we mean: X is bounded in X.*
By X is **topologically bounded***, we mean: $\forall$metric space Y,*
(Y is homeomorphic to X) $\Rightarrow$ (Y is geometrically bounded).

Note that the definition of topologically bounded is universally quantified over metric spaces. This makes it a challenge to study, but study it we will. Moreover, even though it is a topological concept, we will relate it to real analysis through the Extreme Value Theorem.

THEOREM 51.6. *Let $C := S_{\mathbb{R}^2}(0_2,1)$ and $p := (0,1)$.*
Let $X := \{(x,0) \,|\, x \in \mathbb{R}\}$. Let $I := (-1;1)$ and $J := [-1;1]$.
Then I, J C and $C_p^\times$ are all geometrically bounded.
Also, $\mathbb{R}$ and X are both not geometrically bounded.
Also, I, $C_p^\times$, $\mathbb{R}$ and X are all not topologically bounded.

In Theorem 51.6, the fact that I is not topologically bounded follows from Theorem 51.2. In Theorem 51.6, the fact that $C_p^\times$ is not topologically bounded follows from Theorem 51.4.

Let $C := S_{\mathbb{R}^2}(0_2, 1)$ and let $J := [-1; 1]$. We drew pictures of subsets of $\mathbb{R}^2$ that are homeomorphic to J, and noted that they were all geometrically bounded. Some were very big, stretching across several blackboards, but all were geometrically bounded. We drew pictures of subsets of $\mathbb{R}^2$ that are homeomorphic to C, and noted that they were all geometrically bounded. Some were very big, stretching across several blackboards, but all were geometrically bounded. Based on these observations, we speculated that J and C are both topologically bounded. Our intention is to spend the next class or two developing the material necessary to analyze topological boundedness.

DEFINITION 51.7. *Let s and t be sequences.*
By t is a **subsequence** *of s, we mean:*

$$\exists \text{strictly increasing } \ell \in \mathbb{N}^{\mathbb{N}} \text{ s.t. } t = s \circ \ell.$$

The intuition is: The sequence t is obtained from s by dropping some of the terms of s. The terms of s that are NOT dropped must appear in t in exactly the same order as they appear in s.

THEOREM 51.8. *The following are all true:*

$(1/2, 1/4, 1/6, 1/8, \ldots)$ *is a subsequence of* $(1, 1/2, 1/3, 1/4, \ldots)$.
$(4^2, 8^2, 12^2, 16^2, \ldots)$ *is a subsequence of* $(2, 4, 6, 8, \ldots)$.
$(3, 4, 5, 6, \ldots)$ *is a subsequence of* $(1, 2, 3, 4, \ldots)$.
$(2, 1, 3, 4, 5, 6, 7, 8, 9, \ldots)$ *is NOT a subsequence of* $(1, 2, 3, 4, \ldots)$.

All of the following are true:

$(1/2, 1/4, 1/6, 1/8, \ldots) = (1, 1/2, 1/3, 1/4, \ldots) \circ (2, 4, 6, 8, \ldots)$.
$(4^2, 8^2, 12^2, 16^2, \ldots) = (2, 4, 6, 8, \ldots) \circ (8, 32, 72, 128, \ldots)$.
$(3, 4, 5, 6, \ldots) = (1, 2, 3, 4, \ldots) \circ (3, 4, 5, 6, \ldots)$.
$(2, 1, 3, 4, 5, 6, 7, 8, 9, \ldots) = (1, 2, 3, 4, \ldots) \circ (2, 1, 3, 4, 5, \ldots)$.

We noted that $(2, 1, 3, 4, 5, \ldots)$ is not strictly increasing.

DEFINITION 51.9. *Let X be a metric space and $s \in X^{\mathbb{N}}$.*
By s is **subconvergent** *in X, we mean:*

$\exists$*a subsequence t of s s.t. t is convergent in X.*

We sometimes drop "in X" from "subconvergent in X".

THEOREM 51.10. *The following are all true:*

$(-1, 1, -1, 1, -1, 1, -1, 1, \ldots)$ *is NOT convergent in* $\mathbb{R}$.

$(-1,1,-1,1,-1,1,-1,1,\ldots)$ *IS subconvergent in* $\mathbb{R}$.
$(1,2,3,4,\ldots)$ *is NOT subconvergent in* $\mathbb{R}$.

DEFINITION 51.11. *Let X be a metric space.*
By X is **compact**, *we mean:*
$\forall s \in X^{\mathbb{N}}$, *$s$ is subconvergent in X.*
Also, by X is **proper**, *we mean:*
$\forall$*bounded* $s \in X^{\mathbb{N}}$, *s is subconvergent in X.*

Our upcoming goals:
(1) $[-1;1]$ and $S_{\mathbb{R}^2}(0_2,1)$ are both compact.
(2) If a metric space is homeomorphic to a compact metric space, then it is compact.
(3) Any compact metric space is geometrically bounded.
By Definition 51.5 and (2) and (3), we get:
(A) Any compact metric space is topologically bounded.
By (1) and (4), we get our earlier goal:
(B) $[-1;1]$ and $S_{\mathbb{R}^2}(0_2,1)$ are both topologically bounded.
We will see that (1) is hard, (2) is easy and (3) is medium.
A subsequence of a convergent sequence has the same limit:

THEOREM 51.12. *Let X be a metric space, $s \in X^{\mathbb{N}}$ and $p \in X$.*
Let t be a subsequence of s. Assume that $s_\bullet \to p$ in X.
Then $t_\bullet \to p$ in X.

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\ j \geqslant K\ ) \Rightarrow (\ d_X(t_j,p) < \varepsilon\ )$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\ j \geqslant K\ ) \Rightarrow (\ d_X(t_j,p) < \varepsilon\ )$.
Since $s_\bullet \to p$ in X, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\ j \geqslant K\ ) \Rightarrow (\ d_X(s_j,p) < \varepsilon\ )$.
Want: $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ d_X(t_j,p) < \varepsilon\ )\ ]$.
Given $j \in \mathbb{N}$. Want: $(\ j \geqslant K\ ) \Rightarrow (\ d_X(t_j,p) < \varepsilon\ )$.
Assume $j \geqslant K$. Want: $d_X(t_j,p) < \varepsilon$.
Since t is a subsequence of s,
choose a strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
By HW#9-2, we see that $\ell_j \geqslant j$.
Since $\ell_j \geqslant j \geqslant K$, by the choice of K, we get: $d_X(s_{\ell_j},p) < \varepsilon$.
So, since $t_j = (s \circ \ell)_j = s_{\ell_j}$, we get $d_X(t_j,p) < \varepsilon$, as desired. □

DEFINITION 51.13. *Let $s \in \mathbb{R}^{\mathbb{N}}$.*
Then $s_\bullet \to \infty$ in $\mathbb{R}^*$ means: $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\; j \geqslant K \;) \;\Rightarrow\; (\; s_j > M \;).$$
Also, $s_\bullet \to -\infty$ in $\mathbb{R}^*$ means: $\forall N \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\; j \geqslant K \;) \;\Rightarrow\; (\; s_j < N \;).$$

THEOREM 51.14. *Let $s \in \mathbb{R}^{\mathbb{N}}$.*
Then ($s_\bullet \to \infty$ in $\mathbb{R}^*$) $\Rightarrow$ (s is not bounded above in $\mathbb{R}$).
Also, ($s_\bullet \to -\infty$ in $\mathbb{R}^*$) $\Rightarrow$ (s is not bounded below in $\mathbb{R}$).

Proof. Unassigned HW. □

THEOREM 51.15. *Let $s \in \mathbb{R}^{\mathbb{N}}$.*
Let t be a subsequence of s. Assume that $s_\bullet \to \infty$ in $\mathbb{R}^*$.
Then $t_\bullet \to \infty$ in $\mathbb{R}^*$.

Proof. Want: $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\; j \geqslant K \;) \;\Rightarrow\; (\; t_j > M \;).$$
Given $M \in \mathbb{R}$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\; j \geqslant K \;) \;\Rightarrow\; (\; t_j > M \;).$$
Since $s_\bullet \to \infty$ in X, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\; j \geqslant K \;) \;\Rightarrow\; (\; s_j > M \;).$$
Want: $\forall j \in \mathbb{N}$, [($j \geqslant K$) $\Rightarrow$ ($t_j > M$)].
Given $j \in \mathbb{N}$. Want: ($j \geqslant K$) $\Rightarrow$ ($t_j > M$).
Assume $j \geqslant K$. Want: $t_j > M$.
Since t is a subsequence of s,
choose a strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
By HW#9-2, we see that $\ell_j \geqslant j$.
Since $\ell_j \geqslant j \geqslant K$, by the choice of K, we get: $s_{\ell_j} > M$.
So, since $t_j = (s \circ \ell)_j = s_{\ell_j}$, we get $t_j > M$, as desired. □

THEOREM 51.16. *Let $s \in \mathbb{R}^{\mathbb{N}}$.*
Let t be a subsequence of s. Assume that $s_\bullet \to -\infty$ in $\mathbb{R}^*$.
Then $t_\bullet \to -\infty$ in $\mathbb{R}^*$.

Proof. Unassigned HW. □

52. The ε-δ quantified equivalence for continuity

THEOREM 52.1. *Let $a \in \mathbb{R}^{\mathbb{N}}$. Assume: $\forall j \in \mathbb{N}$, $0 \leqslant a_j < 1/j$.*
Then $a_\bullet \to 0$ in $\mathbb{R}$.

Proof. Unassigned HW. □

THEOREM 52.2. *Let X and Y be metric spaces.*
Let $f : X \to Y$ and let $q \in X$.
Assume: $\forall \varepsilon > 0$, $\exists \delta > 0$ *s.t.*, $\forall p \in X$,

$$(d_X(p , q) < \delta) \Rightarrow (d_Y(f(p) , f(q)) < \varepsilon).$$

Then f is continuous at q from X to Y.

Proof. This is HW#10-4. □

The converse of Theorem 52.2 is also true:

THEOREM 52.3. *Let X and Y be metric spaces.*
Let $f : X \to Y$ and let $q \in X$.
Assume that f is continuous at q from X to Y.
Then: $\forall \varepsilon > 0$, $\exists \delta > 0$ *s.t.*, $\forall p \in X$,

$$(d_X(p , q) < \delta) \Rightarrow (d_Y(f(p) , f(q)) < \varepsilon).$$

Proof. Assume: $\exists \varepsilon > 0$ s.t., $\forall \delta > 0$, $\exists p \in X$ s.t.

$$(d_X(p , q) < \delta) \;\&\; (d_Y(f(p) , f(q)) \geqslant \varepsilon).$$

Want: Contradiction.
Choose $\varepsilon > 0$ s.t., $\forall \delta > 0$, $\exists p \in X$ s.t.

$$(d_X(p , q) < \delta) \;\&\; (d_Y(f(p) , f(q)) \geqslant \varepsilon).$$

Define $A : \mathbb{N} \to 2^{\mathbb{R}}$ by

$$A_j = \{ p \in X \mid (d_X(p , q) < 1/j) \,\&\, (d_Y(f(p) , f(q)) \geqslant \varepsilon) \}.$$

Claim 1: $\forall j \in \mathbb{N}$, $A_j \neq \varnothing$.
Proof of Claim 1:
Given $j \in \mathbb{N}$. Want: $A_j \neq \varnothing$.
Since $1/j > 0$, by the choice of ε, choose $p \in X$ s.t.

$$(d_X(p , q) < 1/j) \;\&\; (d_Y(f(p) , f(q)) \geqslant \varepsilon).$$

Then $p \in A_j$, so $A_j \neq \varnothing$, as desired.
End of proof of Claim 1.
Define $s \in X^{\mathbb{N}}$ by $s_j = \mathrm{CH}(A_j)$.

Claim 2: $\forall j \in \mathbb{N}$, $0 \leqslant (d_X(s, q))_j < 1/j$.
Proof of Claim 2:
Given $j \in \mathbb{N}$. Want: $0 \leqslant (d_X(s, q))_j < 1/j$.
Since $(d_X(s, q))_j = d_X(s_j, q)$, we want: $0 \leqslant d_X(s_j, q) < 1/j$.
Since $d_X(s_j, q) \in \mathrm{im}[d_X] \subseteq [0; \infty) \geqslant 0$, want: $d_X(s_j, q) < 1/j$.
Since $A_j \neq \varnothing$, it follows that $\mathrm{CH}(A_j) \in A_j$.
Since $s_j = \mathrm{CH}(A_j) \in A_j$, we conclude that:

$$(\ d_X(\ s_j \ , \ q \ ) \ < \ 1/j \ ) \ \& \ (\ d_Y(\ f(s_j) \ , \ f(q) \ ) \ \geqslant \ \varepsilon \ ).$$

Then $d_X(s_j, q) < 1/j$, as desired.
End of proof of Claim 2.

Claim 3: $\forall j \in \mathbb{N}$, $d_Y(\ (f \circ s)_j \ , \ f(q) \ ) \ \geqslant \ \varepsilon$.
Proof of Claim 3:
Given $j \in \mathbb{N}$. Want: $d_Y(\ (f \circ s)_j \ , \ f(q) \ ) \ \geqslant \ \varepsilon$.
Since $A_j \neq \varnothing$, it follows that $\mathrm{CH}(A_j) \in A_j$.
Since $s_j = \mathrm{CH}(A_j) \in A_j$, we conclude that:

$$(\ d_X(\ s_j \ , \ q \ ) \ < \ 1/j \ ) \ \& \ (\ d_Y(\ f(s_j) \ , \ f(q) \ ) \ \geqslant \ \varepsilon \ ).$$

Then $d_Y(\ (f \circ s)_j \ , \ f(q) \ ) \ = \ d_Y(\ f(s_j) \ , \ f(q) \ ) \ \geqslant \ \varepsilon$, as desired.
End of proof of Claim 3.

By Claim 2 and Theorem 52.1, we see that $(d_X(s, q))_\bullet \to 0$ in $\mathbb{R}$.
Then, by $\Leftarrow$ of HW#10-2, we have: $s_\bullet \to q$ in X.
By assumption, f is continuous at q from X to Y.
Then $(f \circ s)_\bullet \to f(q)$ in Y, so choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\ j \geqslant K \ ) \ \Rightarrow \ (\ d_Y(\ (f \circ s)_j \ , \ f(q) \ ) \ < \ \varepsilon \ ).$$

Let $j := K$.
From Claim 3, we get: $\varepsilon \ \leqslant \ d_Y(\ (f \circ s)_j \ , \ f(q) \ )$.
Since $j \geqslant K$, by the choice of K, we have:

$$d_Y(\ (f \circ s)_j \ , \ f(q) \ ) \ < \ \varepsilon.$$

Then $\varepsilon \ \leqslant \ d_Y(\ (f \circ s)_j \ , \ f(q) \ ) \ < \ \varepsilon$, so $\varepsilon < \varepsilon$. Contradiction. □

53. Distance between a sequence and a point

DEFINITION 53.1. *Let X be a metric space, $s \in X^{\mathbb{N}}$ and $q \in X$. Then $d_X(s, q) \in [0; \infty)^{\mathbb{N}}$ is defined by: $(d_X(s, q))_j = d_X(s_j, q)$.*

The subscript X in "$d_X(s, q)$" is sometimes omitted.

THEOREM 53.2. *Let X be a metric space, $s, t \in X^{\mathbb{N}}$ and $q \in X$.*
Assume that t is a subsequence of s.
Then $d_X(t, q)$ is a subsequence of $d_X(s, q)$.

Proof. Let $a := d_X(s, q)$ and $b := d_X(t, q)$.
Want: b is a subsequence of a.
Want: $\exists$strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $b = a \circ \ell$.
Since t is a subsequence of s,
choose a strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.

Want: $b = a \circ \ell$.
Want: $\forall j \in \mathbb{N}$, $b_j = (a \circ \ell)_j$.
Given $j \in \mathbb{N}$. Want: $b_j = (a \circ \ell)_j$.
Let $k := \ell_j$. Then $a_k = (d_X(s,q))_k = d_X(s_k,q)$.
Since $t_j = (s \circ \ell)_j = s_{\ell_j} = s_k$, we get $d_X(t_j,q) = d_X(s_k,q)$.
Then $b_j = (d_X(t,q))_j = d_X(t_j,q) = d_X(s_k,q) = a_k = a_{\ell_j} = (a \circ \ell)_j$. □

THEOREM 53.3. *Let X be a metric space, $s \in X^{\mathbb{N}}$ and $q \in X$. Assume that s is bounded in X. Then $d_X(s,q)$ bounded in $\mathbb{R}$.*

Proof. Let $\sigma := d_X(s,q)$. Want: σ is bounded in $\mathbb{R}$.
Want: $\mathrm{im}[\sigma]$ is bounded in $\mathbb{R}$. Want: $C \in \mathcal{B}_{\mathbb{R}}$ s.t. $\mathrm{im}[\sigma] \subseteq C$.
Since s is bounded in X, we know that $\mathrm{im}[s]$ is bounded in X.
Then choose $B \in \mathcal{B}_X$ s.t. $\mathrm{im}[s] \subseteq B$.
Choose $p \in X$ and $r > 0$ s.t. $B = B_X(p,r)$. Let $a := d_X(p,q)$.
Since $a = d_X(p,q) \in \mathrm{im}[d_X] \subseteq [0;\infty) \geqslant 0$, we get $a \geqslant 0$.
So, since $r > 0$, we conclude that $a + r > 0$.
Let $C := B_{\mathbb{R}}(0, r+a)$. Then $C \in \mathcal{B}_{\mathbb{R}}$. Want: $\mathrm{im}[\sigma] \subseteq C$.
Want: $\forall z \in \mathrm{im}[\sigma]$, $z \in C$.
Given $z \in \mathrm{im}[\sigma]$. Want: $z \in C$.
Want: $z \in B_{\mathbb{R}}(0, r+a)$. Want: $d_{\mathbb{R}}(z,0) < r + a$.
Since $z \in \mathrm{im}[\sigma]$, choose $j \in \mathbb{N}$ s.t. $z = \sigma_j$.
Then $z = \sigma_j = (d_X(s,q))_j = d_X(s_j,q)$.
By the triangle inequality, $d_X(s_j,q) \leqslant [d_X(s_j,p)] + [d_X(p,q)]$.
We have $s_j \in \mathrm{im}[s] \subseteq B = B_X(p,r)$, so $d_X(s_j,p) < r$.
So, since $d_X(p,q) = a$, we get $[d_X(s_j,p)] + [d_X(p,q)] < r + a$.
Since $a + r > 0$, we see that $-(a+r) < 0$.
Since $d_X(s_j,q) \in \mathrm{im}[d_X] \subseteq [0;\infty) \geqslant 0$,
we get $d_X(s_j,q) \geqslant 0$, and so $0 \leqslant d_X(s_j,q)$.
Then $-(r+a) < 0 \leqslant d_X(s_j,q) = z$, so $-(r+a) < z$.
Also, $z = d_X(s_j,q) \leqslant [d_X(s_j,p)] + [d_X(p,q)] < r + a$, so $z < r + a$.
Then $-(r+a) < z < r + a$, so $|z| < r + a$.
Then $d_{\mathbb{R}}(z,0) = |z - 0| = |z| < r + a$, as desired. □

54. Compact implies geometrically bounded

We finish goal (3):

THEOREM 54.1. *Let X be a nonempty compact metric space. Then X is geometrically bounded.*

Proof. Let $A := X$. Then A is compact.
So, by HW#10-5, we conclude that A is bounded in X.
Then X is bounded in X, so X is geometrically bounded. □

Our remaining goals:
(1) $[-1;1]$ and $S_{\mathbb{R}^2}(0_2, 1)$ are both compact.
(2) If a metric space is homeomorphic to a compact metric space, then it is compact.

55. A continuous image of a compact is compact

THEOREM 55.1. *Let X and Y be sets, $f : X \to Y$ and $s, t \in X^{\mathbb{N}}$. Assume that t is a subsequence of s. Then $f \circ t$ is a subsequence of $f \circ s$.*

Proof. Want: $\exists$strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $f \circ t = (f \circ s) \circ \ell$.
Since t is a subsequence of s,
choose a strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
Want: $f \circ t = (f \circ s) \circ \ell$.
We have $f \circ t = f \circ (s \circ \ell) = (f \circ s) \circ \ell$, as desired. □

THEOREM 55.2. *Let X and Y be metric spaces and let $t \in X^{\mathbb{N}}$. Assume that f is continuous from X to Y. Assume that t is convergent in X. Then $f \circ t$ is convergent in Y.*

Proof. Choose $p \in X$ s.t. $t_\bullet \to p$ in X.
Since $t_\bullet \to p$ in X and since f is continuous at p from X to Y,
it follows that $f \circ t \to f(p)$ in Y.
Then $f \circ t$ is convergent in Y. □

THEOREM 55.3. *Let X and Y be sets. Let $f : X \to\!> Y$. Then, $\forall z \in Y$, $f^*(\{z\}) \neq \varnothing$.*

Proof. Given $z \in Y$. Want: $f^*(\{z\}) \neq \varnothing$.
Since $f : X \to\!> Y$, we see that $\mathrm{im}[f] = Y$.
Then $z \in Y = \mathrm{im}[f]$, so choose $p \in X$ s.t. $f(p) = z$.
Since $f(p) = z \in \{z\}$, it follows that $p \in f^*(\{z\})$.
Then $f^*(\{z\}) \neq \varnothing$, as desired. □

THEOREM 55.4. *Let X and Y be metric spaces. Let $f : X \to> Y$. Assume that f is continuous from X to Y. Asume that X is compact. Then Y is compact.*

Proof. Want: $\forall \sigma \in Y^{\mathbb{N}}$, σ is subconvergent in Y.
Given $\sigma \in Y^{\mathbb{N}}$. Want: σ is subconvergent in Y.

Claim 1: $\forall j \in \mathbb{N}$, $f^*(\{\sigma_j\}) \neq \varnothing$.
Proof of Claim 1:
Given $j \in \mathbb{N}$. Want: $f^*(\{\sigma_j\}) \neq \varnothing$.
By Theorem 55.3 (with z replaced by σ_j),
we see that $f^*(\{\sigma_j\}) \neq \varnothing$, as desired.
End of proof of Claim 1.

By Claim 1, $\forall j \in \mathbb{N}$, $\mathrm{CH}(f^*(\{\sigma_j\})) \in f^*(\{\sigma_j\})$.
Define $s \in X^{\mathbb{N}}$ by $s_j = \mathrm{CH}(f^*(\{\sigma_j\}))$.
Then: $\forall j \in \mathbb{N}$, $s_j \in f^*(\{\sigma_j\})$.
Since X is compact, s is subconvergent in X.
Choose a subsequence t of s s.t. t is convergent in X.
By Theorem 55.2, $f \circ t$ is convergent in Y.
By Theorem 55.1, $f \circ t$ is a subsequence of $f \circ s$.
Then $f \circ s$ is subconvergent in Y.
It therefore suffices to show: $f \circ s = \sigma$.
Want: $\forall j \in \mathbb{N}$, $(f \circ s)_j = \sigma_j$.
Given $j \in \mathbb{N}$. Want: $(f \circ s)_j = \sigma_j$.
We have $s_j \in f^*(\{\sigma_j\})$, so $f(s_j) \in \{\sigma_j\}$, so $f(s_j) = \sigma_j$.
Then $(f \circ s)_j = f(s_j) = \sigma_j$, as desired. □

We finish goal (2):

THEOREM 55.5. *Let X and Y be metric spaces. Assume that X is compact and that X is homeomorphic to Y. Then Y is compact.*

Proof. Since X is homeomorphic to Y,
choose f s.t. f is a homeomorphism from X onto Y.
Then f is continuous from X to Y and $f : X \to> Y$.
So, since X is compact, by Theorem 55.4, we get: Y is compact. □

56. Some topology

Our remaining goal:

(1) $[-1;1]$ and $S_{\mathbb{R}^2}(0_2,1)$ are both compact.

FOR NEXT YEAR: Define $\partial_X A$ as points approached from inside and outside A. Then define $\mathrm{Int}_X A$ as $A \backslash (\partial_X A)$ and $\mathrm{Cl}_X A$ as $A \cup (\partial_X A)$. Then the definitions below will become theorems.

DEFINITION 56.1. *Let X be a metric space and let $A \subseteq X$. Then* $\mathrm{Int}_X A := \{p \in X \,|\, \exists B \in \mathcal{B}_X(p) \text{ s.t. } B \subseteq A\}$. *Also,* $\mathrm{Cl}_X A := \{p \in X \,|\, \exists s \in A^{\mathbb{N}} \text{ s.t. } s_\bullet \to p \text{ in } X\}$.

In Definition 56.1, $\mathrm{Int}_X A$ is called the **interior** in X of A, and $\mathrm{Cl}_X A$ is called the **closure** in X of A. When X is clear, we simply say the interior of A and the closure of A, and we simply write $\mathrm{Int}\, A$ and $\mathrm{Cl}\, A$.

THEOREM 56.2. *Let $I := (-1;1)$, $J := (-1;1]$, $K := [-1;1]$. Then $\mathrm{Int}_{\mathbb{R}} J = I$ and $\mathrm{Cl}_{\mathbb{R}} J = K$ and $\mathrm{Int}_{\mathbb{R}^2} J^2 = I^2$ and $\mathrm{Cl}_{\mathbb{R}^2} J^2 = K^2$.*

THEOREM 56.3. *Let X be a metric space and let $A \subseteq X$. Then:*

$$[\ \mathrm{Int}_X A \subseteq A \subseteq \mathrm{Cl}_X A\ ] \quad \& \\
[\ \mathrm{Int}_X(\mathrm{Int}_X A) = \mathrm{Int}_X A\ ] \quad \& \\
[\ \mathrm{Cl}_X(\mathrm{Cl}_X A) = \mathrm{Cl}_X A\ ] \quad \& \\
[\ \mathrm{Int}_X(X \backslash A) = X \backslash (\mathrm{Cl}_X A)\ ] \quad \& \\
[\ \mathrm{Cl}_X(X \backslash A) = X \backslash (\mathrm{Int}_X A)\ ].$$

DEFINITION 56.4. *Let X be a metric space and let $A \subseteq X$. By A is* **open** *in X, we mean: $\mathrm{Int}_X A = A$. By A is* **closed** *in X, we mean: $\mathrm{Cl}_X A = A$.*

In Definition 56.4, when X is clear, we omit "in X".

DEFINITION 56.5. *Let X be a metric space.*

Then $\mathcal{T}_X := \{U \subseteq X \,|\, U \text{ is open in } X\}$.

Also, $\mathcal{T}'_X := \{C \subseteq X \,|\, C \text{ is closed in } X\}$.

In Definition 56.5, $\mathcal{T}_X$ is called the **topology** on X.

The following theorem gives quantified equivalences for open and closed subsets of a metric space.

THEOREM 56.6. *Let X be a metric space and let $A \subseteq X$. Then: $(A \in \mathcal{T}_X) \Leftrightarrow (\forall p \in A, \exists B \in \mathcal{B}_X(p) \text{ s.t. } B \subseteq A)$. Also: $(A \in \mathcal{T}'_X) \Leftrightarrow (\forall s \in A^{\mathbb{N}}, \forall p \in X, [(s_\bullet \to p \text{ in } X) \Rightarrow (p \in A)])$.*

THEOREM 56.7. *Let* $A := (-1;1)$, $B := (-1;\infty)$,
$$C := [-1;1],\ D := [-1;\infty).$$
Then $A, B \in \mathcal{T}_{\mathbb{R}}$ *and* $C, D \in \mathcal{T}'_{\mathbb{R}}$ *and* $A^2, B^2 \in \mathcal{T}_{\mathbb{R}^2}$ *and* $C^2, D^2 \in \mathcal{T}'_{\mathbb{R}^2}$.

Recall: $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$. Define a function $f : (-1;1) \to \mathbb{R}$ by $f(x) = x/\sqrt{1-x^2}$. Then f is a homeomorphism from $(-1;1)$ onto $\mathbb{R}$. Let $g := \mathrm{adj}_{-1}^{-\infty}(\mathrm{adj}_1^{\infty}(f))$. Then $g : [-1,1] \hookrightarrow\!> \mathbb{R}^*$. Define $d_* \in \mathcal{M}(\mathbb{R}^*)$ by $d_*(p,q) = d_{\mathbb{R}}(g^{-1}(p), g^{-1}(q))$. Then, for example, $d_*(-\infty,\infty) = d_{\mathbb{R}}(-1,1) = |(-1)-1| = 2$. It may seem strange that $-\infty$ should be a finite distance from ∞, and, in fact, we will call d_* the **weirdo metric** on $\mathbb{R}^*$. There are other metrics on $\mathbb{R}^*$, but for any "reasonable" $d \in \mathcal{M}(\mathbb{R}^*)$, we have: $\mathcal{T}_{(\mathbb{R}^*,d)} = \mathcal{T}_{(\mathbb{R}^*,d_*)}$. So, while there is no "standard" metric on $\mathbb{R}^*$, we do have a standard topology on $\mathbb{R}^*$. By Theorem 56.7, $D \in \mathcal{T}'_{\mathbb{R}}$. That is, $[-1;\infty)$ is closed in $\mathbb{R}$. In fact, the closure $\mathrm{Cl}_{\mathbb{R}}B$ in $\mathbb{R}$ of $(-1;\infty)$ is $[-1;\infty)$. It is *NOT* equal to $[-1;\infty]$. This may seem strange, but keep in mind that, since $\infty \notin \mathbb{R}$, we cannot have $\infty \in \mathrm{Cl}_{\mathbb{R}}B$. The set $[-1;\infty)$ is "as closed as it can be", within $\mathbb{R}$. Working in $(\mathbb{R}^*, d_*)$, things are very different. In fact, the closure $\mathrm{Cl}_{(\mathbb{R}^*,d_*)}B$ in $(\mathbb{R}^*, d_*)$ of $(-1;\infty)$ is equal to $[-1;\infty]$.

We drew a few amoeba-like subsets of $\mathbb{R}^2$ and discussed their interiors and closures. Some were bounded, some unbounded. We discussed open amoeba-like subsets of $\mathbb{R}^2$, both bounded and unbounded. We discussed closed amoeba-like subsets of $\mathbb{R}^2$, both bounded and unbounded.

We noted that many subsets of $\mathbb{R}^2$ contain part, but not all, of their boundaries; such sets are neither open nor closed. It can also happen that a set is *both* open and closed:

DEFINITION 56.8. *Let* X *be a metric space and let* $A \subseteq X$.
Then A *is* **clopen** *in* X *means:* $\quad A \in \mathcal{T}'_X \cap \mathcal{T}_X$.

THEOREM 56.9. *Let* X *be a metric space.*
$$\textit{Then:}\quad \varnothing\,,\, X \in \mathcal{T}'_X \cap \mathcal{T}_X.$$

THEOREM 56.10. *Let* $X := [1;2] \cup [3;4]$.
Then $[1;2] \in \mathcal{T}'_X \cap \mathcal{T}_X$. *Also,* $[3;4] \in \mathcal{T}'_X \cap \mathcal{T}_X$.

DEFINITION 56.11. *Let* X *be a metric space.*
Then X *is* **connected** *means:* $\mathcal{T}'_X \cap \mathcal{T}_X = \{\varnothing, X\}$.

That is, a topological space is connected iff it has no clopen sets except for the obvious ones.

THEOREM 56.12. *All of the following are true:*

(1) $\mathbb{R}$ *and* $\mathbb{R}^2$ *are both connected.*
(2) $[1;2] \cup [3;4]$ *is not connected.*
(3) $\forall a \in \mathbb{R}$, $\mathbb{R}_a^\times$ *is not connected.*
(4) $\forall v \in \mathbb{R}^2$, $(\mathbb{R}^2)_v^\times$ *is connected.*

THEOREM 56.13. $\mathbb{R}$ *and* $\mathbb{R}^2$ *are not homeomorphic.*

Proof. Assume that $\mathbb{R}$ and $\mathbb{R}^2$ are homeomorphic.
Want: Contradiction.
Choose f such that f is a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}^2$.
Let $A := \mathbb{R}_0^\times$. Let $B := (\mathbb{R}^2)_{f(0)}^\times$.
Then $f|A$ is a homeomorphism from A onto B.
By (4) of Theorem 56.12, B is connected.
So, since A and B are homeomorphic, A is connected.
By (3) of Theorem 56.12, A is not connected. Contradiction. □

Thus connectedness becomes a topological tool for distinguishing between the Euclidean spaces $\mathbb{R}$ and $\mathbb{R}^2$. There is another tool called "simple connectedness" that is used to distinguish between $\mathbb{R}^2$ and $\mathbb{R}^3$. There are many other tools, but topology is not the focus of our course, so we return to basics.

THEOREM 56.14. *Let* X *be a metric space and let* $A \subseteq X$.
Then: $(\ A \in \mathcal{T}_X\ ) \Leftrightarrow (\ X \backslash A \in \mathcal{T}'_X\ )$.
Also: $(\ A \in \mathcal{T}'_X\ ) \Leftrightarrow (\ X \backslash A \in \mathcal{T}_X\ )$.

That is, a set is open iff its complement is closed, and a set is closed iff its complement is open. In any metric space, singletons are closed:

THEOREM 56.15. *Let* X *be a metric space and let* $p \in X$.
Then $\{x\} \in \mathcal{T}'_X$.

THEOREM 56.16. *Let* $a, b \in \mathbb{R}$. *Then* $[a;b] \in \mathcal{T}'_{\mathbb{R}}$.

Proof. Let $C := [a;b]$. Want: $C \in \mathcal{T}'_{\mathbb{R}}$.
Want: $\forall s \in C^{\mathbb{N}}$, $\forall p \in \mathbb{R}$, $[\ (\ s_\bullet \to p \text{ in } \mathbb{R}\ ) \Rightarrow (\ p \in C\ )\ ]$.
Given $s \in C^{\mathbb{N}}$, $p \in \mathbb{R}$. Want: $(\ s_\bullet \to p \text{ in } \mathbb{R}\ ) \Rightarrow (\ p \in C\ )$.
Assume: $s_\bullet \to p$ in $\mathbb{R}$. Want: $p \in C$.
Since $s \in C^{\mathbb{N}} = [a;b]^{\mathbb{N}}$, we conclude: $\forall j \in \mathbb{N}$, $a \leqslant s_j \leqslant b$.
So, since $s_\bullet \to p$, by HW#8-4 and by unassigned HW, $a \leqslant p \leqslant b$.
Then $p \in [a;b] = C$, as desired. □

THEOREM 56.17. *Let $a \in \mathbb{R}$. Then $[a;\infty) \in \mathcal{T}'_{\mathbb{R}}$.*

Proof. Unassigned HW. □

THEOREM 56.18. *Let $b \in \mathbb{R}$. Then $(-\infty;b] \in \mathcal{T}'_{\mathbb{R}}$.*

Proof. Unassigned HW. □

Our only remaining goal is to show that $[-1;1]$ and $S_{\mathbb{R}^2}(0_2,1)$ are both compact. We can now break this up into various subgoals:

(A) $[-1,1]$ is closed and bounded in $\mathbb{R}$.

(B) $S_{\mathbb{R}^2}(0_2,1)$ is closed and bounded in $\mathbb{R}^2$.

(C) For any subset of a metric space,
(compact) implies (closed and bounded).

(D) For any subset of a *proper* metric space,
(closed and bounded) implies (compact).

(E) $\mathbb{R}$ and $\mathbb{R}^2$ are both proper.

By (C), (D) and (E), we see that
a subset of $\mathbb{R}$ is compact iff it is closed and bounded.
By (C), (D) and (E), we also see that
a subset of $\mathbb{R}^2$ is compact iff it is closed and bounded.
Then, by (A) and (B), $[-1;1]$ and $S_{\mathbb{R}^2}(0_2,1)$ are both compact.
We next work on these five subgoals, (A) to (E).

57. Subgoals (A)-(D)

THEOREM 57.1. *Let X and Y be metric spaces and let $f : X \to Y$. Assume: f is continuous from X to Y. Then: $\forall C \in \mathcal{T}'_Y$, $f^*(C) \in \mathcal{T}'_X$.*

Proof. Given $C \in \mathcal{T}'_Y$. Want: $f^*(C) \in \mathcal{T}'_X$.
Let $A := f^*(C)$. Want: $A \in \mathcal{T}'_X$.
Want: $\forall s \in A^{\mathbb{N}},\ \forall p \in X,\ [\ (\ s_\bullet \to p \text{ in } X\ )\ \Rightarrow\ (\ p \in A\ )\ ]$
Given $s \in A^{\mathbb{N}},\ p \in X$. Want: $(\ s_\bullet \to p \text{ in } X\ )\ \Rightarrow\ (\ p \in A\ )$
Assume $s_\bullet \to p$ in X. Want: $p \in A$.
Let $t := f \circ s$ and let $q := f(p)$.
Since f is continuous at p from X to Y and since $s_\bullet \to p$ in X,
we conclude that $t_\bullet \to q$ in Y.

Claim 1: $t \in C^{\mathbb{N}}$.
Proof of Claim 1:
Want: $\mathrm{dom}[t] = \mathbb{N}$ and $\mathrm{im}[t] \subseteq C$.
Since $A \subseteq X$, we get $A^{\mathbb{N}} \subseteq X^{\mathbb{N}}$.

We have $s \in A^{\mathbb{N}} \subseteq X^{\mathbb{N}}$, so $s : \mathbb{N} \to X$.
Since $s : \mathbb{N} \to X$ and $f : X \to Y$, we see that $t : \mathbb{N} \to Y$.
Then $\mathrm{dom}[t] = \mathbb{N}$. Want: $\mathrm{im}[t] \subseteq C$.
Want: $\forall z \in \mathrm{im}[t]$, $z \in C$.
Given $z \in \mathrm{im}[t]$. Want: $z \in C$.
Since $z \in \mathrm{im}[t]$, choose $j \in \mathbb{N}$ s.t. $z = t_j$.
Since $s \in A^{\mathbb{N}}$, we have $\mathrm{im}[s] \subseteq A$.
Then $s_j \in \mathrm{im}[s] \subseteq A = f^*(C)$, and so $f(s_j) \in C$.
Then $z = t_j = (f \circ s)_j = f(s_j) \in C$, as desired.
End of proof of Claim 1.

Since $t \in C^{\mathbb{N}}$, since $t_\bullet \to q$ in Y and since $C \in \mathcal{T}'_Y$,
we conclude that $q \in C$.
Since $f(p) = q \in C$, we get $p \in f^*(C)$.
Then $p \in f^*(C) = A$, as desired. □

THEOREM 57.2. *Let X and Y be metric spaces and let $f : X \to Y$. Assume: $\forall C \in \mathcal{T}'_Y$, $f^*(C) \in \mathcal{T}'_X$. Then: f is continuous from X to Y.*

Proof. Unassigned HW. □

By Theorem 57.1 and Theorem 57.2, a function is continuous iff
the preimage of any closed set is closed.

THEOREM 57.3. *Let X and Y be metric spaces and let $f : X \to Y$. Assume: f is continuous from X to Y. Then: $\forall U \in \mathcal{T}_Y$, $f^*(U) \in \mathcal{T}_X$.*

Proof. Unassigned HW. □

THEOREM 57.4. *Let X and Y be metric spaces and let $f : X \to Y$. Assume: $\forall U \in \mathcal{T}_Y$, $f^*(U) \in \mathcal{T}_X$. Then: f is continuous from X to Y.*

Proof. Unassigned HW. □

By Theorem 57.3 and Theorem 57.4, a function is continuous iff
the preimage of any open set is open.

It may seem strange that, in
Theorem 57.1, Theorem 57.2,
Theorem 57.3 and Theorem 57.4,
preimages are so important by contrast with forward images. Part

of the explanation is that, *generally*,

points want to go forward,
sets want to go backward and
sequences want to go forward.

Other mathematical objects have a similar propensity to move in one direction or another, and the astute learner will start to track which ones want to do what. Looking at open sets and closed sets, following this philosophy, it makes sense that preimages come up a lot. However, one does sometimes study maps such that the forward image of a closed set is closed, or such that the forward image of an open set is open:

DEFINITION 57.5. *Let X and Y be metric spaces and $f : X \to Y$. By f is a* **closed mapping** *from X to Y, we mean:*

$$\forall C \in \mathcal{T}'_X, \quad f_*(C) \in \mathcal{T}'_Y.$$

By f is an **open mapping** *from X to Y, we mean:*

$$\forall U \in \mathcal{T}_X, \quad f_*(U) \in \mathcal{T}_Y.$$

THEOREM 57.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 + y^2$. Then f is continuous from $\mathbb{R}^2$ to $\mathbb{R}$.*

Proof. Unassigned HW. □

We leave subgoal (A) as an unassigned exercise. For subgoal (B), because $S_{\mathbb{R}^2}(0_2, 1) \subseteq B_{\mathbb{R}^2}(0_2, 2)$, it follows that $S_{\mathbb{R}^2}(0_2, 1)$ is bounded in $\mathbb{R}^2$. To finish subgoal (B), we need only show:

THEOREM 57.7. $S_{\mathbb{R}^2}(0_2, 1) \in \mathcal{T}'_{\mathbb{R}^2}$.

Proof. Let $A := S_{\mathbb{R}^2}(0_2, 1)$. Want: $A \in \mathcal{T}'_{\mathbb{R}^2}$.
Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^2 + y^2$. Then $A = f^*(\{1\})$.
Also, by Theorem 57.6, we see that f is continuous from $\mathbb{R}^2$ to $\mathbb{R}$.
By Theorem 56.15, we conclude that $\{1\} \in \mathcal{T}'_{\mathbb{R}}$.
So, since f is continuous from $\mathbb{R}^2$ to $\mathbb{R}$,
by Theorem 57.1, we get: $f^*(\{1\}) \in \mathcal{T}'_{\mathbb{R}^2}$.
Then $A = f^*(\{1\}) \in \mathcal{T}'_{\mathbb{R}^2}$, as desired. □

To do subgoal (C), we must show that any compact subset of a metric space is closed and bounded. By HW#10-5, any compact subset of a metric space is bounded. The following shows that it's also closed:

THEOREM 57.8. *Let X be a metric space and let $C \subseteq X$. Assume that C is compact. Then $C \in \mathcal{T}'_X$.*

Proof. Want: $\forall s \in C^{\mathbb{N}}$, $\forall p \in X$, $[\ (\ s_\bullet \to p \text{ in } X\ ) \Rightarrow (\ p \in C\ )\ ]$.
Given $s \in C^{\mathbb{N}}$, $p \in X$. Want: $(\ s_\bullet \to p \text{ in } X\ ) \Rightarrow (\ p \in C\ )$.
Assume $s_\bullet \to p$ in X. Want: $p \in C$.
Since C is compact and $s \in C^{\mathbb{N}}$, we get: s is subconvergent in C.
Choose a subsequence t of s s.t. t is convergent in C.
Choose $q \in C$ s.t. $t_\bullet \to q$ in C. Then $t_\bullet \to q$ in X.
Since $s_\bullet \to p$ in X and since t is a subsequence of s,
it follows, from Theorem 51.12, that $t_\bullet \to p$ in X.
Since $t_\bullet \to p$ in X and $t_\bullet \to q$ in X,
it follows, from Theorem 39.12, that $p = q$.
Then $p = q \in C$, as desired. □

THEOREM 57.9. *Let X be a topological space, $A \subseteq X$ and $t \in A^{\mathbb{N}}$. Then: (t is convergent in A) $\Rightarrow$ (t is convergent in X).*

Proof. Unassigned HW. □

THEOREM 57.10. *Let X be a topological space, $A \subseteq X$ and $s \in A^{\mathbb{N}}$. Then: (s is subconvergent in A) $\Rightarrow$ (s is subconvergent in X).*

Proof. Unassigned HW. □

THEOREM 57.11. *Let X be a topological space, $A \in \mathcal{T}'_X$ and $t \in A^{\mathbb{N}}$. Then: (t is convergent in A) $\Leftrightarrow$ (t is convergent in X).*

Proof. By Theorem 57.9, we have:
(t is convergent in A) $\Rightarrow$ (t is convergent in X).
Want: (t is convergent in X) $\Rightarrow$ (t is convergent in A).
Assume: t is convergent in X. Want: t is convergent in A.
Choose $p \in X$ s.t. $t_\bullet \to p$ in X.
Since $A \in \mathcal{T}'_X$ and $t \in A^{\mathbb{N}}$ and $t_\bullet \to p$ in X, it follows that $p \in A$.
Since $t_\bullet \to p$ in X, since $t \in A^{\mathbb{N}}$ and since $p \in A$,
it follows that $t_\bullet \to p$ in A.
Then t is convergent in A, as desired. □

THEOREM 57.12. *Let X be a topological space, $A \in \mathcal{T}'_X$ and $s \in A^{\mathbb{N}}$. Then: (s is subconvergent in A) $\Leftrightarrow$ (s is subconvergent in X).*

Proof. By Theorem 57.10, we have:
(s is subconvergent in A) $\Rightarrow$ (s is subconvergent in X).
Want: (s is subconvergent in X) $\Rightarrow$ (s is subconvergent in A).
Assume: s is subconvergent in X. Want: s is subconvergent in A.

Choose a subsequence t of s s.t. t is convergent in X.
Then, by $\Leftarrow$ of Theorem 57.11, we see that t is convergent in A.
So, since t is a subsequence of s, s is subconvergent in A, as desired. $\square$

We now finish subgoal (D):

THEOREM 57.13. *Let X be a proper metric space and let $A \subseteq X$. Assume: $(\, A \in \mathcal{T}_X' \,) \,\&\, (\, A$ is bounded in $X\,)$. Then A is compact.*

Proof. Want: $\forall s \in A^{\mathbb{N}}$, s is subconvergent in A.
Given $s \in A^{\mathbb{N}}$. Want: s is subconvergent in A.
Since $\mathrm{im}[s] \subseteq A$ and since A is bounded in X,
we conclude that $\mathrm{im}[s]$ is bounded in X.
Then s is bounded in X.
So, since X is a proper metric space, s is subconvergent in X.
By $\Leftarrow$ of Theorem 57.12, s is subconvergent in A, as desired. $\square$

Our only remaining subgoal:
(E) $\mathbb{R}$ and $\mathbb{R}^2$ are both proper.
This is addressed in §58 and §60.

58. Properness and completeness of the line

THEOREM 58.1. *Let S be a set. Then:*
$$(\ \#S \leqslant 1\ ) \ \Leftrightarrow\ (\ \forall x, y \in S,\ x = y\ ).$$

THEOREM 58.2. *Let $A \subseteq \mathbb{R}^*$. Then $\#(A \cap [\mathrm{LB}(A)]) \leqslant 1$*

Proof. Unassigned HW. $\square$

Proof. Let $S := A \cap [\mathrm{LB}(A)]$. Want: $\#S \leqslant 1$.
By Theorem 58.1, want: $\forall x, y \in S$, $x = y$.
Given $x, y \in S$. Want: $x = y$.
Since $S = A \cap [\mathrm{LB}(A)]$, we get: $S \subseteq A$ and $S \subseteq \mathrm{LB}(A)$.
Since $x \in S \subseteq \mathrm{LB}(A)$, it follows that $x \leqslant A$, so $A \geqslant x$.
Then $y \in S \subseteq A \geqslant x$, so $y \geqslant x$. Want: $x \geqslant y$.
Since $y \in S \subseteq \mathrm{LB}(A)$, it follows that $y \leqslant A$, so $A \geqslant y$.
Then $x \in S \subseteq A \geqslant y$, so $x \geqslant y$, as desired. $\square$

THEOREM 58.3. $\forall$*set S, $\forall x$,*
$$[\,(\,\#S = 1\,) \,\&\, (\,x \in S\,)\,] \Rightarrow [\,\mathrm{UE}(S) = x\,].$$

Proof. Unassigned HW. $\square$

THEOREM 58.4. *Let* $A \subseteq \mathbb{R}$. *Assume:*

$(\, A \neq \varnothing \,) \,\&\, (\, A \textit{ is closed in } \mathbb{R} \,) \,\&\, (\, A \textit{ is bounded above in } \mathbb{R} \,)$.

Then $\max A \neq ☹$.

Proof. This is HW#11-3. □

THEOREM 58.5. *Let* $A \subseteq \mathbb{R}$. *Assume:*

$(\, A \neq \varnothing \,) \,\&\, (\, A \textit{ is closed in } \mathbb{R} \,) \,\&\, (\, A \textit{ is bounded below in } \mathbb{R} \,)$.

Then $\min A \neq ☹$.

Proof. Let $B := -A$. Then $\min A = -(\max B)$. Also,

$(\, B \neq \varnothing \,) \,\&\, (\, B \text{ is closed in } \mathbb{R} \,) \,\&\, (\, B \text{ is bounded above in } \mathbb{R} \,)$,

so, by Theorem 58.4, we see that $\max B \neq ☹$.
Then $\max B \in B \subseteq \mathbb{R}$, so $-(\max B) \in \mathbb{R}$, so $-(\max B) \neq ☹$.
Then $\min A = -(\max B) \neq ☹$, as desired. □

THEOREM 58.6. *Let* $A \subseteq \mathbb{R}$. *Assume:* $(\, A \neq \varnothing \,) \,\&\, (\, A \textit{ is compact} \,)$. *Then* $\min A \neq ☹ \neq \max A$.

Proof. By HW#10-5, A is bounded.
By Theorem 57.8, $A \in \mathcal{T}'_{\mathbb{R}}$, so A is closed in $\mathbb{R}$.
Then, by Theorem 58.5 and Theorem 58.4, $\min A \neq ☹ \neq \max A$. □

We consider sequences in metric spaces:

According to HW#6-4, convergent implies bounded.
According to HW#9-5, convergent implies Cauchy.
According to HW#11-4, Cauchy implies bounded.

So, HW#9-5 and HW#11-4, together, prove HW#6-4.

Observe that, in the metric space $\mathbb{Q}$, Cauchy does not imply convergent: Let $s := (1, 1.4, 1.41, 1.414, \ldots)$ be the sequence of decimal approximations to $\sqrt{2}$. Then s is Cauchy, but not convergent.

Observe that, in the metric space $\mathbb{R}$, bounded does not imply Cauchy: The sequence $(-1, 1, -1, 1, -1, 1, \ldots)$ is bounded, but not Cauchy.

DEFINITION 58.7. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Then f *is* **strictly monotone** *means:*

$(\, f \textit{ is strictly increasing} \,) \vee (\, f \textit{ is strictly decreasing} \,)$.

Also, f *is* **semi-monotone** *means:*

$(\, f \textit{ is semi-increasing} \,) \vee (\, f \textit{ is semi-decreasing} \,)$.

THEOREM 58.8. *Let* $a \in \mathbb{R}^{\mathbb{N}}$.

Assume: a *is semi-increasing and bounded above in* $\mathbb{R}$.
Then: a *is convergent in* $\mathbb{R}$.

Proof. This is HW#9-4. □

THEOREM 58.9. *Let $a \in \mathbb{R}^{\mathbb{N}}$.*
Assume: a is semi-decreasing and bounded below in $\mathbb{R}$.
Then: a is convergent in $\mathbb{R}$.

Proof. Let $b := -a$. Then $a = -b$. Want: b is convergent in $\mathbb{R}$.
Since b is semi-increasing and bounded above in $\mathbb{R}$,
it follows, from Theorem 58.8, that b is convergent in $\mathbb{R}$, as desired. □

THEOREM 58.10. *Let $a \in \mathbb{R}^{\mathbb{N}}$.*
Assume: a is semi-monotone and bounded in $\mathbb{R}$.
Then: a is convergent in $\mathbb{R}$.

Proof. Since a is semi-monotone, we know that at least one of the following is true:
(1) a is semi-decreasing or
(2) a is semi-increasing.

Case (1): Since a is bounded, it follows that a is bounded below.
Since a is semi-decreasing and bounded below, by Theorem 58.9,
a is convergent, as desired.
End of Case (1).
Case (2): Since a is bounded, it follows that a is bounded above.
Since a is semi-increasing and bounded above, by Theorem 58.9,
a is convergent, as desired.
End of Case (2). □

The next theorem is fundamental to the area of dynamical systems.

THEOREM 58.11. *Let X be a set, let $f : X \to X$ and let $z \in X$.*
Then: $\forall j \in \mathbb{N}_0$, $f_\circ^j(z) \in X$.

Proof. Unassigned HW. (*Hint:* Use induction on j.) □

THEOREM 58.12. *Let $P \subseteq \mathbb{N}$ and let $m := \max(P_0^+)$.*
Assume $m \neq$ ☹. Then $\forall j \in (m..\infty)$, $j \notin P$.

Proof. Unassigned HW. □

THEOREM 58.13. *Let $s \in \mathbb{R}^{\mathbb{N}}$.*
Then $\exists$subsequence t of s s.t. t is semi-monontone.

Proof. Let $P := \{j \in \mathbb{N} \,|\, \forall q \in (j..\infty),\ s_j \geqslant s_q\}$.
Then one of the following is true:

(1) P is finite or
(2) P is infinite.

Case (1):
Since $P \subseteq \mathbb{N}$, it follows that $P_0^+ \subseteq \mathbb{N}_0$.
Since P is finite, it follows that P_0^+ is finite.
Since $P_0^+ \subseteq \mathbb{R}$ and since P_0^+ is finite, we get $\max(P_0^+) \in P_0^+$.
Let $m := \max(P_0^+)$. Then $m \in P_0^+$.
Then $m \in P_0^+ \subseteq \mathbb{N}_0$, so $m \in \mathbb{N}_0$. Also $m \neq ☹$.
Define $A : \mathbb{N} \to 2^{\mathbb{N}}$ by $A_j := \{q \in (j..\infty) \,|\, s_j < s_q\}$.
Define $f : (m..\infty) \dashrightarrow (m..\infty)$ by $f(j) = \min A_j$.

Claim A: $\forall j \in (m..\infty),\ f(j) \in A_j$.
Proof of Claim A:
Given $j \in (m..\infty)$. Want: $f(j) \in A_j$.
By Theorem 58.12, $j \notin P$.
Then, by definition of P, we get: $\neg(\forall q \in (j..\infty),\ s_j \geqslant s_q)$.
Then $\exists q \in (j..\infty)$ s.t. $s_j < s_q$. Choose $q \in (j..\infty)$ s.t. $s_j < s_q$.
Then $q \in A_j$, so $A_j \neq \varnothing$.
So, since $A_j \subseteq \mathbb{N}$ and since $\mathbb{N}$ is well-ordered,
we conclude that $\min A_j \neq ☹$, and so $\min A_j \in A_j$.
Then $f(j) = \min A_j \in A_j$, as desired.
End of proof of Claim A.

We have $f : (m..\infty) \dashrightarrow (m..\infty)$.
Also, by Claim A, $\forall j \in (m..\infty),\ f(j) \neq ☹$.
Then $f : (m..\infty) \to (m..\infty$.
Then: $\forall j \in \mathbb{N},\ f_\circ^j(m+1) \in (m..\infty) \subseteq \mathbb{N}$.
Define $\ell \in \mathbb{N}^{\mathbb{N}}$ by $\ell_j = f_\circ^j(m+1)$.

Claim B: ℓ is strictly increasing.
Proof of Claim B:
Want: $\forall j \in \mathbb{N},\ \ell_j < \ell_{j+1}$.
Given $j \in \mathbb{N}$. Want: $\ell_j < \ell_{j+1}$.
Let $k := \ell_j$. Want: $k < \ell_{j+1}$.
We have $f(k) = f(\ell_j) = f(f_\circ^j(m+1)) = f_\circ^{j+1}(m+1) = \ell_{j+1}$.

By Claim A, we have $f(k) \in A_k$.
Also, by defininition of A, we have $A_k \subseteq (k..\infty)$.
Then $\ell_{j+1} = f(k) \in A_k \subseteq (k..\infty) > k$.
Then $\ell_{j+1} > k$, so $k < \ell_{j+1}$, as desired.
End of proof of Claim B.

Let $t := s \circ \ell$.
By Claim B, t is a subsequence of s.
Want: t is semi-monotone. Want: t is strictly increasing.
Want: $\forall j \in \mathbb{N}$, $t_j < t_{j+1}$. Given $j \in \mathbb{N}$. Want: $t_j < t_{j+1}$.
Let $k := \ell_j$. By Claim A, we have $f(k) \in A_k$.
Let $q := f(k)$. Then $q \in A_k$.
So, by definition of A, we get $s_k < s_q$.
We have $q = f(k) = f(\ell_j) = f(f_\circ^j(m+1)) = f_\circ^{j+1}(m+1) = \ell_{j+1}$.
Then $t_j = (s \circ \ell)_j = s_{\ell_j} = s_k < s_q = s_{\ell_{j+1}} = (s \circ \ell)_{j+1} = t_{j+1}$.
End of Case (1).

Case (2):
Define $A : \mathbb{N} \to 2^{\mathbb{N}}$ by $A_j = P \cap (j..\infty)$.
Define $f : \mathbb{N} \dashrightarrow \mathbb{N}$ by $f(j) = \min A_j$.

Claim C: $\forall j \in \mathbb{N}$, $f(j) \in A_j$.
Proof of Claim C:
Given $j \in \mathbb{N}$. Want: $f(j) \in A_j$.
Since P is infinite and $[1..j]$ is finite,
we conclude that $P \nsubseteq [1..j]$.
So, since $P \subseteq \mathbb{N} = [1..\infty)$, we get $P \cap (j..\infty) \neq \varnothing$.
Then $A_j = P \cap (j..\infty) \neq \varnothing$.
So, since $A_j \subseteq \mathbb{N}$ and since $\mathbb{N}$ is well-ordered,
we conclude that $\min A_j \neq ☹$, and so $\min A_j \in A_j$.
Then $f(j) = \min A_j \in A_j$, as desired.
End of proof of Claim C.

We have $f : \mathbb{N} \dashrightarrow \mathbb{N}$.
Also, by Claim C, $\forall j \in (m..\infty)$, $f(j) \neq ☹$.
Then $f : \mathbb{N} \to \mathbb{N}$.
Then: $\forall j \in \mathbb{N}$, $f_\circ^j(1) \in \mathbb{N}$.
Define $\ell \in \mathbb{N}^{\mathbb{N}}$ by $\ell_j = f_\circ^j(1)$.

Claim D: ℓ is strictly increasing.
Proof of Claim D:
Want: $\forall j \in \mathbb{N}$, $\ell_j < \ell_{j+1}$.
Given $j \in \mathbb{N}$. Want: $\ell_j < \ell_{j+1}$.
Let $k := \ell_j$. Want: $k < \ell_{j+1}$.
We have $f(k) = f(\ell_j) = f(f_\circ^j(m+1)) = f_\circ^{j+1}(m+1) = \ell_{j+1}$.
By Claim C, we have $f(k) \in A_k$.
Also, by defininition of A, we have $A_k \subseteq (k..\infty)$.
Then $\ell_{j+1} = f(k) \in A_k \subseteq (k..\infty) > k$.
Then $\ell_{j+1} > k$, so $k < \ell_{j+1}$, as desired.
End of proof of Claim D.

Let $t := s \circ \ell$.
By Claim D, t is a subsequence of s.
Want: t is semi-monotone. Want: t is semi-decreasing.
Want: $\forall j \in \mathbb{N}$, $t_j \geqslant t_{j+1}$. Given $j \in \mathbb{N}$. Want: $t_j \geqslant t_{j+1}$.
Let $k := \ell_j$ and let $i := f_\circ^{j-1}(1)$.
Then $k = \ell_j = f_\circ^j(1) = f(f_\circ^{j-1}(1)) = f(i)$.
By Claim C, $f(i) \in A_i$.
Then $k = f(i) \in A_i = P \cap (i..\infty) \subseteq P$, so $k \in P$.
Let $q := f(k)$. Then $q = f(k) = f(\ell_j) = f(f_\circ^j(1)) = f_\circ^{j+1}(1) = \ell_{j+1}$.
By Claim D, $\ell_{j+1} > \ell_j$. Then $q = \ell_{j+1} > \ell_j = k$, so $q \in (k..\infty)$.
So, since $k \in P$, by definition of P, we get: $s_k \geqslant s_q$.
Then $t_j = (s \circ \ell)_j = s_{\ell_j} = s_k \geqslant s_q = s_{\ell_{j+1}} = (s \circ \ell)_{j+1} = t_{j+1}$.
End of Case (2). □

THEOREM 58.14. *$\forall$functions f and g,* $\text{im}[g \circ f] \subseteq \text{im}[g]$.

Proof. Unassigned HW. □

THEOREM 58.15. *$\forall$sequence s, $\forall$subsequence t of s,* $\text{im}[t] \subseteq \text{im}[s]$.

Proof. Given a sequence s and a subsequence t of s.
Want: $\text{im}[t] \subseteq \text{im}[s]$.
Since t is a subsequence of s,
choose a strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
By Theorem 58.14, $\text{im}[s \circ \ell] \subseteq \text{im}[s]$.
Then $\text{im}[t] = \text{im}[s \circ \ell] \subseteq \text{im}[s]$, as desired. □

THEOREM 58.16. *Let X be a metric space and let $s, t \in X^{\mathbb{N}}$.*
Assume: (t is a subsequence of s) & (s is bounded in X).
Then t is bounded in X.

Proof. Since s is bounded in X, we get: $\mathrm{im}[s]$ is bounded in X.
Then choose $B \in \mathcal{B}_X$ s.t. $\mathrm{im}[s] \subseteq B$.
By Theorem 58.15, $\mathrm{im}[t] \subseteq \mathrm{im}[s]$.
Then $\mathrm{im}[t] \subseteq \mathrm{im}[s] \subseteq B$, so $\mathrm{im}[t] \subseteq B$.
So, since $B \in \mathcal{B}_X$, we conclude that $\mathrm{im}[t]$ is bounded in X.
Then t is bounded in X, as desired. □

THEOREM 58.17. *$\mathbb{R}$ is a proper metric space.*

Proof. Want: $\forall$ bounded $s \in \mathbb{R}^{\mathbb{N}}$, s is subconvergent in $\mathbb{R}$.
Given a bounded $s \in \mathbb{R}^{\mathbb{N}}$. Want: s is subconvergent in $\mathbb{R}$.
By Theorem 58.13, choose a subsequence t of s s.t. t is semi-monotone.
Since s is bounded in $\mathbb{R}$ and t is a subsequence of s,
it follows, from Theorem 58.16, that t is bounded in $\mathbb{R}$.
Since t is semi-monotone and bounded in $\mathbb{R}$, by Theorem 58.10,
we conclude that t is convergent in $\mathbb{R}$.
So, since t is a subsequence of s,
we see that s is subconvergent in $\mathbb{R}$, as desired. □

DEFINITION 58.18. *Let X be a metric space, $S \subseteq X$, $\varepsilon > 0$. Then*

$$N_X^{\varepsilon}(S) \quad := \quad \{\, z \in X \mid \exists y \in S \text{ s.t. } d_X(y,z) \;<\; \varepsilon \,\}$$

*is called the ε-**neighborhood** of S.*

We drew a coordinate plane with a short curve S, picked a small distance ε and drew the ε-neighborhood of S.

THEOREM 58.19. *Let X be a metric space, $S \subseteq X$, $\varepsilon > 0$.*
Then $N_X^{\varepsilon}(S) \subseteq X$.

THEOREM 58.20. *Let X be a metric space, $S, T \subseteq X$, $\delta, \varepsilon > 0$.*
Assume $S \subseteq T$ and $\delta \leqslant \varepsilon$.
Then $N_X^{\delta}(S) \subseteq N_X^{\varepsilon}(T)$.

DEFINITION 58.21. *Let X be a metric space, $S \subseteq X$, $\varepsilon > 0$.*
*By S is ε-**dense** in X, we mean: $N_X^{\varepsilon}(S) = X$.*

We noted that $\mathbb{Z}$ is almost $1/2$-dense in $\mathbb{R}$, but not quite. In fact, we have: $\forall a > 1/2$, $\mathbb{Z}$ is a-dense in $\mathbb{R}$. Therefore: $\mathbb{Z}$ is 1-dense in $\mathbb{R}$. Similarly, $(1/2) \cdot \mathbb{Z}$ is $(1/2)$-dense in $\mathbb{R}$. Generally:

THEOREM 58.22. *$\forall j \in \mathbb{N}$, $(1/j) \cdot \mathbb{Z}$ is $(1/j)$-dense in $\mathbb{R}$.*

THEOREM 58.23. *$\forall \varepsilon > 0$, $\mathbb{Q}$ is ε-dense in $\mathbb{R}$.*

Proof. Given $\varepsilon > 0$. Want: $\mathbb{Q}$ is ε-dense in $\mathbb{R}$. Want: $N_{\mathbb{R}}^{\varepsilon}(\mathbb{Q}) = \mathbb{R}$.
Since $N_{\mathbb{R}}^{\varepsilon}(\mathbb{Q}) \subseteq \mathbb{R}$, we want: $\mathbb{R} \subseteq N_{\mathbb{R}}^{\varepsilon}(\mathbb{Q})$.
By the Archimedean Principle, choose $j \in \mathbb{N}$ s.t. $j > 1/\varepsilon$.
Since $j > 1/\varepsilon > 0$, we get $1/j < \varepsilon$.
So, since $(1/j) \cdot \mathbb{Z} \subseteq \mathbb{Q}$, we get $N_{\mathbb{R}}^{1/j}((1/j) \cdot \mathbb{Z}) \subseteq N_{\mathbb{R}}^{\varepsilon}(\mathbb{Q})$.
By Theorem 58.22, we have $N_{\mathbb{R}}^{1/j}((1/j) \cdot \mathbb{Z}) = \mathbb{R}$.
Then $\mathbb{R} = N_{\mathbb{R}}^{1/j}((1/j) \cdot \mathbb{Z}) \subseteq N_{\mathbb{R}}^{\varepsilon}(\mathbb{Q})$, as desired. □

DEFINITION 58.24. *Let X be a metric space and let $S \subseteq X$.
By S is* **dense** *in X, we mean: $\mathrm{Cl}_X S = X$.*

THEOREM 58.25. *Let X be a metric space and let $S \subseteq X$.
Assume: $\forall \varepsilon > 0$, S is ε-dense in X.
Then: S is dense in X.*

Proof. This is HW#12-1. □

THEOREM 58.26. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Proof. By Theorem 58.23, $\forall \varepsilon > 0$, $\mathbb{Q}$ is ε-dense in $\mathbb{R}$.
Then, by Theorem 58.25, $\mathbb{Q}$ is dense in $\mathbb{R}$. □

DEFINITION 58.27. *Let X be a metric space.
By X is* **complete***, we mean: $\forall s \in X^{\mathbb{N}}$,*

(s is Cauchy in X) $\Rightarrow$ (s is convergent in X).

By HW#9-5, we know, for *ANY* metric space X, for any $s \in X^{\mathbb{N}}$, (s is convergent in X) $\Rightarrow$ (s is Cauchy in X).

A metric space that is not proper is said to be **nonproper**.

A metric space that is not complete is said to be **incomplete**.

The intuition behind completeness: Complete means "has no holes", or, equivalently, incomplete means "has holes".

For example, define $X := \mathbb{R}_0^{\times}$. Then the "hole" at 0 allows us to construct Cauchy sequences in X that are not convergent in X. For example, the sequence

$$(1 , 1/2 , 1/3 , 1/4 , \ldots)$$

is Cauchy in X, but is not convergent in X. Consequently, X is an incomplete metric space. The sequence

$$(1 , 1.4 , 1.41 , 1.414 , \ldots)$$

of decimal approximations to $\sqrt{2}$ is Cauchy in $\mathbb{Q}$, but not convergent in $\mathbb{Q}$. Consequently, $\mathbb{Q}$ is an incomplete metric space. Speaking intuitively, X is only slightly incomplete, because there is only one hole. By contrast, $\mathbb{Q}$ is "very" incomplete, with a hole at every irrational.

THEOREM 58.28. *$\mathbb{R}_0^\times$ and $\mathbb{Q}$ are both incomplete.*

Proof. Let $s := (1, 1/2, 1/3, \ldots)$.
Then s is Cauchy in $\mathbb{R}_0^\times$, but not convergent in $\mathbb{R}_0^\times$,
so $\mathbb{R}_0^\times$ is incomplete. Want: $\mathbb{Q}$ is incomplete.
Let $\sigma := (1, 1.4, 1.41, \ldots)$ denote the sequence
of decimal approximations to $\sqrt{2}$.
Then σ is Cauchy in $\mathbb{Q}$, but not convergent in $\mathbb{Q}$,
so $\mathbb{Q}$ is incomplete, as desired. □

For any metric space X, we can "fill in all the holes", and obtain a complete metric space $\overline{X}$:

THEOREM 58.29. *Let X be a metric space.*
Then there exists a metric space $\overline{X}$ s.t.
(*$\overline{X}$ is complete*) & ($X_{\text{set}} \subseteq \overline{X}_{\text{set}}$) & (*X_{set} is dense in $\overline{X}$*).

Proof. Omitted. □

THEOREM 58.30. *Let X be a proper metric space.*
Then X is complete.

Proof. Want: $\forall s \in X^{\mathbb{N}}$,
(s is Cauchy in X) $\Rightarrow$ (s is convergent in X).
Given $s \in X^{\mathbb{N}}$. Want: (s is Cauchy in X) $\Rightarrow$ (s is convergent in X).
Assume s is Cauchy in X. Want: s is convergent in X.
Since s is Cauchy in X, by HW#11-4, we get: s is bounded in X.
So, since X is proper, we see that s is subconvergent in X.
So, since s is Cauchy in X, by HW#11-5,
we conclude that s is convergent in X, as desired. □

THEOREM 58.31. *$\mathbb{R}$ is a complete metric space.*

Proof. By Theorem 58.17, $\mathbb{R}$ is proper.
Then, by Theorem 58.30, $\mathbb{R}$ is complete, as desired. □

THEOREM 58.32. *$\mathbb{R}_0^\times$ and $\mathbb{Q}$ are nonproper metric spaces.*

Proof. By Theorem 58.28, $\mathbb{R}_0^\times$ and $\mathbb{Q}$ are both incomplete.
Then, by Theorem 58.30, $\mathbb{R}_0^\times$ and $\mathbb{Q}$ are both nonproper, as desired. □

By Theorem 58.30, proper implies complete. In the next section, we address why the converse fails.

59. Complete does not imply proper

By Theorem 58.29, any metric space X has a "completion" $\overline{X}$. For example, one completion of $\mathbb{R}_0^\times$ is $\mathbb{R}$. Intuitively, we have "filled in the hole at 0" to go from $\mathbb{R}_0^\times$ to $\mathbb{R}$. For another example, one completion of $\mathbb{Q}$ is $\mathbb{R}$. Intuitively, we have "filled in the all holes at all of the irrationals" to go from $\mathbb{Q}$ to $\mathbb{R}$. In some sense, a metric space with holes is unnatural and, if we encounter such an object, we simply ignore it, and focus on one of its completions.

With experience as my guide, any natural finite dimensional metric space should not just be complete, but proper. For example, the metric spaces $\mathbb{R}$ and $\mathbb{R}^2$ are both proper. On the other hand, with experience as my guide, any natural infinite dimensional metric space is nonproper. Based on this, since we seek a metric space that is complete but non-proper, we should look for some "natural" infinite dimensional metric space. This works, and there are many examples, but the problem is that, in an undergraduate course, it is typical only to look at finite dimenional examples, and to leave infinite dimensions to a graduate course in an area of mathematics called "Functional Analysis".

We therefore turn to "unnatural" finite dimensional metric spaces. We will describe a metric space called $\mathbb{R}_{\leqslant 1}$, and, in Theorem 59.7 below, we will argue that $\mathbb{R}_{\leqslant 1}$ is complete an nonproper.

Recall that the underlying set of a metric space X is denoted X_{set}. We typically omit the subscript "set", but sometimes keep it:

DEFINITION 59.1. *Let X be a metric space.*
Define $d \in \mathcal{M}(X_{\text{set}})$ by $d(p,q) = \min\{d(p,q),1\}$.
Then $X_{\leqslant 1} := (X_{\text{set}}, d)$.

THEOREM 59.2. *Let $Y := \mathbb{R}_{\leqslant 1}$.*
Then $d_Y(2.7, 2.8) = 0.1$ and $d_Y(3, 5000) = 1$.

THEOREM 59.3. *Let X be a metric space, $Y := X_{\leqslant 1}$ and $p \in Y$.*
Then $B_Y(p,2) = Y$.

THEOREM 59.4. *Let X be a nonempty metric space, $Y := X_{\leqslant 1}$.*
Then Y is geometrically bounded.

THEOREM 59.5. *Let X be a metric space, $Y := X_{\leqslant 1}$, $s \in Y^{\mathbb{N}}$. Then s is bounded.*

THEOREM 59.6. *Let X be a metric space and let $s \in X^{\mathbb{N}}$. Then: (s is subconvergent in X) $\Leftrightarrow$ (s is subconvergent in $X_{\leqslant 1}$).*

Proof. Proof of $\Rightarrow$:
Assume: s is subconvergent in X. Want: s is subconvergent in $X_{\leqslant 1}$.
Since s is subconvergent in X,
choose a subsequence t of s s.t. t is convergent in X.
Then, by $\Rightarrow$ of HW#12-3, we see that t is convergent in $X_{\leqslant 1}$.
So, since t is a subsequence of s,
we see that s is subconvergent in $X_{\leqslant 1}$, as desired.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$:
Assume: s is subconvergent in $X_{\leqslant 1}$. Want: s is subconvergent in X.
Since s is subconvergent in $X_{\leqslant 1}$,
choose a subsequence t of s s.t. t is convergent in $X_{\leqslant 1}$.
Then, by $\Leftarrow$ of HW#12-3, we see that t is convergent in X.
So, since t is a subsequence of s,
we see that s is subconvergent in X, as desired.
End of proof of $\Leftarrow$. □

THEOREM 59.7. $\mathbb{R}_{\leqslant 1}$ *is complete and nonproper.*

Proof. This is HW#12-5. □

60. Properness and completeness of the plane

THEOREM 60.1. *Let s, t and u be sequences. Assume that s is a subsequence of t and that t is a subsequence of u. Then s is a subsequence of u.*

Proof. Since s is a subsequence of t,
choose a strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $s = t \circ \ell$.
Since t is a subsequence of u,
choose a strictly increasing $m \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = u \circ m$.
Then $s = t \circ \ell = u \circ m \circ \ell$.
Since $\ell, m \in \mathbb{N}^{\mathbb{N}}$, we see that $m \circ \ell \in \mathbb{N}^{\mathbb{N}}$.
Since ℓ and m are strictly increasing,
we see that $m \circ \ell$ is strictly increasing.

So, since $s = u \circ m \circ \ell$,
we conclude that s is a subsequence of u, as desired. □

THEOREM 60.2. *Let X be a metric space and let $\sigma, s \in X^{\mathbb{N}}$. Assume that σ is a subsequence of s and that σ is subconvergent in X. Then s is subconvergent in X.*

Proof. Since σ is subconvergent in X,
choose a subsequence t of σ s.t. t is convergent in X.
Since t is a subsequence of σ and since σ is a subsequence of s,
by Theorem 60.1, t is a subsequence of s.
So, since t is convergent in X,
we conclude that s is subconvergent in X, as desired. □

THEOREM 60.3. *Let X be a metric space and let $s \in X^{\mathbb{N}}$. Assume that s is subconvergent in X. Then $\exists$strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $s \circ \ell$ is convergent in X.*

Proof. Since s is subconvergent in X,
choose a subsequence t of s s.t. t is convergent in X.
Since t is a subsequence of s,
choose a strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
Want: $s \circ \ell$ is convergent in X.
Since t is convergent in X and since $t = s \circ \ell$,
we conclude that $s \circ \ell$ is convergent in X, as desired. □

THEOREM 60.4. *Let X be a metric space and let $s \in X^{\mathbb{N}}$. Let t be a subsequence of s. Assume: s is bounded in X. Then: t is bounded in X.*

Proof. This is Theorem 58.16. □

THEOREM 60.5. *Let X be a metric space and let $s \in X^{\mathbb{N}}$. Let t be a subsequence of s. Assume: s is convergent in X. Then: t is convergent in X.*

Proof. Since s is convergent in X, choose $p \in X$ s.t. $s_\bullet \to p$ in X.
Then, by Theorem 51.12, we have: $t_\bullet \to p$ in X.
Then t is convergent in X, as desired. □

THEOREM 60.6. *Let* $s \in (\mathbb{R}^2)^{\mathbb{N}}$.
Define $p, q : \mathbb{R}^2 \to \mathbb{R}$ *by* $p(x,y) = x$ *and* $q(x,y) = y$.
Then: $[\ (\ s$ *is bounded in* $\mathbb{R}^2\ )\ \Leftrightarrow$
$(\ p \circ s$ *and* $q \circ s$ *are both bounded in* $\mathbb{R}\ )\ ]$.

Proof. Proof of $\Rightarrow$*:*
Unassigned HW.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$*:*
Assume that $p \circ s$ and $q \circ s$ are both bounded in $\mathbb{R}$.
Want: s is bounded in $\mathbb{R}^2$. Want: $\mathrm{im}[s]$ is bounded in $\mathbb{R}^2$.
Want: $\exists C \in \mathcal{B}_{\mathbb{R}^2}$ s.t. $\mathrm{im}[s] \subseteq C$.
Since $p \circ s$ and $q \circ s$ are both bounded in $\mathbb{R}$,
$\mathrm{im}[p \circ s]$ and $\mathrm{im}[q \circ s]$ are both bounded in $\mathbb{R}$,
so choose $A, B \in \mathcal{B}_{\mathbb{R}}$ s.t. $\mathrm{im}[p \circ s] \subseteq A$ and $\mathrm{im}[q \circ s] \subseteq B$.
Since $A \in \mathcal{B}_{\mathbb{R}}$, choose $a \in \mathbb{R}$ and $\alpha > 0$ s.t. $A = B_{\mathbb{R}}(a, \alpha)$.
Since $B \in \mathcal{B}_{\mathbb{R}}$, choose $b \in \mathbb{R}$ and $\beta > 0$ s.t. $B = B_{\mathbb{R}}(b, \beta)$.
Let $c := (a, b)$ and let $\gamma := \sqrt{\alpha^2 + \beta^2}$.
Let $C := B_{\mathbb{R}^2}(c, \gamma)$. Then $C \in \mathcal{B}_{\mathbb{R}^2}$.
Want: $\mathrm{im}[s] \subseteq C$. Want: $\forall z \in \mathrm{im}[s]$, $z \in C$.
Given $z \in \mathrm{im}[s]$. Want: $z \in C$.
Choose $x, y \in \mathbb{R}$ s.t. $z = (x, y)$. Then $p(z) = x$ and $q(z) = y$.
Since $z \in \mathrm{im}[s]$, choose $j \in \mathbb{N}$ s.t. $z = s_j$.
Then $(p \circ s)_j = p(s_j) = p(z) = x$ and $(q \circ s)_j = q(s_j) = q(z) = y$.
Then $x = (p \circ s)_j \in \mathrm{im}[p \circ s] \subseteq A = B_{\mathbb{R}}(a, \alpha)$, so $d_{\mathbb{R}}(x, a) < \alpha$.
Also, $y = (q \circ s)_j \in \mathrm{im}[q \circ s] \subseteq B = B_{\mathbb{R}}(b, \beta)$, so $d_{\mathbb{R}}(y, b) < \beta$.
Then $|x - a| = d_{\mathbb{R}}(x, a) < \alpha$ and $|y - b| = d_{\mathbb{R}}(y, b) < \beta$.
Then $0 \leqslant |x - a| < \alpha$ and $0 \leqslant |y - b| < \beta$,
so $|x - a|^2 < \alpha^2$ and $|y - b|^2 < \beta^2$.
We have $|x - a|^2 = |(x - a)^2| = (x - a)^2$
and $|y - b|^2 = |(y - b)^2| = (y - b)^2$.
Then $(x - a)^2 + (y - b)^2 = |x - a|^2 + |y - b|^2 < \alpha^2 + \beta^2$.
Since $0 < (x - a)^2 + (y - b)^2 < \alpha^2 + \beta^2$,
we get: $\sqrt{(x - a)^2 + (y - b)^2} < \sqrt{\alpha^2 + \beta^2}$.
Then $d_{\mathbb{R}^2}(z, c)) = |z - c|_2 = |(x - a, y - b)|_2$
$= \sqrt{(x - a)^2 + (y - b)^2} < \sqrt{\alpha^2 + \beta^2} = \gamma$.
Then $z \in B_{\mathbb{R}^2}(c, \gamma) = C$, as desired. *End of proof of* $\Leftarrow$. □

THEOREM 60.7. *Let $s \in (\mathbb{R}^2)^{\mathbb{N}}$.*
Define $p, q : \mathbb{R}^2 \to \mathbb{R}$ by $p(x,y) = x$ and $q(x,y) = y$.
Then: [(s is convergent in $\mathbb{R}^2$) $\Leftrightarrow$
($p \circ s$ and $q \circ s$ are both convergent in $\mathbb{R}$)].

Proof. Proof of $\Rightarrow$:
Unassigned HW.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$:
This is Theorem 42.9.
End of proof of $\Leftarrow$. □

THEOREM 60.8. *$\mathbb{R}^2$ is a proper metric space.*

Proof. Want: $\forall s \in (\mathbb{R}^2)^{\mathbb{N}}$,
(s is bounded in $\mathbb{R}^2$) $\Rightarrow$ (s is subconvergent in $\mathbb{R}^2$).
Given $s \in (\mathbb{R}^2)^{\mathbb{N}}$.
Want: (s is bounded in $\mathbb{R}^2$) $\Rightarrow$ (s is subconvergent in $\mathbb{R}^2$).
Assume s is bounded in $\mathbb{R}^2$. Want: s is subconvergent in $\mathbb{R}^2$.
By Theorem 58.17, $\mathbb{R}$ is proper.
Define $p, q : \mathbb{R}^2 \to \mathbb{R}$ by $p(x,y) = x$ and $q(x,y) = y$.
Since s is bounded in $\mathbb{R}^2$, by $\Rightarrow$ of Theorem 60.6,
$p \circ s$ and $q \circ s$ are both bounded in $\mathbb{R}$.
Since $p \circ s$ is bounded in $\mathbb{R}$ and since $\mathbb{R}$ is proper,
$p \circ s$ is subconvergent in $\mathbb{R}$.
Then, by Theorem 60.3, choose a strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$
s.t. $p \circ s \circ \ell$ is convergent in $\mathbb{R}$.
Let $\sigma := s \circ \ell$. Then $p \circ \sigma$ is convergent in $\mathbb{R}$.
Since σ is a subsequence of s and since s is bounded in $\mathbb{R}^2$,
by Theorem 58.16, σ is bounded in $\mathbb{R}^2$.
By Theorem 60.2, it suffices to show: σ is subconvergent in $\mathbb{R}^2$.
Since σ is bounded in $\mathbb{R}^2$, by $\Rightarrow$ of Theorem 60.6,
$p \circ \sigma$ and $q \circ \sigma$ are both bounded in $\mathbb{R}$.
Since $q \circ \sigma$ is bounded in $\mathbb{R}$ and since $\mathbb{R}$ is proper,
$q \circ \sigma$ is subconvergent in $\mathbb{R}$.
Then, by Theorem 60.3, choose a strictly increasing $m \in \mathbb{N}^{\mathbb{N}}$
s.t. $q \circ \sigma \circ m$ is convergent in $\mathbb{R}$.
Since $p \circ \sigma$ is convergent in $\mathbb{R}$
and since $p \circ \sigma \circ m$ is a subsequence of $p \circ \sigma$,

it follows, from Theorem 60.5, that $p \circ \sigma \circ m$ is convergent in $\mathbb{R}$.
Since $p \circ \sigma \circ m$ and $q \circ \sigma \circ m$ are both convergent in $\mathbb{R}$,
by $\Leftarrow$ of Theorem 60.7, $\sigma \circ m$ is convergent in $\mathbb{R}^2$.
So, since $\sigma \circ m$ is a subsequence of σ,
we see that σ is subconvergent in $\mathbb{R}^2$, as desired. □

THEOREM 60.9. $\mathbb{R}^2$ *is a complete metric space.*

Proof. By Theorem 60.8, $\mathbb{R}^2$ is proper.
Then, by Theorem 58.30, $\mathbb{R}^2$ is complete, as desired. □

THEOREM 60.10. $\forall m \in \mathbb{N}$, $\mathbb{R}^m$ *is a proper metric space.*

Proof. Unassigned HW. □

61. The Extreme Value Theorem

DEFINITION 61.1. $\forall$*functional* f,

$$\max f := \max(\text{im}[f]) \quad \textit{and} \quad \min f := \min(\text{im}[f]).$$

Theorem 61.2, below, is the **Extreme Value Theorem**.

THEOREM 61.2. *Let* X *be a nonempty compact metric space.*
Let $f : X \to \mathbb{R}$. *Assume* f *is continuous from* X *to* $\mathbb{R}$.
Then $\min f \neq ☹ \neq \max f$.

Proof. Let $A := \text{im}[f]$. Want: $\min A \neq ☹ \neq \max A$.
By HW#12-2, we see that f is continuous from X to A.
So, since X is compact and since $f : X \to\!> A$,
by Theorem 55.4, we conclude: A is compact.
Since $X \neq \varnothing$ and $f : X \to A$, we get: $A \neq \varnothing$.
Then, by Theorem 58.6, $\min A \neq ☹ \neq \max A$, as desired. □

62. The Topological Inverse Function Theorem

THEOREM 62.1. *Let* X *be a metric space,* $C \in \mathcal{T}'_X$ *and* $t \in \mathbb{C}^{\mathbb{N}}$.
Assume that t *is convergent in* X.
Then t *is convergent in* C.

Proof. Since t is convergent in X, choose $q \in X$ s.t. $t_\bullet \to q$ in X.
Since $t \in C^{\mathbb{N}}$, since $t_\bullet \to q$ in X and since $C \in \mathcal{T}'_X$,
it follows that $q \in C$.
Then, by Theorem 44.12, $t_\bullet \to q$ in C.
Then t is convergent in C, as desired. □

THEOREM 62.2. *Let X be a metric space, $C \in \mathcal{T}'_X$ and $s \in \mathbb{C}^{\mathbb{N}}$.*
Assume that s is subconvergent in X.
Then s is subconvergent in C.

Proof. Since s is subconvergent in X,
choose a subsequence t of s s.t. t is convergent in X.
By Theorem 62.1, t is convergent in C.
So, since t is a subsequence of s,
we conclude that s is subconvergent in C. □

THEOREM 62.3. *Let X be a compact metric space and let $C \in \mathcal{T}'_X$.*
Then C is compact.

Proof. Want: $\forall s \in C^{\mathbb{N}}$, s is subconvergent in C.
Given $s \in C^{\mathbb{N}}$. Want: s is subconvergent in C.
Since $C \in \mathcal{T}'_X$, we get $C \subseteq X$, and so $C^{\mathbb{N}} \subseteq X^{\mathbb{N}}$.
Since $s \in C^{\mathbb{N}} \subseteq X^{\mathbb{N}}$ and since X is compact,
s is subconvergent in X.
So, since $C \in \mathcal{T}'_X$, by Theorem 62.2,
we see that s is subconvergent in C. □

THEOREM 62.4. *Let X and Y be metric spaces.*
Let $f : X \to Y$, $S \subseteq X$, $A := f_*(S)$.
Assume that f is continuous from X to Y.
Then $f|S$ is continuous from S to A.

Proof. Since $\mathrm{im}[f|S] = f_*(S) = A$, by HW#12-2,
it suffices to show: $f|S$ is continuous from S to Y.
Want: $\forall p \in S$, $f|S$ is continuous at p from S to Y.
Since f is continuous from X to Y,
it follows that f is continuous at p from X to Y.
Then, by Theorem 44.13, f is continuous at p from S to Y. □

THEOREM 62.5. *Let X and Y be metric spaces.*
Let $f : X \to Y$ and let $C \subseteq X$.
Assume that f is continuous from X to Y.
Assume that C is compact. Then $f_*(C)$ is compact.

Proof. Let $A := f_*(C)$. Then $f|C : C \to> A$. Want: A is compact.
By Theorem 62.4, $f|C$ is continuous from C to A.
So, since C is compact, by Theorem 55.4, A is compact. □

THEOREM 62.6. *Let X and Y be sets and let $S \subseteq X$.*
Let $f : X \hookrightarrow\!> Y$. Then $f_*(S) = (f^{-1})^*(S)$.

Proof. Unassigned HW. □

Proof. Unassigned HW. □

THEOREM 62.7. *Let X and Y be sets and let $S \subseteq Y$.*
Let $f : X \hookrightarrow\!> Y$. Then $f^*(S) = (f^{-1})_*(S)$.

Proof. Unassigned HW. □

THEOREM 62.8. *Let $X := [1; 2] \cup (3; 4]$ and $Y := [5; 7]$.*
Define $f : X \to Y$ by $f(x) = \begin{cases} x + 4, & \text{if } 1 \leqslant x \leqslant 2 \\ x + 3, & \text{if } 3 < x \leqslant 4. \end{cases}$
Then $f : X \hookrightarrow\!> Y$ and f is continuous from X to Y.
Also, f^{-1} is not continuous at 6 from Y to X.

Proof. Unassigned HW. □

Theorem 62.8 presents us with a problem. Frequently, we will be given a bijection f between metric spaces, and suppose we want to show that f is a homeomorphism. Say f is given by some formula for $f(x)$ in terms of x. By using continuity of basic functions, together with properties of continuity, we can often verify continuity of f. However, in order to see continuity of f^{-1}, we would need to compute a formula for $f^{-1}(y)$ in terms of y, which may be very difficult.

For example, define $f : [1; 2] \to [2; 10]$ by $f(x) = x + x^3$. Then it is possible to show that f is a bijection from $[1; 2]$ onto $[2; 10]$. Also, it is not hard to show that f is continuous from $[1; 2]$ to $[2; 10]$. However, there is no obvious formula for $f^{-1}(y)$ in terms of y. Even computing $f^{-1}(5)$ involves solving $x+x^3 = 5$ for x, which is difficult. However, the next theorem guarantees that f^{-1} is continuous from $[2; 10]$ to $[1; 2]$.

The next theorem is the **Topological Inverse Function Theorem**.

THEOREM 62.9. *Let X and Y be metric spaces. Let $f : X \hookrightarrow\!> Y$.*
Assume: f is continuous from X to Y and X is compact.
Then: f^{-1} is continuous from Y to X.

Proof. By Theorem 57.2, we want: $\forall C \in \mathcal{T}'_X$, $(f^{-1})^*(C) \in \mathcal{T}'_Y$.
Given $C \in \mathcal{T}'_X$. Want: $(f^{-1})^*(C) \in \mathcal{T}'_Y$.
By Theorem 62.6, $f_*(C) = (f^{-1})^*(C)$. Want: $f_*(C) \in \mathcal{T}'_Y$.
Since X is compact and $C \in \mathcal{T}'_X$, by Theorem 62.3,

we see that C is compact.
Then, by Theorem 62.5, $f_*(C)$ is compact.
Then, by Theorem 57.8, $f_*(C) \in \mathcal{T}'_Y$, as desired. □

63. Limits of functions

DEFINITION 63.1. *Let Y be a metric space, $p \in Y$ and $r > 0$.*
Then $B^\times_Y(p, r) := [B_Y(p, r)]^\times_p$.

DEFINITION 63.2. *Let Y be a metric space and let $p \in Y$.*
Then $\mathcal{B}^\times_Y(p) := \{B^\times_Y(p, r) \,|\, r > 0\}$.

DEFINITION 63.3. *Let Y be a metric space and let $p \in Y$.*
Then $\mathcal{B}^\times_Y := \{B^\times_Y(p, r) \,|\, p \in Y,\ r > 0\}$.

DEFINITION 63.4. *Let Y and Z be metric spaces.*
Let $f : Y \dashrightarrow Z$, let $p \in Y$ and let $q \in Z$.
Then $f \to q$ near p from Y to Z means:
$\mathrm{adj}^q_p f$ *is continuous at p from Y to Z.*

THEOREM 63.5. *Let Y and Z be metric spaces.*
Let $f : Y \dashrightarrow Z$ and let $p \in Y$.
Then: [f is continuous at p from $\mathrm{dom}[f]$ *to Z]*
$\Leftrightarrow$ [$\forall \varepsilon > 0,\ \exists \delta > 0$ *s.t.*, $\forall x \in \mathrm{dom}[f]$,
$(\, d_Y(x, p) < \delta \,) \Rightarrow (\, d_Z(f(x), f(p)) < \varepsilon \,)$]
$\Leftrightarrow$ [$\forall B \in \mathcal{B}_Z(f(p)),\ \exists A \in \mathcal{B}_Y(p)$ *s.t.* $f_*(A) \subseteq B$,]
$\Leftrightarrow$ [$\forall s \in (\mathrm{dom}[f])^{\mathbb{N}},\ (\,[\, s_\bullet \to p$ *in* $Y\,] \Rightarrow [\,(f \circ s)_\bullet \to f(p)$ *in* $Z\,]\,)$].

Proof. Unassigned HW. □

THEOREM 63.6. *Let Y and Z be metric spaces.*
Let $f : Y \dashrightarrow Z$, let $p \in Y$ and let $q \in Z$.
Then: [$f \to q$ near p from Y to Z]
$\Leftrightarrow$ [$\forall \varepsilon > 0,\ \exists \delta > 0$ *s.t.*, $\forall x \in \mathrm{dom}[f]$,
$(\, 0 < d_Y(x, p) < \delta \,) \Rightarrow (\, d_Z(f(x), q) < \varepsilon \,)$]
$\Leftrightarrow$ [$\forall B \in \mathcal{B}_Z(q),\ \exists A \in \mathcal{B}^\times_Y(p)$ *s.t.* $f_*(A) \subseteq B$,]
$\Leftrightarrow$ [$\forall s \in ((\mathrm{dom}[f])^\times_p)^{\mathbb{N}},\ (\,[\, s_\bullet \to p$ *in* $Y\,] \Rightarrow [\,(f \circ s)_\bullet \to q$ *in* $Z\,]\,)$].

Proof. Unassigned HW. □

THEOREM 63.7. *Let f be a function and let $p \in$* $\mathrm{dom}[f]$.
Then $\mathrm{adj}^{f(p)}_p f = f$.

Proof. Unassigned HW. □

THEOREM 63.8. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$ and let $p \in \mathrm{dom}[f]$.
Then: $[\, f \to f(p)$ *near p from X to Y* $]$
$\Leftrightarrow [\, f$ *is continuous at p from* $\mathrm{dom}[f]$ *to Y* $]$.

Proof. Let $g := \mathrm{adj}_p^{f(p)} f$.
By Definition 63.4, we have:
$[\, f \to f(p)$ near p from X to $Y\,]$
$\Leftrightarrow [\, g$ is continuous at p from $\mathrm{dom}[f]$ to $Y\,]$.
By Theorem 63.7, $g = f$.
Then: $[\, f \to f(p)$ near p from X to $Y\,]$
$\Leftrightarrow [\, g$ is continuous at p from $\mathrm{dom}[f]$ to $Y\,]$
$\Leftrightarrow [\, f$ is continuous at p from $\mathrm{dom}[f]$ to $Y\,]$. □

64. Isolated points and limit points

DEFINITION 64.1. *Let T be a metric space.*
Then $\mathrm{Isol}\, T \;:=\; \{q \in T \mid \exists B \in \mathcal{B}_T(q)$ *s.t.* $B = \{q\}\}$.

An element of $\mathrm{Isol}\, T$ is called an **isolated point** of T.

We drew a picture on the board of a subset T of the plane that had two singleton connected components, and two uncountable connected components, each a partial closure of a connected amoeba-like open set. We computed $\mathrm{Isol}\, T$ as the union of the two singleton components.

THEOREM 64.2. *Let $T := [1; 2) \cup \{3, 4\}$. Then* $\mathrm{Isol}\, T = \{3, 4\}$.

Proof. Unassigned HW. □

THEOREM 64.3. *Let $T := \{1, 1/2, 1/3, \ldots\}$. Then* $\mathrm{Isol}\, T = T$.

Proof. Unassigned HW. □

THEOREM 64.4. *Let $A := \{1, 1/2, 1/3, \ldots\}$ and let $T := A_0^+$.*
Then $\mathrm{Isol}\, T = A$.

Proof. Unassigned HW. □

DEFINITION 64.5. *Let Y be a metric space and let $T \subseteq Y$.*
Then $\mathrm{LP}_Y T \;:=\; (\,\mathrm{Cl}_Y T\,) \backslash (\,\mathrm{Isol}\, T\,)$.

An element of $\mathrm{LP}_Y T$ is called a **limit point** in Y of T.

Note: In $\mathrm{Cl}_Y T$, we think of T as a subset of the metric space Y. On the other hand, in $\mathrm{Isol}\, T$, we think of T as a metric space with the relative metric inherited from Y.

We went back to the picture on the board of a subset T of the plane that had two singleton connected components, and two uncountable connected components, each a partial closure of a connected amoeba-like open set. We computed $\mathrm{LP}_{\mathbb{R}^2}T$ as the union of the closures of the two singleton components.

THEOREM 64.6. *Let* $T := [1;2) \cup \{3,4\}$. *Then* $\mathrm{LP}_{\mathbb{R}}T = [1;2]$.

Proof. Unassigned HW. □

THEOREM 64.7. *Let* $T := \{1, 1/2, 1/3, \ldots\}$. *Then* $\mathrm{LP}_{\mathbb{R}}T = \{0\}$.

Proof. Unassigned HW. □

THEOREM 64.8. *Let* $A := \{1, 1/2, 1/3, \ldots\}$ *and let* $T := A_0^+$.
Then $\mathrm{LP}_{\mathbb{R}}T = \{0\}$.

Proof. Unassigned HW. □

THEOREM 64.9. *Let* Y *be a metric space,* $T \subseteq Y$ *and* $q \in Y$.
Then: $[\, q \in \mathrm{Cl}_Y T \,]$
$\Leftrightarrow$ $[\, \forall \delta > 0,\ (\, [B_Y(q,\delta)] \cap T \neq \varnothing \,)\,]$
$\Leftrightarrow$ $[\, \forall B \in \mathcal{B}_Y(q),\ B \cap T \neq \varnothing,]$
$\Leftrightarrow$ $[\, \exists s \in T^{\mathbb{N}}$ *s.t.* $s_\bullet \to q$ *in* $Y\,]$.

Proof. Unassigned HW. □

THEOREM 64.10. *Let* Y *be a metric space,* $T \subseteq Y$ *and* $q \in Y$.
Then: $[\, q \in \mathrm{LP}_Y T \,]$
$\Leftrightarrow$ $[\, \forall \delta > 0,\ (\, [B_Y^\times(q,\delta)] \cap T \neq \varnothing \,)\,]$
$\Leftrightarrow$ $[\, \forall B \in \mathcal{B}_Y^\times(q),\ B \cap T \neq \varnothing,]$
$\Leftrightarrow$ $[\, \exists s \in (T_q^\times)^{\mathbb{N}}$ *s.t.* $s_\bullet \to q$ *in* $Y\,]$.

Proof. Unassigned HW. □

DEFINITION 64.11. *Let* Y *be a metric space,* Z *a set,* $f : Y \dashrightarrow Z$.
Then $\mathrm{LPD}_Y f := \mathrm{LP}_Y(\mathrm{dom}[f])$.

In Definition 64.11, LPD stands for "limit points of the domain".

THEOREM 64.12. *Let* Y *and* Z *be metric spaces.*
Let $f : Y \dashrightarrow Z$, $q \in Y$ *and* $a, b \in Z$.
Assume $f \to a$ *and* $f \to b$ *near* q *from* Y *to* Z.
Then $a = b$.

Proof. Let $D := \mathrm{dom}[f]$.
Since $q \in \mathrm{LP}_Y D$, by Theorem 64.10,
choose $s \in (D_q^\times)^{\mathbb{N}}$ s.t. $s_\bullet \to q$ in Y.
Since $D_q^\times \subseteq D_q^+$, it follows that $(D_q^\times)^{\mathbb{N}} \subseteq (D_q^+)^{\mathbb{N}}$.
Then $s \in (D_q^\times)^{\mathbb{N}} \subseteq (D_q^+)^{\mathbb{N}}$.
So, since $q \in D_q^+$ and since $s_\bullet \to q$ in Y,
it follows that $s_\bullet \to q$ in D_q^+.
Let $g := \mathrm{adj}_q^a f$ and $h := \mathrm{adj}_q^b f$.
Then $g(q) = a$ and $h(q) = b$.
Also, $\mathrm{dom}[g] = D_q^+ = \mathrm{dom}[h]$.
We have: $\forall j \in \mathbb{N}$, $s_j \neq q$.
It follows that: $\forall j \in \mathbb{N}$, $g(s_j) = f(s_j)$ and $h(s_j) = f(s_j)$.
That is, we have: $\forall j \in \mathbb{N}$, $(g \circ s)_j = (f \circ s)_j$ and $(h \circ s)_j = (f \circ s)_j$.
Then $g \circ s = f \circ s$ and $h \circ s = f \circ s$.
Since $f \to a$ and $f \to b$ near q from Y to Z, by Definition 63.4,
it follows that g and h are both continuous at q from D_q^+ to Z.
So, since $s_\bullet \to q$ in D_q^+,
$(g \circ s)_\bullet \to g(q)$ and $(h \circ s)_\bullet \to h(q)$ in Z.
So, since $g \circ s = f \circ s$ and $g(q) = a$ and $h \circ s = f \circ s$ and $h(q) = b$,
$(f \circ s)_\bullet \to a$ and $(f \circ s)_\bullet \to b$ in Z.
Then, by Theorem 39.12, $a = b$, as desired. □

DEFINITION 64.13. *Let ϕ and ψ be functions. Let S be a set. Then, by $\phi = \psi$ on S, we mean: $\forall x \in S$, $\phi(x) = \psi(x)$.*

THEOREM 64.14. *Let Y and Z be metric spaces.*
Let $\phi, \psi : Y \dashrightarrow Z$, $p \in Y$ and $q \in Z$.
Assume $\exists B \in \mathcal{B}_Y^\times(p)$ s.t. $\phi = \psi$ on B.
Assume $\phi \to q$ near p from Y to Z.
Then $\psi \to q$ near p from Y to Z.

Proof. Unassigned HW. □

Let f be a functional and let $k \in \mathbb{N}_0$. Define $p : \mathbb{R} \to \mathbb{R}$ by $p(x) = x^k$. Recall that f^k is defined to be $p \circ f$.

Let $f := 3 \cdot \mathrm{id}_{\mathbb{R}}^7$. Then, for all $x \in \mathbb{R}$, we have $f(x) = 3x^7$.

DEFINITION 64.15. *The function $|\bullet| : \mathbb{R} \to \mathbb{R}$ is defined by*
$$(|\bullet|)(x) = |x|.$$

THEOREM 64.16. *$|\bullet|$ is continuous at 0 from $\mathbb{R}$ to $\mathbb{R}$.*

Proof. Unassigned HW. □

THEOREM 64.17. $|\bullet| \to 0$ *near* 0 *from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. Unassigned HW. □

DEFINITION 64.18. *Let f be a functional. Then* $|f| := (|\bullet|) \circ f$.

THEOREM 64.19. *Let X be a metric space, $f : X \dashrightarrow \mathbb{R}$, $p \in X$. Then:* [$f \to 0$ *near p from X to* $\mathbb{R}$]
$\Leftrightarrow$ [$|f| \to 0$ *near p from X to* $\mathbb{R}$].

Proof. Unassigned HW. □

65. Some sets of functions

DEFINITION 65.1. *Let f be a function, X a metric space and $p \in X$. By f is* **defined near** *p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $B \in$* $\mathrm{dom}[f]$.

NEXT YEAR, just define $\mathcal{O}_j$ as (CVZ) $\cdot$ $(|\bullet|^j)$. See Theorem 68.9

DEFINITION 65.2. $\mathrm{DNZ} := \{\alpha : \mathbb{R} \dashrightarrow \mathbb{R} \,|\, \alpha$ *is defined near* 0 *in* $\mathbb{R}\}$.

THEOREM 65.3. *Both of the following are true:*
(1) [$\forall \alpha, \beta \in \mathrm{DNZ},\ \alpha + \beta, \alpha \cdot \beta \in \mathrm{DNZ}$] *and*
(2) [$\forall c \in \mathbb{R}, \forall \beta \in \mathrm{DNZ},\ c \cdot \alpha \in \mathrm{DNZ}$].

Proof. Unassigned HW. □

DEFINITION 65.4. *Define* $\mathcal{O} : \mathbb{N}_0 \to 2^{\mathrm{DNZ}}$ *by*

$$\mathcal{O}_j = \left\{ \alpha \in \mathrm{DNZ} \,\middle|\, (\,\alpha(0) = 0\,) \,\&\, \left(\frac{\alpha}{|\bullet|^j} \to 0 \textit{ near } 0 \textit{ from } \mathbb{R} \textit{ to } \mathbb{R} \right) \right\}.$$

We argued that $\mathcal{O}_0 \supseteq \mathcal{O}_1 \supseteq \mathcal{O}_2 \supseteq \mathcal{O}_3 \supseteq \cdots$.

We argued (using HW#13-5) that $\mathrm{id}_{\mathbb{R}}^3 \in \mathcal{O}_2$. We also argued that $\mathrm{id}_{\mathbb{R}}^3 \notin \mathcal{O}_3$. This proves that $\mathcal{O}_2 \neq \mathcal{O}_3$.

More generally, we have: $\forall j \in \mathbb{N}$, $\mathrm{id}_{\mathbb{R}}^j \in \mathcal{O}_{j-1}$ and $\mathrm{id}_{\mathbb{R}}^j \notin \mathcal{O}_j$, and so $\mathcal{O}_{j-1} \neq \mathcal{O}_j$. That is, $\mathcal{O}_0 \neq \mathcal{O}_1 \neq \mathcal{O}_2 \neq \mathcal{O}_3 \neq \cdots$.

Then $\mathcal{O}_0 \supsetneq \mathcal{O}_1 \supsetneq \mathcal{O}_2 \supsetneq \mathcal{O}_3 \supsetneq \cdots$.

THEOREM 65.5. *All of the following are true:*
(1) [$\forall j \in \mathbb{N}, \forall \alpha, \beta \in \mathcal{O}_j,\ \alpha + \beta \in \mathcal{O}_j$] *and*
(2) [$\forall j, k \in \mathbb{N}, \forall \alpha \in \mathcal{O}_j, \forall \beta \in \mathcal{O}_k,\ \alpha \cdot \beta \in \mathcal{O}_{j+k}$] *and*
(3) [$\forall j \in \mathbb{N}, \forall c \in \mathbb{R}, \forall \alpha \in \mathcal{O}_j,\ c \cdot \alpha \in \mathcal{O}_j$].

Proof. Unassigned HW. □

DEFINITION 65.6. *Let f and g be functions.*
Let X be a metric space and let $p \in X$.
Then $f = g$ near p in X means: $\exists B \in \mathcal{B}_X(p)$ s.t. $f = g$ on B.

THEOREM 65.7. *Let X and Y be metric spaces.*
Let $\alpha, \beta : X \dashrightarrow Y$, $p \in X$. Assume: $\alpha = \beta$ near p in X.
Assume: α is continuous at p. Then β is continuous at p.

Proof. Unassigned HW. □

THEOREM 65.8. *Let $j \in \mathbb{N}_0$ and let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$.*
Assume $\alpha = \beta$ near 0 in $\mathbb{R}$. Assume $\alpha \in \mathcal{O}_j$. Then $\beta \in \mathcal{O}_j$.

Proof. Unassigned HW. □

DEFINITION 65.9. *Define $\mathcal{H} : \mathbb{N}_0 \to 2^{\mathbb{R}^\mathbb{R}}$ by $\mathcal{H}_J = \{a \cdot \mathrm{id}_\mathbb{R}^j \,|\, a \in \mathbb{R}\}$.*

THEOREM 65.10. *Define $C, L, Q, K : \mathbb{R} \to \mathbb{R}$ by*

$$C(x) = 3, \quad L(x) = 7x, \quad Q(x) = 4x^2, \quad K(x) = 9x^3.$$

Then $C = 3 \cdot \mathrm{id}_\mathbb{R}^0 \in \mathcal{H}_0$ *and* $L = 7 \cdot \mathrm{id}_\mathbb{R}^1 \in \mathcal{H}_1$
and $Q = 4 \cdot \mathrm{id}_\mathbb{R}^2 \in \mathcal{H}_2$ *and* $K = 9 \cdot \mathrm{id}_\mathbb{R}^3 \in \mathcal{H}_3$.

Proof. Unassigned HW. □

THEOREM 65.11. $\forall j \in \mathbb{N}_0$, $\forall f \in \mathcal{H}_j$, $\exists a \in \mathbb{R}$ *s.t.,*

$$\forall x \in \mathbb{R}, \quad f(x) = ax^j.$$

Proof. Unassigned HW. □

DEFINITION 65.12. $\mathbf{0} := C_\mathbb{R}^0$.

THEOREM 65.13. $\forall j, k \in \mathbb{N}_0$, $[\,(\,j \neq k\,) \Rightarrow (\,\mathcal{H}_j \cap \mathcal{H}_k = \{\mathbf{0}\}\,)\,]$.

Proof. Unassigned HW. □

THEOREM 65.14. *All of the following are true:*

(1) $[\,\forall j \in \mathbb{N}, \forall \alpha, \beta \in \mathcal{H}_j,\ \alpha + \beta \in \mathcal{H}_j\,]$ *and*
(2) $[\,\forall j, k \in \mathbb{N}, \forall \alpha \in \mathcal{H}_j, \forall \beta \in \mathcal{H}_k,\ \alpha \cdot \beta \in \mathcal{H}_{j+k}\,]$ *and*
(3) $[\,\forall j \in \mathbb{N}, \forall c \in \mathbb{R}, \forall \alpha \in \mathcal{H}_j,\ c \cdot \alpha \in \mathcal{H}_j\,]$.

Proof. Unassigned HW. □

Proof. Unassigned HW. □

DEFINITION 65.15. $\mathcal{C} := \mathcal{H}_0$, $\mathcal{L} := \mathcal{H}_1$, $\mathcal{Q} := \mathcal{H}_2$, $\mathcal{K} := \mathcal{H}_3$.

THEOREM 65.16. *Define* $C, L, Q, K : \mathbb{R} \to \mathbb{R}$ *by*

$$C(x) = 3, \quad L(x) = 7x, \quad Q(x) = 4x^2, \quad K(x) = 9x^3.$$

Then $C \in \mathcal{C}$ *and* $L \in \mathcal{L}$ *and* $Q \in \mathcal{Q}$ *and* $K \in \mathcal{K}$
and $C \notin \mathcal{O}_0 \supseteq \mathcal{O}_1 \supseteq \mathcal{O}_2 \supseteq \cdots$
and $L \in \mathcal{O}_0$ *and* $L \notin \mathcal{O}_1 \supseteq \mathcal{O}_2 \supseteq \mathcal{O}_3 \supseteq \cdots$
and $Q \in \mathcal{O}_1 \subseteq \mathcal{O}_0$ *and* $Q \notin \mathcal{O}_2 \supseteq \mathcal{O}_3 \supseteq \mathcal{O}_4 \supseteq \cdots$
and $K \in \mathcal{O}_2 \subseteq \mathcal{O}_1 \subseteq \mathcal{O}_0$ *and* $K \notin \mathcal{O}_3 \supseteq \mathcal{O}_4 \supseteq \mathcal{O}_5 \supseteq \cdots$
and $C + Q + L + K \notin \mathcal{O}_0 \supseteq \mathcal{O}_1 \supseteq \mathcal{O}_2 \supseteq \cdots$
and $C + Q + L \notin \mathcal{O}_0 \supseteq \mathcal{O}_1 \supseteq \mathcal{O}_2 \supseteq \cdots$
and $L+Q+K \in \mathcal{O}_0$ *and* $L+Q+K \notin \mathcal{O}_1 \supseteq \mathcal{O}_2 \supseteq \mathcal{O}_3 \supseteq \cdots$
and $Q + K \in \mathcal{O}_1 \subseteq \mathcal{O}_0$ *and* $Q + K \notin \mathcal{O}_2 \supseteq \mathcal{O}_3 \supseteq \mathcal{O}_4 \supseteq \cdots$.

Proof. Unassigned HW. □

THEOREM 65.17. *We have:* $\mathcal{C} \cap \mathcal{O}_0 = \{\mathbf{0}\}$
and $\mathcal{L} \subseteq \mathcal{O}_0$ *and* $\mathcal{L} \cap \mathcal{O}_1 = \{\mathbf{0}\}$
and $\mathcal{Q} \subseteq \mathcal{O}_1$ *and* $\mathcal{Q} \cap \mathcal{O}_2 = \{\mathbf{0}\}$
and $\mathcal{K} \subseteq \mathcal{O}_2$ *and* $\mathcal{K} \cap \mathcal{O}_3 = \{\mathbf{0}\}$.

Proof. Unassigned HW. □

More generally:

THEOREM 65.18. *We have:* $\forall j \in \mathbb{N}, \quad \mathcal{H}_j \subseteq \mathcal{O}_{j-1}$.
Also, we have: $\forall j \in \mathbb{N}_0, \quad \mathcal{H}_j \cap \mathcal{O}_j = \{\mathbf{0}\}$.

Proof. Unassigned HW. □

66. Translations, linearizations and derivatives of functions

DEFINITION 66.1. *Let* S *be a set,* $f : \mathbb{R} \dashrightarrow S$, $p \in \mathbb{R}$.
Then $f(p + \bullet) : \mathbb{R} \dashrightarrow S$ *is defined by:*

$$(f(p + \bullet))(h) = f(p + h).$$

The function $f(p + \bullet)$ is called the **horizontal translate** of f by p.

DEFINITION 66.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$.
Then $f_p^T : \mathbb{R} \dashrightarrow \mathbb{R}$ *is defined by:*

$$f_p^T(h) = [f(p + h)] - [f(p)].$$

The function f_p^T is called the **double translate** of f based at p. Note that, if $p \in \mathbb{R}\backslash(\mathrm{dom}[f])$, then $f_p^T = \varnothing$ is the empty function.

DEFINITION 66.3. *Let f be a function.*

$$\text{Then } f(☹ + \bullet) := ☹ \text{ and } f^T_{☹} = ☹.$$

We drew the graph of a partial function $f : \mathbb{R} \dashrightarrow \mathbb{R}$ whose domain was a half-open interval, and which had an interval of decrease, then one of increase, then one of decrease. We picked a point p on the first interval of decrease. We displayed graphs of $f(p + \bullet)$ and f^T_p.

THEOREM 66.4. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathrm{dom}[f]$. Then $f^T_p(0) = 0$.*

Proof. Unassigned HW. □

The analytic properties of f at/near p are preserved and reflected in the analytic properties of $f(p + \bullet)$ and f^T_p at/near 0. For example:

THEOREM 66.5. *Let $f : \mathbb{R} \to \mathbb{R}$, $p \in \mathbb{R}$.*
Then: (f is defined near p)
$\Leftrightarrow$ *($f(p + \bullet)$ is defined near 0)*
$\Leftrightarrow$ *(f^T_p is defined near 0).*
Also: (f is continuous at p)
$\Leftrightarrow$ *($f(p + \bullet)$ is continuous at 0)*
$\Leftrightarrow$ *(f^T_p is continuous at 0).*

Proof. Unassigned HW. □

DEFINITION 66.6. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and $p \in \mathbb{R}$.*
Then $\mathrm{LINS}_p f := \{L \in \mathcal{L} \mid f^T_p - L \in \mathcal{O}_1\}$.

An element of $\mathrm{LINS}_p f$ is called a **linearization** of f at p. Note that, if $p \in \mathbb{R}\backslash(\mathrm{dom}[f])$, then $\mathrm{LINS}_p f = \varnothing$.

Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, let $p \in \mathrm{dom}[f]$ and let $L \in \mathrm{LINS}_p f$. Assume that the graph of f has a tangent line at a point $(p, f(p))$. Note that the graph of L is typically *NOT* tangent to the graph of f at $(p, f(p))$. Instead, the graph of L is tangent to the graph of f^T_p at $(0, 0)$.

THEOREM 66.7. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.*
Define $L \in \mathcal{L}$ by $L(h) = 6h$. Then $L \in \mathrm{LINS}_3 f$.

Proof. Define $Q \in \mathcal{Q}$ by $Q(h) = h^2$.
We have: $\forall h \in \mathbb{R}$, $f^T_3(h) = [f(3 + h)] - [f(3)] = (3 + h)^2 - 3^2$
$= 9 + 6h + h^2 - 9 = 6h + h^2 = [L(h)] + [Q(h)] = (L + Q)(h)$.
Then $f^T_3 = L + Q$. Also $L - L = \mathbf{0}$.
Then $f^T_3 - L = L + Q - L = \mathbf{0} + Q = Q \in \mathcal{Q} \subseteq \mathcal{O}_1$.
Since $L \in \mathcal{L}$ and $f^T_3 - L \in \mathcal{O}_1$, we get: $L \in \mathrm{LINS}_3 f$, as desired. □

THEOREM 66.8. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.*
Then $\#(\mathrm{LINS}_p f) \leqslant 1$.

Proof. Want: $\forall L, M \in \mathrm{LINS}_p f$, $L = M$.
Given $L, M \in \mathrm{LINS}_p f$. Want: $L = M$.
Since $L, M \in \mathrm{LINS}_p f$, we get $f_p^T - L \in \mathcal{O}_1$ and $f_p^T - M \in \mathcal{O}_1$.
Then $(f_p^T - M) - (f_p^T - L) \in \mathcal{O}_1$.
Since $f_p^T - L \in \mathcal{O}_1 \subseteq \mathrm{DNZ}$, choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $B \subseteq \mathrm{dom}[f_p^T - L]$.
We have $L, M \in \mathrm{LINS}_p f \subseteq \mathcal{L}$.
Since $L, M \in \mathcal{L}$, we get $\mathrm{dom}[L] = \mathbb{R} = \mathrm{dom}[M]$.
Since $L, M \in \mathcal{L}$, we get $L - M \in \mathcal{L}$.
Let $D := \mathrm{dom}[f_p^T]$.
Then, as $\mathrm{dom}[L] = \mathbb{R} = \mathrm{dom}[M]$, we get:
$$\mathrm{dom}[f_p^T - L] = D = \mathrm{dom}[f_p^T - M].$$
Then $(f_p^T - M) - (f_p^T - L) = L - M$ on D.
So, since $B \subseteq \mathrm{dom}[f_p^T - L] = D$, it follows that:
$$(f_p^T - M) - (f_p^T - L) = L - M \text{ on } B.$$
So, since $B \in \mathcal{B}_{\mathbb{R}}(0)$, we conclude that:
$$(f_p^T - M) - (f_p^T - L) = L - M \text{ near } 0 \text{ in } \mathbb{R}.$$
So, since $(f_p^T - M) - (f_p^T - L) \in \mathcal{O}_1$,
by Theorem 65.8, we see that $L - M \in \mathcal{O}_1$.
So, since $L - M \in \mathcal{L}$, we get $L - M \in \mathcal{L} \cap \mathcal{O}_1 = \{\mathbf{0}\}$.
Then $L - M = \mathbf{0}$, so $L = M$, as desired. □

DEFINITION 66.9. *Let $L \in \mathcal{L}$.*
$$\textit{Then } \mathrm{s}\ell_L := \mathrm{UE}\{a \in \mathbb{R} \,|\, L = a \cdot \mathrm{id}_R\}.$$

NEXT YEAR: Easier to define $\mathrm{s}\ell_L := L(1)$. Define $\mathrm{s}\ell_{☹} = ☹$.

THEOREM 66.10. *Define $L \in \mathcal{L}$ by $L(h) = 6h$. Then $\mathrm{s}\ell_L = 6$.*

Proof. Unassigned HW. □

DEFINITION 66.11. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Then $f' : \mathbb{R} \dashrightarrow \mathbb{R}$ is defined by: $f'(p) = \mathrm{UE}\{\mathrm{s}\ell_L \,|\, L \in \mathrm{LINS}_p f\}$.

The function $f' : \mathbb{R} \dashrightarrow \mathbb{R}$ is called the **derivative** of f. Note that, if $p \in \mathbb{R} \backslash (\mathrm{dom}[f])$, then $f'_p = ☹$.

NEXT YEAR: Define $D_p f := \mathrm{UE}(\mathrm{LINS}_p f)$ and $f'_p = \mathrm{s}\ell_{D_p f}$. Also, note that, if $p \in \mathbb{R} \backslash (\mathrm{dom}[f])$, then $D_p f = ☹$. Also, define $\mathrm{s}\ell_{☹} := ☹$.

THEOREM 66.12. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then $f'(3) = 6$.*

Proof. Define $L \in \mathcal{L}$ by $L(h) = 6h$.
By Theorem 66.10, we have: $s\ell_L = 6$.
Let $S := \{s\ell_M \mid M \in \text{LINS}_3 f\}$.
By Definition 66.11, $f'(3) = \text{UE}\, S$.
By Theorem 66.8, $\#(\text{LINS}_3 f) \leqslant 1$.
By Theorem 66.7, $L \in \text{LINS}_3 f$.
Then $\text{LINS}_3 f = \{L\}$.
Then $S = \{s\ell_M \mid M \in \{L\}\} = \{s\ell_L\} = \{6\}$.
Then $f'(3) = \text{UE}\, S = \text{UE}\, \{6\} = 6$, as desired. □

THEOREM 66.13. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$, $L \in \mathcal{L}$. Assume that $f_p^T - L \in \mathcal{O}_1$. Then $f'(p) = s\ell_L$.*

Proof. Let $S := \{s\ell_M \mid M \in \text{LINS}_p f\}$.
By Definition 66.11, $f'(p) = \text{UE}\, S$.
By Theorem 66.8, $\#(\text{LINS}_p f) \leqslant 1$.
By Definition 66.6, $L \in \text{LINS}_p f$.
Then $\text{LINS}_p f = \{L\}$.
Then $S = \{s\ell_M \mid M \in \{L\}\} = \{s\ell_L\}$.
Then $f'(p) = \text{UE}\, S = \text{UE}\, \{s\ell_L\} = s\ell_L$, as desired. □

THEOREM 66.14. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Then $f'(2) = 3 \cdot 2^2$.*

Proof. Define $L \in \mathcal{L}$, $Q \in \mathcal{Q}$, $K \in \mathcal{K}$ by
$$L(h) = 3 \cdot 2^2 \cdot h, \quad Q(h) = 3 \cdot 2 \cdot h^2, \quad K(h) = h^3.$$
Then $L = 3 \cdot 2^2 \cdot \text{id}_{\mathbb{R}}$, so $s\ell_L = 3 \cdot 2^2$.
Also, since $Q \in \mathcal{Q} \subseteq \mathcal{O}_1$ and since $K \in \mathcal{K} \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_1$,
we conclude that $Q + K \in \mathcal{O}_1$.
We have: $\forall h \in \mathbb{R}$,
$$\begin{aligned} f_2^T(h) &= [f(2+h)] - [f(2)] = (2+h)^3 - 2^3 \\ &= 2^3 + 3 \cdot 2^2 \cdot h + 3 \cdot 2 \cdot h^2 + h^3 - 2^3 \\ &= 3 \cdot 2^2 \cdot h + 3 \cdot 2 \cdot h^2 + h^3 \\ &= [L(h)] + [Q(h)] + [K(h)] \\ &= (L + Q + K)(h). \end{aligned}$$
Then $f_2^T = L + Q + K$. Also $L - L = \mathbf{0}$.
Then $f_2^T - L = L + Q + K - L = \mathbf{0} + Q + K = Q + K \in \mathcal{O}_1$.
Then, by Theorem 66.13, we get: $f'(2) = s\ell_L$.
Then $f'(2) = s\ell_L = 3 \cdot 2^2$, as desired. □

THEOREM 66.15. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Then, $\forall x \in \mathbb{R}$, $f'(x) = 3x^2$.*

Proof. Given $x \in \mathbb{R}$. Want: $f'(x) = 3x^2$.
Define $L \in \mathcal{L}$, $Q \in \mathcal{Q}$, $K \in \mathcal{K}$ by
$$L(h) = 3x^2h, \quad Q(h) = 3xh^2, \quad K(h) = h^3.$$
Then $L = 3x^2 \cdot \mathrm{id}_{\mathbb{R}}$, so $\mathrm{s}\ell_L = 3x^2$.
Also, since $Q \in \mathcal{Q} \subseteq \mathcal{O}_1$ and since $K \in \mathcal{K} \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_1$,
we conclude that $Q + K \in \mathcal{O}_1$.
We have: $\forall h \in \mathbb{R}$, $f_x^T(h) = [f(x+h)] - [f(x)] = (x+h)^3 - x^3$
$$= x^3 + 3x^2h + 3xh^2 + h^3 - x^3$$
$$= 3x^2h + 3xh^2 + h^3$$
$$= [L(h)] + [Q(h)] + [K(h)]$$
$$= (L + Q + K)(h).$$
Then $f_x^T = L + Q + K$. Also $L - L = \mathbf{0}$.
Then $f_x^T - L = L + Q + K - L = \mathbf{0} + Q + K = Q + K \in \mathcal{O}_1$.
Then, by Theorem 66.13, we get: $f'(x) = \mathrm{s}\ell_L$.
Then $f'(x) = \mathrm{s}\ell_L = 3x^2$, as desired. □

67. Basic facts about differentiation

THEOREM 67.1. $\forall C \in \mathcal{C}, \quad C' = \mathbf{0}$.

Proof. Unassigned HW. □

THEOREM 67.2. $\forall L \in \mathcal{L}, \quad L' = C_{\mathbb{R}}^{\mathrm{s}\ell_L}$.

Proof. Unassigned HW. □

DEFINITION 67.3. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$.
By f *is* **differentiable at** p, *we mean:* $p \in \mathrm{dom}[f]$.

THEOREM 67.4. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathrm{dom}[f']$.
Then $\exists L \in \mathcal{L}$, $\exists R \in \mathcal{O}_1$ *s.t.* $f_p^T = L + R$.

Proof. Let $S := \{\mathrm{s}\ell_L \mid L \in \mathrm{LINS}_p f\}$.
Then $f'(p) = \mathrm{UE}(S)$.
Since $p \in \mathrm{dom}[f']$, we get ☹ $\neq f'(p)$.
Then $\mathrm{UE}(S) = f'(p) \neq$ ☹ $= \mathrm{UE}(\emptyset)$, so $S \neq \emptyset$.
By Theorem 66.8, $\#(\mathrm{LINS}_p f) \leqslant 1$.
Since $\{\mathrm{s}\ell_L \mid L \in \mathrm{LINS}_p f\} = S \neq \emptyset$, we conclude that $\mathrm{LINS}_p \neq \emptyset$.
Then $\#(\mathrm{LINS}_p f) = 1$.
Let $L := \mathrm{UE}(\mathrm{LINS}_p f)$. Then $\mathrm{LINS}_p f = \{L\}$. Then $L \in \mathrm{LINS}_p f$.
Then $L \in \mathrm{LINS}_p f \subseteq \mathcal{L}$.
Since $L \in \{L\} = \mathrm{LINS}_p f$, we see that $f_p^T - L \in \mathcal{O}_1$.

Let $R := f_p^T - L$. Then $R \in \mathcal{O}_1$.
Then $L \in \mathcal{L}$ and $R \in \mathcal{O}_1$. Want: $f_p^T = R + L$.
Since $R = f_p^T - L$, we get: $R + L = f_p^T - L + L = f_p^T + \mathbf{0} = f_p^T$,
and so $f_p^T = R + L = L + R$, as desired. □

THEOREM 67.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$, $L \in \mathcal{L}$, $R \in \mathcal{O}_1$.
Assume that $f_p^T = L + R$. *Then* $f'(p) = \mathrm{s}\ell_L$.

Proof. Since $f_p^T - L = R \in \mathcal{O}_1$, it follows, from Theorem 66.13, that $f'(p) = \mathrm{s}\ell_L$, as desired. □

THEOREM 67.6. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathrm{dom}[f']$.
Then: (f is defined near p) & (f is continuous at p).

Proof. By Theorem 67.4, choose $L \in \mathcal{L}$ and $R \in \mathcal{O}_1$ s.t. $f_p^T = L + R$.
Since L and R are defined near 0, we get: $L + R$ is defined near 0.
Then f_p^T is defined near 0.
Since L and R are continuous at 0, we get: $L + R$ is continuous at 0.
Then f_p^T is continuous at 0.
Since f_p^T is defined near 0 and f_p^T is continuous at 0, by Theorem 66.5,
we get: (f is defined near p) & (f is continuous at p). □

THEOREM 67.7. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$. *Then* $\mathrm{dom}[f'] \subseteq \mathrm{dom}[f]$.

Proof. Want: $\forall p \in \mathrm{dom}[f']$, $p \in \mathrm{dom}[f]$.
Given $p \in \mathrm{dom}[f']$. Want: $p \in \mathrm{dom}[f]$.
By Theorem 67.6, f is defined near p.
So, by Definition 65.1, choose $B \in \mathcal{B}_X(p)$ s.t. $B \subseteq \mathrm{dom}[f]$.
Since $B \in \mathcal{B}_X(p)$, we get: $p \in B$.
Then $p \in B \subseteq \mathrm{dom}[f]$, as desired. □

DEFINITION 67.8. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $p \in \mathbb{R}$.
Then $\mathrm{SS}_f^p : \mathbb{R} \dashrightarrow \mathbb{R}$ *is defined by*

$$(SS_f^p)(h) = (DQ_f)(p, p + h).$$

We call SS_f^p the **secant slope function** of f based at p. The limit of the secant slope is not always the value of the derivative:

THEOREM 67.9. *Let* $A := \{1, 1/2, 1/3, 1/4, \ldots\} \cup \{0\}$.
Let $f := (\mathrm{id}_{\mathbb{R}}^2)|(A + 3)$.
Then: $f'(3) = ☹$ *and* $\mathrm{SS}_f^3 \to 6$ *near* 0 *from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. Unassigned HW. □

The limit of the secant slope is typically the value of the derivative:

THEOREM 67.10. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $p, m \in \mathbb{R}$.
Assume that f *is defined near* p.
Then: $[\, f'(p) = m \,] \Leftrightarrow [\, \mathrm{SS}_f^p \to m$ *near* 0 *from* $\mathbb{R}$ *to* $\mathbb{R}\,]$.

Proof. Omitted. □

THEOREM 67.11. *Let* $a \in \mathbb{R}$. *Then* $|\bullet| - a \cdot [\mathrm{id}_\mathbb{R}] \notin \mathcal{O}_1$.

Proof. Let $f := |\bullet| - a \cdot [\mathrm{id}_\mathbb{R}]$. Want: $f \notin \mathcal{O}_1$.
Assuume $f \in \mathcal{O}_1$. Want: Contradiction.
Since $f \in \mathcal{O}_1$, we get: $f/[|\bullet|] \to 0$ near 0 from $\mathbb{R}$ to $\mathbb{R}$.
So, by Theorem 63.6, choose $\delta > 0$ s.t., $\forall h \in \mathbb{R}$,

$$(\, 0 < d_\mathbb{R}(h, 0) < \delta \,) \Rightarrow \left(d_\mathbb{R}\left(\left(\frac{f}{|\bullet|} \right)(h), 0 \right) < 1 \right).$$

Then, $\forall h \in \mathbb{R}$, $\left[(\, 0 < |h| < \delta \,) \Rightarrow \left(\left| \left(\frac{f}{|\bullet|} \right)(h) \right| < 1 \right) \right]$.
At least one of the following must be true:

(1) $a \geqslant 0$ or (2) $a \leqslant 0$.

Case (1):
Let $h := -\delta/2$. Since $h < 0$, we get $|h| = -h$.
Since $|h| = \delta/2$ and since $0 < \delta/2 < \delta$, we get: $0 < |h| < \delta$.
Then, by the choice of δ, we have: $\left| \left(\frac{f}{|\bullet|} \right)(h) \right| < 1$.
Let $y := \left(\frac{f}{|\bullet|} \right)(h)$. Then $|y| < 1$. Then $1 > |y|$.
We calculate:

$$y = \left(\frac{f}{|\bullet|} \right)(h) = \frac{f(h)}{|h|} = \frac{|h| - ah}{|h|} = \frac{-h - ah}{-h} = \frac{h + ah}{h} = 1 + a.$$

Since $a \geqslant 0$, we get: $1 + a \geqslant 1$.
So, since $y = 1 + a$, we get: $y \geqslant 1$.
Since $y \geqslant 1 > 0$, we conclude that $y > 0$, and so $|y| = y$.
Then $1 > |y| = y \geqslant 1$, so $1 > 1$. Contradiction.
End of Case (1).

Case (2):
Let $h := \delta/2$. Since $h > 0$, we get $|h| = h$.
Since $|h| = \delta/2$ and since $0 < \delta/2 < \delta$, we get: $0 < |h| < \delta$.

Then, by the choice of δ, we have: $\left|\left(\frac{f}{|\bullet|}\right)(h)\right| < 1$.

Let $y := \left(\frac{f}{|\bullet|}\right)(h)$. Then $|y| < 1$. Then $1 > |y|$.

We calculate: $y = \left(\frac{f}{|\bullet|}\right)(h) = \frac{f(h)}{|h|} = \frac{|h| - ah}{|h|} = \frac{h - ah}{h} = 1 - a$.

Since $a \leqslant 0$, we get: $1 - a \geqslant 1$.

So, since $y = 1 - a$, we get: $y \geqslant 1$.

Since $y \geqslant 1 > 0$, we conclude that $y > 0$, and so $|y| = y$.

Then $1 > |y| = y \geqslant 1$, so $1 > 1$. Contradiction.

End of Case (2). □

THEOREM 67.12. *Let $L \in \mathcal{L}$ and let $a := \mathrm{s}\ell_L$.*
Then $L = a \cdot \mathrm{id}_{\mathbb{R}}$.

Proof. Unassigned HW. □

THEOREM 67.13. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume $f(0) = 0$. Then $f_0^T = f$.

Proof. Unassigned HW. □

THEOREM 67.14. $0 \notin \mathrm{dom}[|\bullet|']$.

Proof. Assume $0 \in \mathrm{dom}[|\bullet|']$. Want: Contradiction.

By Theorem 67.4, choose $L \in \mathcal{L}$ and $R \in \mathcal{O}_1$ s.t. $|\bullet|_0^T = L + R$.

By Theorem 67.13, we have: $|\bullet|_0^T = |\bullet|$.

Then $|\bullet| - L = |\bullet|_0^T - L = L + R - L = R + \mathbf{0} = R \in \mathcal{O}_1$.

Let $a := \mathrm{s}\ell_L$. By Theorem 67.12, we have: $L = a \cdot [\mathrm{id}_R]$.

Then $|\bullet| - a \cdot [\mathrm{id}_{\mathbb{R}}] = |\bullet| - L \in \mathcal{O}_1$. Then $|\bullet| - a \cdot [\mathrm{id}_{\mathbb{R}}] \in \mathcal{O}_1$.

However, by Theorem 67.11, we have: $|\bullet| - a \cdot [\mathrm{id}_{\mathbb{R}}] \notin \mathcal{O}_1$.

Contradiction. □

THEOREM 67.15. *Let $\alpha : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume that $\alpha = \mathbf{0}$ near 0 in $\mathbb{R}$. Then: $\forall j \in \mathbb{N}_0$, $\alpha \in \mathcal{O}_j$.

Proof. Unassigned HW. □

THEOREM 67.16. $(|\bullet|')(2) = 1$.

Proof. Let $m := 1$. Let $L := m \cdot [\mathrm{id}_{\mathbb{R}}]$. Then $\mathrm{s}\ell_L = m$.

We calculate: $\forall h \in (-2; 2)$,

$$\begin{aligned}(|\bullet|_2^T - L)(h) &= |2 + h| - |2| - m \cdot h \\ &= 2 + h - 2 - 1 \cdot h = 0 = \mathbf{0}(h).\end{aligned}$$

Then $|\bullet|_2^T - L = \mathbf{0}$ on $(-2;2)$.
Then $|\bullet|_2^T - L = \mathbf{0}$ near 0 in $\mathbb{R}$.
Then, by Theorem 67.15, $|\bullet|_2^T - L \in \mathcal{O}_1$.
Then, by Theorem 66.13, $(|\bullet|')(2) = \mathrm{s}\ell_L$.
Then $(|\bullet|')(2) = \mathrm{s}\ell_L = m = 1$, as desired.

□

THEOREM 67.17. *$(|\bullet|')(-2) = -1$.*

Proof. Let $m := -1$. Let $L := m \cdot [\mathrm{id}_{\mathbb{R}}]$. Then $\mathrm{s}\ell_L = m$.
We calculate: $\forall h \in (-2;2)$,

$$\begin{aligned}(|\bullet|_{-2}^T - L)(h) &= |-2+h| - |-2| - m \cdot h \\ &= -(-2+h) - 2 - (-1) \cdot h = 0 = \mathbf{0}(h).\end{aligned}$$

Then $|\bullet|_{-2}^T - L = \mathbf{0}$ on $(-2;2)$.
Then $|\bullet|_{-2}^T - L = \mathbf{0}$ near 0 in $\mathbb{R}$.
Then, by Theorem 67.15, $|\bullet|_{-2}^T - L \in \mathcal{O}_1$.
Then, by Theorem 66.13, $(|\bullet|')(-2) = \mathrm{s}\ell_L$.
Then $(|\bullet|')(-2) = \mathrm{s}\ell_L = m = -1$, as desired.

□

THEOREM 67.18. *$\forall x > 0$, $(|\bullet|')(x) = 1$.*

Proof. Let $m := 1$. Let $L := m \cdot [\mathrm{id}_{\mathbb{R}}]$. Then $\mathrm{s}\ell_L = m$.
We calculate: $\forall h \in (-x;x)$,

$$\begin{aligned}(|\bullet|_x^T - L)(h) &= |x+h| - |x| - m \cdot h \\ &= x + h - x - 1 \cdot h = 0 = \mathbf{0}(h).\end{aligned}$$

Then $|\bullet|_x^T - L = \mathbf{0}$ on $(-x;x)$.
Then $|\bullet|_x^T - L = \mathbf{0}$ near 0 in $\mathbb{R}$.
Then, by Theorem 67.15, $|\bullet|_x^T - L \in \mathcal{O}_1$.
Then, by Theorem 66.13, $(|\bullet|')(x) = \mathrm{s}\ell_L$.
Then $(|\bullet|')(x) = \mathrm{s}\ell_L = m = 1$, as desired.

□

THEOREM 67.19. *$\forall x < 0$, $(|\bullet|')(x) = -1$.*

Proof. Let $m := -1$. Let $L := m \cdot [\mathrm{id}_{\mathbb{R}}]$. Then $\mathrm{s}\ell_L = m$.
We calculate: $\forall h \in (x;-x)$,

$$\begin{aligned}(|\bullet|_x^T - L)(h) &= |x+h| - |x| - m \cdot h \\ &= -(x+h) - (-x) - (-1) \cdot h = 0 = \mathbf{0}(h).\end{aligned}$$

Then $|\bullet|_x^T - L = \mathbf{0}$ on $(x;-x)$.
Then $|\bullet|_x^T - L = \mathbf{0}$ near 0 in $\mathbb{R}$.

Then, by Theorem 67.15, $|\bullet|_x^T - L \in \mathcal{O}_1$.
Then, by Theorem 66.13, $(|\bullet|')(x) = \mathrm{s}\ell_L$.
Then $(|\bullet|')(x) = \mathrm{s}\ell_L = m = -1$, as desired.

□

Since $\mathrm{dom}[|\bullet|] = \mathbb{R}$, the next result follows from Theorem 67.14, Theorem 67.18 and Theorem 67.20:

THEOREM 67.20. $\mathrm{dom}[|\bullet|'] = \mathbb{R}_0^\times \subsetneq \mathbb{R} = \mathrm{dom}[|\bullet|]$.

According to Theorem 67.7, $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\mathrm{dom}[f'] \subseteq \mathrm{dom}[f]$. By Theorem 67.20, we see that the inclusion is sometimes proper.

DEFINITION 67.21. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathbb{R}$.*
Then f is **differentiable on** *S means: $S \subseteq \mathrm{dom}[f']$.*

DEFINITION 67.22. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Then f is **differentiable** *means: $\mathrm{dom}[f'] = \mathrm{dom}[f]$.*

Unassigned HW: $\forall f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall p \in \mathbb{R}$, $(f+g)_p^T = [f_p^T] + [g_p^T]$.
Unassigned HW: $\forall c \in \mathbb{R}$, $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall p \in \mathbb{R}$, $(c \cdot f)_p^T = c \cdot [f_p^T]$.
To express these last two unassigned HWs, we sometimes say,
"Double translation is linear."
Unassigned HW: $\forall L, M \in \mathcal{L}$, $\mathrm{s}\ell_{L+M} = [\mathrm{s}\ell_L] + [\mathrm{s}\ell_M]$.
Unassigned HW: $\forall c \in \mathbb{R}$, $\forall L \in \mathcal{L}$, $\mathrm{s}\ell_{c \cdot L} = c \cdot [\mathrm{s}\ell_L]$.
To express these last two unassigned HWs, we sometimes say,
"Computation of slope is linear."
Unassigned HW: $\mathbf{0}' = \mathbf{0}$.
We will sometimes use f_p' instead of $f'(p)$.
According to Theorem 67.14, $(|\bullet|)_0' = ☹$.
Unassigned HW: $(-|\bullet|)_0' = ☹$.
Let $f := |\bullet|$ and $g := -|\bullet|$. Then we have: $f_0' = ☹$ and $g_0' = ☹$. Also, $f + g = \mathbf{0}$. Then $(f+g)' = \mathbf{0}$. It follows that $(f+g)_0' = \mathbf{0}_0 = 0$. Also, $[f_0'] + [g_0'] = ☹ + ☹ = ☹$. Let $p := 0$. Then

$$(f+g)_p' = (f+g)_0' = 0 \neq ☹ = [f_0'] + [g_0'] = [f_p'] + [g_p'],$$

and so $(f+g)_p' \neq [f_p'] + [g_p']$. On the other hand:

THEOREM 67.23. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.*
Then: $(f+g)_p' \;=^*\; [f_p'] + [g_p']$.

Proof. Want: $(\,[f_p'] + [g_p'] \neq ☹\,) \Rightarrow (\,(f+g)_p' = [f_p'] + [g_p']\,)$.
Assume $[f_p'] + [g_p'] \neq ☹$. Want: $(f+g)_p' = [f_p'] + [g_p']$.

Since $[f'_p] + [g'_p] \neq ☹$, we conclude that $f'_p \neq ☹ \neq g'_p$.
Then $p \in \mathrm{dom}[f']$ and $p \in \mathrm{dom}[g']$.
Then, by Theorem 67.4, choose $L, M \in \mathcal{L}$ and $R, S \in \mathcal{O}_1$ s.t.

$$f_p^T = L + R \quad \text{and} \quad g_p^T = M + S.$$

By Theorem 67.5, we have: $\quad f'_p = \mathrm{s}\ell_L \quad$ and $\quad g'_p = \mathrm{s}\ell_M$.
Since $L, M \in \mathcal{L}$, we get: $L + M \in \mathcal{L}$.
Since $R, S \in \mathcal{O}_1$, we get: $R + S \in \mathcal{O}_1$.
Also, $(f+g)_p^T = [f_p^T] + [g_p^T] = [L+R] + [M+S] = [L+M] + [R+S]$.
Then, by Theorem 67.5, $(f+g)'_p = \mathrm{s}\ell_{L+M}$.
Then $(f+g)'_p = \mathrm{s}\ell_{L+M} = [\mathrm{s}\ell_L] + [\mathrm{s}\ell_M] = [f'_p] + [g'_p]$, as desired. □

THEOREM 67.24. *Let $c \in \mathbb{R}$, let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.*
Then $(c \cdot f)'_p \ =^* \ c \cdot [f'_p]$.

Proof. Unassigned HW. □

Unassigned HW: $\forall a, b$, $(\,[\,(\,a \ =^* \ b\,)\,\&\,(\,b \ =^* \ a\,)\,] \Rightarrow [\,a = b\,]\,)$.
Unassigned HW: $\forall a$, $\forall b \frown\!\!\in \mathbb{R}$, $\forall c \in \mathbb{R}$,

$$[\,a \ =^* \ b\,] \Rightarrow [\,c \cdot a \ =^* \ c \cdot b\,].$$

THEOREM 67.25. *Let $c \in \mathbb{R}_0^\times$, let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.*
Then $(c \cdot f)'_p \ = \ c \cdot [f'_p]$.

Proof. By Theorem 67.24, $(c \cdot f)'_p \ =^* \ c \cdot [f'_p]$.
Want: $c \cdot [f'_p] \ =^* \ (c \cdot f)'_p$.
Let $\phi := c \cdot f$ and let $\gamma := 1/c$.
By Theorem 67.24, $(\gamma \cdot \phi)'_p \ =^* \ \gamma \cdot [\phi'_p]$.
So, since $\gamma \cdot \phi = f$ and since $\phi = c \cdot f$, this gives: $f'_p \ =^* \ \gamma \cdot [(c \cdot f)'_p]$.
Multiplying by c, we get: $c \cdot [f'_p] \ =^* \ c \cdot \gamma \cdot [(c \cdot f)'_p]$.
So, since $c \cdot \gamma = 1$, we conclude: $c \cdot [f'_p] \ =^* \ (c \cdot f)'_p$, as desired. □

68. Basic results about big O and little o

DEFINITION 68.1. $\forall$*sets X and Y,* $\quad \mathcal{PF}_{XY} := \bigcup\{Y^D \,|\, D \subseteq X\}$.

THEOREM 68.2. $\forall$*sets X and Y,* $\forall f$,

$$(\ f \in \mathcal{PF}_{XY}\ ) \Leftrightarrow (\ f : X \dashrightarrow Y\ ).$$

DEFINITION 68.3. $\forall$*set X,* $\forall \mathcal{S}, \mathcal{T} \subseteq \mathcal{PF}_{X\mathbb{R}}$,

$$\mathcal{S} + \mathcal{T} := \{f + g \,|\, f \in \mathcal{S}, g \in \mathcal{T}\} \qquad \textit{and}$$
$$\mathcal{S} \cdot \mathcal{T} := \{f \cdot g \,|\, f \in \mathcal{S}, g \in \mathcal{T}\}.$$

DEFINITION 68.4. *$\forall$set X, $\forall f : X \dashrightarrow \mathbb{R}$, $\forall \mathcal{S} \subseteq \mathcal{PF}_{X\mathbb{R}}$,*

$$f + \mathcal{S} := \{f\} + \mathcal{S} =: \mathcal{S} + f \qquad \textit{and}$$
$$f \cdot \mathcal{S} := \{f\} \cdot \mathcal{S} =: \mathcal{S} \cdot f.$$

DEFINITION 68.5. *$\forall$set X, $\forall A \subseteq \mathbb{R}$, $\forall \mathcal{S} \subseteq \mathcal{PF}_{X\mathbb{R}}$,*

$$A \cdot \mathcal{S} := \{c \cdot f \,|\, c \in A,\ f \in \mathcal{S}\} =: \mathcal{S} \cdot A.$$

DEFINITION 68.6. *$\forall$set X, $\forall A \subseteq \mathbb{R}$, $\forall f : X \dashrightarrow \mathbb{R}$,*

$$A \cdot f := A \cdot \{f\} =: f \cdot A.$$

DEFINITION 68.7. *$\forall$set X, $\forall a \in \mathbb{R}$, $\forall \mathcal{S} \subseteq \mathcal{PF}_{X\mathbb{R}}$,*

$$a \cdot \mathcal{S} := \{a\} \cdot \mathcal{S} =: \mathcal{S} \cdot a.$$

Recall Definition 65.2 and Definition 65.4.

DEFINITION 68.8.

$$\text{CVZ} \quad := \quad \{\alpha \in \text{DNZ} \,|\, (\,\alpha \textit{ is continuous at } 0\,) \,\&\, (\,\alpha(0) = 0\,)\}.$$

THEOREM 68.9. *Let $j \in \mathbb{N}_0$. Then $\mathcal{O}_j = (\text{CVZ}) \cdot (|\bullet|^j)$.*

Proof. By HW#1-1, it suffices to show: $\mathcal{O}_j \subseteq (\text{CVZ}) \cdot (|\bullet|^j)$.
Want: $\forall \alpha \in \mathcal{O}_j$, $\alpha \in (\text{CVZ}) \cdot (|\bullet|^j)$.
Given $\alpha \in \mathcal{O}_j$. Want: $\alpha \in (\text{CVZ}) \cdot (|\bullet|^j)$.
Let $\beta := \text{adj}_0^0\left(\dfrac{\alpha}{|\bullet|^j}\right)$. Want: $(\,\beta \in \text{CVZ}\,) \,\&\, (\,\alpha = \beta \cdot (|\bullet|^j)\,)$.
Since $\alpha \in \mathcal{O}_j$, we know:

$$(\,\alpha \in \text{DNZ}\,) \,\&\, (\,\alpha(0) = 0\,) \,\&\, \left(\frac{\alpha}{|\bullet|^j} \to 0 \text{ near } 0 \text{ from } \mathbb{R} \text{ to } \mathbb{R}\right).$$

We have $\text{dom}[\beta] = ((\text{dom}[\alpha])_0^\times)_0^+ \supseteq \text{dom}[\alpha]$,
so, since $\alpha \in \text{DNZ}$, we get: $\beta \in \text{DNZ}$.
Since $\dfrac{\alpha}{|\bullet|^j} \to 0$ near 0 from $\mathbb{R}$ to $\mathbb{R}$, we see that β is continuous at 0.
So, since $\beta \in \text{DNZ}$, it follows that $\beta \in \text{CVZ}$.
Want: $\alpha = \beta \cdot (|\bullet|^j)$.
Want: $\forall x \in \mathbb{R}$, $\alpha(x) = (\beta \cdot (|\bullet|^j))(x)$.
Given $x \in \mathbb{R}$. Want: $\alpha(x) = (\beta \cdot (|\bullet|^j))(x)$.
Want: $[\beta(x)] \cdot [|x|^j] = \alpha(x)$.
Exactly one of the following is true:

(1) $x = 0$ or
(2) $x \neq 0$.

Case (1):
We have $\alpha(x) = \alpha(0) = 0$.

Then $[\beta(x)] \cdot [|x|^j] = [\beta(0)] \cdot [|0|^j] = 0 \cdot 0^j = 0 = \alpha(x)$, as desired.
End of Case (1).

Case (2):
We have $[\beta(x)] \cdot [|x|^j] = \left[\left(\frac{\alpha}{|\bullet|^j} \right)(x) \right] \cdot [|x|^j]$
$= \left[\frac{\alpha(x)}{|x|^j} \right] \cdot [|x|^j] = \alpha(x)$, as desired.
End of Case (2). □

THEOREM 68.10. $\mathcal{O}_0 = \mathrm{CVZ}$.

Proof. By Theorem 68.9, $\mathcal{O}_0 = (\mathrm{CVZ}) \cdot (|\bullet|^0)$.
So, as $|\bullet|^0 = C^1_{\mathbb{R}}$, we get $\mathcal{O}_0 = \mathrm{CVZ}$, as desired. □

NEXT YEAR, just define $\mathcal{O}_j$ as $(\mathrm{CVZ}) \cdot (|\bullet|^j)$.

DEFINITION 68.11.
$$\mathrm{BNZ} := \{\alpha \in \mathrm{DNZ} \,|\, \exists C \in \mathcal{B}_{\mathbb{R}}(0) \textit{ s.t. } \alpha_*(C) \textit{ is bounded in } \mathbb{R}\}.$$

We graphed $x \mapsto x^{-2} : \mathbb{R} \dashrightarrow \mathbb{R}$. This is in DNZ, but not in CVZ and not in BNZ.

We graphed $x \mapsto (x-1)^{-2} : \mathbb{R} \dashrightarrow \mathbb{R}$. This is in DNZ and in BNZ, but not in CVZ.

We graphed $\left(x \mapsto \begin{cases} 1, & \text{if } x < 0 \\ x, & \text{if } x \geqslant 0 \end{cases} \right) : \mathbb{R} \to \mathbb{R}$. This is in DNZ and in BNZ, but not in CVZ.

By definition of CVZ, we have DNZ $\subseteq$ CVZ.

According to HW#1-2, we have CVZ $\subseteq$ BNZ.

THEOREM 68.12. $(\mathrm{CVZ}) \cdot (\mathrm{CVZ}) \subseteq \mathrm{CVZ}$.

Proof. By HW#1-2, CVZ $\subseteq$ BNZ, so $(\mathrm{CVZ}) \cdot (\mathrm{CVZ}) \subseteq (\mathrm{BNZ}) \cdot (\mathrm{CVZ})$.
According to HW#1-4, we have $(\mathrm{BNZ}) \cdot (\mathrm{CVZ}) \subseteq \mathrm{CVZ}$.
Then $(\mathrm{CVZ}) \cdot (\mathrm{CVZ}) \subseteq (\mathrm{BNZ}) \cdot (\mathrm{CVZ}) \subseteq \mathrm{CVZ}$, as desired. □

DEFINITION 68.13. *Define* $\widehat{\mathcal{O}} : \mathbb{N}_0 \to 2^{\mathrm{DNZ}}$ *by* $\widehat{\mathcal{O}}_j = (\mathrm{BNZ})(|\bullet|^j)$.

THEOREM 68.14. $\widehat{\mathcal{O}}_0 = \mathrm{BNZ}$.

Proof. By Theorem 68.13, $\widehat{\mathcal{O}}_0 = (\mathrm{BNZ}) \cdot (|\bullet|^0)$.
So, as $|\bullet|^0 = C^1_{\mathbb{R}}$, we get $\widehat{\mathcal{O}}_0 = \mathrm{BNZ}$, as desired. □

THEOREM 68.15. $\forall j \in \mathbb{N}_0, \quad \mathbb{R} \cdot \widehat{\mathcal{O}}_j \subseteq \widehat{\mathcal{O}}_j.$
Also, $\forall j \in \mathbb{N}_0, \quad \widehat{\mathcal{O}}_j + \widehat{\mathcal{O}}_j \subseteq \widehat{\mathcal{O}}_j.$
Also, $\forall j, k \in \mathbb{N}_0, \quad \widehat{\mathcal{O}}_j \cdot \widehat{\mathcal{O}}_k \subseteq \widehat{\mathcal{O}}_{j+k}.$

Proof. Unassigned homework. □

THEOREM 68.16. *Let* $j \in \mathbb{N}$. *Then* $\widehat{\mathcal{O}}_j \subseteq o_{j-1}$.

Proof. Since $|\bullet| \in \text{CVZ}$,
we get $(\text{BNZ}) \cdot (|\bullet|) \cdot (|\cdot|^{j-1}) \in (\text{BNZ}) \cdot (\text{CVZ}) \cdot (|\cdot|^{j-1})$.
By HW#1-4, we have $(\text{BNZ}) \cdot (\text{CVZ}) \subseteq \text{CVZ}$,
and so $(\text{BNZ}) \cdot (\text{CVZ}) \cdot (|\bullet|^{j-1}) \subseteq (\text{CVZ}) \cdot (|\bullet|^{j-1})$.
Then $\widehat{\mathcal{O}}_j = (\text{BNZ}) \cdot (|\bullet|^j)$
$= (\text{BNZ}) \cdot (|\bullet|) \cdot (|\bullet|^{j-1})$
$\in (\text{BNZ}) \cdot (\text{CVZ}) \cdot (|\bullet|^{j-1})$
$\subseteq (\text{CVZ}) \cdot (|\bullet|^{j-1}) = o_{j-1}$, as desired. □

We define $0.5 = 1/2$. For any $f : \mathbb{R} \dashrightarrow \mathbb{R}$, for any $j \in \mathbb{N}_0$, we define $f^{n+0.5} : \mathbb{R} \dashrightarrow \mathbb{R}$ by $f^{n+0.5}(x) = [f^n(x)] \cdot [\sqrt{f(x)}]$.

We displayed all the $\widehat{\mathcal{O}}_j$s and all the o_js in a chain:

$$\widehat{\mathcal{O}}_0 \supseteq o_0 \supseteq \widehat{\mathcal{O}}_1 \supseteq o_1 \supseteq \widehat{\mathcal{O}}_2 \supseteq o_2 \supseteq \widehat{\mathcal{O}}_3 \supseteq o_3 \supseteq \widehat{\mathcal{O}}_4 \supseteq o_4 \supseteq \cdots.$$

Elements of $\widehat{\mathcal{O}}_0$ are said to have **constant order**.
Elements of o_0 are said to be **subconstant**.
Elements of $\widehat{\mathcal{O}}_1$ are said to have **linear order**.
Elements of o_1 are said to be **sublinear**.
Elements of $\widehat{\mathcal{O}}_2$ are said to have **quadratic order**.
Elements of o_2 are said to be **subquadratic**.
Elements of $\widehat{\mathcal{O}}_3$ are said to have **cubic order**.
Elements of o_3 are said to be **subcubic**.
Elements of $\widehat{\mathcal{O}}_4$ are said to have **quartic order**.
Elements of o_4 are said to be **subquartic**.

We have: $|\bullet|^0$ has constant order, but is not subconstant.
We have: $|\bullet|^{0.5}$ is subconstant, but is not of linear order.
We have: $|\bullet|^1$ has linear order, but is not sublinear.
We have: $|\bullet|^{1.5}$ is sublinear, but is not of quadratic order.
We have: $|\bullet|^2$ has quadratic order, but is not subquadratic.
We have: $|\bullet|^{2.5}$ is subquadratic, but is not of cubic order.
We have: $|\bullet|^3$ has cubic order, but is not subcubic.

We have: $|\bullet|^{3.5}$ is subcubic, but is not of quartic order.
We have: $|\bullet|^4$ has quartic order, but is not subquartic.

THEOREM 68.17. *Let $j, k \in \mathbb{N}_0$. Then $\mathcal{O}_k \cdot \widehat{\mathcal{O}}_j = \widehat{\mathcal{O}}_j \cdot \mathcal{O}_k \subseteq \mathcal{O}_{j+k}$.*

Proof. By commutativity of multiplication, $\mathcal{O}_k \cdot \widehat{\mathcal{O}}_j = \widehat{\mathcal{O}}_j \cdot \mathcal{O}_k$.
It remains to show: $\widehat{\mathcal{O}}_j \cdot \mathcal{O}_k \subseteq \mathcal{O}_{j+k}$.
By HW#1-4, we have (BNZ) $\cdot$ (CVZ) $\subseteq$ CVZ.
Then $\widehat{\mathcal{O}}_j \cdot \mathcal{O}_k = (\text{BNZ}) \cdot (|\bullet|^j) \cdot (\text{CVZ}) \cdot (|\bullet|^k)$
$\subseteq (\text{CVZ}) \cdot (|\bullet|^{j+k}) = \mathcal{O}_{j+k}$, as desired. ☐

THEOREM 68.18. *Let $j, k \in \mathbb{N}_0$. Then $\mathcal{O}_j \cdot \mathcal{O}_k \subseteq \mathcal{O}_{j+k}$.*

Proof. By HW#1-5, we have $\mathcal{O}_j \subseteq \widehat{\mathcal{O}}_j$. Then $\mathcal{O}_j \cdot \mathcal{O}_k \subseteq \widehat{\mathcal{O}}_j \cdot \mathcal{O}_k$.
By Theorem 68.17, we have $\widehat{\mathcal{O}}_j \cdot \mathcal{O}_k \subseteq \mathcal{O}_{j+k}$.
Then $\mathcal{O}_j \cdot \mathcal{O}_k \subseteq \widehat{\mathcal{O}}_j \cdot \mathcal{O}_k \subseteq \mathcal{O}_{j+k}$, as desired. ☐

THEOREM 68.19. *Let $j, k \in \mathbb{N}_0$. Then $\widehat{\mathcal{O}}_j \cdot \widehat{\mathcal{O}}_k \subseteq \widehat{\mathcal{O}}_{j+k}$.*

Proof. By HW#1-3, we have (BNZ) $\cdot$ (BNZ) $\subseteq$ BNZ.
Then $\widehat{\mathcal{O}}_j \cdot \widehat{\mathcal{O}}_k = (\text{BNZ}) \cdot (|\bullet|^j) \cdot (\text{BNZ}) \cdot (|\bullet|^k)$
$\subseteq (\text{BNZ}) \cdot (|\bullet|^{j+k}) = \widehat{\mathcal{O}}_{j+k}$, as desired. ☐

Recall: $\forall j \in \mathbb{N}$, $\forall P \in \mathcal{H}_j$, $\exists c \in \mathbb{R}$ s.t., $\forall x \in \mathbb{R}$, $P(x) = cx^j$.
Also, $\forall j \in \mathbb{N}$, elements of $\mathcal{H}_j$ are said to be homogeneous polynomials of degree j.
Recall: $\mathcal{C} = \mathcal{H}_0$, $\mathcal{L} = \mathcal{H}_1$, $\mathcal{Q} = \mathcal{H}_2$, $\mathcal{K} = \mathcal{H}_3$.
Elements of $\mathcal{C}$ are said to be **constant**.
Elements of $\mathcal{L}$ are said to be **(homogeneous) linear**.
Elements of $\mathcal{Q}$ are said to be **(homogeneous) quadratic**.
Elements of $\mathcal{K}$ are said to be **(homogeneous) cubic**.

THEOREM 68.20. *Let $L \in \mathcal{L}$. Then $L(1) = \text{s}\ell_L$.*

Proof. Since $L \in \mathcal{L}$, choose $m \in \mathbb{R}$ s.t., $\forall x \in \mathbb{R}$, $L(x) = mx$.
Then $\text{s}\ell_L = m$. Then $L(1) = m \cdot 1 = m = \text{s}\ell_l$, as desired. ☐

NOTE: Next year, we'll just use $L(1)$ as the definition of $\text{s}\ell_L$.
We recalled the definition of f_p^T, see Definition 66.2.
We recalled the definition of $\text{LINS}_p f$, see Definition 66.6.
We recalled the definition of f', see Definition 66.11.
The next theorem asserts: "Any linear function $\mathbb{R} \to \mathbb{R}$ has linear order."

THEOREM 68.21. $\mathcal{L} \subseteq \widehat{\mathcal{O}}_1$.

Proof. Want: $\forall L \in \mathcal{L}$, $L \in \widehat{\mathcal{O}}_1$.
Given $L \in \mathcal{L}$. Want: $L \in \widehat{\mathcal{O}}_1$. Want: $L \in (\text{BNZ}) \cdot (|\bullet|)$.
Let $\gamma := \text{adj}_0^j \left(\frac{L}{|\bullet|} \right)$. Want: $\gamma \in \text{BNZ}$ and $L = \gamma \cdot (|\bullet|)$.
Know: $\forall x \in \mathbb{R}_0^\times$, $\gamma(x) = \left(\frac{L}{|\bullet} \right)(x)$. Also, $\gamma(0) = 0$.
Also, $\text{dom}[\gamma] = ((\text{dom}[L])_0^\times)_0^+ = (\mathbb{R}_0^\times)_0^+ = \mathbb{R}$, so $B_\mathbb{R}(0,1) \subseteq \text{dom}[\gamma]$, so $\gamma \in \text{DNZ}$.
Since $L \in \mathcal{L}$, choose $m \in \mathbb{R}$ s.t., $\forall x \in \mathbb{R}$, $L(x) = mx$.
Let $a := |m|$. Then $0 \leqslant a$.
Know: $\forall x \in \mathbb{R}_0^\times$,

$$|\gamma(x)| = \left| \left(\frac{L}{|\bullet|} \right)(x) \right| = \left| \frac{L(x)}{|x|} \right| = \frac{|L(x)|}{|x|} = \frac{|m| \cdot |x|}{|x|} = |m| = a.$$

So, since $|\gamma(0)| = |0| = 0 \leqslant a$, we conclude: $\forall x \in \mathbb{R}$, $|\gamma(x)| \leqslant a$.
Then, $\forall x \in \mathbb{R}$, we have: $\gamma(x) \in [-a; a]$.
Then $\text{im}[\gamma] \subseteq [-a; a]$.
Since $\gamma_*(B_\mathbb{R}(0,1)) \subseteq \text{im}[\gamma] \subseteq [-a; a]$,
and since $[-a; a]$ is bounded in $\mathbb{R}$,
we conclude that $\gamma_*(B_\mathbb{R}(0,1))$ is bounded in $\mathbb{R}$.
So, since $\gamma \in \text{DNZ}$, we get $\gamma \in \text{BNZ}$.
Want: $L = \gamma \cdot (|\bullet|)$. Want: $\forall x \in \mathbb{R}$, $L(x) = (\gamma \cdot (|\bullet|))(x)$.
Given $x \in \mathbb{R}$. Want: $L(x) = (\gamma \cdot (|\bullet|))(x)$.
Want: $[\gamma(x)] \cdot [|x|] = L(x)$.
One of the following is true:
(1) $x = 0$ or (2) $x \neq 0$.

Case (1):
We have $[\gamma(x)] \cdot [|x|] = [\alpha(0)] \cdot [|0|] = 0 \cdot 0 = 0 = m \cdot 0 = L(0) = L(x)$.
End of Case (1).

Case (2):
We have $[\gamma(x)] \cdot [|x|] = \left[\left(\frac{L}{|\bullet|} \right)(x) \right] \cdot [|x|] = \frac{L(x)}{|x|} \cdot |x| = L(x)$.
End of Case (2). □

THEOREM 68.22. $\forall k \in \mathbb{N}_0$, $\mathcal{H}_k \subseteq \widehat{\mathcal{O}}_k$.

Proof. Unassigned HW. □

THEOREM 68.23. $(\mathcal{C} \subseteq \widehat{\mathcal{O}}_0)$ & $(\mathcal{L} \subseteq \widehat{\mathcal{O}}_1 \subseteq \mathcal{o}_0)$
& $(\mathcal{Q} \subseteq \widehat{\mathcal{O}}_2 \subseteq \mathcal{o}_1)$ & $(\mathcal{K} \subseteq \widehat{\mathcal{O}}_3 \subseteq \mathcal{o}_2)$
& $(\forall k \in \mathbb{N},\ \mathcal{H}_k \subseteq \widehat{\mathcal{O}}_k \subseteq \mathcal{o}_{k-1})$.

Proof. Unassigned HW. □

We recalled Theorem 65.17.
To summarize the preceding discussion:
Any constant has constant order.
However, any nonzero constant is not subconstant.
Any linear function has linear order (hence is subconstant).
However, any nonzero linear function is not sublinear.
Any quadratic function has quadratic order (hence is sublinear).
However, any nonzero quadratic function is not subquadratic.
Any cubic function has cubic order (hence is subquadratic).
However, any nonzero cubic function is not subcubic.
Etc.

THEOREM 68.24. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathrm{dom}[f']$.
Then $\exists L \in \mathrm{LINS}_p f$, $\exists R \in \mathcal{o}_1$ *s.t.* $(L + R = f_p^T)\,\&\,(f'_p = \mathrm{s}\ell_L)$.

Proof. Since $p \in \mathrm{dom}[f']$, we get $f'_p \neq ☹$.
So, since $f'_p = \mathrm{UE}\{\mathrm{s}\ell_M \,|\, M \in \mathrm{LINS}_p f\}$, we get: $\mathrm{LINS}_p f \neq ☹$.
Choose $L \in \mathrm{LINS}_p f$. Then, by Theorem 66.8, $\mathrm{LINS}_p f = \{L\}$.
Then $f'_p = \mathrm{UE}\{\mathrm{s}\ell_L\} = \mathrm{s}\ell_L$. Want: $L + R = f_p^T$.
Since $L \in \mathcal{L}$, we see that $L - L = \mathbf{0}$.
Then $L + R = L + f_p^T - L = f_p^T + \mathbf{0} = f_p^T$, as desired. □

THEOREM 68.25. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathrm{dom}[f']$.
Then $\exists L \in \mathrm{LINS}_p f$, $\exists R \in \mathcal{o}_1$ *s.t.* $[\,(\,L + R = f_p^T\,)\,\&\,(\,f'_p = \mathrm{s}\ell_L\,)\,]$.

Proof. Since $p \in \mathrm{dom}[f']$, we get $f'_p \neq ☹$.
Then $\mathrm{UE}\{\mathrm{s}\ell_M \,|\, M \in \mathrm{LINS}_p f\} = f'_p \neq ☹$, and so $\mathrm{LINS}_p f \neq \varnothing$.
Choose $L \in \mathrm{LINS}_p f$. Let $R := f_p^T - L$.
Then, by definition of $\mathrm{LINS}_p f$, we have: $R \in \mathcal{o}_1$.
Want: $(\,L + R = f_p^T\,)\,\&\,(\,f'_p = \mathrm{s}\ell_L\,)$.
By Theorem 66.8, $\mathrm{LINS}_p f = \{L\}$.
Then $\{\mathrm{s}\ell_M \,|\, M \in \mathrm{LINS}_p f\} = \{\mathrm{s}\ell_L\}$.
Then $f'_p = \mathrm{UE}\{\mathrm{s}\ell_M \,|\, M \in \mathrm{LINS}_p f\} = \mathrm{UE}\{\mathrm{s}\ell_L\} = \mathrm{s}\ell_L$.
Want: $L + R = f_p^T$.

Since $L \in \text{LINS}_p f \subseteq \mathcal{L}$, we get $L - L = \mathbf{0}$.
Then $L + R = L + (f_p^T - L) = f_p^T + \mathbf{0} = f_p^T$, as desired. □

THEOREM 68.26. *Let $L \in \mathcal{L}$. Then L is continuous.*

Proof. Unassigned HW. □

THEOREM 68.27. *Let $k \in \mathbb{N}_0$, $R \in \mathcal{O}_k$. Then R is continuous at 0.*

Proof. By Theorem 68.10, $\mathcal{O}_0 = \text{CVZ}$.
We have $R \in \mathcal{O}_k \subseteq \mathcal{O}_0 = \text{CVZ}$,
and so R is continuous at 0. □

THEOREM 68.28. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$. Then*
$$(\, f \text{ is continuous at } p \,) \Leftrightarrow (\, f_p^T \text{ is continuous at } 0 \,).$$

Proof. Unassigned HW. □

THEOREM 68.29. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \text{dom}[f']$. Then f is continuous at p.*

Proof. By Theorem 68.25, choose $L \in \mathcal{L}$ and $R \in \mathcal{O}_1$ s.t.
$$(\, f_p^T = L + R \,) \,\&\, (\, f_p' = \text{s}\ell_L \,).$$
By Theorem 68.26, L is continuous at 0.
By Theorem 68.27, R is continuous at 0.
Then by Theorem 44.22, $L + R$ is continuous at 0.
So, since $f_p^T = L + R$, we conclude that f_p^T is continuous at 0.
Then, by $\Leftarrow$ of Theorem 68.28, f is continuous at p, as desired. □

THEOREM 68.30. *Let $f : \mathbb{R} \to \mathbb{R}$, $p \in \mathbb{R}$ and $L \in \text{LINS}_p f$.*
$$\textit{Then } f_p' = \text{s}\ell_L.$$

Proof. By Theorem 39.12, $\text{LINS}_p f = \{L\}$.
Then $f_p' = \text{UE}\{\text{s}\ell_M \,|\, M \in \text{LINS}_p f\} = \text{UE}\{\text{s}\ell_L\} = \text{s}\ell_L$. □

69. The Product and Chain Rules

THEOREM 69.1. *Let $L \in \mathcal{L}$, $b \in \mathbb{R}$.*
$$\textit{Then } \text{s}\ell_{Lb} = (\text{s}\ell_l) \cdot b.$$

Proof. We have $\text{s}\ell_{Lb} = (Lb)(1) = [L(1)] \cdot b = (\text{s}\ell_l) \cdot b$, as desired. □

THEOREM 69.2. *Let $L, M \in \mathcal{L}$.*
$$\textit{Then } \text{s}\ell_{L+M} = (\text{s}\ell_l) + (\text{s}\ell_M).$$

Proof. We have $\mathrm{s}\ell_{L+M} = (L + M)(1)$.
$= [L(1)] + [M(1)] = (\mathrm{s}\ell_l) + (\mathrm{s}\ell_M)$, as desired. □

Theorem 69.3, below, is the **High School Product Rule**.

THEOREM 69.3. *Let $a, A, b, B \,\frown\!\in\, \mathbb{R}$.*
Then $AB - ab \;=\; a \cdot [B - b] \,+\, b \cdot [A - a] \,+\, [A - a] \cdot [B - b]$.

Proof. Unassigned HW. □

Theorem 69.4, below, is the **Precalculus Product Rule**.

THEOREM 69.4. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$, $a := f_p$, $b := g_p$.*
Then $(fg)_p^T = a \cdot [g_p^T] + b \cdot [f_p^T] + [f_p^T] \cdot [g_p^T]$.

Proof. Want: $\forall h \in \mathbb{R}$,
$$(fg)_p^T(h) = ((f_p^T) \cdot (g_p) + (f_p) \cdot (g_p^T) + (f_p^T) \cdot (g_p^T))(h).$$
Given $h \in \mathbb{R}$.
Want: $(fg)_p^T(h) = ((f_p^T) \cdot (g_p) + (f_p) \cdot (g_p^T) + (f_p^T) \cdot (g_p^T))(h)$.
Let $A := f_{p+h}$, $B := g_{p+h}$.
Then $A - a = f_{p+h} - f_p = f_p^T(h)$.
Also, $B - b = g_{p+h} - g_p = g_p^T(h)$.
By Theorem 69.3, $AB - ab \;=\; a \cdot [B - b] + b \cdot [A - a] + [A - a] \cdot [B - b]$.
Then $(fg)_p^T(h) = [(fg)_{p+h}] - [(fg)_p] = f_{p+h} \cdot g_{p+h} - f_p \cdot g_p$
$= AB - ab = (A - a)b + a(B - b) + (A - a)(B - b)$
$= [f_p^T(h)] \cdot [g(p)] + [f(p)] \cdot [g_p^T(h)] + [f_p^T(h)] \cdot [g_p^T(h)]$
$= ((f_p^T) \cdot (g_p) + (f_p) \cdot (g_p^T) + (f_p^T) \cdot (g_p^T))(h)$,
as desired. □

Theorem 69.5, below, is the **Product Rule**.

THEOREM 69.5. *Let $f, g \in \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$.*
Then $(fg)'_p \;=^*\; (f'_p) \cdot (g_p) + (f_p) \cdot (g'_p)$.

Proof. Want: $[(f'_p) \cdot (g_p) + (f_p) \cdot (g'_p) \neq ☹] \Rightarrow [(fg)'_p = (f'_p) \cdot (g_p) + (f_p) \cdot (g'_p)]$.
Assume: $(f'_p) \cdot (g_p) + (f_p) \cdot (g'_p) \neq ☹$.
Want: $(fg)'_p = (f'_p) \cdot (g_p) + (f_p) \cdot (g'_p)$.
Since $[(f'_p) \cdot (g_p) + (f_p) \cdot (g'_p) \neq ☹]$,
we get: $f_p \neq ☹ \neq g_p$ and $f'_p \neq ☹ \neq g'_p$.
Since $f'_p \neq ☹$, we see that $p \in \mathrm{dom}[f']$.
Then, by Theorem 68.25, choose $L \in \mathrm{LINS}_p f$ and $R \in \mathcal{O}_1$ s.t.
$f_p^T = L + R$ and $f'_p = \mathrm{s}\ell_L$.
Since $g'_p \neq ☹$, we see that $p \in \mathrm{dom}[g']$.

Then, by Theorem 68.25, choose $M \in \mathrm{LINS}_p f$ and $S \in \mathcal{O}_1$ s.t.

$$g_p^T = M + S \qquad \text{and} \qquad g'_p = \mathrm{s}\ell_M.$$

Let $a := f_p$ and $b := g_p$.

By the Precalculus Product Rule (Theorem 69.4),

we have: $(fg)_p^T = a \cdot [g_p^T] + b \cdot [f_p^T] + [f_p^T] \cdot [g_p^T]$.

Then $(fg)_p^T = a \cdot [M + S] + b \cdot [L + R] + [L + R] \cdot [M + S]$

$= (b \cdot L + a \cdot M) + (a \cdot S + b \cdot R + [L + R] \cdot [M + S])$.

We have $b \cdot L + a \cdot M \in \mathcal{L}$. Then $(b \cdot L + a \cdot M) - (b \cdot L + a \cdot M) = \mathbf{0}$.

Then $(fg)_p^T - (b \cdot L + a \cdot M) = a \cdot S + b \cdot R + [L + R] \cdot [M + S]$.

Since $L + R, M + S \in \mathcal{L} + \mathcal{O}_1 \subseteq \widehat{\mathcal{O}}_1 + \widehat{\mathcal{O}}_1 \subseteq \widehat{\mathcal{O}}_1$,

we get: $[L + R] \cdot [M + S] \in \widehat{\mathcal{O}}_1 \cdot \widehat{\mathcal{O}}_1 \subseteq \widehat{\mathcal{O}}_2$.

Then $(fg)_p^T - (b \cdot L + a \cdot M) \in \mathbb{R} \cdot \mathcal{O}_1 + \mathbb{R} \cdot \mathcal{O}_1 + \widehat{\mathcal{O}}_2$

$\subseteq \mathcal{O}_1 + \mathcal{O}_1 + \mathcal{O}_1 \subseteq \mathcal{O}_1$.

So, since $b \cdot L + a \cdot M \in \mathcal{L}$, we see that $b \cdot L + a \cdot M \in \mathrm{LINS}_p(fg)$.

Then: $(fg)'_p = \mathrm{s}\ell_{b \cdot L + a \cdot M} = b \cdot (\mathrm{s}\ell_L) + a \cdot (\mathrm{s}\ell_M)$

$= (\mathrm{s}\ell_L) \cdot b + a \cdot (\mathrm{s}\ell_M) = (f'_p) \cdot (g_p) + (f_p) \cdot (g'_p)$, as desired. □

DEFINITION 69.6. *Let X, Y and Z be sets.*
Let $\mathcal{S} \subseteq \mathcal{PF}_{XY}$, $\mathcal{T} \subseteq \mathcal{PF}_{YZ}$. Then $\mathcal{T} \circ \mathcal{S} := \{g \circ f \,|\, g \in \mathcal{T},\ f \in \mathcal{S}\}$.

DEFINITION 69.7. *Let X, Y and Z be sets.*
Let $f \in \mathcal{PF}_{XY}$, $\mathcal{T} \subseteq \mathcal{PF}_{YZ}$. Then $\mathcal{T} \circ f := \mathcal{T} \circ (\{f\})$.

DEFINITION 69.8. *Let X, Y and Z be sets.*
Let $\mathcal{S} \subseteq \mathcal{PF}_{XY}$, $g \in \mathcal{PF}_{YZ}$. Then $g \circ \mathcal{S} := (\{g\}) \circ \mathcal{S}$.

THEOREM 69.9. (CVZ) $\circ$ (CVZ) $\subseteq$ CVZ
and (BNZ) $\circ$ (CVZ) $\subseteq$ BNZ.

Proof. Unassigned HW: (CVZ) $\circ$ (CVZ) $\subseteq$ CVZ.

Want: (BNZ) $\circ$ (CVZ) $\subseteq$ BNZ.

By HW#2-2, (BNZ) $\circ$ (CVZ) $\subseteq$ BNZ, as desired. □

DEFINITION 69.10. *Let X be a set and let $f : X \dashrightarrow \mathbb{R}$.*
Then $|f| : X \dashrightarrow \mathbb{R}$ is defined by $(|f|)(x) = |f(x)|$.

DEFINITION 69.11. *Let X be a set and let $\mathcal{S} \subseteq \mathcal{PF}_{X\mathbb{R}}$.*
Then $|\mathcal{S}| := \{|f| \text{ s.t. } f \in \mathcal{S}\}$.

THEOREM 69.12. $|\mathrm{BNZ}| \subseteq \mathrm{BNZ}$ *and* $|\mathrm{CVZ}| \subseteq \mathrm{CVZ}$.

Proof. Unassigned HW: $|\mathrm{BNZ}| \subseteq \mathrm{BNZ}$.

Want: $|\mathrm{CVZ}| \subseteq \mathrm{CVZ}$.

Unassigned HW: $|\bullet| \in$ CVZ. By Theorem 69.9, (CVZ)$\circ$(CVZ) $\subseteq$ CVZ.
Then $|$CVZ$| = (|\bullet|) \circ$ (CVZ) $\subseteq$ (CVZ) $\circ$ (CVZ) $\subseteq$ CVZ, as desired. □

THEOREM 69.13. *Let $\psi, \psi : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $j, k \in \mathbb{N}_0$.*
Let $\beta := \phi \cdot (|\bullet|^j)$ and let $\gamma := \psi \cdot (|\bullet|^k)$.
Then $\beta \circ \gamma = (\phi \circ \gamma) \cdot (|\psi|^j) \cdot (|\bullet|^{jk})$.

Proof. Want: $\forall x \in \mathbb{R}$, $(\beta \circ \gamma)(x) = ((\phi \circ \gamma) \cdot (|\psi|^j) \cdot (|\bullet|^{jk}))(x)$.
Given $x \in \mathbb{R}$. Want: $(\beta \circ \gamma)(x) = ((\phi \circ \gamma) \cdot (|\psi|^j) \cdot (|\bullet|^{jk}))(x)$.
We compute: $(\beta \circ \gamma)(x) = (\beta(\gamma(x)) = (\phi \cdot (|\bullet|^j))(\gamma(x))$
$= [\phi(\gamma(x))] \cdot [|\ \gamma(x)\ |^j]$
$= [(\phi \circ \gamma)(x)] \cdot [|\ (\psi \cdot (|\bullet|^k))(x)\ |^j]$
$= [(\phi \circ \gamma)(x)] \cdot [|\ [\psi(x)] \cdot [|x|^k]\ |^j]$
$= [(\phi \circ \gamma)(x)] \cdot [|\psi(x)|] \cdot [|x|^{jk}]$
$= ((\phi \circ \gamma) \cdot (|\psi|^j) \cdot (|\bullet|^{jk}))(x)$, as desired. □

THEOREM 69.14. *Let $\alpha \in$ BNZ, $j \in \mathbb{N}_0$. Then $\alpha^j \in$ BNZ.*

Proof. Unassigned HW. □

THEOREM 69.15. *Let $\alpha \in$ CVZ, $j \in \mathbb{N}$. Then $\alpha^j \in$ CVZ.*

Proof. Unassigned HW. □

THEOREM 69.16. *Let $j, k \in \mathbb{N}$.*
Then $\mathcal{O}_j \circ \widehat{\mathcal{O}}_k \subseteq \mathcal{O}_{jk}$.

Proof. Want: $\forall \alpha \in \mathcal{O}_j \circ \widehat{\mathcal{O}}_k$, $\alpha \in \mathcal{O}_{jk}$.
Given $\alpha \in \mathcal{O}_j \circ \widehat{\mathcal{O}}_k$. Want: $\alpha \in \mathcal{O}_{jk}$.
Since $\alpha \in \mathcal{O}_j \circ \widehat{\mathcal{O}}_k$, choose $\beta \in \mathcal{O}_j$, $\gamma \in \widehat{\mathcal{O}}_k$ s.t. $\alpha = \beta \circ \gamma$.
Want: $\beta \circ \gamma \in \mathcal{O}_{jk}$.
Since $\beta \in \mathcal{O}_j =$ (CVZ) $\cdot (|\bullet|^j)$, choose $\phi \in$ CVZ s.t. $\beta = \phi \cdot (|\bullet|^j)$.
Since $\gamma \in \widehat{\mathcal{O}}_k =$ (BNZ) $\cdot (|\bullet|^k)$, choose $\psi \in$ BNZ s.t. $\gamma = \psi \cdot (|\bullet|^k)$.
By Theorem 69.13, $\beta \circ \gamma = (\phi \circ \gamma) \cdot (|\psi|^j) \cdot (|\bullet|^{jk})$.
Want: $\beta \circ \gamma \in$ (CVZ) $\cdot (|\bullet|^{jk})$. Want: $(\phi \circ \gamma) \cdot (|\psi|^j) \in$ CVZ.
Since $k \in \mathbb{N}$, we get $\widehat{\mathcal{O}}_k \subseteq \widehat{\mathcal{O}}_1$. By Theorem 68.16, $\widehat{\mathcal{O}}_1 \subseteq \mathcal{O}_0$.
By Theorem 68.10, $\mathcal{O}_0 =$ CVZ. Then $\gamma \in \widehat{\mathcal{O}}_k \subseteq \widehat{\mathcal{O}}_1 \subseteq \mathcal{O}_0 =$ CVZ.
By Theorem 69.9, (CVZ) $\circ$ (CVZ) $\subseteq$ CVZ.
Then $\phi \circ \gamma \in$ (CVZ) $\circ$ (CVZ) $\subseteq$ CVZ.
Also, $|\psi| \in |$BNZ$| \subseteq$ BNZ, so, by Theorem 69.14, $|\psi|^j \in$ BNZ.
By HW#1-4, (BNZ) $\cdot$ (CVZ) $\subseteq$ CVZ.
Then $(\phi \circ \gamma) \cdot (|\psi|^j) \in$ (CVZ) $\cdot$ (BNZ) $=$ (BNZ) $\cdot$ (CVZ) $\subseteq$ BNZ. □

THEOREM 69.17. *Let $j, k \in \mathbb{N}$.*
Then $\widehat{\mathcal{O}}_j \circ \mathcal{o}_k \subseteq \mathcal{o}_{jk}$.

Proof. Unassigned HW. $\square$

THEOREM 69.18. *Let $j, k \in \mathbb{N}$.*
Then $\mathcal{o}_j \circ \mathcal{o}_k \subseteq \mathcal{o}_{jk}$.

Proof. By Theorem 69.16, $\mathcal{o}_j \circ \widehat{\mathcal{O}}_k \subseteq \mathcal{o}_{jk}$.
Also, $\mathcal{o}_k \subseteq \widehat{\mathcal{O}}_k$.
Then $\mathcal{o}_j \circ \mathcal{o}_k \subseteq \mathcal{o}_j \circ \widehat{\mathcal{O}}_k \subseteq \mathcal{o}_{jk}$, as desired. $\square$

THEOREM 69.19. *Let $j, k \in \mathbb{N}$.*
Then $\widehat{\mathcal{O}}_j \circ \widehat{\mathcal{O}}_k \subseteq \widehat{\mathcal{O}}_{jk}$.

Proof. Unassigned HW. $\square$

Theorem 69.20, below, is the **Precalculus Chain Rule**.

THEOREM 69.20. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$, $q := f_p$.*
$$\textit{Then } (g \circ f)_p^T = (g_q^T) \circ (f_p^T).$$

Proof. Want: $\forall h \in \mathbb{R}$, $((g \circ f)_p^T)(h) = ((g_q^T) \circ (f_p^T))(h)$.
Given $h \in \mathbb{R}$. Want: $((g \circ f)_p^T)(h) = ((g_q^T) \circ (f_p^T))(h)$.
We compute $((g \circ f)_p^T)(h) = [(g \circ f)(p + h)] - [(g \circ f)(p)]$
$= [g(f_{p+h})] - [g(f_p)] = [g(f_{p+h})] - [g(q)]$.
Exactly one of the following is true:
(1) $q = ☹$ or (2) $q \neq ☹$.

Case (1):
Since $q = ☹$, we get:
both $[g(f_{p+h})] - [g(q)] = ☹$ and $((g_q^T) \circ (f_p^T))(h) = ☹$.
Then $((g \circ f)_p^T)(h) = [g(f_{p+h})] - [g(q)] = ☹ = ((g_q^T) \circ (f_p^T))(h)$.
End of Case (1).

Case (2):
Since $f_p = q \neq ☹$, we get $f_p \in \mathrm{im}[f] \subseteq \mathbb{R}$.
Then $q = f_p \in \mathbb{R}$, so $q - q = 0$.
Let $k := (f_p^T)(h)$. Then $k = f_{p+h} - f_p = f_{p+h} - q$.
Then $q + k = q + (f_{p+h} - q) = f_{p+h} + (q - q) = f_{p+h} + 0 = f_{p+h}$.
Then $((g \circ f)_p^T)(h) = [g(f_{p+h})] - [g(q)] = [g(q + k)] - [g(q)]$
$= g_q^T(k) = (g_q^T)((f_p^T)(h)) = ((g_q^T) \circ (f_p^T))(h)$,
as desired. *End of Case (2).* $\square$

THEOREM 69.21. *Let $M \in \mathcal{L}$, $x \in \mathbb{R}$.*
Then $M(x) = (\mathrm{s}\ell_M) \cdot x$.

Proof. Since $M \in \mathcal{L}$, we get $M(x \cdot 1) = x \cdot [M(1)]$.
Then $M(x) = M(x \cdot 1) = x \cdot [M(1)] = x \cdot (\mathrm{s}\ell_M) = (\mathrm{s}\ell_M) \cdot x$. □

THEOREM 69.22. *Let $L, M \in \mathcal{L}$.*
Then $\mathrm{s}\ell_{M \circ L} = (\mathrm{s}\ell_M) \cdot (\mathrm{s}\ell_L)$.

Proof. Let $x := \mathrm{s}\ell_L$. Then $x = L(1)$.
Also, by Theorem 69.21, we have $M(x) = (\mathrm{s}\ell_M) \cdot x$.
Then $\mathrm{s}\ell_{M \circ L} = (M \circ L)(1) = M(L(1))$
$= M(x) = (\mathrm{s}\ell_M) \cdot x = (\mathrm{s}\ell_M) \cdot (\mathrm{s}\ell_L)$, as desired. □

THEOREM 69.23. *Let f and g be functionals and let h be a function.*
Then $(f + g) \circ h = (f \circ h) + (g \circ h)$.

Proof. Want: $\forall x$, $((f + g) \circ h)(x) = ((f \circ h) + (g \circ h))(x)$.
Given x. Want: $((f + g) \circ h)(x) = ((f \circ h) + (g \circ h))(x)$.
We have $((f + g) \circ h)(x) = (f + g)(h_x) = f(h_x) + g(h_x)$
$= [(f \circ h)(x)] + [(g \circ h)(x)] = ((f \circ h) + (g \circ h))(x)$. □

Theorem 69.23 asserts: "$\circ$ is additive on the left."
WARNING: $\circ$ is not additive on the right.
Let $f := \mathrm{id}_{\mathbb{R}}$, $g := \mathrm{id}_{\mathbb{R}}$ and $h := (\mathrm{id}_{\mathbb{R}})^2$.
Then $h \circ (f + g) = (f + g)^2 = f^2 + 2fg + g^2 \neq f^2 + g^2 = (h \circ f) + (h \circ g)$.
However, if we replace h by a homogeneous linear function, then we do get additivity on the right:

THEOREM 69.24. *Let $L \in \mathcal{L}$. Let f and g be functionals.*
Then $L \circ (f + g) = (L \circ f) + (L \circ g)$.

Proof. Want: $\forall x$, $(L \circ (f + g))(x) = ((L \circ f) + (L \circ g))(x)$.
Given x. Want: $(L \circ (f + g))(x) = ((L \circ f) + (L \circ g))(x)$.
Since $L \in \mathcal{L}$, we get: $L(f_x + g_x) = [L(f_x)] + [L(g_x)]$.
We have $(L \circ (f + g))(x) = L((f + g)_x) = L(f_x + g_x) = [L(f_x)] + [L(g_x)]$
$= [(L \circ f)(x)] + [(L \circ g)(x)] = ((L \circ f) + (L \circ g))(x)$. □

Theorem 69.25, below, is the **Chain Rule**.

THEOREM 69.25. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathbb{R}$, $q := f_p$.*
Then $(g \circ f)'_p \;=^*\; (g'_q) \cdot (f'_p)$.

Proof. Want: $[(g'_q)\cdot(f'_p) \neq ☹] \Rightarrow [(g\circ f)'_p = (g'_q)\cdot(f'_p)]$.
Assume: $(g'_q)\cdot(f'_p) \neq ☹$. Want: $(g\circ f)'_p = (g'_q)\cdot(f'_p)$.
Since $(g'_q)\cdot(f'_p) \neq ☹$, we see:

both $g'_q \neq ☹$ and $f'_p \neq ☹$.

Since $f'_p \neq ☹$, we get $p \in \mathrm{dom}[f']$.
Then, by Theorem 68.25, choose $L \in \mathrm{LINS}_p f$ and $R \in \mathcal{O}_1$ s.t.

$$f_p^T = L + R \qquad \text{and} \qquad f'_p = \mathrm{s}\ell_L.$$

Since $g'_q \neq ☹$, we get $q \in \mathrm{dom}[g']$.
Then, by Theorem 68.25, choose $M \in \mathrm{LINS}_q g$ and $S \in \mathcal{O}_1$ s.t.

$$g_q^T = M + S \qquad \text{and} \qquad g'_q = \mathrm{s}\ell_M.$$

By the Precalculus Chain Rule (Theorem 69.20), $(g\circ f)_p^T = (g_q^T)\circ(f_p^T)$.
Then $(g\circ f)_p^T = (g_q^T)\circ(f_p^T) = (M+S)\circ(L+R)$
$= [M\circ(L+R)] + [S\circ(L+R)]$
$= [M\circ L] + [M\circ R] + [S\circ(L+R)]$.
Since $L, M \in \mathcal{L}$, we get $M\circ L \in \mathcal{L}$.
Then $(M\circ L) - (M\circ L) = \mathbf{0}$.
Then $(g\circ f)_p^T - (M\circ L) = [M\circ R] + [S\circ(L+R)]$.
We have $L + R \in \mathcal{L} + \mathcal{O}_1 \subseteq \widehat{\mathcal{O}}_1 + \widehat{\mathcal{O}}_1 \subseteq \widehat{\mathcal{O}}_1$. Also, $M \in \mathcal{L} \subseteq \widehat{\mathcal{O}}_1$.
Then $(g\circ f)_p^T - (M\circ L) \in [\widehat{\mathcal{O}}_1 \circ \mathcal{O}_1] + [\mathcal{O}_1 \circ \widehat{\mathcal{O}}_1] \subseteq \mathcal{O}_1 + \mathcal{O}_1 \subseteq \mathcal{O}_1$.
Then $M\circ L \in \mathrm{LINS}_p(g\circ f)$.
Then $(g\circ f)'_p = \mathrm{s}\ell_{M\circ L} = (\mathrm{s}\ell_M)\cdot(\mathrm{s}\ell_L) = (g'_q)\cdot(f'_p)$, as desired. □

70. The Quotient Rule

THEOREM 70.1. *Define $r:\mathbb{R}\to\mathbb{R}$ by $r_x = 1/x$.*
Then, $\forall x\in\mathbb{R}$, $r'_x = -1/(x^2)$.

Proof. Unassigned HW. See HW#3-2. □

Theorem 70.2, below, is the **Quotient Rule**.

THEOREM 70.2. *Let $f, g:\mathbb{R}\to\mathbb{R}$ and let $x\in\mathbb{R}$.*
Then $\displaystyle\left(\frac{f}{g}\right)'_x =^* \frac{(g_x)\cdot(f'_x) - (f_x)\cdot(g'_x)}{(g_x)^2}$.

Proof. Define $r:\mathbb{R}\to\mathbb{R}$ by $r_x = 1/x$. Let $y := g_x$.
By Theorem 69.25, we have: $(r\circ g)'_x =^* (r'_y)\cdot(g'_x)$.
By Theorem 70.1, we have: $r'_y = -1/(y^2)$.
Then $\displaystyle\left(\frac{1}{g}\right)'_x = (r\circ g)'_x =^* (r'_y)\cdot(g'_x)$

$$= \left(-\frac{1}{y^2}\right) \cdot (g'_x) = -\frac{g'_x}{y^2} = -\frac{g'_x}{(g_x)^2}.$$

By Theorem 69.5, $\left(f \cdot \left(\frac{1}{g}\right)\right)'_x =^* (f'_x) \cdot \left(\left(\frac{1}{g}\right)_x\right) + (f_x) \cdot \left(\left(\frac{1}{g}\right)'_x\right)$.

Then $\left(\frac{f}{g}\right)'_x = \left(f \cdot \left(\frac{1}{g}\right)\right)'_x =^* (f'_x) \cdot \left(\left(\frac{1}{g}\right)_x\right) + (f_x) \cdot \left(\left(\frac{1}{g}\right)'_x\right)$

$$=^* \frac{f'_x}{g_x} + (f_x) \cdot \left(-\frac{g'_x}{(g_x)^2}\right) = \frac{(g_x) \cdot (f'_x) - (f_x) \cdot (g'_x)}{(g_x)^2}.$$ □

71. Fermat's Theorem

DEFINITION 71.1. *Let f and g be functionals and let S be a set.*
Then $f < g$ on S means: $\forall x \in S$, $f(x) < g(x)$.
Also, $f \leqslant g$ on S means: $\forall x \in S$, $f(x) \leqslant g(x)$.
Then $f > g$ on S means: $\forall x \in S$, $f(x) > g(x)$.
Also, $f \geqslant g$ on S means: $\forall x \in S$, $f(x) \geqslant g(x)$.
Then $f = g$ on S means: $\forall x \in S$, $f(x) = g(x)$.

Keep in mind Axiom 10.25. From that axiom,
we get: $\forall y$, $\neg(☹ < y)$. Also, $\forall y$, $\neg(y < ☹)$.
Also, $\forall y$, $\neg(☹ < y)$. Also, $\forall y$, $\neg(☹ > y)$.
However, $☹ = ☹$, so both $☹ \leqslant ☹$ and $☹ \geqslant ☹$.

DEFINITION 71.2. *Let f be a functional, $a \in \mathbb{R}$, S a set.*
Then $f < a$ on S means: $f < C_S^a$ on S.
Also, $f \leqslant a$ on S means: $f \leqslant C_S^a$ on S.
Then $f > a$ on S means: $f > C_S^a$ on S.
Also, $f \geqslant a$ on S means: $f \geqslant C_S^a$ on S.
Also, $f = a$ on S means: $f = C_S^a$ on S.
Then $a < f$ on S means: $C_S^a < f$ on S.
Also, $a \leqslant f$ on S means: $C_S^a \leqslant f$ on S.
Then $a > f$ on S means: $C_S^a > f$ on S.
Also, $a \geqslant f$ on S means: $C_S^a \geqslant f$ on S.
Also, $a = f$ on S means: $C_S^a = f$ on S.

THEOREM 71.3. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $a \in \mathbb{R}_0^\times$.*
Then $(af)' = a \cdot [f']$.

Proof. Want: $\forall p \in \mathbb{R}$, $(af)'_p = (a \cdot [f'])_p$.
Given $p \in \mathbb{R}$. Want: $(af)'_p = (a \cdot [f'])_p$. Want: $(af)'_p = a \cdot [f'_p]$.

By Theorem 67.24, we have: $(af)'_p \;\; =^* \;\; a \cdot [f'_p]$.
It suffices to show: $(af)'_p \;\; {}^*\!= \;\; a \cdot [f'_p]$.
Let $g := af$ and $b := 1/a$. Then $ba = 1$ and $bg = f$.
By Theorem 67.24, we have: $(bg)'_p \;\; =^* \;\; b \cdot [g'_p]$.
Reversing this, we get: $b \cdot [g'_p] \;\; {}^*\!= \;\; (bg)'_p$.
Multiplying by a, we get: $ab \cdot [g'_p] \;\; {}^*\!= \;\; a \cdot [(bg)'_p]$.
So, since $ab = 1$ and since $bg = f$, we get: $[g'_p] \;\; {}^*\!= \;\; a \cdot [f'_p]$.
So, since $g = af$, we get: $[(af)'_p] \;\; {}^*\!= \;\; a \cdot [f'_p]$, as desired.

□

THEOREM 71.4. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \mathrm{dom}[f']$. Assume $f'_p > 0$. Then $\exists \delta > 0$ s.t. (1) $f > f_p$ on $(p; p + \delta)$ and (2) $f < f_p$ on $(p - \delta; p)$.*

Proof. Since $p \in \mathrm{dom}[f']$, by Theorem 68.25, choose $L \in \mathrm{LINS}_p f$ and $R \in \mathcal{O}_1$ s.t.

$$f_p^T = L + R \qquad \text{and} \qquad f'_p = \mathrm{s}\ell_L.$$

Since $R \in \mathcal{O}_1$ and $L/2 \in \mathcal{L}$, by HW#2-5,
choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $|R| \leqslant |L/2|$ on B.
Since $B \in \mathcal{B}_{\mathbb{R}}(0)$, choose $\delta > 0$ s.t. $B = B_{\mathbb{R}}(0, \delta)$.
Want: (1) and (2).

Proof of (1):
By HW-2-1, Want: $f_p^T > 0$ on $(0; \delta)$.
We have $\mathrm{s}\ell_L = f'_p > 0$.
Then, $\forall x > 0$, we have $L(x) = (\mathrm{s}\ell_L) \cdot x > 0$.
That is, $L > 0$ on $(0; \infty)$. Then $L > 0$ on $(0; \delta)$.
Then $L/2 > 0$ on $(0; \delta)$. Also, $|L/2| = L/2$ on $(0; \delta)$.
Since $(0; \delta) \subseteq (-\delta; \delta) = B_{\mathbb{R}}(0, \delta) = B$ and since $|R| \leqslant |L/2|$ on B,
we get: $|R| \leqslant |L/2|$ on $(0; \delta)$.
So, as $|L/2| = L/2$ on $(0; \delta)$, we conclude: $|R| \leqslant L/2$ on $(0; \delta)$.
Then $-L/2 \leqslant R \leqslant L/2$ on $(0; \delta)$.
Adding L, this yields: $L/2 \leqslant L + R \leqslant 3L/2$ on $(0; \delta)$.
So, since $L + R = f_p^T$, we get: $L/2 \leqslant f_p^T \leqslant 3L/2$ on $(0; \delta)$.
Then $f_p^T \geqslant L/2 > 0$ on $(0; \delta)$, as desired. *End of proof of (1).*

Proof of (2): Unassigned HW. *End of proof of (2).* □

THEOREM 71.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $p \in \text{dom}[f']$. *Assume* $f'_p < 0$.
Then $\exists \delta > 0$ *s.t.* *(1)* $f < f_p$ *on* $(p; p + \delta)$ *and*
(2) $f > f_p$ *on* $(p - \delta; p)$.

Proof. Let $g := -f$. By Theorem 71.3, $g' = -f'$, so $g'_p = -f'_p$.
By Theorem 71.4, choose $\delta > 0$ s.t.
$g > g_p$ on $(p; p + \delta)$ and
$g < g_p$ on $(p - \delta; p)$.
Want: (1) and (2).
From the choice of δ, we get:
$-g < -g_p$ on $(p; p + \delta)$ and
$-g > -g_p$ on $(p - \delta; p)$.
So, since $-g = f$ and $-g_p = f_p$, we get: (1) and (2), as desired. □

DEFINITION 71.6. *Let* X *be a metric space.*
Let $f : X \dashrightarrow \mathbb{R}$ *and let* $p \in \text{dom}[f]$.
Then f *has a* **local minimum** *at* p *in* X *means:*
$\exists B \in \mathcal{B}_{\mathbb{R}}(p)$ *s.t.* $f \geqslant f_p$ *on* B.
Also, f *has a* **local maximum** *at* p *in* X *means:*
$\exists B \in \mathcal{B}_{\mathbb{R}}(p)$ *s.t.* $f \leqslant f_p$ *on* B.

THEOREM 71.7. *Let* X *be a metric space.*
Let $f : X \dashrightarrow \mathbb{R}$ *and let* $p \in \text{dom}[f]$.
Then: (f *has a local maximum at* p *in* $\mathbb{R}$)
$\Leftrightarrow$ ($-f$ *has a local minimum at* p *in* $\mathbb{R}$).
Also: (f *has a local minimum at* p *in* $\mathbb{R}$)
$\Leftrightarrow$ ($-f$ *has a local maximum at* p *in* $\mathbb{R}$).

Proof. Unassigned HW. □

DEFINITION 71.8. *Let* X *be a metric space.*
Let $f : X \dashrightarrow \mathbb{R}$ *and let* $p \in \text{dom}[f]$.
Then f *has a* **local extremum** *at* p *in* X *means:*
either f *has a local minimum at* p *in* X
or f *has a local maximum at* p *in* X.

THEOREM 71.9. *Let* X *be a metric space.*
Let $f : X \dashrightarrow \mathbb{R}$ *and let* $p \in \text{dom}[f]$.
Assume that f *has a local extremum at* p *in* X.
Then $\exists g \in \{f, -f\}$ *s.t.* g *has a local maximum at* p *in* X.

Proof. Follows from Theorem 71.7. Unassigned HW. □

THEOREM 71.10. *Let $p \in \mathbb{R}$, let $\delta > 0$ and let $B \in \mathcal{B}_{\mathbb{R}}(p)$.*
Then: $B \cap (p-\delta;p) \neq \varnothing \neq B \cap (p;p+\delta)$.

Proof. Since $B \in \mathcal{B}_{\mathbb{R}}(p)$, choose $r > 0$ s.t. $B = B_{\mathbb{R}}(p,r)$.
Let $\alpha := \min\{r,\delta\}$. Then $\alpha > 0$.
Let $y := p - (\alpha/2)$ and $z := p + (\alpha/2)$.
Then $y, z \in B_{\mathbb{R}}(p,\alpha)$ and $y \in (p-\alpha;p)$ and $z \in (p;p+\alpha)$.
Also, since $\alpha < r$, we have $B_{\mathbb{R}}(p,\alpha) \subseteq B_{\mathbb{R}}(p,r)$.
Then $y, z \in B_{\mathbb{R}}(p,\alpha) \subseteq B_{\mathbb{R}}(p,r) = B$.
Also, since $\alpha < \delta$, we have $(p-\alpha;p) \subseteq (p-\delta;p)$.
Then $y \in (p-\alpha;p) \subseteq (p-\delta;p)$.
Also, since $\alpha < \delta$, we have $(p;p+\alpha) \subseteq (p;p+\delta)$.
Then $z \in (p;p+\alpha) \subseteq (p;p+\delta)$.
Since $y \in B$ and $y \in (p-\delta;p)$, we get $y \in B \cap (p-\delta;p)$.
Then $B \cap (p-\delta;p) \neq \varnothing$. Want: $B \cap (p;p+\delta) \neq \varnothing$.
Since $z \in B$ and $z \in (p;p+\delta)$, we get $z \in B \cap (p;p+\delta)$.
Then $B \cap (p,p+\delta) \neq \varnothing$, as desired. □

Theorem 71.11, below, is called **Fermat's Theorem**.

THEOREM 71.11. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathrm{dom}[f']$.*
Assume that f has a local extremum at p in $\mathbb{R}$. *Then* $f'_p = 0$.

Proof. By Theorem 71.9, choose $g \in \{f,-f\}$ s.t.
g has a local maximum at p in $\mathbb{R}$.
Since $g \in \{f,-f\}$, it follows that $f \in \{g,-g\}$.
Then, by Theorem 71.3, $f' \in \{g',-g'\}$.
Then $f'_p \in \{g'_p, -g'_p\}$. Want: $g'_p = 0$.
Assume that $g'_p \neq 0$. Want: Contradiction.
Since g has a local maximum at p in $\mathbb{R}$,
choose $B \in \mathcal{B}_{\mathbb{R}}(p)$ s.t. $g \leqslant g_p$ on B.
Since $g'_p \neq 0$, exactly one of the following must be true:
(1) $g'_p > 0$ or (2) $g'_p < 0$.

Case (1):
By Theorem 71.4, choose $\delta > 0$ s.t.
$g > g_p$ on $(p;p+\delta)$ and
$g < g_p$ on $(p-\delta;p)$.
By Theorem 71.10, $B \cap (p;p+\delta) \neq \varnothing$.

Choose $x \in B \cap (p; p+\delta) \neq \varnothing$. Then $x \in B$ and $x \in (p; p+\delta)$.
Since $x \in B$, by choice of B, we get $g_x \leqslant g_p$.
Since $x \in (p; p+\delta)$, by choice of δ, we get $g_x > g_p$. Then $g_p < g_x$.
Then $g_p < g_x \leqslant g_p$, so $g_p < g_p$. Contradiction.
End of Case (1).

Case (2): Unassigned HW. *End of Case (2).* □

72. The Mean Value Theorem

DEFINITION 72.1. *Let X be a metric space and let Y be a set.*
Let $\lambda, \mu : X \dashrightarrow Y$ and let $p \in X$.
Then $\lambda = \mu$ near p means: $\exists B \in \mathcal{B}_X(p)$ s.t. $\lambda = \mu$ on B.
Also, $\lambda < \mu$ near p means: $\exists B \in \mathcal{B}_X(p)$ s.t. $\lambda < \mu$ on B.
Also, $\lambda \leqslant \mu$ near p means: $\exists B \in \mathcal{B}_X(p)$ s.t. $\lambda \leqslant \mu$ on B.
Also, $\lambda > \mu$ near p means: $\exists B \in \mathcal{B}_X(p)$ s.t. $\lambda > \mu$ on B.
Also, $\lambda \geqslant \mu$ near p means: $\exists B \in \mathcal{B}_X(p)$ s.t. $\lambda \geqslant \mu$ on B.

THEOREM 72.2. *Let $\lambda, \mu : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume $\mu \in \mathrm{DNZ}$. Assume $\lambda = \mu$ near 0.
Then $\lambda \in \mathrm{DNZ}$.

Proof. Unassigned HW. □

THEOREM 72.3. *Let $\lambda, \mu : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume $\mu \in \mathrm{BNZ}$. Assume $\lambda = \mu$ near 0.
Then $\lambda \in \mathrm{BNZ}$.

Proof. Unassigned HW. □

THEOREM 72.4. *Let $\lambda, \mu : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume $\mu \in \mathrm{CVZ}$. Assume $\lambda = \mu$ near 0.
Then $\lambda \in \mathrm{CVZ}$.

Proof. Unassigned HW. □

THEOREM 72.5. *Let $\alpha : \mathbb{R} \dashrightarrow \mathbb{R}$, $k \in \mathbb{N}_0$ and $\lambda := \mathrm{adj}_0^0\left(\dfrac{\alpha}{|\bullet|^k}\right)$.*
Assume $\alpha_0 = 0$. Then $\alpha = \lambda \cdot (|\bullet|^k)$.

Proof. Want: $\forall x \in \mathbb{R}$, $\alpha_x = (\lambda \cdot (|\bullet|^k))_x$.
Given $x \in \mathbb{R}$. Want: $\alpha_x = (\lambda \cdot (|\bullet|^k))_x$. Want: $\alpha_x = (\lambda_x) \cdot (|x|^k)$.

Exactly one of the following is true:

(1) $x = 0$ or (2) $x \neq 0$.

Case (1):

We have $\lambda_x = \lambda_0 = \left(\mathrm{adj}_0^0\left(\frac{\alpha}{|\bullet|^k}\right)\right)_0 = 0$. Also, $|x|^k = |0|^k = 0$.

Then $\alpha_x = \alpha_0 = 0 = 0 \cdot 0 = (\lambda_x) \cdot (|x|^k)$, as desired.

End of Case (1).

Case (2):

We have $\lambda_x = \left(\mathrm{adj}_0^0\left(\frac{\alpha}{|\bullet|^k}\right)\right)_x = \left(\frac{\alpha}{|\bullet|^k}\right)_x = \frac{\alpha_x}{|x|^k}$.

Then $\alpha_x = \left(\frac{\alpha_x}{|x|^k}\right) \cdot (|x|^k) = (\lambda_x) \cdot (|x|^k)$, as desired. *End of Case (2).*

□

THEOREM 72.6. *Let $k \in \mathbb{N}_0$, $\alpha \in \mathrm{DNZ}$ and $\lambda := \mathrm{adj}_0^0\left(\frac{\alpha}{|\bullet|^k}\right)$.*

Then: $\mathrm{dom}[\lambda] = \mathrm{dom}[\alpha]$ *and* $\lambda \in \mathrm{DNZ}$.

Proof. Since $\alpha \in \mathcal{O}_k \subseteq \mathrm{DNZ}$, we see that $\alpha \in \mathrm{DNZ}$. Then $0 \in \mathrm{dom}[\alpha]$.

It follows that $((\mathrm{dom}[\alpha])_0^\times)_0^+ = \mathrm{dom}[\alpha]$.

Then $\mathrm{dom}[\lambda] = \left(\mathrm{dom}\left[\frac{\alpha}{|\bullet|^k}\right]\right)_0^+ = ((\mathrm{dom}[\alpha])_0^\times)_0^+ = \mathrm{dom}[\alpha]$.

It remains to show: $\lambda \in \mathrm{DNZ}$.

Since $\alpha \in \mathrm{DNZ}$ and since $\mathrm{dom}[\lambda] = \mathrm{dom}[\alpha]$, we see that $\lambda \in \mathrm{DNZ}$, as desired.

□

THEOREM 72.7. *Let $k \in \mathbb{N}_0$, $\alpha \in \mathcal{O}_k$ and $\lambda := \mathrm{adj}_0^0\left(\frac{\alpha}{|\bullet|^k}\right)$.*

Then $\lambda \in \mathrm{CVZ}$.

Proof. By Theorem 72.6, we see that $\lambda \in \mathrm{DNZ}$.

Want: $(\lambda_0 = 0)\,\&\,(\lambda$ is continuous at $0)$.

We have $\lambda_0 = \left(\mathrm{adj}_0^0\left(\frac{\alpha}{|\bullet|^k}\right)\right)_0 = 0$. Want: λ is continuous at 0.

Since $\alpha \in \mathcal{O}_k$, we see that $\frac{\alpha}{|\bullet|^k} \to 0$ near 0.

Then $\mathrm{adj}_0^0\left(\frac{\alpha}{|\bullet|^k}\right)$ is continuous at 0.

Then λ is continuous at 0, as desired. □

THEOREM 72.8. *Let* $k \in \mathbb{N}_0$, $\alpha \in \widehat{\mathcal{O}}_k$ *and* $\lambda := \mathrm{adj}_0^0\left(\dfrac{\alpha}{|\bullet|^k}\right)$.

Then $\lambda \in \mathrm{BNZ}$.

Proof. This HW#4-1. □

THEOREM 72.9. *Let* $\alpha, \beta : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $k \in \mathbb{N}_0$.
Assume: $(\beta \in \mathcal{O}_k)\,\&\,(\alpha = \beta$ *near* $0)$. *Then* $\alpha \in \mathcal{O}_k$.

Proof. Since $\beta \in \mathcal{O}_k \subseteq \mathcal{O}_0 = \mathrm{CVZ}$, we get: $\beta_0 = 0$.
Since $\alpha = \beta$ near 0, we get $\alpha_0 = \beta_0$.
Let $\lambda := \mathrm{adj}_0^0\left(\dfrac{\alpha}{|\bullet|^k}\right)$ and $\mu := \mathrm{adj}_0^0\left(\dfrac{\beta}{|\bullet|^k}\right)$.
Since $\alpha = \beta$ near 0, it follows that $\lambda = \mu$ near 0.
Since $\alpha_0 = \beta_0 = 0$, by Theorem 72.5, we see that $\alpha = \lambda \cdot (|\bullet|^k)$.
Since $\beta \in \mathcal{O}_k$, by Theorem 72.7, we see that $\mu \in \mathrm{CVZ}$.
So, since $\lambda = \mu$ near 0, by Theorem 72.4, we see that $\lambda \in \mathrm{CVZ}$.
Then $\alpha = \lambda \cdot (|\bullet|^k) \in (\mathrm{CVZ}) \cdot (|\bullet|^k) = \mathcal{O}_k$, as desired. □

DEFINITION 72.10. *Let* $f : \mathbb{R} \to \mathbb{R}$ *and let* $S \subseteq \mathrm{dom}[f]$.
Then f **is constant on** S **means:** $\forall x, y \in S$**,** $f_x = f_y$**.**

DEFINITION 72.11. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
By f *is* **constant**, *we mean:*

f *is constant on* $\mathrm{dom}[f]$.

Recall Definition 48.1:

DEFINITION 72.12. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $S \subseteq \mathrm{dom}[f]$.
By f *is* **strictly increasing** *on* S, *we mean:*

$\forall t, u \in S,\ [\ (\ t < u\ )\ \Rightarrow\ (\ f(t) < f(u)\ )\ ]$.

By f *is* **strictly decreasing** *on* S, *we mean:*

$\forall t, u \in S,\ [\ (\ t < u\ )\ \Rightarrow\ (\ f(t) > f(u)\ )\ ]$.

By f *is* **semi-increasing** *on* S, *we mean:*

$\forall t, u \in S,\ [\ (\ t \leqslant u\ )\ \Rightarrow\ (\ f(t) \leqslant f(u)\ )\ ]$.

By f *is* **semi-decreasing** *on* S, *we mean:*

$\forall t, u \in S,\ [\ (\ t \leqslant u\ )\ \Rightarrow\ (\ f(t) \geqslant f(u)\ )\ ]$.

THEOREM 72.13. *Let* $m \in \mathbb{R}$. *Define* $L \in \mathcal{L}$ *by* $L_x = mx$.

Then $L' = C_{\mathbb{R}}^m$.

Proof. Unassigned HW. See HW#3-3. □

Let X be a metric space and let $S \subseteq X$. Recall:

$$\begin{aligned} & \mathrm{Int}_X S = \{p \in X \,|\, \exists B \in \mathcal{B}_X(p) \text{ s.t. } B \subseteq S\} \\ \text{and} \quad & \mathrm{Cl}_X S = \{p \in X \,|\, \exists q \in X^{\mathbb{N}} \text{ s.t. } q_\bullet \to p \text{ in } X\} \\ \text{and} \quad & (\, S \text{ is open in X} \,) \Leftrightarrow (\, \mathrm{Int}_X S = S \,) \\ \text{and} \quad & (\, S \text{ is closed in X} \,) \Leftrightarrow (\, \mathrm{Cl}_X S = S \,). \end{aligned}$$

THEOREM 72.14. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$, $p \in \mathrm{dom}[f]$, $S \subseteq \mathbb{R}$ and $q := f_p$.*
Show: $(\, f = a$ on $S \,) \Leftrightarrow (\, f_p^T = a - q$ on $S - p \,)$.

Proof. Unassigned HW. See HW#2-1. □

We drew a graph of a function on the board to show that, in the next theorem, the assumption that S is open in $\mathbb{R}$ is needed.

THEOREM 72.15. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathrm{dom}[f]$.*
Assume: $(\, S$ is open in $\mathbb{R}\,) \,\&\, (\, f$ is constant on $S\,)$. Then $f' = 0$ on S.

Proof. Want: $\forall p \in S$, $f'_p = 0$. Given $p \in S$. Want: $f'_p = 0$.
Since f is constant on S and $p \in S \subseteq \mathrm{dom}[f]$, we get: $f = f_p$ on S.
Let $q := f_p$. Then $f = q$ on S.
Since S is open in $\mathbb{R}$, we get: $S = \mathrm{Int}_{\mathbb{R}} S$.
Since $p \in S = \mathrm{Int}_{\mathbb{R}} S$, choose $B \in \mathcal{B}_{\mathbb{R}}(p)$ s.t. $B \subseteq S$.
Since $B \in \mathcal{B}_{\mathbb{R}}(p)$, we get: $B - p \in \mathcal{B}_{\mathbb{R}}(0)$.
Since $f = q$ on S and since $B \subseteq S$, we get: $f = q$ on B.
By Theorem 72.14, $f_p^T = q - q$ on $B - p$.
Then $f_p^T = 0$ on $B - p$. Then $f_p^T = \mathbf{0}$ on $B - p$.
So, since $B - p \in \mathcal{B}_{\mathbb{R}}(0)$, we get: $f_p^T = \mathbf{0}$ near 0.
So, since $\mathbf{0} \in \mathcal{O}_1$, by Theorem 72.9, we get: $f_p^T \in \mathcal{O}_1$.
Since $f_p^T - \mathbf{0} = f_p^T \in \mathcal{O}_1$ and $\mathbf{0} \in \mathcal{L}$, it follows that $\mathbf{0} \in \mathrm{LINS}_p f$.
Then, by Theorem 68.30, we conclude that $f'_p = \mathrm{s}\ell_{\mathbf{0}}$.
Then $f'_p = \mathrm{s}\ell_{\mathbf{0}} = \mathbf{0}(1) = 0$, as desired. □

DEFINITION 72.16. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathbb{R}$.*
Then f is **c/d** *on S means:*

$$(\, f \textit{ is continuous on } S \,) \quad \& \quad (\, \mathrm{Int}_{\mathbb{R}} S \subseteq \mathrm{dom}[f'] \,).$$

That is, f is c/d on S means:
f is continuous on S and differentiable on the interior of S.

Theorem 72.17, below, is called **Rolle's Theorem**.

THEOREM 72.17. *Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume $a < b$. Let $I := (a; b)$ and let $J := [a; b]$.
Assume f is c/d on J and $f_a = f_b$. Then $\exists c \in I$ s.t. $f'_c = 0$.

Proof. Let $g := f|J$. By Theorem 62.4, g is continuous.
Then, by the Extreme Value Theorem (Theorem 61.2),
$\min g \neq ☹ \neq \max g$.
Let $y := \min g$ and $z := \max g$. Then $y \neq ☹ \neq z$.
Also, $y \leqslant f \leqslant z$ on J.
Exactly one of the following is true:
(1) $y = z$ or (2) $y \neq z$.

Case (1):
Let $c := (a + b)/2$. Since $a < b$, we get $c \in (a; b)$.
Then $c \in (a; b) = I$. Want: $f'_c = 0$.
Since $I = \mathrm{Int}_{\mathbb{R}} J$, it follows that I is open in $\mathbb{R}$.
Since $y \leqslant f \leqslant z$ on J and since $y = z$,
it follows that f is constant on J.
So, since $I = (a; b) \subseteq [a; b] = J$, we see that f is constant on I.
So, since I is open in $\mathbb{R}$, by Theorem 72.15, $f' = 0$ on I.
So, since $c \in I$, we get $f'_c = 0$, as desired.
End of Case (1).

Case (2):
Let $u := f_a$. Since $y \neq z$, we get: $(\, u \neq y \,) \vee (\, u \neq z \,)$.
Choose $v \in \{y, z\}$ s.t. $u \neq v$. Let $Q := \mathrm{im}[g]$.
Then $y = \min g = \min Q$ and $z = \max g = \max Q$. Since $\min Q = y \neq ☹$, we get $\min Q \in Q$.
Since $\max Q = z \neq ☹$, we get $\max Q \in Q$.
Then $y, z \in Q$. Then $v \in \{y, z\} \subseteq Q$.
Since f is c/d on J, we see that f is continuous on J, and so $J \in \mathrm{dom}[f]$.
So, since $g = f|J$, we get $\mathrm{dom}[g] = J$.
So, since $v \in Q = \mathrm{im}[g]$, choose $c \in J$ s.t. $v = g_c$.
Since $c \in J$, we get $(f|J)_c = f_c$. Then $g_c = (f|J)_c = f_c$.
Since $f_a = u \neq v = f_c$, we get $f_a \neq f_c$, and so $a \neq c$.
Since $f_b = f_a \neq f_c$, we get $f_b \neq f_c$, and so $b \neq c$.
Then $c \in J \backslash \{a, b\} = [a; b] \backslash \{a, b\} = (a; b) = I$. Want: $f'_c = 0$.
By Fermat's Theorem (Theorem 71.11),

it suffices to show: f has a local extremum at c in $\mathbb{R}$.
Since I is open in $\mathbb{R}$, we have $I = \mathrm{Int}_{\mathbb{R}} I$.
Since $c \in I = \mathrm{Int}_{\mathbb{R}} I$, choose $S \in \mathcal{B}_{\mathbb{R}}(c)$ s.t. $S \subseteq I$.
Then $S \subseteq I = (a;b) \subseteq [a;b] = J$.
Recall that $y \leqslant f \leqslant z$ on J.
So, since $S \subseteq J$, we see that $y \leqslant f \leqslant z$ on S.
So, since $v \in \{y, z\}$, at least one of the following must be true:
$(\, v \leqslant f \leqslant z \text{ on } S\,)$ or $(\, y \leqslant f \leqslant v \text{ on } S\,)$.
So, since $v = f_c$, at least one of the following must be true:
$(\, f_c \leqslant f \leqslant z \text{ on } S\,)$ or $(\, y \leqslant f \leqslant f_c \text{ on } S\,)$.
Then at least one of the following must be true:
$(\, f_c \leqslant f \text{ on } S\,)$ or $(\, f \leqslant f_c \text{ on } S\,)$.
So, since $S \in \mathcal{B}_{\mathbb{R}}(c)$, at least one of the following must be true:
f has a local minimum at c in $\mathbb{R}$ or
f has a local maximum at c in $\mathbb{R}$.
Then f has a local extremum at c in $\mathbb{R}$, as desired. *End of Case (2).* □

Theorem 72.18, below, is called the **Mean Value Theorem**.

THEOREM 72.18. *Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume $a < b$. Let $I := (a;b)$ and let $J := [a;b]$. Assume f is c/d on J. Then $\exists c \in I$ s.t. $f'_c = \mathrm{DQ}_f(a,b)$.*

Proof. Let $m := \mathrm{DQ}_f(a,b)$. Then $m = \dfrac{f_b - f_a}{b - a}$, so $m \cdot (b-a) = f_b - f_a$.
Define $L \in \mathcal{L}$ by $L(x) = mx$. By Theorem 72.13, $L' = C^m_{\mathbb{R}}$.
Then $\mathrm{dom}[L'] = \mathrm{dom}[C^m_{\mathbb{R}}] = \mathbb{R}$. Then L is continuous on $\mathbb{R}$.
Then L is continuous on J and $I \subseteq \mathrm{dom}[L']$.
By assumption, f is c/d on J.
Then f is continuous on J and $I \subseteq \mathrm{dom}[f']$.
So, since L is continuous on J and $I \subseteq \mathrm{dom}[L']$,
we see that $f - L$ is continuous on J and that $I \subseteq \mathrm{dom}[(f-L)']$.
Let $g := f - L$. Then g is continuous on J and $I \subseteq \mathrm{dom}[g']$.
It follows that g is c/d on J.
Also, $g_b = f_b - L_b = f_b - mb$ and $g_a = f_a - L_a = f_a - ma$.
Then $g_b - g_a = (f_b - mb) - (f_a - ma) = (f_b - f_a) - m \cdot (b-a)$.
So, since $m \cdot (b-a) = f_b - f_a$, we see that $g_b - g_a = 0$, and so $g_a = g_b$.
So, since g is c/d on J, by Theorem 72.17, choose $c \in I$ s.t. $g'_c = 0$.
Want: $f'_c = \mathrm{DQ}_f(a,b)$. Since $g = f - L$, we see that $g'_c =^* f'_c - L'_c$.
Since $c \in I \subseteq \mathrm{dom}[f']$, we get: $f'_c \in \mathrm{im}[f] \subseteq \mathbb{R}$.

So, as $L'_c = C^m_{\mathbb{R}}(c) = m \in \mathbb{R}$, we have $f'_c - L'_c \in \mathbb{R}$, and so $f'_c - L'_c \neq ☹$.
Since $g'_c =^* f'_c - L'_c \neq ☹$, we get: $g'_c = f'_c - L'_c$.
Then $0 = g'_c = f'_c - L'_c$, so $f'_c = L'_c$.
So, since $L'_c = m = \mathrm{DQ}_f(a,b)$, we get $f'_c = \mathrm{DQ}_f(a,b)$, as desired. □

Recall: $\forall a,b \in \mathbb{R}$, $[a|b] = [a;b] \cup [b;a] = [\ \min\{a,b\}\ ,\ \max\{a,b\}\ ]$.

DEFINITION 72.19. *Let $J \subseteq \mathbb{R}$. By J is an* **interval**, *we mean:*
$$\forall a,b \in J, \quad [a|b] \subseteq J.$$

The bounded intervals come in four flavors:
$\varnothing$, open bounded nonempty,
half-open bounded nonempty, closed bounded nonempty.
The unbounded intevals come in three flavors:
open nonclosed unbounded, closed nonopen unbounded, $\mathbb{R}$.
The set of open bounded nonempty intervals is
$$\{(a;b)\,|\,a,b \in \mathbb{R}, a < b\}.$$
The set of half-open bounded nonempty intervals is
$$\{[a;b)\,|\,a,b \in \mathbb{R}, a < b\} \cup \{(a;b]\,|\,a,b \in \mathbb{R}, a < b\}.$$
The set of closed bounded nonempty intervals is
$$\{[a;b]\,|\,a,b \in \mathbb{R}, a \leqslant b\}.$$
The set of open nonclosed unbounded intervals is
$$\{(a,\infty)\,|\,a \in \mathbb{R}\} \cup \{(-\infty,b)\,|\,b \in \mathbb{R}\}.$$
The set of closed nonopen unbounded intervals is
$$\{[a,\infty)\,|\,a \in \mathbb{R}\} \cup \{(-\infty,b)\,|\,b \in \mathbb{R}\}.$$

THEOREM 72.20. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and $a,b \in \mathbb{R}$.*
Then $\mathrm{DQ}_f(a,b) = \mathrm{DQ}_f(b,a)$.

Proof. We have $\mathrm{DQ}_f(a,b) = \dfrac{f_b - f_a}{b-a} = \dfrac{f_a - f_b}{a-b} = \mathrm{DQ}_f(b,a)$. □

THEOREM 72.21. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and $a,b \in \mathbb{R}$.*
Let $\alpha := \min\{a,b\}$ and $\beta := \max\{a,b\}$.
Then $\mathrm{DQ}_f(\alpha,\beta) = \mathrm{DQ}_f(a,b)$.

Proof. By Theorem 72.20, we have $\mathrm{DQ}_f(\alpha,\beta) = \mathrm{DQ}_f(a,b)$.
Then $\{\,\mathrm{DQ}_f(a,b)\,,\,\mathrm{DQ}_f(b,a)\,\} = \{\,\mathrm{DQ}_f(a,b)\,\}$.
As $(\alpha,\beta) \in \{(a,b),(b,a)\}$, we get $\mathrm{DQ}_f(\alpha,\beta) \in \{\,\mathrm{DQ}_f(a,b)\,,\,\mathrm{DQ}_f(b,a)\}$.
Then $\mathrm{DQ}_f(\alpha,\beta) \in \{\,\mathrm{DQ}_f(a,b)\,,\,\mathrm{DQ}_f(b,a)\,\} = \{\,\mathrm{DQ}_f(a,b)\,\}$,
so $\mathrm{DQ}_f(\alpha,\beta) = \mathrm{DQ}_f(a,b)$, as desired. □

THEOREM 72.22. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let J be an interval.*
Let $a, b \in J$ and let $I := \mathrm{Int}_{\mathbb{R}} J$.
Assume that $a \neq b$ and that f is c/d on J.
Then $\mathrm{DQ}_f(a, b) \in f'_*(I)$.

Proof. Since $a, b \in J$ and since J is an interval, we get: $[a|b] \subseteq J$.
Let $\alpha := \min\{a, b\}$ and $\beta := \max\{a, b\}$. Then $[a|b] = [\alpha; \beta]$.
Since $a \neq b$, we get $\alpha < \beta$.
Let $P := (\alpha; \beta)$ and $Q := [\alpha; \beta]$. Then $Q = [\alpha; \beta] = [a|b] \subseteq J$.
By HW#3-4, we see that $P = \mathrm{Int}_{\mathbb{R}} Q$.
By Theorem 72.21, we have: $\mathrm{DQ}_f(\alpha, \beta) = \mathrm{DQ}_f(a, b)$.
Since $Q \subseteq J$, we see that $\mathrm{Int}_{\mathbb{R}} Q \subseteq \mathrm{Int}_{\mathbb{R}} J$, so $P \subseteq I$.
Since f is c/d on J, we get: f is continuous on J and $I \subseteq \mathrm{dom}[f']$.
So, since $Q \subseteq J$ and $P \subseteq I$, we conclude:
f is continuous on Q and $P \subseteq \mathrm{dom}[f']$.
Then f is c/d on Q.
So, by the Mean Value Theorem (Theorem 72.18),
choose $c \in P$ s.t. $f'_c = \mathrm{DQ}_f(\alpha, \beta)$.
As $c \in P \subseteq I \subseteq \mathrm{dom}[f']$, we get $c \in \mathrm{dom}[f']$ and $c \in I$, so $f'_c \in f'_*(I)$.
Then $\mathrm{DQ}_f(a, b) = \mathrm{DQ}_f(\alpha, \beta) = f'_c \in f'_*(I)$, as desired. □

THEOREM 72.23. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let J be an interval.*
Assume f is c/d on J. Let $I := \mathrm{Int}_{\mathbb{R}} J$, $T := f'_*(I)$. Then
(1) $(\, 0 \notin T \,) \Rightarrow (\, f|J$ is $1-1 \,)$
and (2) $(\, T > 0 \,) \Rightarrow (\, f$ is strictly increasing on $J \,)$
and (3) $(\, T < 0 \,) \Rightarrow (\, f$ is strictly decreasing on $J \,)$
and (4) $(\, T = \{0\} \,) \Leftrightarrow (\, f$ is constant on $J \,)$
and (5) $(\, T \geqslant 0 \,) \Leftrightarrow (\, f$ is semi-increasing on $J \,)$
and (6) $(\, T \leqslant 0 \,) \Leftrightarrow (\, f$ is semi-decreasing on $J \,)$.

Proof. Proof of (1):
Assume $0 \notin T$. Want: $f|T$ is 1-1.
Want: $\forall a, b \in J$, $[\, (\, f_a = f_b \,) \Rightarrow (\, a = b \,)\,]$.
Given $a, b \in J$. Want: $(\, f_a = f_b \,) \Rightarrow (\, a = b \,)$.
Assume that $f_a = f_b$. Want $a = b$.
Assume $a \neq b$. Want: Contradiction.
Since $a \neq b$ and $f_a = f_b$, we get $\dfrac{f_b - f_a}{b - a} = 0$.
By Theorem 72.22, $\mathrm{DQ}_f(a, b) \in f'_*(I)$.

Then $0 = \dfrac{f_b - f_a}{b - a} = \mathrm{DQ}_f(a,b) \in f'_*(I) = T$, so $0 \in T$.
However, by assumption, $0 \notin T$. Contradiction.
End of proof of (1).

Proof of (2)-(6): Unassigned HW. *End of proof of (2)-(6).* □

73. Taylor's Formula to order 2

THEOREM 73.1. *Let X be a metric space. Then $\mathcal{B}_X \subseteq \mathcal{T}_X$.*

Proof. Want: $\forall B \in \mathcal{B}_X$, $B \in \mathcal{T}_X$.
Given $B \in \mathcal{B}_X$. Want: $B \in \mathcal{T}_X$. Want: $\mathrm{Int}_X B = B$.
Since $\mathrm{Int}_X B \subseteq B$, it suffices to prove: $B \subseteq \mathrm{Int}_X B$.
Want: $\forall p \in B$, $p \in \mathrm{Int}_X B$. Given $p \in B$. Want: $p \in \mathrm{Int}_X B$.
Want: $\exists A \in \mathcal{B}_X(p)$ s.t. $A \subseteq B$.
By Theorem 38.16, $\exists A \in \mathcal{B}_X(p)$ s.t. $A \subseteq B$, as desired. □

DEFINITION 73.2. *Let X be a metric space and let f be a function. Assume* $\mathrm{dom}[f] \subseteq X$*.
Then* $\mathrm{IntD}_X f := \mathrm{Int}_X(\mathrm{dom}[f])$*.*

THEOREM 73.3. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathrm{dom}[f']$.
Then $0 \in \mathrm{IntD}_{\mathbb{R}}(f_P^T)$.*

Proof. Want: $0 \in \mathrm{Int}_{\mathbb{R}}(\mathrm{dom}[f_P^T])$.
Want: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $B \subseteq \mathrm{dom}[f_p^T]$.
By Theorem 68.25, choose $L \in \mathrm{LINS}_p f$ and $R \in \mathcal{O}_1$ s.t.

$$f_p^T = L + R \qquad \text{and} \qquad f'_p = \mathrm{s}\ell_L.$$

Since $f_p^T = L + R$, we get: $\mathrm{dom}[f_p^T] = (\mathrm{dom}[L]) \cap (\mathrm{dom}[R])$.
Since $R \in \mathcal{O}_1 \subseteq \mathrm{DNZ}$, choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $B \subseteq \mathrm{dom}[R]$.
Want: $B \subseteq \mathrm{dom}[f_p^T]$.
Since $L \in \mathcal{L}$, we have: $\mathrm{dom}[L] = \mathbb{R}$.
Since $\mathrm{dom}[R] \subseteq \mathbb{R}$, we get $\mathbb{R} \cap (\mathrm{dom}[R]) = \mathrm{dom}[R]$.
Then $\mathrm{dom}[f_p^T] = (\mathrm{dom}[L]) \cap (\mathrm{dom}[R]) = \mathbb{R} \cap (\mathrm{dom}[R]) = \mathrm{dom}[R]$.
Then $B \subseteq \mathrm{dom}[R] = \mathrm{dom}[f_p^T]$, as desired. □

THEOREM 73.4. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $a, b \in \mathbb{R}$.
Assume that $[a|b] \subseteq \mathrm{dom}[f']$.
Then $\exists c \in [a|b]$ s.t. $[f'_c] \cdot [b - a] = f_b - f_a$.*

Proof. Exactly one of the following is true:

(1) $a = b$ or (2) $a \neq b$.

Case (1):
Let $c := a$. Then $c \in [a|b]$.
Want: $[f'_c] \cdot [b - a] = f_b - f_a$.
Since $a = b$, we get $b - a = 0$ and $f_b - f_a = 0$.
Since $c \in [a|b] \subseteq \mathrm{dom}[f'] \subseteq \mathrm{dom}[f]$, we get $f'_c \in \mathrm{im}[f]$.
Since $f'_c \in \mathrm{im}[f] \subseteq \mathbb{R}$ and since $b - a = 0$, we get $[f'_c] \cdot [b - a] = 0$.
Then $[f'_c] \cdot [b - a] = 0 = f_b - f_a$, as desired..
End of Case (1).

Case (2):
Let $\alpha := \min\{a, b\}$ and $\beta := \max\{a, b\}$. Then $[a|b] = [\alpha; \beta]$.
Then $[\alpha; \beta] = [a|b] \subseteq \mathrm{dom}[f']$. Then f is c/d on $[\alpha; \beta]$.
By the MVT (Theorem 72.18), choose $c \in (\alpha; \beta)$ s.t. $f'_c = \mathrm{DQ}_f(\alpha, \beta)$.
Then $c \in (\alpha; \beta) \subseteq [\alpha; \beta] = [a|b]$. Want: $[f'_c] \cdot [b - a] = f_b - f_a$.
By Theorem 72.21, $\mathrm{DQ}_f(\alpha, \beta) = \mathrm{DQ}_f(a, b)$.
Then $f'_c = \mathrm{DQ}_f(\alpha, \beta) = \mathrm{DQ}_f(a, b) = \dfrac{f_b - f_a}{b - a}$.
Then $[f'_c] \cdot [b - a] = \left[\dfrac{f_b - f_a}{b - a}\right] \cdot [b - a] = f_b - f_a$, as desired.
End of Case (2). □

THEOREM 73.5. *Let $x \in \mathbb{R}$ and let $w \in [0|x]$.*
Then $\exists t \in [0; 1]$ s.t. $tx = w$.

Proof. Unassigned HW. □

THEOREM 73.6. *Let $B \in \mathcal{B}_{\mathbb{R}}$. Then B is an interval.*

Proof. Unassigned HW. □

Theorem 73.7, below, is called the **Choice MVT**.

THEOREM 73.7. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $B \in \mathcal{B}_{\mathbb{R}}(0)$.*
Assume that $f_0 = 0$ and that $B \subseteq \mathrm{dom}[f']$.
Then $\exists \alpha : B \to [0; 1]$ s.t., $\forall x \in B$, $f_x = [f'(\alpha_x \cdot x)] \cdot x$.

Proof. Define $S : B \to 2^{[0;1]}$ by $S_x = \{t \in [0; 1] \,|\, f_x = [f'(\alpha_x \cdot x)] \cdot x\}$.

Claim: $\forall x \in B$, $S_x \neq \varnothing$.

Proof of Claim:
Given $x \in B$. Want: $S_x \neq \emptyset$.
Since B is an interval and since $0, x \in B$, we get: $[0|x] \subseteq B$.
Then $[0|x] \subseteq B \subseteq \mathrm{dom}[f']$, so, by Theorem 73.4, choose $w \in [0|x]$ s.t. $[f'_w] \cdot [x - 0] = f_x - f_0$.
Since $w \in [0|x]$, choose $t \in [0;1]$ s.t. $tx = w$.
Since $f_x = f_x - f_0 = [f'_w] \cdot [x - 0] = [f'(tx)] \cdot x$, we get: $t \in S_x$.
Then $S_x \neq \emptyset$, as desired.
End of proof of Claim.

Define $\alpha : B \to [0;1]$ by $\alpha_x = \mathrm{CH}(S_x)$.
Want: $\forall x \in B, \quad f_x = [f'(\alpha_x \cdot x)] \cdot x$.
Given $x \in B$. Want: $f_x = [f'(\alpha_x \cdot x)] \cdot x$.
By the Claim, $S_x \neq \emptyset$, and so $\mathrm{CH}(S_x) \in S_x$.
Since $\alpha_x = \mathrm{CH}(S_x) \in S_x$, we get $f_x = [f'(\alpha_x \cdot x)] \cdot x$, as desired. □

THEOREM 73.8. *Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume $0 = g_0$. Then $g_0^T = g$.*

Proof. Want: $\forall h \in \mathbb{R}$, $(g_0^T)(h) = g(h)$.
Given $h \in \mathbb{R}$. Want: $(g_0^T)(h) = g(h)$.
We have $(g_0^T)(h) = g_{0+h} - g_0 = g_h - 0 = g_h$, as desired. □

THEOREM 73.9. *Let $L \in \mathcal{L}$. Assume $\mathrm{s}\ell_L = 0$. Then $L = \mathbf{0}$.*

Proof. Want: $\forall h \in \mathbb{R}$, $L_h = \mathbf{0}_h$.
Given $h \in \mathbb{R}$. Want: $L_h = \mathbf{0}_h$.
We have $L_h = (\mathrm{s}\ell_L) \cdot h = 0 \cdot h = 0 = \mathbf{0}_h$, as desired. □

The next result asserts that any partial function $\mathbb{R} \dashrightarrow \mathbb{R}$ that vanishes to order one at 0 is necessarily sublinear.

THEOREM 73.10. *Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume that $0 = g_0 = g'_0$. Then $g \in \mathcal{O}_1$.*

Proof. Since $g'_0 = 0$, we conclude that $0 \in \mathrm{dom}[g']$.
By Theorem 68.25, choose $L \in \mathrm{LINS}_0 g$ and $R \in \mathcal{O}_1$ s.t.
$$g_0^T = L + R \qquad \text{and} \qquad g'_0 = \mathrm{s}\ell_L.$$
By assumption, $g'_0 = 0$. Then $\mathrm{s}\ell_L = g'_0 = 0$, and so $L = \mathbf{0}$.
By assumption $0 = g_0$. Then $g_0^T = g$.
Then $g = g_0^T = L + R = \mathbf{0} + R = R \in \mathcal{O}_1$, as desired. □

THEOREM 73.11. *Let $B \in \mathcal{B}_{\mathbb{R}}(0)$, $\alpha : B \to [0;1]$. Then $\alpha \in \mathrm{BNZ}$.*

Proof. Unassigned HW. □

The next result asserts that any partial function $\mathbb{R} \dashrightarrow \mathbb{R}$ that vanishes to order two at 0 is necessarily subquadratic.

It is an unassigned induction exercise to show: $\forall k \in \mathbb{N}_0$, any partial function $\mathbb{R} \dashrightarrow \mathbb{R}$ that vanishes to order k at 0 is necessarily "$\mathcal{o}_k$".

THEOREM 73.12. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume $0 = f_0 = f'_0 = f''_0$. Then $f \in \mathcal{o}_2$.*

Proof. Let $g := f'$. Then $0 = g_0 = g'_0$, so, by Theorem 73.10, $g \in \mathcal{o}_1$.
Since $0 = g_0$, we see that $g_0^T = g$.
Since $g'_0 = 0$, we conclude that $0 \in \mathrm{dom}[g']$.
Then, by Theorem 73.3, we get: $0 \in \mathrm{IntD}_{\mathbb{R}} g$.
Since $0 \in \mathrm{IntD}_{\mathbb{R}} g = \mathrm{Int}_{\mathbb{R}}(\mathrm{dom}[g])$, choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $B \subseteq \mathrm{dom}[g]$.
Then by the Choice MVT (Theorem 73.7),
choose $\alpha : B \to [0;1]$ s.t., $\forall x \in B$, $f_x = [f'(\alpha_x \cdot x] \cdot x$.
Let $i := \mathrm{id}_{\mathbb{R}}$. Then $i \in \mathcal{L} \subseteq \widehat{\mathcal{O}}_1$.
Since, $\forall x \in B$, we have $f_x = [f'(\alpha_x \cdot x] \cdot x = ([f' \circ (\alpha \cdot i] \cdot i)_x$,
it follows that $f = [f \circ (\alpha \cdot i)] \cdot i$ on B.
So, since $B \in \mathcal{B}_{\mathbb{R}}(0)$, we get: $f = [f \circ (\alpha \cdot i)] \cdot i$ near 0.
Since $B \in \mathcal{B}_{\mathbb{R}}(0)$ and $\alpha : B \to [0;1]$, we see that $\alpha \in \mathrm{BNZ}$.
Since $\alpha \in \mathrm{BNZ} = \widehat{\mathcal{O}}_0$ and $i \in \widehat{\mathcal{O}}_1$, we see that $\alpha \cdot i \in \widehat{\mathcal{O}}_0 \cdot \widehat{\mathcal{O}}_1$.
Since $\alpha \cdot i \in \widehat{\mathcal{O}}_0 \cdot \widehat{\mathcal{O}}_1 \subseteq \widehat{\mathcal{O}}_{0+1} = \widehat{\mathcal{O}}_1$ and $f' = g \in \mathcal{o}_1$,
we see that $f' \circ (\alpha \cdot i) \in \mathcal{o}_1 \circ \widehat{\mathcal{O}}_1$.
Since $f' \circ (\alpha \cdot i) \in \mathcal{o}_1 \circ \widehat{\mathcal{O}}_1 \subseteq \mathcal{o}_{1 \cdot 1} = \mathcal{o}_1$ and $i \in \widehat{\mathcal{O}}_1$,
we see that $[f \circ (\alpha \cdot i)] \cdot i \in \mathcal{o}_1 \cdot \widehat{\mathcal{O}}_1$.
Then $[f \circ (\alpha \cdot i)] \cdot i \in \mathcal{o}_1 \cdot \widehat{\mathcal{O}}_1 \subseteq \mathcal{o}_{1+1} = \mathcal{o}_2$.
So, since $f = [f \circ (\alpha \cdot i)] \cdot i$ near 0, we get $f \in \mathcal{o}_2$, as desired. □

THEOREM 73.13. *Let $L : \mathbb{R} \to \mathbb{R}$ and let $m \in \mathbb{R}$.
Assume: $\forall h \in \mathbb{R}$, $L_h = mh$.
Then, $\forall h \in \mathbb{R}$, $L'_h = m$. Also, $\forall h \in \mathbb{R}$, $L''_h = 0$.*

Proof. Unassigned HW. □

THEOREM 73.14. *Let $Q : \mathbb{R} \to \mathbb{R}$ and let $a \in \mathbb{R}$.
Assume: $\forall h \in \mathbb{R}$, $Q_h = ah^2/2$.
Then, $\forall h \in \mathbb{R}$, $Q'_h = ah$. Also, $\forall h \in \mathbb{R}$, $Q''_h = a$.*

Proof. Unassigned HW. □

Theorem 73.15, below, is **Taylor's Theorem to order 2**.

THEOREM 73.15. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $p \in \mathrm{dom}[f'']$.
Let $m := f'_p$ *and let* $a := f''_p$.
Deinfe $L \in \mathcal{L}$ *and* $Q \in \mathcal{Q}$ *by* $L_h = mh$ *and* $Q_h = ah^2/2$.
Let $R := f_p^T - (L + Q)$. *Then* $R \in \mathcal{O}_2$.

Proof. By Theorem 73.12, it suffices to show: $0 = R_0 = R'_0 = R''_0$.
We have: $\forall h \in \mathbb{R}$, $L_h = mh$ and $Q_h = ah^2/2$.
Then, $\forall h \in \mathbb{R}$, $L'_h = m$ and $Q'_h = ah$.
Also, $\forall h \in \mathbb{R}$, $L''_h = 0$ and $Q''_h = a$.
Then $R_0 = f_p - (a + 0) = 0 - 0 = 0$.
Also, $R'_0 \;=^*\; f'_p - (m + 0) = m - m = 0 \neq ☹$, so $R'_0 = 0$.
Also, $R''_0 \;=^*\; f''_p - (0 + a) = a - a = 0 \neq ☹$, so $R''_0 = 0$.
Then $0 = R_0 = R'_0 = \mathbb{R}''_0$, as desired. ☐

THEOREM 73.16. *Let* $R \in \mathcal{O}_2$ *and* $Q \in \mathcal{Q}\backslash\{\mathbf{0}\}$.
Then $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ *s.t.* $|R| \leqslant |Q|$ *on* B.

Proof. Since $Q \in \mathcal{Q}$, choose $a \in \mathbb{R}$ s.t., $\forall h \in \mathbb{R}$, $Q_h = ah^2$.
Since $Q \neq \mathbf{0}$, we conclude that $a \neq 0$.
Let $\varepsilon := |a|$. Then $\varepsilon > 0$.
Since $R \in \mathcal{O}_2 = (\mathrm{CVZ}) \cdot (|\bullet|^2)$, choose $\phi \in \mathrm{CVZ}$ s.t. $R = \phi \cdot (|\bullet|^2)$.
Since $\phi \in \mathrm{CVZ} \subseteq \mathrm{DNZ}$, choose $C \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $C \subseteq \mathrm{dom}[\phi]$.
Since $\phi \in \mathrm{CVZ}$, we see that ϕ is continuous at 0,
so choose $D \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\phi_*(D) \subseteq B_{\mathbb{R}}(\phi_0, \varepsilon)$.
Since $\phi \in \mathrm{CVZ}$, we get $\phi_0 = 0$. Then $\phi_*(D) \subseteq B_{\mathbb{R}}(\phi_0, \varepsilon) = B_{\mathbb{R}}(0, \varepsilon)$.
Let $B := C \cap D$. Then $B \in \{C, D\} \subseteq \mathcal{B}_{\mathbb{R}}(0)$.
Want: $|R| \leqslant |Q|$ on B. Want: $\forall h \in B$, $|R|_h \leqslant |Q|_h$.
Given $h \in B$. Want: $|R|_h \leqslant |Q|_h$.
We have $h \in B = C \cap D$, so $h \in C$ and $h \in D$.
By the choice of C, $C \subseteq \mathrm{dom}[\phi]$.
Since $h \in C \subseteq \mathrm{dom}[\phi]$ and since $h \in D$, we get $\phi_h \in \phi_*(D)$.
Since $\phi_h \in \phi_*(D) \subseteq B_{\mathbb{R}}(0, \varepsilon)$, we get $d_{\mathbb{R}}(\phi_h, 0) < \varepsilon$.
Then $|\phi_h| = |\phi_h - 0| = d_{\mathbb{R}}(\phi_h, 0) < \varepsilon$.
Since $|\phi_h| < \varepsilon$ and $h^2 \geqslant 0$, we get $|\phi_h| \cdot h^2 \leqslant \varepsilon \cdot h^2$.
Since $|h|^2 \geqslant 0$, it follows that $|\,|h|^2\,| = |h|^2$.
Since $R = \phi \cdot (|\bullet|^2)$, we conclude that $R_h = (\phi_h) \cdot (|h|^2)$.
Then $|R|_h = |R_h| = |\,(\phi_h) \cdot (|h|^2)\,| = |\phi_h| \cdot |\,|h|^2\,| = |\phi_h| \cdot |h|^2$
$\leqslant \varepsilon \cdot |h|^2 = |a| \cdot |h|^2 = |ah^2| = |Q_h| = |Q|_h$, as desired. ☐

The logic of Theorem 73.16 shows: Any subquadratic is dominated, near 0, by any nonzero quadratic polynomial, even inhomogeneous.

In the notation of Theorem 73.15, we showed that, $\forall x \in \mathbb{R}$,

$$f_x = f_p + (f'_p) \cdot (x-p) + (f''_p/2) \cdot (x-p)^2 + R_{x-p}.$$

The remainder term R_{x-p} is subquadratic in $x-p$.
Also, the rest, $f_p + (f'_p) \cdot (x-p) + (f''_p/2) \cdot (x-p)^2$,
is a quadratic polynomial in $x-p$.
Consequently, unless $0 = f_p = f'_p = f''_p$,
the remainder is negligible (for $x \approx p$) compared to the rest.

74. The Second Derivative Test

THEOREM 74.1. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.*
Then: $[\ (f_p^T)'_0 = f'_p\ ]\ \&\ [\ (f_p^T)''_0 = f''_p\ ]$.

Proof. Let $q := f_p$. Then $f_p^T = [f(p+\bullet)] - [C_{\mathbb{R}}^q]$.
Differentiating this, we get: $(f_p^T)' = ([f(p+\bullet)] - [C_{\mathbb{R}}^q])'$.
By HW#4-5, we have: $([f(p+\bullet)] - [C_{\mathbb{R}}^q])' = (f(p+\bullet))'$.
By HW#4-4, we have: $(f(p+\bullet))' = f'(p+\bullet)$.
Then $(f_p^T)' = ([f(p+\bullet)] - [C_{\mathbb{R}}^q])' = (f(p+\bullet))' = f'(p+\bullet)$.
Contracting this, we get: $(f_p^T)' = f'(p+\bullet)$.
Evaluating this at 0, we get: $(f_p^T)'_0 = (f'(p+\bullet))_0$.
Then $(f_p^T)'_0 = (f'(p+\bullet))_0 = f'(p+0) = f'(p) = f'_p$.
It remains to show: $(f_p^T)''_0 = f''_p$.
Differentiating $(f_p^T)' = f'(p+\bullet)$, we get: $(f_p^T)'' = (f'(p+\bullet))'$.
By HW#4-4 (with f replaced by f'),
we have: $(f'(p+\bullet))' = f''(p+\bullet)$.
Then $(f_p^T)'' = (f'(p+\bullet))' = f''(p+\bullet)$.
Evaluating this at 0, we get: $(f_p^T)''_0 = (f''(p+\bullet))_0$.
Then $(f_p^T)''_0 = (f''(p+\bullet))_0 = f''(p+0) = f''(p) = f''_p$, as desired. □

DEFINITION 74.2. *Let X be a metric space.*
Let $f : X \dashrightarrow \mathbb{R}$ and let $p \in \mathrm{dom}[f]$.
By f has a **strict local minimum** *at p in X, we mean:*
$\exists B \in \mathcal{B}_X(p)$ *s.t.* $f > f_p$ *on* $B_p^\times$.
By f has a **strict local maximum** *at p in X, we mean:*
$\exists B \in \mathcal{B}_X(p)$ *s.t.* $f < f_p$ *on* $B_p^\times$.

Recall Theorem 74.1.

THEOREM 74.3. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.*
Assume $f'_p = 0$ and $f''_p > 0$.
Then f has a strict local minimum at p in $\mathbb{R}$.

Proof. By HW#4-5, it suffices to show:
f_p^T has a strict local minimum at 0 in $\mathbb{R}$.
Let $g := f_p^T$. Want: g has a strict local minimum at 0 in $\mathbb{R}$.
Want: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $g > g_0$ on $B_0^\times$.
By assumption $f'_p = 0$ and $f''_p > 0$.
Let $m := f'_p$ and let $a := f''_p$. Then $m = 0$ and $a > 0$.
Define $L \in \mathcal{L}$ and $Q \in \mathcal{Q}$ by $L_h = mh$ and $Q_h = ah^2/2$.
Then $(\, L = \mathbf{0}\,)$ and $(\, Q > \mathbf{0}$ on $\mathbb{R}_0^\times\,)$.
By Theorem 73.15, $f_p^T - (L + Q) \in \mathcal{O}_2$. Let $R := f_p^T - (L + Q)$.
Then $R \in \mathcal{O}_2$. Also, $R = f_p^T - (L + Q) = g - (\mathbf{0} + Q) = g - Q$.
Since $Q > \mathbf{0}$ on $\mathbb{R}_0^\times$, we get: $Q/2 > \mathbf{0}$ on $\mathbb{R}_0^\times$.
Then $Q/2 \neq \mathbf{0}$. So, as $Q/2 \in \mathcal{Q}$, we get: $Q/2 \in \mathcal{Q}\backslash\{\mathbf{0}\}$.
Then, by Theorem 73.16, choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $|R| \leqslant |Q/2|$ on B.
Want: $g > g_0$ on $B_0^\times$.
Since $f'_p = 0 \neq ☹$, we get: $p \in \mathrm{dom}[f']$.
So, since $\mathrm{dom}[f'] \subseteq \mathrm{dom}[f]$, we see that $p \in \mathrm{dom}[f]$.
Then $(f_p^T)_0 = 0$. Then $g_0 = (f_p^T)_0 = 0$.
Want: $g > 0$ on $B_0^\times$.
Since $Q \in \mathcal{Q}$, we get: $Q - Q = \mathbf{0}$.
Then $Q + R = Q + (g - Q) = g + \mathbf{0} = g$.
Want: $Q + R > 0$ on $B_0^\times$.
Since $Q/2 > \mathbf{0} = 0$ on $\mathbb{R}_0^\times$, we get: $Q/2 > 0$ on $\mathbb{R}$.
We have $(Q/2)_0 = Q_0/2 = (a \cdot 0^2/2)/2 = 0$.
Since $Q/2 > 0$ on $\mathbb{R}_0^\times$ and since $(Q/2)_0 = 0$,
we get: $Q/2 \geqslant 0$ on $\mathbb{R}$.
Then $|Q/2| = Q/2$ on $\mathbb{R}$. Then $|R| \leqslant |Q/2| = Q/2$ on B.
Since $|R| \leqslant Q/2$ on B, we get: $-Q/2 \leqslant R \leqslant Q/2$ on B.
Since $R \geqslant -Q/2$ on B, we get: $Q + R \geqslant Q - (Q/2)$ on B.
So, since $Q - (Q/2) = Q/2$, we get: $Q + R \geqslant Q/2$ on B.
So, since $B_0^\times \subseteq B$, we get: $Q + R \geqslant Q/2$ on $B_0^\times$.
So, since $Q/2 > 0$ on $B_0^\times$, we get: $Q + R > 0$ on $B_0^\times$, as desired. □

Theorem 74.3 is the **Second Derivative Test for Local Minima**.

THEOREM 74.4. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.*
Assume $f'_p = 0$ and $f''_p < 0$.
Then f has a strict local maximum at p in $\mathbb{R}$.

Proof. Unassigned HW. (*Hint:* Let $g := -f$ and use Theorem 74.3.) □

Theorem 74.4 is the **Second Derivative Test for Local Maxima**.

75. Tensors, sets of tensors and tensor spaces

NOTE TO SELF:
Next year, define $\mathcal{I}_{\mathbb{R}} := \{\varnothing\}$.
Keep in mind that $\mathbb{R} \neq \mathbb{R}^{\mathcal{I}_{\mathbb{R}}}$,
so we only have: $\forall V \in \mathrm{TNSR}$, $V = \mathbb{R}^{\mathcal{I}_V}$;
this does not extend to TNSR^+.
Also, $\forall x \in \mathbb{R}$, define $x_\varnothing := x$.
Also, define $\varepsilon^{\mathbb{R}}_{\varnothing} := 1$.
Also, define $\pi^{\mathbb{R}}_{\varnothing} = \mathrm{id}_{\mathbb{R}}$.
Also, $\forall V \in \mathrm{TNSR}^+$, $\forall f : \mathbb{R} \dashrightarrow V$, define $\partial^{\mathbb{R}V}_{\varnothing} f = f'$. Also, $\forall \sigma \in \mathbb{N}$, $\forall k \in \mathbb{N}^\sigma$, define $k \| \varnothing := k$ and $\varnothing \| k := k$.
Also, define $\varnothing \| \varnothing := \varnothing$.

Recall: Let X be a set. Then, $\forall m \in \mathbb{N}$, we defined $X^m = X^{[1..m]}$,
so X^m denotes the set of all functions $\{1, \ldots, m\} \to X$.
Moreover, $\forall a, b \in X$, $(a, b) \in X^2 = X^{[1..2]} = X^{\{1,2\}}$,
$(a, b)_1 = a$ and $(a, b)_2 = b$.
There are simliar definitions for (a, b, c), *etc.*

DEFINITION 75.1. $\mathrm{EUCL} := \{\mathbb{R}^m \,|\, m \in \mathbb{N}\}$.

We have: $\mathrm{EUCL} := \{\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots\}$.
WARNING: $(5) \in \mathbb{R}^1 = \mathbb{R}^{[1..1]} = \mathbb{R}^{\{1\}}$ and $(5)_1 = 5$.
However, $(5) \neq 5$ and, in fact, $5 \notin \mathbb{R}^1$.
Thus, $\mathbb{R} \neq \mathbb{R}^1$, and, in fact, $\mathbb{R} \notin \mathrm{EUCL}$.
In Definition 75.2 below, any element of $\mathbb{N}^\rho$ is called a ρ-**shape**.
Then m is a ρ-shape. Also, any element of $[m]$ is called a ρ-**index**.

DEFINITION 75.2. *Let $\rho \in \mathbb{N}$ and let $m \in \mathbb{N}^\rho$.*
Then $[m] := [1..m_1] \times \cdots \times [1..m_\rho]$.
Also, we define $m_1 \times \cdots \times m_\rho := m$.

Following Definition 75.2, we see, $\forall \rho \in \mathbb{N}$, $\forall m \in \mathbb{N}^\rho$,

$$m_1 \times \cdots \times m_\rho \quad = \quad (m_1, \ldots, m_\rho).$$

THEOREM 75.3. *We have* $2 \times 3 = (2,3)$.
Also, $[2 \times 3] = [1..2] \times [1..3] = \{1,2\} \times \{1,2,3\}$.
Also, $[2 \times 3] = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$.

Here are some 2-shapes: 5×7, 6×4, 2×3, 1×9, 1×1.
Here are all of the (2×3)-indices:

$$(1,1)\ ,\ (1,2)\ ,\ (1,3)\ ,\ (2,1)\ ,\ (2,2)\ ,\ (2,3).$$

DEFINITION 75.4. *Let X be a set, $\rho \in \mathbb{N}$, $m \in \mathbb{N}^\rho$.*
Then $X^m := X^{[m]}$.

In Definition 75.4, the elements of X^m are called m-shaped ρ-**tensors** with entries in X. A 1-tensor is sometimes called a **tuple**.
A 2-tensor is sometimes called a **matrix**.

THEOREM 75.5. $\forall$*set X, we have*

$$X^{2\times 3} = X^{[2\times 3]} = X^{\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}}.$$

Some notation: Let X be a set and let $a,b,c,d,e,f \in X$. Then

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \in \quad X^{2\times 3}$$

is defined by

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{(1,1)} = a, \quad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{(1,2)} = b, \quad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{(1,3)} = c,$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{(2,1)} = d, \quad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{(2,2)} = e, \quad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{(2,3)} = f.$$

The index subscripts

$$(1,1)\ ,\ (1,2)\ ,\ (1,3)\ ,\ (2,1)\ ,\ (2,2)\ ,\ (2,3)$$

are generally abbreviated: $\quad 11\ ,\ 12\ ,\ 13\ ,\ 21\ ,\ 22\ ,\ 23.$
This notation is for (2×3)-shaped matrices, but, $\forall p,q \in \mathbb{N}$, we have a similar notation for $(p \times q)$-shaped matrices.

DEFINITION 75.6. *Let X be a set, $\rho \in \mathbb{N}$, $m \in \mathbb{N}^\rho$, $Y := X^m$, $j \in [m]$.*
Then $\pi_j^Y : Y \to X$ is defined by: $\pi_j^Y(z) \;=\; z_j$.

THEOREM 75.7. *Let* $S := \mathbb{R}^2$ *and let* $z := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

$$\text{Then } \pi^S_{22}(z) = z_{22} = 5.$$

WARNING: $\mathbb{R}^{2\times 3} \neq \mathbb{R}^2 \times \mathbb{R}^3$.

In fact, $\forall a,b,c,d,e,f \in \mathbb{R}$, $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in \mathbb{R}^{2\times 3}$.

Also, $\forall a,b,c,d,e \in \mathbb{R}$, $((a,b),(c,d,e)) \in \mathbb{R}^2 \times \mathbb{R}^3$.

Also, $\forall a,b,c,d,e,f \in \mathbb{R}$, $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \notin \mathbb{R}^2 \times \mathbb{R}^3$.

Also, $\forall a,b,c,d,e \in \mathbb{R}$, $((a,b),(c,d,e)) \notin \mathbb{R}^{2\times 3}$.

WARNING: The sets

$$\mathbb{R}, \quad \mathbb{R}^1, \quad \mathbb{R}^{1\times 1}, \quad \mathbb{R}^{1\times 1\times 1}, \quad \ldots$$

are all distinct; no two are equal. However, as vector spaces (if you know what that means), they are all one-dimesional, and so are all vector space isomorphic to one another. Keep in mind, though, that $\mathbb{R}^{2\times 3}$ and $\mathbb{R}^2 \times \mathbb{R}^3$ are not even vector space isomorphic; in fact, the vector space $\mathbb{R}^{2\times 3}$ is six dimensional, while $\mathbb{R}^2 \times \mathbb{R}^3$ is only five-dimensional.

THEOREM 75.8. *We have* $\mathbb{R}^3 = \mathbb{R}^{\{1,2,3\}}$.

$$\text{Also, } \mathbb{R}^{1\times 3\times 1} = \mathbb{R}^{\{(1,1,1),(1,2,1),(1,3,1)\}}.$$

WARNING: Let $m \in \mathbb{N}$. Then

$$\begin{array}{lll} \mathbb{R}^m, & & \\ \mathbb{R}^{1\times m}, & \mathbb{R}^{m\times 1}, & \\ \mathbb{R}^{1\times 1\times m}, & \mathbb{R}^{1\times m\times 1}, & \mathbb{R}^{m\times 1\times 1} \end{array}$$

are all distinct sets, but are all vector space isomorphic to one another.

DEFINITION 75.9. *Let* $\rho \in \mathbb{N}$ *and* $m \in \mathbb{N}^\rho$. *Then*

$$\forall j \in \mathbb{N}, \quad j \times m := (j, m_1, \ldots, m_\rho) \quad \text{and}$$
$$m \times j := (m_1, \ldots, m_\rho, j).$$

Also, $\forall j,k \in \mathbb{N}$,

$$j \times k \times m := (j, k, m_1, \ldots, m_\rho) \quad \text{and}$$
$$j \times m \times k := (j, m_1, \ldots, m_\rho, k) \quad \text{and}$$
$$m \times j \times k := (m_1, \ldots, m_\rho, j, , k).$$

WARNING: Let $\rho \in \mathbb{N}$ and $m \in \mathbb{N}^\rho$. Then

$$\begin{array}{lll} \mathbb{R}^m, & & \\ \mathbb{R}^{1\times m}, & \mathbb{R}^{m\times 1}, & \\ \mathbb{R}^{1\times 1\times m}, & \mathbb{R}^{1\times m\times 1}, & \mathbb{R}^{m\times 1\times 1} \end{array}$$

are all distinct sets, but are all vector space isomorphic to one another.

Let $k, \ell, m \in \mathbb{N}$. Then $[(k, \ell, m)] = [1..k] \times [1..\ell] \times [1..m]$.
Let $\ell, m \in \mathbb{N}$. Then $[(\ell, m)] = [1..\ell] \times [1..m]$.
Let $m \in \mathbb{N}$. Then $[(m)] = [1..m]$. Also, we have

$$(m) \in \mathbb{N}^1 = \mathbb{N}^{\{1\}} \quad \text{and} \quad (m)_1 = m.$$

While $(m) \neq m$, in the next theorem, we will prove:

$$\forall \text{set } X, \qquad X^{(m)} = X^m.$$

THEOREM 75.10. *Let X be a set and let $m \in \mathbb{N}$. Then $X^{(m)} = X^m$.*

Proof. We have $X^{(m)} = X^{[(m)]} = X^{[1..m]} = X^m$. □

DEFINITION 75.11. $\forall \rho \in \mathbb{N}, \quad \text{TNSR}_\rho := \{\mathbb{R}^m \mid m \in \mathbb{N}^\rho\}$.
Also, $\text{TNSR} \;:=\; \text{TNSR}_1 \cup \text{TNSR}_2 \cup \text{TNSR}_3 \cup \cdots$.

An element of TNSR will be called a **tensor space**.
For any $\rho \in \mathbb{N}$, an element of TNSR_ρ will be called a ρ-**tensor space**.

THEOREM 75.12. *We have* $\text{TNSR}_1 = \text{EUCL}$.

Proof. By Theorem 75.10,

$$\mathbb{R}^{(1)} = \mathbb{R}^1 \quad \text{and} \quad \mathbb{R}^{(2)} = \mathbb{R}^2 \quad \text{and} \quad \mathbb{R}^{(3)} = \mathbb{R}^3 \quad \text{and} \quad \cdots.$$

Then $\text{TNSR}_1 = \{\mathbb{R}^{(1)}, \mathbb{R}^{(2)}, \mathbb{R}^{(3)}, \ldots\} = \{\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots\} = \text{EUCL}$. □

We have $\mathbb{R}^{2\times3}, \mathbb{R}^{5\times7}, \mathbb{R}^{7\times9\times4}, \mathbb{R}^{2\times2\times2}, \mathbb{R}^{2\times2\times2\times2} \in \text{TNSR}$.
Also, $\forall v \in \mathbb{R}^{2\times3}$, $\exists a, b, c, d, e, f \in \mathbb{R}$ s.t. $v = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$.
That is, each element of $\mathbb{R}^{2\times3}$, as a 2-tensor,
is pictured as a 2-dimensional array, fitting nicely on the page.
Picturing elements of $\mathbb{R}^{5\times7}$ is slightly harder,
requiring room for a 5×7 array, but is still 2-dimensional.
By contrast, the elements of $\mathbb{R}^{7\times9\times4}$ are all 3-tensors,
and it's hard to put a 3-dimensional array onto a page.
The tensor space $\mathbb{R}^{2\times2\times2}$ is only 8-dimensional,
but is still difficult to picture its elements on a page.
Nevertheless, we can at least imagine each element of $\mathbb{R}^{2\times2\times2}$,
because we live in a three-dimensional world.
By contrast, elements of $\mathbb{R}^{2\times2\times2\times2}$ are hard even to imagine,
without living in a four-dimensional space.

THEOREM 75.13. *Let $S \in \text{TNSR}$. Then*

$$\forall v \in S, \quad v \text{ is a functional} \qquad \textit{and}$$
$$\forall v, w \in S, \quad v + w \in S \qquad \textit{and}$$
$$\forall c \in \mathbb{R}, \forall v \in S, \quad cv \in S.$$

DEFINITION 75.14. *Let X be a set, $\rho \in \mathbb{N}$, $m \in \mathbb{N}^\rho$, $Y := X^m$. Then $\mathcal{I}_Y := [m]$.*

Following Definition 75.14, we have $Y = X^m = X^{[m]} = X^{\mathcal{I}_Y}$.

THEOREM 75.15. *Let $S := \mathbb{R}^{2\times 3\times 4}$.*
Then $\mathcal{I}_S = [2\times 3\times 4] = [1..2]\times[1..3]\times[1..4]$
$= \{1,2\}\times\{1,2,3\}\times\{,1,2,3,4\}$.

Let $S := \mathbb{R}^{2\times 3\times 4}$. Then $S = \mathbb{R}^{[2\times 3\times 4]} = \mathbb{R}^{\mathcal{I}_S}$. More generally:

THEOREM 75.16. *Let $S \in$ TNSR. Then $S = \mathbb{R}^{\mathcal{I}_S}$.*

DEFINITION 75.17. *Let $S \in$ TNSR. Then*

$$\forall v, w \in S, \qquad v \bullet_S w := \sum_{j\in\mathcal{I}_S} v_j w_j.$$

Also, $$\forall v \in S, \qquad |v|_S := \sqrt{v \bullet_S v}.$$

Following Definition 75.17, if S is obvious,
we may sometimes omit "S" from the subscript in $v \bullet_S w$,
and simply write $v \bullet w$.
Also, following Definition 75.17, if S is obvious,
we may sometimes omit "S" from the subscript in $|v|_S$,
and simply write $|v|$.

THEOREM 75.18. *We have*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \bullet \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$
$$= \quad 1\cdot 7 + 2\cdot 8 + 3\cdot 9 + 4\cdot 10 + 5\cdot 11 + 6\cdot 12.$$

Also, $$\left| \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \right| = \sqrt{1^2+2^2+3^2+4^2+5^2+6^2}.$$

DEFINITION 75.19. *Let $S \in$ TNSR.*
Define $\delta \in \mathcal{M}(S)$ by $\delta(v,w) = |w - v|_S$.
Then δ is called the **standard metric** *on S.*
Also, the metric space (S,δ) is denoted S.

Let $S \in$ TNSR and let δ be the standard metric on S. Then $d_S = \delta$.

THEOREM 75.20. *Let* $S := \mathbb{R}^{2\times 3}$.

Then $d_S\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 \\ 9 & 8 & 2 \end{bmatrix}\right) = \sqrt{(1-0)^2+(2-1)^2+(3-3)^2+(4-9)^2+(5-8)^2+(6-2)^2}$.

DEFINITION 75.21. *Let* X *be a set,* $\rho \in \mathbb{N}$, $m \in \mathbb{N}^\rho$, $Y := X^m$.
Then $\Pi_Y := \{\pi_i^Y \,|\, i \in [m]\}$.

DEFINITION 75.22. *We define* $\Pi_{\mathbb{R}} := \{\mathrm{id}_{\mathbb{R}}\}$.

Let $S \in \mathrm{TNSR}^+$ and let $p \in \Pi_S$.
Then $p : S \to \mathbb{R}$ is distance semi-decreasing, *i.e.*, Lipschitz-1.
It follows that p is continuous from S to $\mathbb{R}$.

THEOREM 75.23. *Let* $S := \mathbb{R}^{2\times 3}$.
Then $\Pi_S = \{\pi_{11}^S, \pi_{12}^S, \pi_{13}^S, \pi_{21}^S, \pi_{22}^S, \pi_{23}^S\}$.

DEFINITION 75.24. *We define:* $\mathrm{EUCL}^+ := \{\mathbb{R}\} \cup \mathrm{EUCL}$.
We also define: $\mathrm{TNSR}^+ := \{\mathbb{R}\} \cup \mathrm{TNSR}$.

DEFINITION 75.25. *Let* $S \in \mathrm{TNSR}^+$.
Then $\mathcal{M}_0^S := \{C_S^1\}$.
Also, $\forall j \in \mathbb{N}$, $\mathcal{M}_j^S := \{p_1 \cdots p_j \,|\, p_1, \ldots, p_j \in \Pi_S\}$.

For any $S \in \mathrm{TNSR}^+$, for any $j \in \mathbb{N}$, the elements of $\mathcal{M}_j^S$ are called **monomials** of degree j on S. Note that, $\forall j \in \mathbb{N}$, $\mathcal{M}_j^{\mathbb{R}} = \{\mathrm{id}_{\mathbb{R}}^j\}$, so there is only one monomial of degree j on $\mathbb{R}$, namely $\mathrm{id}_{\mathbb{R}}^j$.

Since every projection is continuous, it follows that every monomial is continuous.

DEFINITION 75.26. *Let* $S \in \mathrm{TNSR}$ *and let* $j \in \mathbb{N}$.

Then $\mathcal{H}_j^S := \{\, a_1\mu_1 + \cdots + a_\ell\mu_\ell \mid \ell \in \mathbb{N},\ a_1, \ldots, a_\ell \in \mathbb{R},\ \mu_1, \ldots, \mu_\ell \in \mathcal{M}_j^S \}$.

For any $S \in \mathrm{TNSR}^+$, for any $j \in \mathbb{N}$, the elements of $\mathcal{H}_j^S$ are called **homogeneous polynomial functionals** of degree j on S. Thus, a homogeneous polynomial of degree j is a finite linear combination of monomials, all of degree j. The word "homogeneous" expresses the idea that the monomials being used all have the same degree.

Since every monomial is continuous, it follows that every homogeneous polynomial functional is continuous.

THEOREM 75.27. $\forall j \in \mathbb{N}_0$, $\mathcal{H}_j^{\mathbb{R}} = \mathcal{H}_j$.

THEOREM 75.28. *Let $S := \mathbb{R}^{2\times 3}$. Define $P, Q : S \to \mathbb{R}$ by*
$$P\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = bcde \quad \text{and} \quad Q\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = 4bcde - \sqrt{2}f^4.$$
Then $P = (\pi_{12}^S)\cdot(\pi_{13}^S)\cdot(\pi_{21}^S)\cdot(\pi_{22}^S) \in \mathcal{M}_4^S \subseteq \mathcal{H}_4^S$. Also, $Q \in \mathcal{H}_4^S$.

DEFINITION 75.29. *Let $S, T \in \mathrm{TNSR}^+$ and $j \in \mathbb{N}_0$.*
$$\text{Then:} \quad \mathcal{H}_j^{ST} := \{ f : S \to T \mid \forall p \in \Pi_T,\ p \circ f \in \mathcal{H}_j^S \}.$$

For any $S, T \in \mathrm{TNSR}^+$, for any $j \in \mathbb{N}$, the elements of $\mathcal{H}_j^{ST}$ are called **homogeneous polynomials** of degree j on S.

Let $S, T \in \mathrm{TNSR}^+$ and $f : S \to T$.
Then (f is continuous) $\Leftrightarrow$ ($\forall p \in \Pi_T, p \circ f$ is continuous).
So, since every homogeneous polynomial functional is continuous,
it follows that every homogeneous polynomial is continuous.

THEOREM 75.30. *Let $S \in \mathrm{TNSR}^+$ and $j \in \mathbb{N}_0$.*
$$\text{Then } \mathcal{H}_j^{S\mathbb{R}} = \mathcal{H}_j^S.$$

DEFINITION 75.31. *Let $S, T \in \mathrm{TNSR}^+$.*
Then $\mathcal{C}_{ST} := \mathcal{H}_0^{ST}$ and $\mathcal{L}_{ST} := \mathcal{H}_1^{ST}$ and $\mathcal{Q}_{ST} := \mathcal{H}_2^{ST}$ and $\mathcal{K}_{ST} := \mathcal{H}_3^{ST}$.

THEOREM 75.32. *Let $S \in \mathrm{TNSR}^+$ and $k \in \mathbb{N}_0$.*
$$\text{Then } H_k^{S\mathbb{R}} = \mathcal{H}_k^S.$$

THEOREM 75.33. *Let $k \in \mathbb{N}_0$. Then $H_k^{\mathbb{R}\mathbb{R}} = \mathcal{H}_k$.*

DEFINITION 75.34. *Let $S \in \mathrm{TNSR}$.*
$$\text{Then } 0_S := C_{\mathcal{I}_S}^0. \quad \text{Also } \mathbf{0}_S := C_S^0.$$

In Definition 75.34, we have $0_S \in \mathbb{R}^{\mathcal{I}_S} = S$ and $\mathbf{0}_S : S \to \mathbb{R}$.

DEFINITION 75.35. *We define: $0_{\mathbb{R}} := 0$ and $\mathbf{0}_{\mathbb{R}} := \mathbf{0}$.*

DEFINITION 75.36. *Let $S \in \mathrm{TNSR}^+$.*
Then $|\bullet|_S : S \to [0;\infty)$ is defined by $|v|_S = \sqrt{v \bullet v}$.

In Definition 75.36, comparing with Definition 75.17, we see: $\forall v \in S$, we have $|v|_S = |v|$.

76. Some function spaces between tensor spaces

Let $S \in \mathrm{TNSR}$. Recall, from Definition 75.19, that, $\forall v, w \in S$,
$$d_S(v,w) \quad = \quad |v - w|_S \quad = \quad \sqrt{\sum_{i \in \mathcal{I}_s} (v_i - w_i)^2}.$$

DEFINITION 76.1. *Let $S, T \in \mathrm{TNSR}^+$. Then*

$\mathrm{DNZ}_{ST} := \{\alpha : S \dashrightarrow T \,|\, \alpha \text{ is defined near } 0\}$,

$\mathrm{BNZ}_{ST} := \{\alpha \in \mathrm{DNZ}_{ST} \,|\, \exists B \in \mathcal{B}_S(0_S) \text{ s.t. } \alpha_*(B) \text{ is bounded in } T\}$,

$\mathrm{CVZ}_{ST} := \{\alpha \in \mathrm{DNZ}_{ST} \,|\, [\,\alpha(0_S) = 0_T\,] \,\&\, [\,\alpha \text{ is continuous at } 0_S\,]\}$.

DEFINITION 76.2. *Let X be a set, $T \in \mathrm{TNSR}$, $f, g : X \dashrightarrow T$. Then $f + g : X \dashrightarrow T$ is defined by $(f + g)_z = (f_z) + (g_z)$.*

DEFINITION 76.3. *Let X be a set, $T \in \mathrm{TNSR}$, $c \in \mathbb{R}$, $f : X \dashrightarrow T$. Then $c \cdot f : X \dashrightarrow T$ is defined by $(c \cdot f)_z = c \cdot (f_z)$.*

DEFINITION 76.4. *Let X be a set, $T \in \mathrm{TNSR}$.*
Let $f : X \dashrightarrow \mathbb{R}$, $g : X \dashrightarrow T$.
Then $f \cdot g : X \dashrightarrow T$ is defined by $(f \cdot g)_z = (f_z) \cdot (g_z)$.
Also, $g \cdot f := f \cdot g$.

DEFINITION 76.5. *Let $S, T \in \mathrm{TNSR}^+$ and $j \in \mathbb{N}_0$. Then:*

$$\mathcal{O}_j^{ST} := \left\{ \alpha \in \mathrm{DNZ} \,\middle|\, [\,\alpha(0_S) = 0_T\,] \,\&\, \left[\frac{\alpha}{|\bullet|_S^j} \to 0_T \text{ near } 0_S \right] \right\}$$

and $\widehat{\mathcal{O}}_j^{ST} := (\mathrm{BNZ}_{ST}) \cdot (|\bullet|_S^j)$.

THEOREM 76.6. *Let $S, T \in \mathrm{TNSR}^+$ and let $j \in \mathbb{N}_0$. Then*

$$\mathcal{O}_j^{ST} = (\mathrm{CVZ}_{ST}) \cdot (|\bullet|_S^j).$$

THEOREM 76.7. *We have:* $\mathrm{DNZ}_{\mathbb{R}\mathbb{R}} = \mathrm{DNZ}$,
$\mathrm{BNZ}_{\mathbb{R}\mathbb{R}} = \mathrm{BNZ}$ *and* $\mathrm{CVZ}_{\mathbb{R}\mathbb{R}} = \mathrm{CVZ}$.

77. Basic properties of tensor spaces

DEFINITION 77.1. $\forall x, y \in \mathbb{R}$, $x \bullet_{\mathbb{R}} y := xy$.
$\forall x \in \mathbb{R}$, $|x|_{\mathbb{R}} := |x|$.

Following Definition 77.1,
we may sometimes omit "$\mathbb{R}$" from the subscript in $x \bullet_{\mathbb{R}} y$,
and simply write $x \bullet y$.

THEOREM 77.2. *Let $S \in \mathrm{TNSR}^+$. Then*

$\forall v, w \in S$, $|v \bullet_S w|_{\mathbb{R}} \leqslant |v|_S \cdot |w|_S$ *and*

$\forall v, w \in S$, $|v + w|_S \leqslant |v|_S + |w|_S$ *and*

$\forall v, w \in S$, $|\,|v|_S - |w|_S\,|_{\mathbb{R}} \leqslant |\,v - w\,|_S$.

Let $S \in \mathrm{TNSR}^+$ and let $v, w \in S$. Then

$$d_{\mathbb{R}}(|v|_S, |w|_S) \;=\; |\,|v|_S \,-\, |w|_S\,|_{\mathbb{R}} \;\leqslant\; |\,v \,-\, w\,|_S = d_S(v,w).$$

That is, $|\bullet|_S : S \to \mathbb{R}$ is distance-semidecreasing, *i.e.*, Lipschitz-1. It follows that $|\bullet|_S : S \to \mathbb{R}$ is continuous.

THEOREM 77.3. *Let* $S, T \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$.
Then: $(f \in \mathcal{L}_{ST}) \quad \Leftrightarrow \quad (\; [\,\forall v, w \in S,\; f_{v+w} = f_v + f_w\,] \;\;\&\;\; [\,\forall c \in \mathbb{R},\, \forall v \in S,\; f_{cv} = c \cdot (f_v)\,] \;)$.

That is, f is linear iff f is "algebraically linear".

DEFINITION 77.4. $\forall S, T \in \mathrm{TNSR}^+, \quad \mathbf{0}_{ST} \;:=\; C_S^{0_T}$.

THEOREM 77.5. *Let* $S, T \in \mathrm{TNSR}^+$, *let* $j \in \mathbb{N}_0$ *and let* $f \in \mathcal{H}_j^{ST}$. *Let* $c \in \mathbb{R}$ *and* $v \in S$. *Then* $f(cv) = c^j \cdot [f(v)]$.

THEOREM 77.6. *Let* $S, T \in \mathrm{TNSR}^+$.
Then: $(\mathcal{Q}_{ST} \subseteq \widehat{\mathcal{O}}_2^{ST}) \,\&\, (\mathcal{Q}_{ST} \cap \mathcal{o}_2^{ST} = \{\mathbf{0}_{ST}\})$.

THEOREM 77.7. *Let* $S, T \in \mathrm{TNSR}^+$ *and let* $j \in \mathbb{N}_0$.
Then: $(\mathcal{H}_j^{ST} \subseteq \widehat{\mathcal{O}}_j^{ST}) \,\&\, (\mathcal{H}_j^{ST} \cap \mathcal{o}_j^{ST} = \{\mathbf{0}_{ST}\})$.

THEOREM 77.8. $\forall S, T \in \mathrm{TNSR}^+, \quad \mathrm{CVZ}_{ST} \subseteq \mathrm{BNZ}_{ST}$.

THEOREM 77.9. *Let* $S, T \in \mathrm{TNSR}^+$ *and let* $j \in \mathbb{N}_0$.
Then $\widehat{\mathcal{O}}_{j+1}^{ST} \subseteq \mathcal{o}_j^{ST} \subseteq \widehat{\mathcal{O}}_j^{ST}$.

Proof. We have:

$$\begin{aligned} \widehat{\mathcal{O}}_{j+1}^{ST} &= (\mathrm{BNZ}_{ST}) \cdot (|\bullet|_S^{j+1}) \\ &= (\mathrm{BNZ}_{ST}) \cdot (|\bullet|_S) \cdot (|\bullet|_S^j) \\ &\subseteq (\mathrm{BNZ}_{ST}) \cdot (\mathrm{CVZ}_{ST}) \cdot (|\bullet|_S^j) \\ &\subseteq (\mathrm{CVZ}_{ST}) \cdot (|\bullet|_S^j) = \mathcal{o}_j^{ST}. \end{aligned}$$

Want: $\mathcal{o}_j^{ST} \subseteq \widehat{\mathcal{O}}_j^{ST}$.

We have: $\mathcal{o}_j^{ST} = (\mathrm{CVZ}_{ST}) \cdot (|\bullet|_S^j) \subseteq (\mathrm{BNZ}_{ST}) \cdot (|\bullet|_S^j) = \widehat{\mathcal{O}}_j^{ST}$. □

THEOREM 77.10. $\forall S \in \mathrm{TNSR}^+, \qquad |\bullet|_S \in \mathrm{CVZ}$.

78. Double translation

DEFINITION 78.1. *Let* $R, S \in \mathrm{TNSR}^+$, $f : R \dashrightarrow S$.
Then, $\forall p \in \mathbb{R}$, ${}_{RS}f_p^T : R \dashrightarrow S$ *is defined by* $({}_{RS}f_p^T)(h) = f_{p+h} - f_p$.
Also, ${}_{RS}f_{☹}^T \;:=\; ☹$.

In Definition 78.1, we will almost always omit the pre-subscript RS,
and write f_p^T instead of ${}_{RS}f_p^T$,
and, also, $f_{☹}^T$ instead of ${}_{RS}f_{☹}^T$.
We have: $\forall R,S \in \text{TNSR}$, $\forall f : R \dashrightarrow S$, $\forall p \in R \backslash (\text{dom}[f])$,

$$f_p^T = \varnothing \neq ☹.$$

DEFINITION 78.2. *Let X be a set and let $S \in \text{TNSR}^+$. Then*
$\forall f, g : X \dashrightarrow S,\quad f \bullet_S g : X \dashrightarrow \mathbb{R}$ *is defined by* $(f \bullet_S g)_p = (f_p) \bullet_S (g_p)$
and $\forall v \in S,\quad \forall g : X \dashrightarrow S,\quad v \bullet_S g := (C_X^v) \bullet_S g$
and $\forall f : X \dashrightarrow S,\quad \forall w \in S,\quad f \bullet_S w := f \bullet_S (C_X^w)$.

DEFINITION 78.3. *Let $S,T \in \text{TNSR}^+$, $f : S \dashrightarrow T$ and $p \in S$.*
Then $\text{LINS}_p^{ST} := \{ L \in \mathcal{L}_{ST} \mid f_p^T - L \in \mathcal{O}_1^{ST} \}$
and $D_p^{ST} f := \text{UE}(\text{LINS}_p^{ST} f)$.

THEOREM 78.4. *Let $S,T \in \text{TNSR}^+$, $f : S \dashrightarrow T$ and $p \in S$.*
Then $\#(\text{LINS}_p^{ST}) \leqslant 1$.

79. Tensor products of tensor spaces

DEFINITION 79.1. *Let $\sigma, \tau \in \mathbb{N}$, let $m \in \mathbb{N}^\sigma$ and let $n \in \mathbb{N}^\tau$.*
Then $m \| n := (m_1, \ldots, m_\sigma, n_1, \ldots, n_\tau)$.

DEFINITION 79.2. *Let $\sigma \in \mathbb{N}$, let $m \in \mathbb{N}^\sigma$ and let $n \in \mathbb{N}$.*
Then $m \| n := m \| (n)$ *and* $n \| m := (n) \| m$.

DEFINITION 79.3. $\forall m, n \in \mathbb{N},\quad m \| n := (m) \| (n)$.

DEFINITION 79.4. *Let $\sigma, \tau \in \mathbb{N}$, let $m \in \mathbb{N}^\sigma$ and let $n \in \mathbb{N}^\tau$.*
Then $\mathbb{R}^m \otimes \mathbb{R}^n := \mathbb{R}^{m \| n}$.

DEFINITION 79.5. $\forall S \in \text{TNSR},\quad S \otimes \mathbb{R} := S$ *and* $\mathbb{R} \otimes S := S$.

DEFINITION 79.6. *We define:* $\mathbb{R} \otimes \mathbb{R} := \mathbb{R}$.

Recall that $\mathbb{R}^{2 \times 3} \neq \mathbb{R}^2 \times \mathbb{R}^3$. On the other hand, we have:

THEOREM 79.7. $\mathbb{R}^2 \otimes \mathbb{R}^3 = \mathbb{R}^{(2)} \otimes \mathbb{R}^{(3)} = \mathbb{R}^{(2,3)} = \mathbb{R}^{2 \times 3}$.

THEOREM 79.8. *We have:*

$$\mathbb{R}^1 \otimes \mathbb{R}^{3 \times 5} = \mathbb{R}^{1 \times 3 \times 5} \neq \mathbb{R}^{3 \times 5} \quad \textit{and}$$
$$\mathbb{R} \otimes \mathbb{R}^{3 \times 5} = \mathbb{R}^{3 \times 5} \quad \textit{and}$$
$$\mathbb{R}^{6 \times 8 \times 2} \otimes \mathbb{R} = \mathbb{R}^{6 \times 8 \times 2} \quad \textit{and}$$
$$\mathbb{R}^{6 \times 8 \times 2} \otimes \mathbb{R}^{3 \times 5} = \mathbb{R}^{6 \times 8 \times 2 \times 3 \times 5}.$$

80. Slopes of linear maps between tensor spaces

Let $S, T \in$ TNSR. According to HW#5-5, $\forall i \in \mathcal{I}_S$, $\forall j \in \mathcal{I}_T$,
we have $i \| j \in \mathcal{I}_{S \otimes T}$.

Also, we have the following:

THEOREM 80.1. *Let $S, T \in$ TNSR, $k \in \mathcal{I}_{S \otimes T}$.
Then $\exists 1\ i \in \mathcal{I}_S$, $\exists 1\ j \in \mathcal{I}_T$ s.t. $k = i \| j$.*

DEFINITION 80.2. *Let $\sigma, \tau \in \mathbb{N}$.
Let $m \in \mathbb{N}^\sigma$, $n \in \mathbb{N}^\tau$. Let $S := \mathbb{R}^m$, $T := \mathbb{R}^n$. Let $k \in \mathcal{I}_{S \otimes T}$.
Then $k_S := (k_1, \ldots, k_\sigma)$ and $k_T := (k_{\sigma+1}, \ldots, k_{\sigma+\tau})$.*

DEFINITION 80.3. *Let $S, T \in$ TNSR and let $L \in \mathcal{L}_{ST}$.
Then $\mathrm{s}\ell_L^{ST} \in \mathbb{R}^{\mathcal{I}_{T \otimes S}}$ is defined by*

$$(\mathrm{s}\ell_L^{ST})_k \quad = \quad \pi_{k_T}^T\,(\,L\,(\,\varepsilon_{k_S}^S\,)\,).$$

We have: $\forall S, T \in$ TNSR, $\forall L \in \mathcal{L}_{ST}$, $\mathrm{s}\ell_L^{ST} \in \mathbb{R}^{\mathcal{I}_{T \otimes S}} = T \otimes S$.

THEOREM 80.4. *Let $\sigma := 1$, $\tau = 1$, $m = (3)$, $n = (2)$.
Let $S := \mathbb{R}^3$ and $T := \mathbb{R}^2$.
Define $L \in \mathcal{L}_{ST}$ by*

$$L(x, y, z) = (\ 3x - 2y + 4z\ ,\ \ 7x + \ y - 5z\ ).$$

Then: (1) $(\mathrm{s}\ell_L^{ST})_{21} = 7$ and (2) $\mathrm{s}\ell_L^{ST} = \begin{bmatrix} 3 & -2 & 4 \\ 7 & 1 & -5 \end{bmatrix}$.

Proof. Proof of (1):
We calculate $(\mathrm{s}\ell_L^{ST})_{21} = \pi_2^T(L(\varepsilon_1^S)) = \pi_2^T(L(1, 0, 0)) = \pi_2^T(3, 7) = 7$.
End of proof of (1).

Proof of (2): Unassigned HW. *End of proof of (2).* □

DEFINITION 80.5. *Let $S \in$ TNSR and let $L \in \mathcal{L}_{S\mathbb{R}}$.
Then $\mathrm{s}\ell_L^{S\mathbb{R}} \in \mathbb{R}^{\mathcal{I}_S}$ is defined by*

$$(\mathrm{s}\ell_L^{S\mathbb{R}})_k \quad = \quad L\,(\,\varepsilon_k^S\,).$$

We have: $\forall S \in$ TNSR, $\forall L \in \mathcal{L}_{S\mathbb{R}}$, $\mathrm{s}\ell_L^{S\mathbb{R}} \in \mathbb{R}^{\mathcal{I}_S} = S = \mathbb{R} \otimes S$.

DEFINITION 80.6. *Let $T \in$ TNSR$^+$ and let $L \in \mathcal{L}_{\mathbb{R}T}$.
Then $\mathrm{s}\ell_L^{\mathbb{R}T} := L(1)$.*

We have: $\forall T \in$ TNSR, $\forall L \in \mathcal{L}_{\mathbb{R}T}$, $\mathrm{s}\ell_L^{\mathbb{R}T} \in T = T \otimes \mathbb{R}$.
Also, $\forall L \in \mathcal{L}$, both $L \in \mathcal{L}_{\mathbb{R}\mathbb{R}}$ and $\mathrm{s}\ell_L^{\mathbb{R}\mathbb{R}} = \mathrm{s}\ell_L$.

THEOREM 80.7. *Let* $S, T \in \mathrm{TNSR}^+$ *and let* $L \in \mathcal{L}_{ST}$.
Then $\mathrm{s}\ell_L^{ST} \in T \otimes S$.

DEFINITION 80.8. *Let* $S, T \in \mathrm{TNSR}^+$. *Then* $\mathrm{s}\ell_{☹}^{ST} = ☹$.

DEFINITION 80.9. *Let* $S, T \in \mathrm{TNSR}^+$. *Then* $\mathrm{s}\ell_\bullet^{ST} : \mathcal{L}_{ST} \to T \otimes S$ *is defined by* $(\mathrm{s}\ell_\bullet^{ST})(L) = \mathrm{s}\ell_L^{ST}$.

DEFINITION 80.10. *Let* $S, T \in \mathrm{TNSR}^+$ *and let* $f : S \dashrightarrow T$. *Then* ${}_{ST}f' : S \dashrightarrow T \otimes S$ *is defined by* ${}_{ST}f'_p = \mathrm{s}\ell_\bullet^{ST}(D_p^{ST} f)$.

In Definition 80.10, we will almost always omit the pre-subscript ST, and write f'_p instead of ${}_{ST}f'_p$.

THEOREM 80.11. *Let* $f : \mathbb{R}^{3\times2} \dashrightarrow \mathbb{R}^{5\times7}$.
Then $f' : \mathbb{R}^{3\times2} \dashrightarrow \mathbb{R}^{5\times7\times3\times2}$ *and*
$f'' : \mathbb{R}^{3\times2} \dashrightarrow \mathbb{R}^{5\times7\times3\times2\times3\times2}$ *and*
$f''' : \mathbb{R}^{3\times2} \dashrightarrow \mathbb{R}^{5\times7\times3\times2\times3\times2\times3\times2}$.

THEOREM 80.12. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}^{5\times7}$.
Then $f' : \mathbb{R} \dashrightarrow \mathbb{R}^{5\times7}$ *and*
$f'' : \mathbb{R} \dashrightarrow \mathbb{R}^{5\times7}$ *and*
$f''' : \mathbb{R} \dashrightarrow \mathbb{R}^{5\times7}$.

THEOREM 80.13. *Let* $f : \mathbb{R}^1 \dashrightarrow \mathbb{R}^{5\times7}$.
Then $f' : \mathbb{R}^1 \dashrightarrow \mathbb{R}^{5\times7\times1}$ *and*
$f'' : \mathbb{R}^1 \dashrightarrow \mathbb{R}^{5\times7\times1\times1}$ *and*
$f''' : \mathbb{R}^1 \dashrightarrow \mathbb{R}^{5\times7\times1\times1\times1}$.

THEOREM 80.14. *Let* $f : \mathbb{R}^{3\times2} \dashrightarrow \mathbb{R}$.
Then $f' : \mathbb{R}^{3\times2} \dashrightarrow \mathbb{R}^{3\times2}$ *and*
$f'' : \mathbb{R}^{3\times2} \dashrightarrow \mathbb{R}^{3\times2\times3\times2}$ *and*
$f''' : \mathbb{R}^{3\times2} \dashrightarrow \mathbb{R}^{3\times2\times3\times2\times3\times2}$.

81. Linearity of multivariable differentiation

THEOREM 81.1. *Let* $S, T \in \mathrm{TNSR}^+$, $f, g : S \dashrightarrow T$ *and* $p \in S$.
Then $D_p^{ST}(f + g) =^* D_p^{ST} f + D_p^{ST} g$.

THEOREM 81.2. *Let* $S, T \in \mathrm{TNSR}^+$, $c \in \mathbb{R}$, $f : S \dashrightarrow T$ *and* $p \in S$.
Then $D_p^{ST}(c \cdot f) =^* c \cdot D_p^{ST} f$.

THEOREM 81.3. *Let* $S, T \in \mathrm{TNSR}^+$, $f, g : S \dashrightarrow T$ *and* $p \in S$.
Then $(f + g)'_p =^* f'_p + g'_p$.

THEOREM 81.4. *Let $S, T \in \mathrm{TNSR}^+$, $c \in \mathbb{R}$, $f : S \dashrightarrow T$ and $p \in S$.*

$$\text{Then } (c \cdot f)'_p =^* c \cdot f'_p.$$

82. The multivariable product rule

DEFINITION 82.1. *Let X, Y and Z be sets. Let $f : X \times Y \dashrightarrow Z$. Then, $\forall a \in X$, $f(a, \bullet) : Y \dashrightarrow Z$ is defined by $(f(a, \bullet))(b) = f(a, b)$. Also, $\forall b \in Y$, $f(\bullet, b) : X \dashrightarrow Z$ is defined by $(f(\bullet, b))(a) = f(a, b)$.*

DEFINITION 82.2. *Let $S, T, U \in \mathrm{TNSR}^+$.*

Then $\mathcal{B}_{ST}^U := \{ B : S \times T \to U \quad | \quad (\forall v \in S,\, B(v, \bullet) \in \mathcal{L}_{TU}) \;\&\; (\forall w \in T,\, B(\bullet, w) \in \mathcal{L}_{SU}) \}$.

Elements of $\mathcal{B}_{ST}^U$ are called **bilinear** functions from S times T to U.

NOTE TO SELF: We need to establish that every bilinear is bounded, which we plan to make HW#7-4. This is needed to prove $\widehat{\mathcal{O}} * \widehat{\mathcal{O}} \subseteq \widehat{\mathcal{O}}$, etc., which, in turn is needed for the Product Rule (Theorem 82.10).

DEFINITION 82.3. *Let X, Y and Z be sets.*

$$\text{Let } * : X \times Y \dashrightarrow Z, \quad a \in X \quad \text{and} \quad b \in Y.$$
$$\text{Then} \quad a * b := *(a, b).$$

Every dot product is bilinear:

THEOREM 82.4. *Let $S \in \mathrm{TNSR}^+$. Define $* : S \times S \to \mathbb{R}$ by $v * w = v \bullet_S w$. Then $* \in \mathcal{B}_{SS}^{\mathbb{R}}$.*

DEFINITION 82.5. *Let $R, S, T, U \in \mathrm{TNSR}^+$ and let $* \in \mathcal{B}_{ST}^U$.*

Then: $\forall f : R \dashrightarrow S$, $\forall g : R \dashrightarrow T$,

$f * g : R \dashrightarrow U$ *is defined by* $(f * g)_p = f_p * g_p$.

Also, $\forall v \in S$, $\forall g : R \dashrightarrow T$, $v * g := C_R^v * g$.

Also, $\forall f : R \dashrightarrow S$, $\forall w \in T$, $f * w := f * C_R^w$.

NOTE TO SELF: Need to define bilinear products of sets of functions, and, also, compositions of sets of functions.

The following is proved in the same was as for functions $\mathbb{R} \dashrightarrow \mathbb{R}$. Needed: Boundedness of bilinear maps between tensor spaces, which we plan to make HW#7-4.

THEOREM 82.6. *Let $U, W, X, Z \in \mathrm{TNSR}^+$, $* \in \mathcal{B}_{WX}^Z$, $j, k \in \mathbb{N}_0$.*

Then $\widehat{\mathcal{O}}_j^{UW} * \mathcal{O}_k^{UX} \subseteq \mathcal{O}_{j+k}^{UZ}$.

Also, $\mathcal{O}_j^{UW} * \widehat{\mathcal{O}}_k^{UX} \subseteq \mathcal{O}_{j+k}^{UZ}$.

Also, $\widehat{\mathcal{O}}_j^{UW} * \widehat{\mathcal{O}}_k^{UX} \subseteq \widehat{\mathcal{O}}_{j+k}^{UZ}$.

The following is proved in the same was as for functions $\mathbb{R} \dashrightarrow \mathbb{R}$.

THEOREM 82.7. *Let* $V, W, X \in \mathrm{TNSR}^+$, $j, k \in \mathbb{N}$.
Then $\widehat{\mathcal{O}}_j^{WX} \circ \mathcal{o}_k^{VW} \subseteq \mathcal{o}_{jk}^{VX}$.
Also, $\mathcal{o}_j^{WX} \circ \widehat{\mathcal{O}}_k^{VW} \subseteq \mathcal{o}_{jk}^{VX}$.
Also, $\widehat{\mathcal{O}}_j^{WX} \circ \widehat{\mathcal{O}}_k^{VW} \subseteq \widehat{\mathcal{O}}_{jk}^{VX}$.

Theorem 82.8, below, is called the **Multivariable Precalculus Product Rule**.

THEOREM 82.8. *Let* $R, S, T, U \in \mathrm{TNSR}^+$ *and let* $* \in \mathcal{B}_{ST}^U$.
Let $f : R \dashrightarrow S$, $g : R \dashrightarrow T$ *and* $p \in R$.
Then $(f * g)_p^T \;=\; (f_p^T) * g_p \;+\; f_p * (g_p^T) \;+\; (f_p^T) * (g_p^T)$.

Proof. It is an unassigned HW problem to modify the proof of Theorem 69.4 to prove this theorem. □

THEOREM 82.9. *Let* $R, S, T, U \in \mathrm{TNSR}^+$ *and let* $* \in \mathcal{B}_{ST}^U$.
Then: $\forall L \in \mathcal{L}_{RS}$, $\forall w \in T$, $L * w \in \mathcal{L}_{RU}$.
Also, $\forall v \in S$, $\forall M \in \mathcal{L}_{RT}$, $v * M \in \mathcal{L}_{RU}$.

Theorem 82.10, below, is called the **Linearization Product Rule**.

THEOREM 82.10. *Let* $R, S, T, U \in \mathrm{TNSR}^+$ *and let* $* \in \mathcal{B}_{ST}^U$.
Let $f : R \dashrightarrow S$, $g : R \dashrightarrow T$ *and* $p \in R$.
Then $D_p^{RU}(f * g) \;=^*\; (D_p^{RS} f) * g_p \;+\; f_p * (D_p^{RT} g)$.

Proof. It is an unassigned HW problem to modify the proof of Theorem 69.5 to prove this theorem. Needed: Boundedness of bilinear maps between tensor spaces, which we plan to make HW#7-4. □

Theorem 82.11, below, is called the **Tensorial Product Rule**.

THEOREM 82.11. *Let* $R, S, T, U \in \mathrm{TNSR}^+$ *and let* $* \in \mathcal{B}_{ST}^U$.
Let $f : R \dashrightarrow S$, $g : R \dashrightarrow T$ *and* $p \in R$.
Let $S' := S \otimes R$ *and* $T' := T \otimes R$ *and* $U' := U \otimes R$.
Define $*_L \in \mathcal{B}_{S'T}^{U'}$ *by* $A \;*_L\; b = \mathrm{s}\ell_\bullet^{RU}(\mathrm{Lin}_A^{RS} * b)$.
Define $*_R \in \mathcal{B}_{ST'}^{U'}$ *by* $a \;*_R\; B = \mathrm{s}\ell_\bullet^{RU}(a * \mathrm{Lin}_B^{RS})$.
Then $(f * g)'_p \;=^*\; f'_p \;*_L\; g_p \;+\; f_p \;*_R\; g'_p$.

Proof. It is an unassigned HW problem to modify the proof of Theorem 69.5 to prove this theorem. Needed: Boundedness of bilinear maps between tensor spaces, which we plan to make HW#7-4. □

DEFINITION 82.12. *Let* $S, T \in \mathrm{TNSR}^+$.
Then $\mathrm{s}\ell_\bullet^{ST} : \mathcal{L}_{ST} \to T \otimes S$ *is defined by* $(\mathrm{s}\ell_\bullet^{ST})_L = \mathrm{s}\ell_L^{ST}$.

THEOREM 82.13. *Let* $S, T \in \mathrm{TNSR}^+$.
Then $\mathrm{s}\ell_\bullet^{ST} : \mathcal{L}_{ST} \hookrightarrow\!\!> T \otimes S$.

In fact, not only is $\mathrm{s}\ell_\bullet^{ST} : \mathcal{L}_{ST} \to T \otimes S$ bijective, but, also, it is algebraically linear, and its inverse is algebraically linear, too. We discussed all this in class.

DEFINITION 82.14. *Let* $S, T \in \mathrm{TNSR}^+$.
Then, $\forall L \in \mathcal{L}_{ST}$, $\quad [L]_{ST} := \mathrm{s}\ell_L^{ST}$.
Also, $\forall A \in T \otimes S$, $\quad \mathrm{Lin}_A^{ST} := (\mathrm{s}\ell_\bullet^{ST})_A^{-1}$.

THEOREM 82.15. *Let* $S, T \in \mathrm{TNSR}^+$.
Then, $\forall A \in T \otimes S$, $\quad [\mathrm{Lin}_A^{ST}]_{ST} = A$.
Also, $\forall L \in \mathcal{L}_{ST}$, $\quad \mathrm{Lin}_{[L]_{ST}}^{ST} = L$.

THEOREM 82.16. *Let* $S, T \in \mathrm{TNSR}^+$.
Then, $\forall L, M \in \mathcal{L}_{ST}$, $\quad [L + M]_{ST} = [L]_{ST} + [M]_{ST}$.
Also, $\forall c \in \mathbb{R}$, $\forall L \in \mathcal{L}_{ST}$, $\quad [c \cdot L]_{ST} = c \cdot ([L]_{ST})$.
Also, $\forall A, B \in T \otimes S$, $\quad \mathrm{Lin}_{A+B} = \mathrm{Lin}_A + \mathrm{Lin}_B$.
Also, $\forall c \in \mathbb{R}$, $\forall A \in T \otimes S$, $\quad \mathrm{Lin}_{c \cdot A} = c \cdot (\mathrm{Lin}_A)$.

DEFINITION 82.17. *Let* $R, S, T, U \in \mathrm{TNSR}^+$, $* \in \mathcal{B}_{ST}^U$.
Then, $\forall A \in S \otimes R$, $\forall y \in T$, $\quad A * y := [L_A^{RS} * y]_{RU}$.
Also, $\forall x \in S$, $\forall B \in T \otimes R$, $\quad x * B := [x * L_B^{RT}]_{RU}$.

THEOREM 82.18. *Let* $R, S, T, U \in \mathrm{TNSR}^+$, $* \in \mathcal{B}_{ST}^U$.
Let $f : R \dashrightarrow S$, $\quad g : R \dashrightarrow T$, $\quad p \in R$.
Then: $\quad f_p' \;=^*\; f_p' * g_p \;+\; f_p * g_p'$.

83. The multivariable Chain Rule

Theorem 83.1, below, is called the **Linearization Chain Rule**.

THEOREM 83.1. *Let* $S, T, U \in \mathrm{TNSR}^+$.
Let $f : S \dashrightarrow T$, $\quad g : T \dashrightarrow U$, $\quad p \in S$, $\quad q := f_p$.
Then: $\quad D_p^{SU}(g \circ f) \;=^*\; (D_q^{TU} g) \circ (D_p^{ST} f)$.

Proof. It is an unassigned HW problem to modify the proof of Theorem 69.25 to prove this theorem. ∎

DEFINITION 83.2. *Let* $S, T, U \in \mathrm{TNSR}^+$.

Then $*_{STU} \in \mathcal{B}^{U\otimes S}_{U\otimes T, T\otimes S}$ *is defined by*

$$B \;\; *_{STU} \;\; A \quad = \quad [\;(\mathrm{Lin}^{TU}_B) \;\circ\; (\mathrm{Lin}^{ST}_A)\;]_{SU}.$$

Theorem 83.3, below, is called the **Tensorial Chain Rule**.

THEOREM 83.3. *Let* $S, T, U \in \mathrm{TNSR}^+$.

Let $f : S \dashrightarrow T$, $\quad g : T \dashrightarrow U$, $\quad p \in S$, $\quad q := f_p$.

Then: $\quad (g \circ f)'_p \quad =^* \quad g'_q \;\; *_{STU} \;\; f'_p$.

THEOREM 83.4. *Let* $T \in \mathrm{TNSR}$, $v \in T$.

$$\textit{Then:} \quad v \quad = \quad \sum_{j\in\mathcal{I}_T} v_j \varepsilon^T_j.$$

Proof. This is HW#6-1. ∎

THEOREM 83.5. *Let* $S, T, U \in \mathrm{TNSR}$, $A \in T \otimes S$,

$L := \mathrm{Lin}^{ST}_A$, $i \in \mathcal{I}_S$, $j \in \mathcal{I}_T$.

Then $A_{j\|i} = (L(\varepsilon^S_i))_j$.

Proof. Let $k := j\|i$. Then $k_T = j$ and $k_S = i$.
Since $L = \mathrm{Lin}^{ST}_A$, we get: $[L]_{ST} = A$. Then $A = [L]_{ST} = \mathrm{s}\ell^{ST}_L$.
Then $A_{j\|i} = A_k = (\mathrm{s}\ell^{ST}_L)_k = \pi^T_{k_T}(L(\varepsilon^S_{k_S})) = \pi^T_j(L(\varepsilon^S_i)) = (L(\varepsilon^S_i))_j$. ∎

THEOREM 83.6. *Let* $S, T, U \in \mathrm{TNSR}$.

Let $\quad B \in U \otimes T$, $\quad A \in T \otimes S$, $\quad k \in \mathcal{I}_U$ $\quad$ *and* $\quad i \in \mathcal{I}_S$.

$$\textit{Then:} \quad (B \;\; *_{STU} \;\; A)_{k\|i} \quad = \quad \sum_{j\in\mathcal{I}_T} (\,B_{k\|j}\,) \cdot (\,A_{j\|i}\,).$$

Proof. Let $L := \mathrm{Lin}^{ST}_A$ and let $M := \mathrm{Lin}^{TU}_B$.
Then $L \in \mathcal{L}_{ST}$ and $M \in \mathcal{L}_{TU}$, so $M \circ L \in \mathcal{L}_{SU}$. Let $C \;:=\; B \;\; *_{STU} \;\; A$.
Then, by Definition 83.2, we have $C = [M \circ L]_{SU}$.
Let $N := \mathrm{Lin}^{SU}_C$. By Theorem 83.5, we have: $\quad C_{k\|i} = (N(\varepsilon^S_i))_k$.
Also, by Theorem 83.5, we have: $\quad \forall j \in \mathcal{I}_T, \quad B_{k\|j} = (M(\varepsilon^T_j))_k$.
By HW#6-2, we have: $L(\varepsilon^S_i) \;=\; \sum_{j\in\mathcal{I}_T} (\,A_{j\|i}\,) \cdot (\,\varepsilon^T_j\,)$.
Since $C = [M \circ L]_{SU}$, we get $\mathrm{Lin}^{SU}_C = M \circ L$. Then $N = M \circ L$.
Then $(B \;\; *_{STU} \;\; A)_{k\|i} \;=\; C_{k\|i} \;=\; (N(\varepsilon^S_i))_k \;=\; ((M \circ L)(\varepsilon^S_i))_k$

$$= \; (M(L(\varepsilon^S_i)))_k \; = \; \left(M \left(\sum_{j\in\mathcal{I}_T} (\,A_{j\|i}\,) \cdot (\,\varepsilon^T_j\,) \right) \right)_k$$

$$= \left(\sum_{j \in \mathcal{I}_T} (\, A_{j\|i} \,) \cdot [\, M(\varepsilon_j^T) \,] \right)_k = \sum_{j \in \mathcal{I}_T} (\, A_{j\|i} \,) \cdot [\, (\, M(\varepsilon_j^T) \,)_k \,]$$
$$= \sum_{j \in \mathcal{I}_T} (\, A_{j\|i} \,) \cdot (\, B_{k\|j} \,) = \sum_{j \in \mathcal{I}_T} (\, B_{k\|j} \,) \cdot (\, A_{j\|i} \,),$$

as desired. □

THEOREM 83.7. *Let* $\ell, m, n \in \mathbb{N}$, $B \in \mathbb{R}^{n \times m}$, $A \in \mathbb{R}^{m \times \ell}$,

$S := \mathbb{R}^\ell$, $T := \mathbb{R}^m$, $U := \mathbb{R}^n$.

Then $B \ *_{STU} \ A = BA$.

Proof. We have $B \ *_{STU} \ A \ , \ BA \in \mathbb{R}^{[1..n] \times [1..\ell]}$.
Want: $\forall k \in [1..n]$, $\forall i \in [1..\ell]$, $(\, B \ *_{STU} \ A \,)_{ki} = (\, BA \,)_{ki}$.
Given $k \in [1..n]$, $i \in [1..\ell]$. Want: $(\, B \ *_{STU} \ A \,)_{ki} = (\, BA \,)_{ki}$.
We have $T = \mathbb{R}^m = \mathbb{R}^{(m)}$, so $\mathcal{I}_T = [(m)] = [1..m]$.
By Theorem 83.6, $(\, B \ *_{STU} \ A \,)_{ki} = \sum_{j \in \mathcal{I}_T} B_{k\|j} \cdot A_{j\|i}$.

$$\text{Then } (\, B \ *_{STU} \ A \,)_{ki} = \sum_{j \in \mathcal{I}_T} B_{k\|j} \cdot A_{j\|i}$$
$$= \sum_{j \in [1..m]} B_{kj} \cdot A_{ji} = (\, BA \,)_{ki}, \text{ as desired.}$$ □

THEOREM 83.8. *Let* $\ell, m, n \in \mathbb{N}$, $f : \mathbb{R}^\ell \to \mathbb{R}^m$ *and* $g : \mathbb{R}^m \to \mathbb{R}^n$. *Let* $p \in \mathbb{R}^\ell$ *and* $q := f_p$. *Then* $(g \circ f)'_p =^* (g'_q) \cdot (f'_p)$.

Proof. Let $S := \mathbb{R}^\ell$, $T := \mathbb{R}^m$ and $U := \mathbb{R}^n$.
Want: $(\, (g'_q) \cdot (f'_p) \neq ☹ \,) \Rightarrow (\, (g \circ f)'_p = (g'_q) \cdot (f'_p) \,)$.
Assume: $(g'_q) \cdot (f'_p) \neq ☹$ Want: $(g \circ f)'_p = (g'_q) \cdot (f'_p)$.
Since $(g'_q) \cdot (f'_p) \neq ☹$, we get $g'_q \in \mathbb{R}^{n \times m}$ and $f'_p \in \mathbb{R}^{m \times \ell}$,
Then, by Theorem 83.7, $(g'_q) \ *_{STU} \ (f'_p) = (g'_q) \cdot (f'_p)$.
By Theorem 83.3, we have $(g \circ f)'_p =^* g'_q \ *_{STU} \ f'_p$.
So, since $(g'_q) \ *_{STU} \ (f'_p) = (g'_q) \cdot (f'_p) \neq ☹$,
we get: $(g \circ f)'_p = g'_q \ *_{STU} \ f'_p$.
Then: $(g \circ f)'_p = g'_q \ *_{STU} \ f'_p = (g'_q) \cdot (f'_p)$. □

84. Basic results in tensor algebra

DEFINITION 84.1. *Let* $S, T \in \mathrm{TNSR}^+$.
Then $\mathcal{C}_S^T := \mathcal{C}_{ST}$ *and*
$\mathcal{L}_S^T := \mathcal{L}_{ST}$ *and*
$\mathcal{Q}_S^T := \mathcal{Q}_{ST}$ *and*
$\mathcal{K}_S^T := \mathcal{K}_{ST}$ *and*

$$\mathbf{0}_S^T \;:=\; \mathbf{0}_{ST} \qquad \textit{and}$$
$$\forall L \in \mathcal{L}_{ST}, \qquad [L]_{ST} \;:=\; [L]_{ST}.$$

DEFINITION 84.2. *Let X, Y and Z be sets.*
Let $f : X \times Y \dashrightarrow Z$.
Then $f(\bullet,\bullet) : X \cap Y \dashrightarrow Z$ is defined by $(f(\bullet,\bullet))(s) = f(s,s)$.

DEFINITION 84.3. *Let W, X, Y and Z be sets.*
Let $f : W \times X \times Y \dashrightarrow Z$.
Then $\forall a \in W$, $\forall b \in X$, $f(a,b,\bullet) : Y \dashrightarrow Z$ is defined by $(f(a,b,\bullet))(s) = f(a,b,s)$.
Also, $\forall a \in W$, $\forall c \in Y$, $f(a,\bullet,c) : X \dashrightarrow Z$ is defined by $(f(a,\bullet,c))(s) = f(a,s,c)$.
Also, $\forall b \in X$, $\forall c \in Y$, $f(\bullet,b,c) : W \dashrightarrow Z$ is defined by $(f(\bullet,b,c))(s) = f(s,b,c)$.
Also, $\forall a \in W$, $f(a,\bullet,\bullet) : X \cap Y \dashrightarrow Z$ is defined by $(f(a,\bullet,\bullet))(s) = f(a,s,s)$.
Also, $\forall a \in W$, $f(a,\bullet,\bullet\bullet) : X \times Y \dashrightarrow Z$ is defined by $(f(a,\bullet,\bullet\bullet))(s,t) = f(a,s,t)$.
Also, $\forall b \in X$, $f(\bullet,b,\bullet) : W \cap Y \dashrightarrow Z$ is defined by $(f(\bullet,b,\bullet))(s) = f(s,b,s)$.
Also, $\forall b \in X$, $f(\bullet,b,\bullet\bullet) : W \times Y \dashrightarrow Z$ is defined by $(f(\bullet,b,\bullet\bullet))(s,t) = f(s,b,t)$.
Also, $\forall c \in Y$, $f(\bullet,\bullet,c) : W \cap X \dashrightarrow Z$ is defined by $(f(\bullet,\bullet,c))(s) = f(s,s,c)$.
Also, $\forall c \in Y$, $f(\bullet,\bullet\bullet,c) : W \times X \dashrightarrow Z$ is defined by $(f(\bullet,\bullet\bullet,c))(s,t) = f(s,t,c)$.
Also, $f(\bullet,\bullet,\bullet) : W \cap X \cap Y \dashrightarrow Z$ is defined by $(f(\bullet,\bullet,\bullet))(s) = f(s,s,s)$.

Recall, from Definition 82.2, that $\mathcal{B}_{TU}^V$ denotes the set of bilinear maps $T \times U \to V$. We next introduce a notation for the set of symmetric bilinear maps $U \times U \to V$, as follows:

DEFINITION 84.4. *Let $U, V \in \mathrm{TNSR}^+$.*
Then $\mathcal{SB}_U^V \;:=\; \{F \in \mathcal{B}_{UU}^V \,|\, \forall a,b \in U,\ F(a,b) = F(b,a)\}$.

We next introduce trilinear and symmetric trilinear notation:

DEFINITION 84.5. *Let $V, W, X, Y \in \mathrm{TNSR}^+$.*
Then $\mathcal{T}_{VWX}^Y \;:=\; \{\ F : V \times W \times X \to Y\ |$

$$\begin{aligned}&\forall a\in V,\,\forall b\in W, \quad f(a,b,\bullet)\in\mathcal{L}_X^Y \qquad \textit{and}\\ &\forall a\in V,\,\forall c\in X, \quad f(a,\bullet,c)\in\mathcal{L}_W^Y \qquad \textit{and}\\ &\forall b\in W,\,\forall c\in X, \quad f(\bullet,b,c)\in\mathcal{L}_V^Y\ \}.\end{aligned}$$

DEFINITION 84.6. *Let* $X, Y \in \mathrm{TNSR}^+$.
Then $\mathcal{ST}_X^Y \ := \ \{ \ F \in \mathcal{T}_{XXX}^Y \mid \forall a, b, c \in X,$

$$F(a,b,c) = F(a,c,b) = F(b,a,c) = F(b,c,a) = F(c,a,b) = F(c,b,a) \ \}.$$

THEOREM 84.7. *Let* $V, W \in \mathrm{TNSR}^+$ *and let* $B \in \mathcal{B}_{VV}^W$.
Then, $\forall x \in V, \quad B(x,\bullet)\,,\ B(\bullet,x) \ \in \ \mathcal{L}_V^W$. *Also,* $B(\bullet,\bullet) \in \mathcal{Q}_V^W$.

THEOREM 84.8. *Let* $V, W \in \mathrm{TNSR}^+$ *and let* $T \in \mathcal{T}_{VVV}^W$.
Then, $\forall x, y \in V, \quad T(x,y,\bullet)\,,\ T(x,\bullet,y)\,,\ T(\bullet,x,y) \ \in \ \mathcal{L}_V^W$.
Also, $\forall x \in V, \quad T(x,\bullet,\bullet)\,,\ T(\bullet,x,\bullet)\,,\ T(\bullet,\bullet,x) \ \in \ \mathcal{Q}_V^W$.
Also, $\forall x \in V, \quad T(x,\bullet,\bullet\bullet)\,,\ T(\bullet,x,\bullet\bullet)\,,\ T(\bullet,\bullet\bullet,x) \ \in \ \mathcal{B}_{VV}^W$.
Also, $T(\bullet,\bullet,\bullet) \in \mathcal{K}_V^W$.

THEOREM 84.9. *Let* $V, W \in \mathrm{TNSR}^+$, $K \in \mathcal{K}_V^W$ *and* $F \in \mathcal{ST}_V^W$.
Assume $K = F(\bullet,\bullet,\bullet)$. *Then,* $\forall x \in V$, $\quad D_x^{VW} K = 3\cdot(F(x,x,\bullet))$.

Proof. Given $x \in V$. Want: $D_x K = 3 \cdot (F(x,x,\bullet))$.
We have: $\forall h \in S$,

$$\begin{aligned} K_x^T(h) &= (K(x+h)) - (K(x)) \\ &= (F(x+h,x+h,x+h)) - (F(x,x,x)) \\ &= (F(x,x,x)) + 3\cdot(F(x,x,h)) + 3\cdot(F(x,h,h)) + (F(h,h,h)) \\ &\qquad\qquad - (F(x,x,x)) \\ &= 3\cdot(F(x,x,h)) + 3\cdot(F(x,h,h)) + (F(h,h,h)). \end{aligned}$$

Then $K_x^T = 3\cdot(F(x,x,\bullet)) + 3\cdot(F(x,\bullet,\bullet)) + \cdot(F(\bullet,\bullet,\bullet))$.
Let $L := 3\cdot(F(x,x,\bullet))$ and let $R := 3\cdot(F(x,\bullet,\bullet)) + \cdot(F(\bullet,\bullet,\bullet))$.
Then $K_x^T = L + R$ and $L \in \mathcal{L}_V^W$.
We have $F(x,\bullet,\bullet) \in \mathcal{Q}_V^W \subseteq \widehat{\mathcal{O}}_2^{VW} \subseteq \mathcal{o}_1^{VW}$.
Also, $F(\bullet,\bullet,\bullet) \in \mathcal{K}_V^W \subseteq \widehat{\mathcal{O}}_3^{VW} \subseteq \mathcal{o}_2^{VW} \subseteq \mathcal{o}_1^{VW}$.
Then $R = 3\cdot(F(x,\bullet,\bullet)) + \cdot(F(\bullet,\bullet,\bullet)) \in \mathcal{o}_1^{VW}$.
Then $K_x^T - L = L + R - L = R + \mathbf{0}_V^W = R \in \mathcal{o}_1^{VW}$.
Then $L \in \mathrm{LINS}_p K$, and so $D_p^{VW} K = L$.
Then $D_p^{VW} K = L = 3\cdot(F(x,x,\bullet))$, as desired. □

THEOREM 84.10. *Let* $V, W \in \mathrm{TNSR}$, $L \in \mathcal{L}_V^W$, $A := [L]_V^W$, $x \in V$.

$$\textit{Then } L(x) = \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} A_{j\|i}\cdot x_i\cdot \varepsilon_j^W.$$

Proof. By HW#6-2, we have: $\forall i \in \mathcal{I}_V,\quad L(\varepsilon_i^V) = \sum_{j\in\mathcal{I}_W} A_{j\|i} \cdot \varepsilon_j^W.$

We compute:

$$\begin{aligned} L(x) = L\left(\sum_{i\in\mathcal{I}_V} x_i\varepsilon_i^V\right) &= \sum_{i\in\mathcal{I}_V} x_i \cdot (L(\varepsilon_i^V)) \\ &= \sum_{i\in\mathcal{I}_V} x_i \cdot \left(\sum_{j\in\mathcal{I}_W} A_{j\|i} \cdot \varepsilon_j^W\right) \\ &= \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} x_i \cdot A_{j\|i} \cdot \varepsilon_j^W \\ &= \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} A_{j\|i} \cdot x_i \cdot \varepsilon_j^W, \qquad \text{as desired.} \end{aligned}$$

□

The following theorem is the **Cauchy-Schwarz inequality**:

THEOREM 84.11. *Let I be a finite set and let $x, y \in \mathbb{R}^I$.*

$$\textit{Then } \sum_{j\in I} x_j y_j \leqslant \sqrt{\sum_{j\in I} x_j^2} \cdot \sqrt{\sum_{j\in I} y_j^2}.$$

Proof. Let $a := \sqrt{\sum_{j\in I} x_j^2}$ and let $b := \sqrt{\sum_{j\in I} y_j^2}$.

Then $a^2 = \sum_{j\in I} x_j^2$ and $b^2 = \sum_{j\in I} y_j^2$. Want: $\sum_{j\in I} x_j y_j \leqslant ab$.

At least one of the following must be true:

(1) $a = 0$ or (2) $b = 0$ or (3) $a \neq 0 \neq b$.

Case (1):

We have: $\sum_{j\in I} x_j^2 = a^2 = 0^2 = 0$. Then: $\forall j \in I$, $x_j = 0$.

Then $\sum_{j\in I} x_j y_j = \sum_{j\in I} 0 \cdot y_j = 0 \leqslant 0 = 0 \cdot b = ab$, as desired.

End of Case (1).

Case (2):

We have $\sum_{j\in I} y_j^2 = b^2 = 0^2 = 0$. Then: $\forall j \in I$, $y_j = 0$.

Then $\sum_{j\in I} x_j y_j = \sum_{j\in I} x_j \cdot 0 = 0 \leqslant 0 = a \cdot 0 = ab$, as desired.

End of Case (2).

Case (3):

Let $u := \dfrac{x}{a}$ and $v := \dfrac{y}{b}$.

Then, $\forall j \in I$, $u_j = \dfrac{x_j}{a}$ and $v_j = \dfrac{y_j}{b}$.

Then, $\forall j \in I$, $au_j = x_j$ and $bv_j = y_j$.

We have: $\forall t \in \mathbb{R}$, $0 \leqslant t^2$.

Then $\forall j \in I$, $0 \leqslant (u_j - v_j)^2$, and so $0 \leqslant \sum_{j \in I} (u_j - v_j)^2$.

$$\text{Then } 0 \leqslant \sum_{j\in I}(u_j - v_j)^2 = \sum_{j\in I}(u_j^2 - 2u_jv_j + v_j^2)$$
$$= \left(\sum_{j\in I} u_j^2\right) - 2\cdot\left(\sum_{j\in I} u_jv_j\right) + \left(\sum_{j\in I} v_j^2\right).$$

$$\text{Then } 2\cdot\left(\sum_{j\in I} u_jv_j\right) \leqslant \left(\sum_{j\in I} u_j^2\right) + \left(\sum_{j\in I} v_j^2\right)$$
$$= \left(\sum_{j\in I}\left(\frac{x_j}{a}\right)^2\right) + \left(\sum_{j\in I}\left(\frac{y_j}{b}\right)^2\right)$$
$$= \left(\sum_{j\in I}\frac{x_i^2}{a^2}\right) + \left(\sum_{j\in I}\frac{y_j^2}{b^2}\right)$$
$$= \left(\frac{1}{a^2}\right)\cdot\left(\sum_{j\in I} x_i^2\right) + \left(\frac{1}{b^2}\right)\cdot\left(\sum_{j\in I} y_j^2\right)$$
$$= \left(\frac{1}{a^2}\right)\cdot a^2 + \left(\frac{1}{b^2}\right)\cdot b^2 = 1 + 1 = 2.$$

Mulitplying this by $\dfrac{ab}{2}$, we get: $ab \cdot \sum_{j\in I} u_jv_j \leqslant ab$.

Then $\sum_{j\in I} x_jy_j = \sum_{j\in I}(au_j)\cdot(bv_j) = ab\cdot\sum_{j\in I} u_jv_j \leqslant ab$, as desired.

End of Case (3). □

THEOREM 84.12. *Let* $U, V, W, X \in \mathrm{TNSR}$ *and let* $T \in \mathcal{T}^X_{UVW}$.
Then $\exists C \geqslant 0$ *s.t.,* $\forall p \in U$, $\forall q \in V$, $\forall r \in W$,
$$|T(p,q,r)|_X \leqslant C \cdot |p|_U \cdot |q|_V \cdot |r|_W.$$

Proof. Unassigned HW, modeled on HW#7-4. □

THEOREM 84.13. *Let* $U, V \in \mathrm{TNSR}$ *and let* $L \in \mathcal{L}^V_U$.
Then $\exists C \geqslant 0$ *s.t.,* $\forall p \in U$,
$$|L(p)|_V \leqslant C \cdot |p|_U.$$

Proof. Unassigned HW, modeled on HW#7-4. □

DEFINITION 84.14. *Let* $V, W, X \in \mathrm{TNSR}$ *and let* $B \in \mathcal{B}_{VW}^X$.
Then $[B]_{VW}^X \in X \otimes V \otimes W$ *is defined by:*
$$\forall i \in \mathcal{I}_V,\ \forall j \in \mathcal{I}_W,\ \forall k \in \mathcal{I}_X$$
$$([B]_{VW}^X)_{k\|i\|j} = \pi_k^X(\,B(\varepsilon_i^V, \varepsilon_j^W)\,).$$

Unassigned HW: Extend Definition 84.14 to $V, W, X \in \mathrm{TNSR}^+$.

For example: Let $V, X \in \mathrm{TNSR}$, $W := \mathbb{R}$ and let $B \in \mathcal{B}_{VW}^X$. Then $[B]_{VW}^X \in X \otimes V$ is defined by:
$$\forall i \in \mathcal{I}_V,\ \forall k \in \mathcal{I}_X$$
$$([B]_{VW}^X)_{k\|j} = \pi_k^X(\,B(\varepsilon_i^V, 1)\,).$$
Note that $[B]_{VW}^X \in X \otimes V = X \otimes V \otimes \mathbb{R} = X \otimes V \otimes W$.

DEFINITION 84.15. *Let* $U, V, W, X \in \mathrm{TNSR}$ *and let* $T \in \mathcal{T}_{UVW}^X$.
Then $[T]_{UVW}^X \in X \otimes U \otimes V \otimes W$ *is defined by:*
$$\forall i \in \mathcal{I}_U,\ \forall j \in \mathcal{I}_V,\ \forall k \in \mathcal{I}_W,\ \forall \ell \in \mathcal{I}_X$$
$$([T]_{UVW}^X)_{\ell\|i\|j\|k} = \pi_\ell^X(\,T(\varepsilon_i^U, \varepsilon_j^V, \varepsilon_k^W)\,).$$

Unassigned HW: Extend Definition 84.15 to $U, V, W, X \in \mathrm{TNSR}^+$.

For example: Let $U, V, W \in \mathrm{TNSR}$, $X := \mathbb{R}$ and let $T \in \mathcal{T}_{UVW}^X$. Then $[T]_{UVW}^X \in U \otimes V \otimes W$ is defined by:
$$\forall i \in \mathcal{I}_U,\ \forall j \in \mathcal{I}_V,\ \forall k \in \mathcal{I}_W,$$
$$([T]_{UVW}^X)_{i\|j\|k} = T(\varepsilon_i^U, \varepsilon_j^V, \varepsilon_k^W).$$
Note that $[T]_{UVW}^X \in U \otimes V \otimes W = \mathbb{R} \otimes U \otimes V \otimes W = X \otimes U \otimes V \otimes W$.

NOTE TO SELF:
Next year, define $\mathcal{I}_\mathbb{R} := \{\varnothing\}$.
Keep in mind that $\mathbb{R} \neq \mathbb{R}^{\mathcal{I}_\mathbb{R}}$,
so we only have: $\forall V \in \mathrm{TNSR}$, $V = \mathbb{R}^{\mathcal{I}_V}$;
this does not extend to TNSR^+.
Also, $\forall x \in \mathbb{R}$, define $x_\varnothing := x$.
Also, define $\varepsilon_\varnothing^\mathbb{R} := 1$.
Also, define $\pi_\varnothing^\mathbb{R} = \mathrm{id}_\mathbb{R}$.
Also, $\forall V \in \mathrm{TNSR}^+$, $\forall f : \mathbb{R} \dashrightarrow V$, define $\partial_\varnothing^{\mathbb{R}V} f = f'$. Also, $\forall \sigma \in \mathbb{N}$, $\forall k \in \mathbb{N}^\sigma$, define $k\|\varnothing := k$ and $\varnothing\|k := k$.
Also, define $\varnothing\|\varnothing := \varnothing$.

85. Derivatives of homogeneous cubics

NOTE: The second midterm in Spring 2019 will cover everything up to and including (2) of the next result, Theorem 85.1.

THEOREM 85.1. *Let* $V, W \in$ TNSR*,* $K \in \mathcal{K}_V^W$*,* $F \in \mathcal{ST}_V^W$*.*
Assume: $\forall x \in V,\ K(x) = F(x,x,x)$*.*
Define $Q \in \mathcal{Q}_V^{W\otimes V}$*,* $L \in \mathcal{L}_V^{W\otimes V\otimes V}$*,* $C \in \mathcal{C}_V^{W\otimes V\otimes V\otimes V}$ *by:*
$$Q_x = [F(x,x,\bullet)]_V^W, \qquad L_x = [F(x,\bullet,\bullet\bullet)]_{VV}^W, \qquad C_x = [F]_{VVV}^W.$$
Then: (1) $K' = 3Q$, (2) $K'' = 6L$, (3) $K''' = 6C$.

Proof. Proof of (1):
Want: $\forall x \in V,\ K'_x = 3 \cdot Q_x$.
Given $x \in V$. Want: $K'_x = 3 \cdot Q_x$.
By Theorem 84.9, we have: $D_x K = 3 \cdot (F(x,x,\bullet))$.
Then $K'_x = [D_x K]_V^W = 3 \cdot [F(x,x,\bullet)]_V^W = 3 \cdot Q_x$, as desired.
End of proof of (1).

Proof of (2):
Want: $\forall x \in V,\ K''_x = 6 \cdot L_x$.
Given $x \in V$. Want: $K''_x = 6 \cdot L_x$.
By (1) of Theorem 85.1, we have: $K' = 3 \cdot Q$.
Then, by Theorem 81.4, we get: $K''_x \ =^* \ 3 \cdot Q'_x$.
Want: $3 \cdot Q'_x = 6 \cdot L_x$.
Define $B \in \mathcal{SB}_V^{W\otimes V}$ by $B(x,y) = [F(x,y,\bullet)]_V^W$.
Then $Q = B(\bullet,\bullet)$, so, by HW#6-5, we have: $D_x^{VW} Q = 2 \cdot (B(x,\bullet))$.
Since $D_x^{VW} Q = 2 \cdot (B(x,\bullet)) \in \mathcal{L}_V^W$,
we get: $Q'_x = 2 \cdot [B(x,\bullet)]_V^W \in W \otimes V \otimes V$.
Also, by definition of L, we have: $L_x = [F(x,\bullet,\bullet\bullet)]_{VV}^W$.
Then $L_x = [F(x,\bullet,\bullet\bullet)]_{VV}^W \in W \otimes V \otimes V$.
Want: $3 \cdot 2 \cdot [B(x,\bullet)]_V^W = 6 \cdot [F(x,\bullet,\bullet\bullet)]_{VV}^W$.
Want: $[B(x,\bullet)]_V^W = [F(x,\bullet,\bullet\bullet)]_{VV}^W$.
Want: $\forall k \in \mathcal{I}_W,\ \forall i,j \in \mathcal{I}_V,\ (\,[B(x,\bullet)]_V^W\,)_{k\|i\|j} \ = \ (\,[F(x,\bullet,\bullet\bullet)]_{VV}^W\,)_{k\|i\|j}$.
Given $k \in \mathcal{I}_W,\ i,j \in \mathcal{I}_V$.
Want: $(\,[B(x,\bullet)]_V^W\,)_{k\|i\|j} \ = \ (\,[F(x,\bullet,\bullet\bullet)]_{VV}^W\,)_{k\|i\|j}$.
We have $(\,[F(x,\bullet,\bullet\bullet)]_{VV}^W\,)_{k\|i\|j} = \pi_k^W(F(x,\varepsilon_i^V,\varepsilon_j^V))$.
Also, $(\,[B(x,\bullet)]_V^W\,)_{k\|i\|j} \ = \ \pi_{k\|i}^{W\otimes V}(B(x,\varepsilon_j^V))$.
Also, $(\,[F(x,\varepsilon_j,\bullet)]_V^W\,)_{k\|i} = \pi_k^W(F(x,\varepsilon_j^V,\varepsilon_i^V))$.
By definition of B, we have: $B(x,\varepsilon_j^V) = [F(x,\varepsilon_j,\bullet)]_V^W$.
By symmetry of F, we have: $F(x,\varepsilon_j^V,\varepsilon_i^V) = F(x,\varepsilon_i^V,\varepsilon_j^V)$.
Then
$$\begin{aligned}(\,[B(x,\bullet)]_V^W\,)_{k\|i\|j} &= \pi_{k\|i}^{W\otimes V}(B(x,\varepsilon_j^V)) \ = \ (B(x,\varepsilon_j^V))_{k\|i} \\ &= (\,[F(x,\varepsilon_j,\bullet)]_V^W\,)_{k\|i} \ = \ \pi_k^W(F(x,\varepsilon_j^V,\varepsilon_i^V)) \\ &= \pi_k^W(F(x,\varepsilon_i^V,\varepsilon_j^V)) \ = \ (\,[F(x,\bullet,\bullet\bullet)]_{VV}^W\,)_{k\|i\|j},\end{aligned}$$
as desired.

End of proof of (2).

Proof of (3):
Want: $\forall x \in V$, $K'''_x = 6 \cdot C_x$.
Given $x \in V$. Want: $K'''_x = 6 \cdot C_x$.
By (1) of Theorem 85.1, we have: $K'' = 6 \cdot L$.
Then, by Theorem 81.4, we get: $K'''_x =^* 6 \cdot L'_x$.
Want: $6 \cdot L'_x = 6 \cdot C_x$. Want: $L'_x = C_x$.
By HW#6-4, we have $D_x L = L$.
Since $D_x L = L \in \mathcal{L}_V^{W \otimes V \otimes V}$,
we get: $L'_x = [L]_V^{W \otimes V \otimes V} \in W \otimes V \otimes V \otimes V$.
Also, by definition of C, we have: $C_x = [F]^W_{VVV}$.
Then $C_x = [F]^W_{VVV} \in W \otimes V \otimes V \otimes V$.
Want: $[L]_V^{W \otimes V \otimes V} = [F]^W_{VVV}$.
Want: $\forall i, j, k \in \mathcal{I}_V, \forall \ell \in \mathcal{I}_W$, $(\, [L]_V^{W \otimes V \otimes V}\,)_{\ell \| i \| j \| k} = (\, [F]^W_{VVV}\,)_{\ell \| i \| j \| k}$.
Given $i, j, k \in \mathcal{I}_V$, $\ell \in \mathcal{I}_W$.
Want: $(\, [L]_V^{W \otimes V \otimes V}\,)_{\ell \| i \| j \| k} = (\, [F]^W_{VVV}\,)_{\ell \| i \| j \| k}$.
We have: $(\, [F]^W_{VVV}\,)_{\ell \| i \| j \| k} = \pi^W_\ell(\, F(\varepsilon^V_i, \varepsilon^V_j, \varepsilon^V_k)\,)$.
Also, $(\, [L]_V^{W \otimes V \otimes V}\,)_{\ell \| i \| j \| k} = \pi^W_{\ell \| i \| j}(\, L(\varepsilon^V_k)\,)$.
Also, $(\, [\, F(\varepsilon^V_k, \bullet, \bullet\bullet)\,]^W_{VV}\,)_{\ell \| i \| j} = \pi^W_\ell(\, F(\varepsilon^V_k, \varepsilon^V_i, \varepsilon^V_j)\,)$.
By definition of L, we have $L(\varepsilon^V_k) = [\, F(\varepsilon^V_k, \bullet, \bullet\bullet)\,]^W_{VV}$.
By symmetry of F, we have $F(\varepsilon^V_k, \varepsilon^V_i, \varepsilon^V_j) = F(\varepsilon^V_i, \varepsilon^V_j, \varepsilon^V_k)$.
Then $(\, [L]_V^{W \otimes V \otimes V}\,)_{\ell \| i \| j \| k} = \pi^W_{\ell \| i \| j}(\, L(\varepsilon^V_k)\,) = (\, L(\varepsilon^V_k)\,)_{\ell \| i \| j}$
$= (\, [\, F(\varepsilon^V_k, \bullet, \bullet\bullet)\,]^W_{VV}\,)_{\ell \| i \| j} = \pi^W_\ell(\, F(\varepsilon^V_k, \varepsilon^V_i, \varepsilon^V_j)\,)$
$= \pi^W_\ell(\, F(\varepsilon^V_i, \varepsilon^V_j, \varepsilon^V_k)\,) = (\, [F]^W_{VVV}\,)_{\ell \| i \| j \| k}$, as desired.
End of proof of (3). □

THEOREM 85.2. *Let $S \in \mathrm{TNSR}^+$, $v \in S$, $t \in \mathbb{R}$, $M := \mathrm{Lin}^{\mathbb{R}S}_v$.*
Then $M_t = tv$.

Proof. Since $\mathrm{Lin}^{\mathbb{R}S}_\bullet$ and $[\bullet]_{\mathbb{R}S}$ are inverses, we get: $[\mathrm{Lin}^{\mathbb{R}S}_v]_{\mathbb{R}S} = v$.
By Definition 80.6, we have $\mathrm{s}\ell^{\mathbb{R}S}_M = M(1)$.
Then $M(1) = \mathrm{s}\ell^{\mathbb{R}S}_M = [M]_{\mathbb{R}S} = [\mathrm{Lin}^{\mathbb{R}S}_v]_{\mathbb{R}S} = v$.
Then $M_t = M(t) = M(t \cdot 1) = t \cdot (M(1)) = tv$, as desired. □

THEOREM 85.3. *Let $v \in \mathbb{R}$ and $L := \mathrm{Lin}^{\mathbb{R}\mathbb{R}}_x$.*
Then $L_1 = v$.

Proof. Replace S by $\mathbb{R}$ and t by 1 in Theorem 85.2. □

THEOREM 85.4. *Let* $S \in \mathrm{TNSR}^+$, $v \in S$, $t \in \mathbb{R}$.

Then $v \;\; *_{\mathbb{R}\mathbb{R}S} \;\; t \;\; = \;\; tv$.

Proof. Let $L := \mathrm{Lin}_t^{\mathbb{R}\mathbb{R}}$ and $M := \mathrm{Lin}_v^{\mathbb{R}S}$.
By Theorem 85.3 (with v replaced by t), we have $L_1 = t$.
By Theorem 85.2, we have $M_t = tv$.
By Definition 83.2, we have $v \;\; *_{\mathbb{R}\mathbb{R}S} \;\; t \;\; = \;\; [M \circ L]_{\mathbb{R}S}$.
By Definition 80.6, we have $\mathrm{s}\ell_{L \circ M}^{\mathbb{R}S} = (L \circ M)(1)$.
Then $v \;\; *_{\mathbb{R}\mathbb{R}S} \;\; t \;\; = \;\; [M \circ L]_{\mathbb{R}S} \;\; = \;\; \mathrm{s}\ell_{M \circ L}^{\mathbb{R}S} \;\; = \;\; (M \circ L)(1)$
$= \;\; M(L_1) \;\; = \;\; M(t) \;\; = \;\; M_t \;\; = \;\; tv$, as desired. □

THEOREM 85.5. *Let* $x, y \in \mathbb{R}$.

Then $x \;\; *_{\mathbb{R}\mathbb{R}\mathbb{R}} \;\; y \;\; = \;\; xy$.

Proof. Replacing S by $\mathbb{R}$ and v by x and t by y, we get: $x \;\; *_{\mathbb{R}\mathbb{R}\mathbb{R}} \;\; y \;\; = \;\; yx$.
Then $x \;\; *_{\mathbb{R}\mathbb{R}\mathbb{R}} \;\; y \;\; = \;\; yx \;\; = \;\; xy$, as desired.

□

Recall: Let $L \in \mathcal{L}$, $m := \mathrm{s}\ell_L$, $x \in \mathbb{R}$. Then $L_x = mx$.
Then next result is the tensor analogue for that statement:

THEOREM 85.6. *Let* $S, T \in \mathrm{TNSR}^+$, $L \in \mathcal{L}_S^T$, $A := \mathrm{s}\ell_L^{ST}$, $x \in S$.

Then $L(x) \;\; = \;\; A \;\; *_{\mathbb{R}ST} \;\; x$.

Proof. Let $F := \mathrm{Lin}_x^{\mathbb{R}S}$ and $G := \mathrm{Lin}_A^{ST}$.
By Definition 83.2, we have $A \;\; *_{\mathbb{R}ST} \;\; x \;\; = \;\; [G \circ F]_{\mathbb{R}T}$.
By Definition 80.6, we have $\mathrm{s}\ell_{G \circ F}^{\mathbb{R}T} = (G \circ F)(1)$.
Then $A \;\; *_{\mathbb{R}ST} \;\; x \;\; = \;\; [G \circ F]_{\mathbb{R}T} \;\; = \;\; \mathrm{s}\ell_{G \circ F}^{\mathbb{R}T} \;\; = \;\; (G \circ F)(1) \;\; = \;\; G(F_1)$.
By Theorem 85.2, $\forall t \in \mathbb{R}$, $F_t = tx$. Then $F_1 = 1 \cdot x = x$.
We have $A = \mathrm{s}\ell_L^{ST} = [L]_{ST}$.
So, since $\mathrm{Lin}_\bullet^{ST}$ and $[\bullet]_{ST}$ are inverses, we get: $\mathrm{Lin}_A^{ST} = L$.
Then $G = \mathrm{Lin}_A^{ST} = L$.
Then $L(x) \;\; = \;\; G(x) \;\; = \;\; G(F_1) \;\; = \;\; A \;\; *_{\mathbb{R}ST} \;\; x$, as desired. □

86. Tensor products of tensors

DEFINITION 86.1. *Let* $V, W \in \mathrm{TNSR}^+$, $x \in V$, $y \in W$.
Then $y \otimes x \;\; := \;\; y \;\; *_{V\mathbb{R}W} \;\; x$.

THEOREM 86.2. . *Let* $V, W \in \mathrm{TNSR}^+$, $x \in V$, $y \in W$.
Let $i \in \mathcal{I}_V$, $j \in \mathcal{I}_W$. *Then* $\pi_{j\|i}^{W \otimes V}(y \otimes x) = y_j x_i$.

Proof. Let $L := \mathrm{Lin}_x^{V\mathbb{R}}$ and let $M := \mathrm{Lin}_y^{\mathbb{R}W}$.
Then $y \otimes x = [M \circ L]_{VW}$. Also $[L]_{V\mathbb{R}} = x$.
Then $\pi_{j\|i}^{W\otimes V}(y \otimes x) = \pi_{j\|i}^{W\otimes V}([M \circ L]_{VW}) = \pi_j^W((M \circ L)(\varepsilon_i^V))$.
By Definition 80.5, $(\mathrm{s}\ell_L^{V\mathbb{R}})_i = L(\varepsilon_i^V)$.
Then $x_i = ([L]_{V\mathbb{R}})_i = (\mathrm{s}\ell_L^{V\mathbb{R}})_i = L(\varepsilon_i^V)$.
Then $(M \circ L)(\varepsilon_i^V) = M(L(\varepsilon_i^V)) = M(x_i)$.
By Theorem 85.2, we have: $\forall t \in \mathbb{R}$, $M_t = ty$.
Then $M(x_i) = M_{x_i} = x_i y$, so $\pi_j^W(M(x_i)) = x_i \cdot (\pi_j^W(y)) = x_i y_j$.
Then $\pi_{j\|i}^{W\otimes V}(y\otimes x) = \pi_j^W((M\circ L)(\varepsilon_i^V)) = \pi_j^W(M(x_i)) = x_i y_j = y_j x_i$. ∎

THEOREM 86.3. *Let* $V := \mathbb{R}^3$, $W := \mathbb{R}^2$.
Let $x := (2, 3, 4)$, $y := (10, 20)$. *Then* $y \otimes x = \begin{bmatrix} 20 & 30 & 40 \\ 40 & 60 & 80 \end{bmatrix}$.

Proof. This follows from Theorem 86.2. ∎

87. Polarizing polynomials and linearizing multilinears

In the next result, students provided the coefficients on Q, and, after hearing them, I found the coefficients on F.

THEOREM 87.1. *Let* $V := \mathbb{R}^2$ *and* $X := \mathbb{R}$.
Define $Q \in \mathcal{Q}_V^X$ *by* $Q(t, u) = 8t2 + 3tu + 4u^2$.
Define $F \in \mathcal{B}_{VV}^X$ *by* $F((r, s), (t, u)) = 8rt + (3/2)ru + (3/2)st + 4su$.
Then $F(\bullet, \bullet) = Q$.

Proof. Want: $\forall t, u \in \mathbb{R}$, $F((t, u), (t, u)) = Q(t, u)$.
Given $t, u \in \mathbb{R}$. Want: $F((t, u), (t, u)) = Q(t, u)$.
We have $F((t, u), (t, u)) = 8tt + (3/2)tu + (3/2)ut + 4uu$
$= 8t^2 + 3tu + 4u^2 = Q(x, y)$, as desired. ∎

The technique of going from Q to F generalizes, yielding:

THEOREM 87.2. *Let* $V, W \in \mathrm{TNSR}^+$, $Q \in \mathcal{Q}_V^W$.
Then $\exists 1 F \in \mathcal{SB}_V^W$ *s.t.* $F(\bullet, \bullet) = Q$.

Proof. Unassigned HW. ∎

THEOREM 87.3. *Let* $V, X \in \mathrm{TNSR}^+$.
Define $\Delta : \mathcal{B}_{VV}^X \to \mathcal{Q}_V^X$ *by* $\Delta(B) = B(\bullet, \bullet)$. *Let* $\Phi := \Delta|(\mathcal{SB}_V^X)$.
Then $\Delta : \mathcal{B}_{VV}^X \to\!> \mathcal{Q}_V^X$ *and* $\Phi : \mathcal{SB}_V^X \hookrightarrow\!> \mathcal{Q}_V^X$.

Proof. The existence part of Theorem 87.2 shows: $\Phi : \mathcal{SB}_V^X \to\!> \mathcal{Q}_V^X$.
It follows that: $\Delta : \mathcal{B}_{VV}^X \to\!> \mathcal{Q}_V^X$. Want: $\Phi : \mathcal{SB}_V^X \hookrightarrow\!> \mathcal{Q}_V^X$.

The uniqueness part of Theorem 87.2 shows: $\Phi : \mathcal{SB}_V^X \hookrightarrow \mathcal{Q}_V^X$.
Then $\Phi : \mathcal{SB}_V^X \hookrightarrow\!> \mathcal{Q}_V^X$, as desired. □

THEOREM 87.4. *Let* $V, W, X \in \mathrm{TNSR}^+$.
Define $\Psi : \mathcal{B}_{VW}^X \to X \otimes V \otimes W$ *by* $\Psi(B) = [B]_{VW}^X$.
Then $\Psi : \mathcal{B}_{VW}^X \hookrightarrow\!> X \otimes V \otimes W$.

Proof. Unassigned HW. □

DEFINITION 87.5. *Let* $V, W, X \in \mathrm{TNSR}^+$, $A \in X \otimes V \otimes W$.
Define $\Psi : \mathcal{B}_{VW}^X \hookrightarrow\!> X \otimes V \otimes W$ *by* $\Psi(B) = [B]_{VW}^X$.
Then $\mathrm{Bilin}_A^{VWX} := \Psi^{-1}(A)$.

THEOREM 87.6. *Let* $U, V, W, X \in \mathrm{TNSR}^+$.
Define $\Psi : \mathcal{T}_{UVW}^X \to X \otimes U \otimes V \otimes W$ *by* $\Psi(T) = [T]_{UVW}^X$.
Then $\Psi : \mathcal{T}_{UVW}^X \hookrightarrow\!> X \otimes U \otimes V \otimes W$.

Proof. Unassigned HW. □

DEFINITION 87.7. *Let* $U, V, W, X \in \mathrm{TNSR}^+$.
Define $\Psi : \mathcal{T}_{UVW}^X \hookrightarrow\!> X \otimes U \otimes V \otimes W$ *by* $\Psi(T) = [T]_{UVW}^X$.
Then $\mathrm{Trilin}_A^{UVWX} := \Psi^{-1}(A)$.

Let $U, V, W \in \mathrm{TNSR}^+$ and let $B \in \mathcal{B}_{UV}^W$.
Then $[B]_{UV}^W \in W \otimes U \otimes V$.
So, since $\mathcal{L}_{U \otimes V}^W \hookrightarrow\!> W \otimes U \otimes V$,
we can choose $L \in \mathcal{L}_{U \otimes V}^W$ s.t. $[L]_{U \otimes V}^W = [B]_{UV}^W$.
We "tensorize" B to get $[B]_{UV}^W$,
and then "linearize" $[B]_{UV}^W$ to get L.

In this way,
any bilinear map can be converted into a linear map,
and so "bilinear algebra" is simply a part of linear algebra.

Let $U, V, W, X \in \mathrm{TNSR}^+$ and let $T \in \mathcal{B}_{UVW}^X$.
Then $[T]_{UVW}^X \in X \otimes U \otimes V \otimes W$.
So, since $\mathcal{L}_{U \otimes V \otimes W}^X \hookrightarrow\!> X \otimes U \otimes V \otimes W$,
we can choose $L \in \mathcal{L}_{U \otimes V \otimes W}^X$ s.t. $[L]_{U \otimes V \otimes W}^X = [T]_{UVW}^X$.
We "tensorize" T to get $[T]_{UVW}^X$,
and then "linearize" $[T]_{UVW}^X$ to get L.

In this way,
any trilinear map can be converted into a linear map,
and so "trilinear algebra" is simply a part of linear algebra.

Let $V, W \in \mathrm{TNSR}^+$ and let $Q \in \mathcal{Q}_V^W$.
Then, since $\mathcal{SB}_V^W \hookrightarrow\!> \mathcal{Q}_V^W$,
we can choose $B \in \mathcal{SB}_V^W$ s.t. $Q = B(\bullet, \bullet)$.
Then $[B]_{VV}^W \in W \otimes V \otimes V$.
So, since $\mathcal{L}_{V \otimes V}^W \hookrightarrow\!> W \otimes V \otimes V$,
we can choose $L \in \mathcal{L}_{V \otimes V}^W$ s.t. $[L]_{V \otimes V}^W = [B]_{VV}^W$.
We "polarize" Q to get B,
"tensorize" B to get $[B]_{VV}^W$,
and then "linearize" $[B]_{VV}^W$ to get L.

In this way,
any homogeneous quadratic can be converted into a linear map,
and so "quadratic algebra" is simply a part of linear algebra.

Let $V, W \in \mathrm{TNSR}^+$ and let $K \in \mathcal{K}_V^W$.
Then, since $\mathcal{ST}_V^W \hookrightarrow\!> \mathcal{K}_V^W$,
we can choose $T \in \mathcal{ST}_V^W$ s.t. $K = T(\bullet, \bullet, \bullet)$.
Then $[T]_{VVV}^W \in W \otimes V \otimes V \otimes V$.
So, since $\mathcal{L}_{V \otimes V \otimes V}^W \hookrightarrow\!> W \otimes V \otimes V \otimes V$,
we can choose $L \in \mathcal{L}_{V \otimes V \otimes V}^W$ s.t. $[L]_{V \otimes V \otimes V}^W = [T]_{VVV}^W$.
We "polarize" K to get T,
"tensorize" T to get $[T]_{VVV}^W$,
and then "linearize" $[T]_{VVV}^W$ to get L.

In this way,
any homogeneous cubic can be converted into a linear map,
and so "cubic algebra" is simply a part of linear algebra.

In Theorem 89.1 below, we will see that, with some hypotheses,
any function can be approximated by a polynomial.
Any polynomial can be broken up into homogeneous parts,
and each part can then be converted into a linear map.
So linear algebra is the study of EVERYTHING!

88. Computing multilinear from the tensor

THEOREM 88.1. *Let* $V, W \in \mathrm{TNSR}^+$, $A \in U \otimes V$.
Then, $\forall x \in U$, $\quad \mathrm{Lin}_A^{VW}(x) \; = \; A \; *_{\mathbb{R}VW} \; x$.

Proof. This is a restatement of Theorem 85.6. $\square$

THEOREM 88.2. *Let* $U, V, W \in \mathrm{TNSR}^+$*,* $A \in W \otimes U \otimes V$*.*
Let $S := U \otimes V$*.* *Then,* $\forall x \in U$*,* $\forall y \in V$*,*

$$\mathrm{Bilin}_A^{UVW}(x, y) \; = \; A \; *_{\mathbb{R}SW} \; (x \otimes y).$$

Proof. Unassigned HW. Similar to proof of Theorem 85.6. □

THEOREM 88.3. *Let* $T, U, V, W \in \mathrm{TNSR}^+$*,* $A \in W \otimes T \otimes U \otimes V$*.*
Let $S := T \otimes U \otimes V$*.* *Then,* $\forall x \in T$*,* $\forall y \in U$*,* $\forall z \in V$*,*

$$\mathrm{Trilin}_A^{TUVW}(x, y) \; = \; A \; *_{\mathbb{R}SW} \; (x \otimes y \otimes z).$$

Proof. Unassigned HW. Similar to proof of Theorem 85.6. □

89. The cubic approximation theorem

THEOREM 89.1. *Let* $S, Z \in \mathrm{TNSR}^+$*,* $f : S \dashrightarrow Z$*,* $p \in \mathrm{dom}[f''']$*.*
Let $L := \mathrm{Lin}_{f'_p}^{SZ}$*,* $Q := \mathrm{Bilin}_{f''_p}^{SSZ}(\bullet, \bullet)$*,* $K := \mathrm{Trilin}_{f'''_p}^{SSSZ}(\bullet, \bullet, \bullet)$*.*
Then:

$$(\, f_p^T \,) \; - \; \left(L \; + \; \frac{Q}{2!} \; + \; \frac{K}{3!} \right) \; \in \; \mathcal{O}_3^{SZ}.$$

Proof. This proof is deferred until later. □

Let $S, Z \in \mathrm{TNSR}^+$, $f : S \dashrightarrow Z$, $p \in \mathrm{dom}[f''']$, $h \in S$.
Let $T := S \otimes S$, $U := S \otimes S \otimes S$.
Then $f' : S \dashrightarrow Z \otimes S$ and $f'' : S \dashrightarrow Z \otimes T$ and $f''' : S \dashrightarrow Z \otimes U$.
Then $f_p \in Z$, $f'_p \in Z \otimes S$ and $f''_p \in Z \otimes T$ and $f'''_p \in Z \otimes U$.
The idea of Theorem 89.1 is:
Up to a subcubic error (in h), we have:

$$f_{p+h} - f_p \;\approx\; L_h \;+\; \frac{Q_h}{2!} \;+\; \frac{K_h}{3!},$$

so

$$f_{p+h} \;\approx\; f_p \;+ \mathrm{Lin}_{f'_p}^{SZ}(h) \;+\; \frac{\mathrm{Bilin}_{f''_p}^{SSZ}(h, h)}{2!} \;+\; \frac{\mathrm{Trilin}_{f'''_p}^{SSSZ}(h, h, h)}{3!},$$

so

$$f_{p+h} \;\approx\; f_p \;+\; [\, (f'_p) \; *_{\mathbb{R}SZ} \; h \,]$$

$$+ \;\; \frac{(f''_p) \; *_{\mathbb{R}TZ} \; (h \otimes h)}{2!} \;\; + \;\; \frac{(f'''_p) \; *_{\mathbb{R}UZ} \; (h \otimes h \otimes h)}{3!}.$$

DEFINITION 89.2. *Let* $S \in \mathrm{TNSR}^+$*.*
Then $S^{\otimes 0} := \mathbb{R}$ *and* $S^{\otimes 1} := S$ *and* $S^{\otimes 2} := S \otimes S$ *and* $S^{\otimes 3} := S \otimes S \otimes S$*.*
Also, $\forall h \in S$*,* $h^{\otimes 0} := 1$ *and* $h^{\otimes 1} := h$ *and* $h^{\otimes 2} := h \otimes h$
and $h^{\otimes 3} := h \otimes h \otimes h$*.*

Let $S, Z \in \mathrm{TNSR}^+$, $f : S \dashrightarrow Z$, $p \in \mathrm{dom}[f''']$, $x \in S$.
Let $T := S^{\otimes 2}$, $U := S^{\otimes 3}$.
Then $f' : S \dashrightarrow Z \otimes S$ and $f'' : S \dashrightarrow Z \otimes T$ and $f''' : S \dashrightarrow Z \otimes U$.
Then $f_p \in Z$, $f'_p \in Z \otimes S$ and $f''_p \in Z \otimes T$ and $f'''_p \in Z \otimes U$.
The idea of Theorem 89.1 is:
Up to a subcubic error (in $x - p$), we have:

$$f_x \quad \approx \quad f_p \quad + \quad [\ (f'_p)\ \ *_{\mathbb{R}SZ}\ \ (x-p)\ ]$$
$$+ \quad \frac{(f''_p)\ \ *_{\mathbb{R}TZ}\ \ ((x-p)^{\otimes 2})}{2!} \quad + \quad \frac{(f'''_p)\ \ *_{\mathbb{R}UZ}\ \ ((x-p)^{\otimes 3})}{3!}.$$

90. Directional and partial derivatives

THEOREM 90.1. *Let* $S, T \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$, $p \in S$.
Then $\mathrm{Lin}^{ST}_{f'_p} = D^{ST}_p f$.

Proof. Since $f'_p = (\mathrm{s}\ell^{ST}_\bullet)(D^{ST}_p f)$, we get $(\mathrm{s}\ell^{ST}_\bullet)^{-1}(f'_p) = D^{ST}_p f$.
Then $\mathrm{Lin}^{ST}_{f'_p} = (\mathrm{s}\ell^{ST}_\bullet)^{-1}(f'_p) = D^{ST}_p f$, as desired. □

THEOREM 90.2. *Let* $S, T \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$, $p \in \mathrm{dom}[f']$.
Then $f^T_p - (D^{ST}_p f) \in \mathcal{O}^{ST}_1$.

Proof. Since $p \in \mathrm{dom}[f']$, we get $f'_p \neq$ ☹.
We have $(\mathrm{s}\ell_\bullet)(D^{ST}_p f) = f'_p \neq$ ☹, so $D^{ST}_p f \neq$ ☹.
Since ☹ $\neq D^{ST}_p f = \mathrm{UE}(\mathrm{LINS}_p f)$, we get $D^{ST}_p f \in \mathrm{LINS}_p f$.
Then $f^T_p - (D^{ST}_p f) \in \mathcal{O}^{ST}_1$, as desired. □

THEOREM 90.3. *Let* $S, T \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$.
Assume $f_{0_S} = 0_T$. *Then* $f^T_{0_S} = f$.

Proof. Want: $\forall h \in S$, $f^T_{0_S}(h) = f(h)$.
Given $h \in S$. Want: $f^T_{0_S}(h) = f(h)$.
We have $f^T_{0_S}(h) = f_{0_S + h} - f_{0_S} = f_h - 0_T = f_h = f(h)$, as desired. □

THEOREM 90.4. *Let* $S, T \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$, $p, q \in S$.
Then $D^{ST}_q(f(p + \bullet)) = D^{ST}_{p+q} f$.

Proof. Want: $\mathrm{LINS}^{ST}_q(f(p + \bullet)) = \mathrm{LINS}^{ST}_{p+q} f$.
Want: $\forall L \in \mathcal{L}^T_S$, $[\ (\ L \in \mathrm{LINS}^{ST}_q(f(p{+}\bullet))\ ) \Leftrightarrow (\ L \in \mathrm{LINS}^{ST}_{p+q} f\ )\ ]$.
Given $L \in \mathcal{L}^T_S$. Want: $(\ L \in \mathrm{LINS}^{ST}_q(f(p{+}\bullet))\ ) \Leftrightarrow (\ L \in \mathrm{LINS}^{ST}_{p+q} f\ )$.
We have: $(\ L \in \mathrm{LINS}^{ST}_q(f(p + \bullet))\ ) \Leftrightarrow (\ (f(p + \bullet))^T_q - L \in \mathcal{O}^{ST}_1\ )$.

Also, we have: $(\ L \in \mathrm{LINS}^{ST}_{p+q} f\ ) \Leftrightarrow (\ f^T_{p+q} - L \in \mathcal{O}^{ST}_1\ )$.
Want: $(\ f(p+\bullet)\ )^T_q = f^T_{p+q}$.
Want: $\forall h \in S$, $(\ f(p+\bullet)\ )^T_q(h) = f^T_{p+q}(h)$.
Given $h \in S$. Want: $(\ f(p+\bullet)\ )^T_q(h) = f^T_{p+q}(h)$.
We have $(\ f(p+\bullet)\ )^T_q(h) = (\ (f(p+\bullet))_{q+h}\ ) - (\ (f(p+\bullet))_q\ )$
$= (\ f(p+q+h)\ ) - (\ f(p+q)\ ) = f^T_{p+q}(h)$. □

THEOREM 90.5. *Let* $S \in \mathrm{TNSR}^+$, $u, v \in S$.
Then $u \ast_{\mathbb{R}S\mathbb{R}} v = u \bullet_S v$.

Proof. By Theorem 83.6, $u \ast_{\mathbb{R}S\mathbb{R}} v = \sum_{j \in \mathcal{I}_S} u_j v_j$.

By Definition 75.17, $u \bullet_S v = \sum_{j \in \mathcal{I}_S} u_j v_j$.

Then $u \ast_{\mathbb{R}S\mathbb{R}} v = \sum_{j \in \mathcal{I}_S} u_j v_j = u \bullet_S v$. □

DEFINITION 90.6. *Let* $S \in \mathrm{TNSR}^+$, $p, v \in S$.
Then $i^v_p : \mathbb{R} \to S$ *is defined by* $\imath^v_p(t) = p + vt$.

DEFINITION 90.7. *Let* $S \in \mathrm{TNSR}^+$, $v \in S$, $f : S \dashrightarrow T$.
Then $\partial^{ST}_v f : S \dashrightarrow T$ *is defined by* $(\partial^{ST}_v f)_p = (f \circ (i^v_p))'_0$.

THEOREM 90.8. *Let* $S \in \mathrm{TNSR}^+$, $f : S \dashrightarrow \mathbb{R}$, $p, v \in S$.
Then $(\partial^{ST}_v f)_p =^* (f'_p) \bullet_S v$.

Proof. By HW#8-4, $(\partial^{ST}_v f)_p =^* (D^{ST}_p f)_v$.
By Theorem 90.1, $(D^{ST}_p f)(v) = (\mathrm{Lin}^{ST}_{f'_p})(v)$.
By Theorem 88.1, $(\mathrm{Lin}^{ST}_{f'_p})(v) = f'_p \ast_{\mathbb{R}S\mathbb{R}} v$.
By Theorem 90.5, $f'_p \ast_{\mathbb{R}S\mathbb{R}} v = (f'_p) \bullet_S v$.
Then $(\partial^{ST}_v f)_p =^* (D^{ST}_p f)_v = (D^{ST}_p f)(v) = (\mathrm{Lin}^{ST}_{f'_p})(v)$
$= f'_p \ast_{\mathbb{R}S\mathbb{R}} v = (f'_p) \bullet_S v$. □

DEFINITION 90.9. *Let* $S \in \mathrm{TNSR}$, $T \in \mathrm{TNSR}^+$.
Let $f : S \dashrightarrow T$, $j \in \mathcal{I}_S$. *Then* $\partial^{ST}_j f := \partial^{ST}_{\varepsilon^S_j} f$.

THEOREM 90.10. *Let* $S \in \mathrm{TNSR}$, $v \in S$, $j \in \mathcal{I}_S$.
Then $v \bullet_S \varepsilon^S_j = v_j$.

Proof. Unassigned HW. □

DEFINITION 90.11. *Let* S *be a set*, $W \in \mathrm{TNSR}^+$.
Let $f : S \dashrightarrow W$, $p, q \in S$.
Then $f|^q_p := f_q - f_p.a$

THEOREM 90.12. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $a, b \in \mathbb{R}$.
Assume that $[a|b] \subseteq \mathrm{dom}[f']$.
Then $\exists c \in [a|b]$ *s.t.* $f|_a^b = (f'_c) \cdot (b - a)$.

Proof. This is a restatement of Theorem 73.4. □

THEOREM 90.13. *Let* $V := \mathbb{R}^2$, $W := \mathbb{R}$, $R : V \dashrightarrow W$.
Let $\delta > 0$, $J := (-\delta; \delta)$, $x, y \in J$. *Assume* $J^2 \subseteq \mathrm{dom}[R']$.
Then $\exists \beta \in [0|y]$ *s.t.* $R|_{(x,0)}^{(x,y)} = (\,(\partial_2^{VW} R)(x, \beta)\,) \cdot y$.

Proof. Let $i := i_{(x,0)}^{(0,1)}$. By HW#8-3, we have: $i' = C_{\mathbb{R}}^{(0,1)}$.
Also, we have: $\forall t \in \mathbb{R}$, $i_t = (x, 0) + t \cdot (0, 1) = (x, t)$.
Let $f := R \circ i$.

Claim: $\forall t \in J$, $(\partial_2^{VW} R)_{i_t} = f'_t \neq ☹$.
Proof of Claim:
Given $t \in J$. Want: $(\partial_2^{VW} R)_{i_t} = f'_t \neq ☹$.
Since $f = R \circ i$, by the Chain Rule,
we have: $D_t^{\mathbb{R}W} f =^* (D_{i_t}^{VW} R) \circ (D_t^{\mathbb{R}V} i)$.
We have $i_t = (x, t) \in J^2 \subseteq \mathrm{dom}[R']$. Then $D_{i_t}^{VW} R \in \mathcal{L}_V^W$.
We have $t \in \mathbb{R} = \mathrm{dom}[i']$. Then $D_t^{\mathbb{R}V} i \in \mathcal{L}_{\mathbb{R}}^V$.
It follows that $(D_{i_t}^{VW} R) \circ (D_t^{\mathbb{R}V} i) \in \mathcal{L}_{\mathbb{R}}^W$,
and so $(D_{i_t}^{VW} R) \circ (D_t^{\mathbb{R}V} i) \neq ☹$.
Since $D_t^{\mathbb{R}W} f =^* (D_{i_t}^{VW} R) \circ (D_t^{\mathbb{R}V} i) \neq ☹$,
we get: $D_t^{\mathbb{R}W} f = (D_{i_t}^{VW} R) \circ (D_t^{\mathbb{R}V} i)$.
Since $D_t^{\mathbb{R}W} f = (D_{i_t}^{VW} R) \circ (D_t^{\mathbb{R}V} i) \in \mathcal{L}_{\mathbb{R}}^W$, we get $(D_t^{\mathbb{R}W} f)(1) \in W$.
Also, $f'_t = \mathrm{s}\ell_\bullet^{\mathbb{R}W}(D_t^{\mathbb{R}W} f) = (D_t^{\mathbb{R}W} f)(1)$, so $f'_t = (D_t^{\mathbb{R}W} f)(1)$.
Then $f'_t = (D_t^{\mathbb{R}W} f)(1) \in W$.
Then $f'_t \neq ☹$. Want: $(\partial_2^{VW} R)_{i_t} =^* f'_t$.
By HW#8-4, $(\partial_{\varepsilon_2^V}^{VW} R)_{i_t} =^* (D_{i_t}^{VW} R)_{\varepsilon_2^V}$.
So, since $(\partial_2^{VW} R)_{i_t} = (\partial_{\varepsilon_2^V}^{VW} R)_{i_t}$, we get $(\partial_2^{VW} R)_{i_t} =^* (D_{i_t}^{VW} R)_{\varepsilon_2^V}$.
Want: $f'_t = (D_{i_t}^{VW} R)_{\varepsilon_2^V}$.
We have $i'_t = \mathrm{s}\ell_\bullet^{\mathbb{R}V}(D_t^{\mathbb{R}V} i) = (D_t^{\mathbb{R}V} i)(1)$.
Since $D_t^{\mathbb{R}W} f = (D_{i_t}^{VW} R) \circ (D_t^{\mathbb{R}V} i)$ and since $(D_t^{\mathbb{R}V} i)(1) = i'_t$,
we get $(D_t^{\mathbb{R}W} f)(1) = (D_{i_t}^{VW} R)(i'_t)$.
Recall that $f'_t = (D_t^{\mathbb{R}W} f)(1)$.
Also, $i'_t = i'(t) = (C_{\mathbb{R}}^{(0,1)})(t) = (0, 1) = \varepsilon_2^V$.
Then $f'_t = (D_t^{\mathbb{R}W} f)(1) = (D_{i_t}^{VW} R)(i'_t)$
$= (D_{i_t}^{VW} R)(\varepsilon_2^V) = (D_{i_t}^{VW} R)_{\varepsilon_2^V}$, as desired.

End of proof of Claim.

By the Claim, $J \subseteq \mathrm{dom}[f']$.
Since $0, y \in J$ and since J is an interval, we get $[0|y] \subseteq J$.
Then $[0|y] \subseteq J \subseteq \mathrm{dom}[f']$.
So, by Theorem 90.12, choose $\beta \in [0|y]$ s.t. $f|_0^y \;=\; (f'_\beta) \cdot (y - 0)$.
Want: $R|_{(x,0)}^{(x,y)} \;=\; (\,(\partial_2^{VW} R)(x, \beta)\,) \cdot y$.
We have $f_y = (R \circ i)_y = R(i_y) = R(x, y)$.
Also, $f_0 = (R \circ i)_0 = R(i_0) = R(x, 0)$.
Then $f|_0^y = f_y - f_0 = [R(x, y)] - [R(x, 0)] = R|_{(x,0)}^{(x,y)}$.
By the Claim, $(\partial_2^{VW} R)_{i_\beta} \;=\; f'_\beta$. Also, $i_\beta = (x, \beta)$.
Then $f'_\beta = (\partial_2^{VW} R)_{i_\beta} = (\partial_2^{VW} R)(i_\beta) = (\partial_2^{VW} R)(x, \beta)$.

$$\begin{aligned}\text{Then } R|_{(x,0)}^{(x,y)} \;&=\; f|_0^y \;=\; (f'_\beta) \cdot (y - 0)\\ &=\; (f'_\beta) \cdot y \;=\; (\partial_2^{VW} R)(x, \beta), \qquad \text{as desired.}\end{aligned}$$ □

THEOREM 90.14. *Let $V := \mathbb{R}^2$ and $W := \mathbb{R}$. Define $f : V \to W$ by*

$$f(x, y) = \begin{cases} 1, & \text{if } (x = 0) \vee (y = 0) \\ 0, & \text{if } x \neq 0 \neq y. \end{cases}$$

Then $0_2 \in \mathrm{dom}[\partial_1 f]$, $0_2 \in \mathrm{dom}[\partial_2 f]$, and f is not continuous at 0_2.

Proof. Unassigned HW. □

THEOREM 90.15. *Let f and g be functions. Then:*

$$\begin{aligned}(\; g \supseteq f \;) \quad &\Leftrightarrow \quad (\;(\,\mathrm{dom}[g] \supseteq \mathrm{dom}[f]\,) \;\&\; (\,g|(\mathrm{dom}[f]) = f\,)\;)\\ &\Leftrightarrow \quad (\; \forall p, \quad g_p \;=^*\; f_p \;).\end{aligned}$$

Proof. Unassigned HW. □

THEOREM 90.16. *Let f and g be functions and let S be a set. Assume that $S \supseteq \mathrm{dom}[f]$. Then:*

$$(\; g \supseteq f \;) \quad \Leftrightarrow \quad (\; \forall p \in S, \quad g_p \;=^*\; f_p \;).$$

Proof. Unassigned HW. □

On the board, we graphed two partial functions $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ s.t. $g \supseteq f$, and discussed Theorem 90.15 and Theorem 90.16 in the context of those two functions.

THEOREM 90.17. *Let $S, T \in \mathrm{TNSR}^+$ and let $f, g : S \dashrightarrow T$. Assume that $g \supseteq f$. Then:*

(1) $(\; f \in \mathrm{DNZ}_S^T \;) \;\Rightarrow\; (\; g \in \mathrm{DNZ}_S^T \;)$ *and*
(2) $(\; f \in \mathcal{O}_1 \;) \;\Rightarrow\; (\; g \in \mathcal{O}_1 \;)$ *and*

(3) $\forall p \in S$, [$g_p^T \supseteq f_p^T$ *and*
$\mathrm{LINS}_p^{ST} g \supseteq \mathrm{LINS}_p^{ST} f$ *and*
$D_p^{ST} g =^* D_p^{ST} f$ *and*
$g_p' =^* f_p'$] *and*

(4) $g' \supseteq f'$ *and*

(5) $\forall v \in S$, $\partial_v^{ST} g \supseteq \partial_v^{ST} f$.

Proof. Unassigned HW. □

THEOREM 90.18. *Let* $S, T \in \mathrm{TNSR}^+$ *and let* $f, g : S \dashrightarrow T$.
Then $(f+g)' \supseteq f' + g'$.

Proof. This follows from Theorem 81.3. □

THEOREM 90.19. *Let* $S, T \in \mathrm{TNSR}^+$, *let* $c \in \mathbb{R}$ *and let* $f : S \dashrightarrow T$.
Then $(c \cdot f)' \supseteq c \cdot f'$.

Proof. This follows from Theorem 81.4. □

Let S and T be tensor spaces, let $f : S \dashrightarrow T$ and let $k \in \mathbb{N}_0$. Then the kth derivative of f is denoted $f^{(k)}$. Then $f^{(k)} : S \dashrightarrow T \otimes (S^{\otimes k})$.

Let S and T be tensor spaces and let $f : S \dashrightarrow T$. Then $f^{(1)} = f'$ and $f^{(2)} = f''$ and $f^{(3)} = f'''$. By convention, $f^{(0)} = f$.

THEOREM 90.20. *Let* $S, T \in \mathrm{TNSR}^+$, $f, g : S \dashrightarrow T$, $q \in S$, $\ell \in \mathbb{N}_0$. *Assume* $q \in \mathrm{dom}[f^{(\ell)}]$ *and* $q \in \mathrm{dom}[g^{(\ell)}]$. *Then* $q \in \mathrm{dom}[(f+g)^{(\ell)}]$.

Proof. Unassigned HW. □

THEOREM 90.21. *Let* $S, T \in \mathrm{TNSR}^+$, $\ell \in \mathbb{N}_0$, $f : S \dashrightarrow T$, $q \in S$. *Let* $c \in \mathbb{R}$. *Assume* $q \in \mathrm{dom}[f^{(\ell)}]$. *Then* $q \in \mathrm{dom}[(c \cdot f)^{(\ell)}]$.

Proof. Unassigned HW. □

THEOREM 90.22. *Let* $R, S, T, U \in \mathrm{TNSR}^+$, $* \in \mathcal{B}_{ST}^U$,
$f : R \dashrightarrow S$ *and* $g : R \dashrightarrow T$.
Then: $(f * g)' \supseteq (f' * g) + (f * g')$ *and*
$(f * g)'' \supseteq (f'' * g) + 2 \cdot (f' * g') + (f * g'')$.

Proof. This follows from Theorem 82.11, but requires careful definitions of the various bilinear multiplications, all of which are called $*$. □

We will not require Theorem 90.22, so we have not carefully defined all the multiplications appearing in it, and we omit a detailed proof. Instead, more important to us is:

THEOREM 90.23. *Let* $R, S, T, U \in \text{TNSR}^+$, $\quad * \in \mathcal{B}_{ST}^{U}$,

$$f : R \dashrightarrow S, \quad g : R \dashrightarrow T, \quad p \in R \quad \textit{and} \quad k \in \mathbb{N}_0.$$

Assume that $p \in \text{dom}[f^{(k)}]$ *and that* $p \in \text{dom}[g^{(k)}]$.
Then $p \in \text{dom}[(f * g)^{(k)}]$.

Proof. We have: $p \in (\,\text{dom}[f^{(k)}]\,) \cap (\,\text{dom}[g^{(k)}]\,)$.
Want: $(\,\text{dom}[f^{(k)}]\,) \cap (\,\text{dom}[g^{(k)}]\,) \subseteq \text{dom}[(f * g)^{(k)}]$.
Let I denote the set of all $\ell \in \mathbb{N}_0$ s.t.

$$\forall V, W, X, Y \in \text{TNSR}^+, \quad \forall \odot \in \mathcal{B}_{WX}^{Y}, \quad \forall \phi : V \dashrightarrow W, \quad \forall \psi : V \dashrightarrow X, \quad (\,\text{dom}[\phi^{(\ell)}]\,) \cap (\,\text{dom}[\psi^{(\ell)}]\,) \subseteq \text{dom}[(\phi \odot \psi)^{(\ell)}].$$

Want: $k \in I$. Want: $I = \mathbb{N}_0$.
We have: $\forall V, W, X, Y \in \text{TNSR}^+$, $\forall \odot \in \mathcal{B}_{WX}^{Y}$,

$$\forall \phi : V \dashrightarrow W, \quad \forall \psi : V \dashrightarrow X, \quad (\,\text{dom}[\phi]\,) \cap (\,\text{dom}[\psi]\,) = \text{dom}[\phi \odot \psi].$$

It follows that $0 \in I$. Want: $\forall \ell \in I$, $\ell + 1 \in I$.
Given $\ell \in I$. Want: $\ell + 1 \in I$.
Know: $\forall V, W, X, Y \in \text{TNSR}^+$, $\forall \odot \in \mathcal{B}_{WX}^{Y}$,

$$\forall \phi : V \dashrightarrow W, \quad \forall \psi : V \dashrightarrow X, \quad (\,\text{dom}[\phi^{(\ell)}]\,) \cap (\,\text{dom}[\psi^{(\ell)}]\,) \subseteq \text{dom}[(\phi \odot \psi)^{(\ell)}].$$

Want: $\forall V, W, X, Y \in \text{TNSR}^+$, $\forall \odot \in \mathcal{B}_{WX}^{Y}$,

$$\forall \phi : V \dashrightarrow W, \quad \forall \psi : V \dashrightarrow X, \quad (\,\text{dom}[\phi^{(\ell+1)}]\,) \cap (\,\text{dom}[\psi^{(\ell+1)}]\,) \subseteq \text{dom}[(\phi \odot \psi)^{(\ell+1)}].$$

Given $V, W, X, Y \in \text{TNSR}^+$, $\odot \in \mathcal{B}_{WX}^{Y}$, $\phi : V \dashrightarrow W$, $\psi : V \dashrightarrow X$.
Want: $(\,\text{dom}[\phi^{(\ell+1)}]\,) \cap (\,\text{dom}[\psi^{(\ell+1)}]\,) \subseteq \text{dom}[(\phi \odot \psi)^{(\ell+1)}]$.
Want: $\forall q \in (\,\text{dom}[\phi^{(\ell+1)}]\,) \cap (\,\text{dom}[\psi^{(\ell+1)}]\,)$, $\quad q \in \text{dom}[(\phi \odot \psi)^{(\ell+1)}]$.
Given $q \in (\,\text{dom}[\phi^{(\ell+1)}]\,) \cap (\,\text{dom}[\psi^{(\ell+1)}]\,)$.
Want: $q \in \text{dom}[(\phi \odot \psi)^{(\ell+1)}]$.
Let $\quad S' := S \otimes R, \quad T' := T \otimes R, \quad U' := U \otimes R.$
Define $\odot_L : S' \times T \to U'$ by $A \odot_L b = \text{s}\ell_{\bullet}^{RT}((\text{Lin}_A^{RS}) \odot b)$.
Define $\odot_R : S \times T' \to U'$ by $a \odot_R B = \text{s}\ell_{\bullet}^{RT}(a \odot (\text{Lin}_B^{RS}))$.
Then, by Theorem 82.11, $\quad (\phi \odot \psi)' \supseteq (\phi' \odot_L \psi) + (\phi \odot_R \psi')$.
Then: $\quad (\,(\phi \odot \psi)'\,)^{(\ell)} \supseteq (\,(\phi' \odot_L \psi) + (\phi \odot_R \psi')\,)^{(\ell)}$.
So, since $\quad (\,(\phi \odot \psi)'\,)^{(\ell)} = (\phi \odot \psi)^{(\ell+1)}$, $\quad$ we see that:

$$(\phi \odot \psi)^{(\ell+1)} \supseteq (\,(\phi' \odot_L \psi) + (\phi \odot_R \psi')\,)^{(\ell)}.$$

Then: $\text{dom}[\,(\phi \odot \psi)^{(\ell+1)}\,] \supseteq \text{dom}[\,(\,(\phi' \odot_L \psi) + (\phi \odot_R \psi')\,)^{(\ell)}\,]$.
We have $q \in \text{dom}[\phi^{(\ell+1)}]$ and $\phi^{(\ell+1)} = (\phi')^{(\ell)}$, so $q \in \text{dom}[(\phi')^{(\ell)}]$.
Also, $q \in \text{dom}[\psi^{(\ell+1)}] \subseteq \text{dom}[\psi^{(\ell)}]$, so $q \in \text{dom}[\psi^{(\ell)}]$.

Since $\quad q \in \mathrm{dom}[(\phi')^{(\ell)}]$, $\quad$ since $q \in \mathrm{dom}[\psi^{(\ell)}]$,
$\quad$ and $\quad$ since $\ell \in I$, $\quad$ we get: $\quad q \in \mathrm{dom}[(\phi' \odot_L \psi)^{(\ell)}]$.
We have $q \in \mathrm{dom}[\phi^{(\ell+1)}] \subseteq \mathrm{dom}[\phi^{(\ell)}]$, so $q \in \mathrm{dom}[\phi^{(\ell)}]$.
Also, $q \in \mathrm{dom}[\psi^{(\ell+1)}]$ and $\psi^{(\ell+1)} = (\psi')^{(\ell)}$, so $q \in \mathrm{dom}[(\psi')^{(\ell)}]$.
Since $\quad q \in \mathrm{dom}[\phi^{(\ell)}]$, $\quad$ since $q \in \mathrm{dom}[(\psi')^{(\ell)}]$,
$\quad$ and $\quad$ since $\ell \in I$, $\quad$ we get: $\quad q \in \mathrm{dom}[(\phi \odot_R \psi')^{(\ell)}]$.
Since $\quad q \in \mathrm{dom}[(\phi' \odot_L \psi)^{(\ell)}]$ $\quad$ and $\quad q \in \mathrm{dom}[(\phi \odot_R \psi')^{(\ell)}]$,
$\quad$ by Theorem 90.21, we get: $q \in \mathrm{dom}[(\phi' \odot_L \psi + \phi \odot_R \psi')^{(\ell)}]$.

$$\text{Then: } q \in \mathrm{dom}[\,(\,(\phi' \odot_L \psi) + (\phi \odot_R \psi')\,)^{(\ell)}\,] \subseteq \mathrm{dom}[\,(\phi \odot \psi)^{(\ell+1)}\,], \quad \text{as desired.}$$ □

Theorem 90.23 above is called the **Product Derivatives Domain Theorem**.

THEOREM 90.24. *Let* $S, T, U \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$, $g : T \dashrightarrow U$.
Then: $\quad (g \circ f)' \supseteq (g' \circ f) * f'$ $\quad$ *and*
$$(g \circ f)'' \supseteq ((g'' \circ f) * f') * f' + (g' \circ f) * f''.$$

Proof. This follows from Theorem 83.3, but requires careful definitions of the various bilinear multiplications, all of which are called $*$. □

We will not require Theorem 90.24, so we have not carefully defined all the multiplications appearing in it, and we omit a detailed proof. Instead, more important to us is:

THEOREM 90.25. *Let* $S, T, U \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$, $g : T \dashrightarrow U$. *Assume that* $p \in \mathrm{dom}[f^{(k)}]$ *and that* $f_p \in \mathrm{dom}[g^{(k)}]$.
Then $p \in \mathrm{dom}[(g \circ f)^{(k)}]$.

Proof. We have: $p \in (\,\mathrm{dom}[f^{(k)}]\,) \cap (\,f^*(\,\mathrm{dom}[g^{(k)}]\,)\,)$.
Want: $(\,\mathrm{dom}[f^{(k)}]\,) \cap (\,f^*(\mathrm{dom}[g^{(k)}]\,)\,) \subseteq \mathrm{dom}[(g \circ f)^{(k)}]$.
Let I denote the set of all $\ell \in \mathbb{N}_0$ s.t.
$$\forall V, W, X \in \mathrm{TNSR}^+,\ \forall \phi : V \dashrightarrow W,\ \forall \psi : W \dashrightarrow X,$$
$$(\,\mathrm{dom}[\phi^{(\ell)}]\,) \cap (\,\phi^*(\,\mathrm{dom}[\psi^{(\ell)}]\,)\,) \subseteq \mathrm{dom}[(\psi \circ \phi)^{(\ell)}].$$
Want: $k \in I$. $\qquad$ Want: $I = \mathbb{N}_0$.
We have: $\forall V, W, X \in \mathrm{TNSR}^+$, $\forall \phi : V \dashrightarrow W$, $\forall \psi : W \dashrightarrow X$,
$$(\,\mathrm{dom}[\phi]\,) \cap (\,\phi^*(\,\mathrm{dom}[\psi]\,)\,) = \mathrm{dom}[\psi \circ \phi].$$
It follows that $0 \in I$. $\qquad$ Want: $\forall \ell \in I$, $\ell + 1 \in I$.
Given $\ell \in I$. $\qquad$ Want: $\ell + 1 \in I$.
Know: $\quad \forall V, W, X \in \mathrm{TNSR}^+$, $\forall \phi : V \dashrightarrow W$, $\forall \psi : W \dashrightarrow X$,
$$(\,\mathrm{dom}[\phi^{(\ell)}]\,) \cap (\,\phi^*(\,\mathrm{dom}[\psi^{(\ell)}]\,)\,) \subseteq \mathrm{dom}[(\psi \circ \phi)^{(\ell)}].$$

Want: $\forall V, W, X \in \text{TNSR}^+$, $\forall \phi : V \dashrightarrow W$, $\forall \psi : W \dashrightarrow X$,

$$(\,\text{dom}[\phi^{(\ell+1)}]\,) \cap (\,\phi^*(\,\text{dom}[\psi^{(\ell+1)}]\,)\,) \subseteq \text{dom}[(\psi\circ\phi)^{(\ell+1)}].$$

Given $V, W, X \in \text{TNSR}^+$, $\phi : V \dashrightarrow W$, $\psi : W \dashrightarrow X$.

Want: $(\,\text{dom}[\phi^{(\ell+1)}]\,) \cap (\,\phi^*(\,\text{dom}[\psi^{(\ell+1)}]\,)\,) \subseteq \text{dom}[(\psi \circ \phi)^{(\ell+1)}]$.

Want: $\forall q \in (\,\text{dom}[\phi^{(\ell+1)}]\,) \cap (\,\phi^*(\,\text{dom}[\psi^{(\ell+1)}]\,)\,)$,
$q \in \text{dom}[(\psi \circ \phi)^{(\ell+1)}]$.

Given $q \in (\,\text{dom}[\phi^{(\ell+1)}]\,) \cap (\,\phi^*(\,\text{dom}[\psi^{(\ell+1)}]\,)\,)$,

Want: $q \in \text{dom}[(\psi \circ \phi)^{(\ell+1)}]$.

By Theorem 83.3, $(\psi \circ \phi)' \supseteq (\psi' \circ \phi) \;\,{*_{VWX}}\;\, \phi'$.

Then $(\,(\psi \circ \phi)'\,)^{(\ell)} \supseteq (\,(\psi' \circ \phi) \;\,{*_{VWX}}\;\, \phi'\,)^{(\ell)}$.

So, since $(\,(\psi \circ \phi)'\,)^{(\ell)} = (\psi \circ \phi)^{(\ell+1)}$, we see that:
$(\psi \circ \phi)^{(\ell+1)} \supseteq (\,(\psi' \circ \phi) \;\,{*_{VWX}}\;\, \phi'\,)^{(\ell)}$.

Then: $\text{dom}[\,(\psi \circ \phi)^{(\ell+1)}\,] \supseteq \text{dom}[\,(\,(\psi' \circ \phi) \;\,{*_{VWX}}\;\, \phi'\,)^{(\ell)}\,]$.

We have $q \in \text{dom}[\phi^{(\ell+1)}]$ and $\phi^{(\ell+1)} = (\phi')^{(\ell)}$, so $q \in \text{dom}[(\phi')^{(\ell)}]$.

Also, $q \in \text{dom}[\psi^{(\ell+1)}] \subseteq \text{dom}[\psi^{(\ell)}]$, so $q \in \text{dom}[\psi^{(\ell)}]$.

Since $q \in \text{dom}[(\phi')^{(\ell)}]$, since $q \in \text{dom}[\psi^{(\ell)}]$,
and since $\ell \in I$, we get: $q \in \text{dom}[(\psi' \circ \phi)^{(\ell)}]$.

We have $q \in \text{dom}[\psi^{(\ell+1)}]$ and $\psi^{(\ell+1)} = (\psi')^{(\ell)}$, so $q \in \text{dom}[(\psi')^{(\ell)}]$.

Since $q \in \text{dom}[(\psi' \circ \phi)^{(\ell)}]$ and $q \in \text{dom}[(\psi')^{(\ell)}]$,
by Theorem 90.23, we get: $q \in \text{dom}[\,(\,(\psi' \circ \phi) \;\,{*_{VWX}}\;\, \phi'\,)^{(\ell)}\,]$.

Then: $q \in \text{dom}[\,(\,(\psi' \circ \phi) \;\,{*_{VWX}}\;\, \phi'\,)^{(\ell)}\,]$
$\subseteq \text{dom}[\,(\psi \circ \phi)^{(\ell+1)}\,]$, as desired. □

Theorem 90.25 above is called the **Composition Deriviatives Domain Theorem**.

91. An infintely directionally differentiable function that is not continuous

You are expected to know the definition of "infinitely directionally differentiable", and the fact that there is an infinitely directionally differentiable function that is not continuous. You are neither expected to know the proof of that fact nor expected to know the other facts covered in this section.

Let $V := \mathbb{R}^2$. For all $k \in \mathbb{N}$, for all $s \in V^k$, for all $\phi : V \dashrightarrow \mathbb{R}$, let $\partial_s \phi := \partial_{s_1} \cdots \partial_{s_k} \phi$. Let $S := V \cup V^2 \cup V^3 \cup \cdots$. We will say that a function $\phi : V \to \mathbb{R}$ is **infinitely directionally differentiable** if, for all $s \in S$, we have $\text{dom}[\partial_s \phi] = V$.

Choose an infinitely differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that ($g = 0$ on $(-\infty, 1] \cup [3, \infty)$) and ($g(2) = 1$) and ($\mathrm{im}[g] \subseteq [0, 1]$). Define $f : V \to \mathbb{R}$ by $f(x, y) = \begin{cases} g(y/x^2), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

FACT 91.1. *The function $f : V \to \mathbb{R}$ is infinitely directionally differentiable, and is discontinuous at 0_2.*

Proof. We define

$$\begin{aligned} U &:= \{(x, y) \in V \mid x^2 < y < 3x^2\} \qquad \text{and} \\ \overline{U} &:= \{(x, y) \in V \mid x^2 \leqslant y \leqslant 3x^2\}. \end{aligned}$$

Then $0_V = (0, 0) \in \overline{U}$. Also, U is the interior in V of $\overline{U}$, so $V \backslash U$ is the closure in V of $V \backslash \overline{U}$. Let Φ be the set of all $\phi : V \to \mathbb{R}$ such that

- $\phi = 0$ on $V \backslash U$ and
- for all $s \in S$, $\partial_s \phi$ is continuous on $V \backslash \{0_V\}$.

Claim 1: $f \in \Phi$. *Proof of Claim 1:* By construction, $f = 0$ on $V \backslash U$. Let $s \in S$ be given. We wish to show that $\partial_s \phi$ is continuous on $V \backslash \{0_V\}$.

By the Multivariable Chain Rule (to be proved later), $\forall x, y \in \mathbb{R}$, we have: if $x \neq 0$, then $\partial_s f$ is continuous at (x, y). Let $Y := \{(0, y) | y \in \mathbb{R}\}$ denote the y-axis. Then $\partial_s f$ is continuous on $V \backslash Y$. It therefore suffices to show that $\partial_s \phi$ is continuous on $Y \backslash \{0_V\}$. Let $v \in Y \backslash \{0_V\}$ be given. We wish to show: $\partial_s \phi$ is continuous at v.

Let $V_0 := V \backslash \overline{U}$. We have $v \in Y \backslash \{0_V\} \subseteq V \backslash \overline{U} = V_0$. So, since V_0 is open in V, we see that V_0 is an open nbd in V of v. Since $V_0 \subseteq V \backslash U$ and since $f = 0$ on $V \backslash U$, we see that $f = 0$ on V_0. So, since V_0 is open in V, it follows that $\partial_s \phi = 0$ on V_0. So, since V_0 is an open nbd in V of v, it follows that $\partial_s \phi$ is continuous at v. *End of proof of Claim 1.*

Claim 2: Let $\phi \in \Phi$ and let $v \in V$. Then we have $(\partial_v \phi)(0_V) = 0$. *Proof of Claim 2:* Let $L := \mathbb{R}v \subseteq V$ denote the line through v in V, and give L the relative topology inherited from V. Choose a nbd L_0 of 0_V in L such that $L_0 \subseteq V \backslash U$. Since $\phi = 0$ on $V \backslash U$, we conclude that $\phi = 0$ on L_0. Then $(\partial_v \phi)(0_V) = 0$, as desired. *End of proof of Claim 2.*

Claim 3: Let $\phi \in \Phi$, let $v \in V$ and let $s \in S$. Then $\partial_s \partial_v \phi$ is continuous on $V \backslash \{0_V\}$. *Proof of Claim 3:* Because $s \in S = V \cup V^2 \cup V^3 \cup \cdots$, choose $k \in \mathbb{N}$ such that $s \in V^k$. Let $t := (s_1, \ldots, s_k, v)$. Then we have $t \in V^{k+1} \subseteq S$ and $\partial_t \phi = \partial_s \partial_v \phi$. Since $\phi \in \Phi$, we see that $\partial_t \phi$ is continuous on $V \backslash \{0_V\}$. Then $\partial_s \partial_v \phi$ is continuous on $V \backslash \{0_V\}$, as desired. *End of proof of Claim 3.*

Claim 4: Let $\phi \in \Phi$ and let $v \in V$. Then we have $\partial_v \phi \in \Phi$. *Proof of Claim 4:* By Claim 3, it suffices to show that $\partial_v \phi = 0$ on $V \backslash U$. Since $\phi \in \Phi$, we know that $\phi = 0$ on $V \backslash U$. So, since $V \backslash \overline{U} \subseteq V \backslash U$, it follows that $\phi = 0$ on $V \backslash \overline{U}$. So since $V \backslash \overline{U}$ is open in V, we see that $\partial_v \phi = 0$ on $V \backslash \overline{U}$. By Claim 2, $(\partial_v \phi)(0_V) = 0$. Let $U^* := \overline{U} \backslash (U \cup \{0_V\})$. It remains to show that $\partial_v \phi = 0$ on U^*. Let $x \in U^*$ be given. We wish to show that $(\partial_v \phi)(x) = 0$.

As $\phi \in \Phi$ and $v \in V \subseteq S$, we know that $\partial_v \phi$ is continuous on $V \backslash \{0_V\}$. We have $x \in U^* \subseteq V \backslash \{0_V\}$. Then $\partial_v \phi$ is continuous at x. Also, $x \in \overline{U} \backslash U \subseteq V \backslash U$, and $V \backslash U$ is the closure in V of $V \backslash \overline{U}$. Then x is an element of the closure in V of $V \backslash \overline{U}$, so, since $\partial_v \phi = 0$ on $V \backslash \overline{U}$ and since $\partial_v \phi$ is continuous at x, we get $(\partial_v \phi)(x) = 0$. *End of proof of Claim 4.*

Claim 5: Let $\phi \in \Phi$. Then we have: $\forall k \in \mathbb{N}$, $\forall s \in V^k$, $\partial_s \phi \in \Phi$. *Proof of Claim 5:* We argue by induction on k. By Claim 4, we have: $\forall s \in V$, $\partial_s \phi \in \Phi$, proving the base case. Let $k \in \mathbb{N}$ be given and make the induction assumption: $\forall s \in V^k$, $\partial_s \phi \in \Phi$. We wish to show: $\forall s \in V^{k+1}$, $\partial_s \phi \in \Phi$. Let $s \in V^{k+1}$ be given. We wish to show: $\partial_s \phi \in \Phi$.

Let $r := (s_2, \ldots, s_{k+1}) \in V^k$ and $\psi = \partial_r \phi$. Then $\partial_s \phi = \partial_{s_1} \partial_r \phi = \partial_{s_1} \psi$. By the induction assumption, $\partial_r \phi \in \Phi$. That is, we have $\psi \in \Phi$. Then, by Claim 4, $\partial_{s_1} \psi \in \Phi$. Then $\partial_s \phi = \partial_{s_1} \psi \in \Phi$. *End of proof of Claim 5.*

Let $C := \{(x, y) \in V \,|\, y = 2x^2\}$. By construction, know both that $f = 1$ on $C \backslash \{0_V\}$ and that $f(0_V) = 0$. Thus f is discontinuous at 0_V. It remains to show that f is infinitely directionally differentiable. Let $s \in S$ be given. We wish to show that $\mathrm{dom}[\partial_s \phi] = V$.

Since $s \in S = V \cup V^2 \cup V^3 \cup \cdots$, choose $k \in \mathbb{N}$ such that $s \in V^k$. By Claim 1, we have $f \in \Phi$. Then, by Claim 5, we have $\partial_s \phi \in \Phi$. Then, by definition of Φ, we see that $\mathrm{dom}[\partial_s \phi] = V$, as desired. □

We say a function $\phi : V \to \mathbb{R}$ is **directionally differentiable** if, for all $v \in V$, $\mathrm{dom}[\partial_v \phi] = V$. Recall: $f : V \to \mathbb{R}$ is defined by

$$f(v) = \begin{cases} g(y/x^2), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Let $\sigma : V \to \mathbb{R}$ be an infinitely differentiable function satisfying $\sigma = 0$ on $V \backslash (B(0_2, 1))$ and $\sigma(0_2) = 1$ and $\mathrm{im}[\sigma] \subseteq [0, 1]$. For all $\varepsilon > 0$, define $\sigma_\varepsilon : V \to \mathbb{R}$ by $\sigma_\varepsilon(x) = \varepsilon \cdot [\sigma(x/\varepsilon)]$; then $\sigma_\varepsilon = 0$ on $V \backslash (B(0_2, \varepsilon))$ and $\sigma_\varepsilon(0_2) = \varepsilon$ and $\mathrm{im}[\sigma_\varepsilon] \subseteq [0, \varepsilon]$. For all $\varepsilon > 0$, for all $p \in V$, we define $f_{p,\varepsilon} : V \to \mathbb{R}$ by $f_{p,\varepsilon}(x) = [\sigma_\varepsilon(x - p)][f(x - p)]$; then

- $f_{p,\varepsilon} = 0$ on $V\backslash(B(p,\varepsilon))$,
- $\mathrm{im}[f_{p,\varepsilon}] \subseteq [0,\varepsilon]$,
- $f_{p,\varepsilon}$ is infinitely directionally differentiable and
- $f_{p,\varepsilon}$ is discontinuous at p.

For any sequence $p_1, p_2, \ldots$ of distinct points in V, there is a sequence $\varepsilon_1, \varepsilon_2, \ldots$ of positive real numbers s.t. $f_{p_1,\varepsilon_1} + f_{p_2,\varepsilon_2} + \cdots$ is both directionally differentiable and discontinuous at each point of $\{p_1, p_2, \ldots\}$. In particular, there is a directionally differentiable $g : V \to \mathbb{R}$ such that g is discontinuous at each point of $\mathbb{Q}^2$.

A subset S of a topological space X is said to be **interior-free in** X if the interior in X of S is empty. A subset S of a topological space X is said to be **meager in** X if S is a subset of a countable union of closed interior-free subsets of X. A subset S of a topological space X is said to be **comeager in** X if $X\backslash S$ is meager in X. By the Baire Category Theorem, for any $n \in \mathbb{N}$, any comeager subset of $\mathbb{R}^n$ is to be dense in $\mathbb{R}^n$; in particular, a comeager subset of $\mathbb{R}^n$ is nonempty.

For any $\phi : V \to \mathbb{R}$, define $C_\phi := \{x \in V \,|\, \phi \text{ is continuous at } x\}$, and let $D_\phi := V\backslash C_\phi$ be the set of points of discontinuity of ϕ.

FACT 91.2. *Let* $\phi : V \to \mathbb{R}$ *and assume:* $\forall i \in \{1,2\}$, $\mathrm{dom}[\partial_i\phi] = V$. *Then* C_ϕ *is comeager in* V.

Proof. For all $x \in \mathbb{R}$, the maps $\phi(x,\bullet) : \mathbb{R} \to \mathbb{R}$ and $\phi(\bullet,x) : \mathbb{R} \to \mathbb{R}$ are both differentiable, hence continuous. Then, by *e.g.*, Namioka, Separate continuity and joint continuity, Pacific Journal of Mathematics, Volume 51, Number 2, 1974, we see that C_ϕ is comeager in V. □

FACT 91.3. *Let* $\phi : V \to \mathbb{R}$. *Assume:* $\forall i,j \in \{1,2\}$, $\mathrm{dom}[\partial_i\partial_j\phi] = V$. *Then* C_ϕ *contains a dense open subset of* V.

Proof. Since $D_\phi = V\backslash C_\phi$, we wish to show that D_ϕ is nowhere dense in V. Let $\overline{D}$ be the closure in V of D_ϕ. Let U be the interior in V of $\overline{D}$. We wish to show: $U = \varnothing$. Assume, for a contradiction, that $U \neq \varnothing$.

For all $j \in \{1,2\}$, let $\psi_j := \partial_j\phi$. For all $i \in \{1,2\}$, $\mathrm{dom}[\partial_i\psi_1] = V$, so, by Fact 91.2, we see that C_{ψ_1} is comeager in V. For all $i \in \{1,2\}$, $\mathrm{dom}[\partial_i\psi_2] = V$, so, by Fact 91.2, we see that C_{ψ_2} is comeager in V. Let $C := C_{\psi_1} \cap C_{\psi_2}$. Then C is comeager in V, so, by the Baire Category Theorem, C is dense in V. So, as U is a nonempty open subset of V, we conclude that $C \cap U \neq \varnothing$. Choose $x \in C \cap U$. Since $x \in C = C_{\psi_1} \cap C_{\psi_2}$, we see that ψ_1 and ψ_2 are both continuous at x.

For all $j \in \{1,2\}$, choose an open neighborhood U_j in U of x such that ψ_j is bounded on U_j. Then $U_1 \cap U_2$ is an open neighborhood in U of x, so choose open intervals $I, J \subseteq \mathbb{R}$ such that $x \in I \times J \subseteq U_1 \cap U_2$. Then ψ_1 and ψ_2 are bounded on $I \times J$. That is, $\partial_1 \phi$ and $\partial_2 \phi$ are bounded on $I \times J$. Then, by the Mean Value Theorem, ϕ is continuous on $I \times J$. Let $W := I \times J$. Then $W \subseteq C_\phi$, so $W \cap D_\phi = \varnothing$.

We have both $x \in I \times J = W$ and $W = I \times J \subseteq U_1 \cap U_2$. Also, $U_1 \subseteq U$ and $U_2 \subseteq U$. Also, by definition of U, we have $U \subseteq \overline{D}$. Then $x \in W \subseteq U_1 \cap U_2 \subseteq U \subseteq \overline{D}$. Since I and J are open intervals in $\mathbb{R}$ and since $W = I \times J$, it follows that W is an open subset of V. So, since $x \in W$, we see that W is an open neighborhood in V of x. So, since $x \in \overline{D}$ and since $\overline{D}$ is the closure in V of D_ϕ, it follows that $W \cap D_\phi \neq \varnothing$. However, we showed that $W \cap D_\phi = \varnothing$. Contradiction. □

92. A tensor computation

DEFINITION 92.1. *Let* $\mathcal{I}_{\mathbb{R}} := \{\varnothing\}$.
$\forall x \in \mathbb{R}$, *let* $x_\varnothing := x$. *Let* $\pi^{\mathbb{R}}_\varnothing := \mathrm{id}_{\mathbb{R}}$ *and let* $\varepsilon^{\mathbb{R}}_\varnothing := 1$.
$\forall \sigma \in \mathbb{N}$, $\forall m \in \mathbb{N}^\sigma$, *let* $m \| \varnothing := m$ *and* $\varnothing \| m := m$.
Define $\varnothing \| \varnothing := \varnothing$.

Note: $\mathbb{R} \neq \mathbb{R}^{\{\varnothing\}} = \mathbb{R}^{\mathcal{I}_{\mathbb{R}}}$. We therefore cannot extend the statement

$$\forall S \in \mathrm{TNSR}, \qquad S = \mathbb{R}^{\mathcal{I}_S}$$

to TNSR^+. On the other hand, we do have a quantified equivalence for equality of tensors:

THEOREM 92.2. $\forall S \in \mathrm{TNSR}^+$, $\forall v, w \in S$, *we have:*

$$(\, v = w \,) \qquad \Leftrightarrow \qquad (\, \forall j \in \mathcal{I}_S,\ v_j = w_j \,).$$

THEOREM 92.3. *Let* $V, W, X \in \mathrm{TNSR}^+$, $A \in W \otimes V$, $B \in X \otimes W$. *Let* $i \in \mathcal{I}_V$, $k \in \mathcal{I}_W$. *Then:* $(B \ *_{VWX}\ A)_{k\|i} \ = \sum\limits_{j \in \mathcal{I}_W} (B_{k\|j}) \cdot (A_{j\|i})$.

Proof. Unassigned HW. □

THEOREM 92.4. *Let* $V, W \in \mathrm{TNSR}^+$, $A \in W \otimes V$.
Let $i \in \mathcal{I}_V$ *and* $j \in \mathcal{I}_W$. *Then:* $(A \ *_{\mathbb{R}VW}\ \varepsilon^V_i) \ *_{\mathbb{R}W\mathbb{R}}\ \varepsilon^W_j \ = \ A_{j\|i}$.

Proof. Let $x \ := \ A \ *_{\mathbb{R}VW}\ \varepsilon^V_i$. Want: $x \ *_{\mathbb{R}W\mathbb{R}}\ \varepsilon^W_j \ = \ A_{j\|i}$. We have:

$$x \ *_{\mathbb{R}W\mathbb{R}}\ \varepsilon^W_j \quad = \quad (x \ *_{\mathbb{R}W\mathbb{R}}\ \varepsilon^W_j)_{\varnothing\|\varnothing} \quad = \quad \sum_{k \in \mathcal{I}_W} (x_{\varnothing\|k}) \cdot (\varepsilon^W_j)_{k\|\varnothing}$$

$$= \quad \sum_{k \in \mathcal{I}_W} x_k \cdot (\varepsilon^W_j)_k \quad = \quad (\, x_j \cdot (\varepsilon^W_j)_j \,) + 0 \ = \ x_j \ = \ x_{j\|\varnothing}$$

$$= \quad (A \ *_{\mathbb{R}VW} \ \varepsilon_i^V)_{j\|\varnothing} \quad = \quad \sum_{\ell \in \mathcal{I}_V} (\, A_{j\|\ell} \,) \cdot (\, (\varepsilon_i^V)_{\ell\|\varnothing} \,)$$

$$= \quad \sum_{\ell \in \mathcal{I}_V} (\, A_{j\|\ell} \,) \cdot (\, (\varepsilon_i^V)_\ell \,) \quad = \quad (\, (\, A_{j\|i} \,) \cdot (\, (\varepsilon_i^V)_i \,)\,) \ + \ 0$$

$$= \quad A_{j\|i}, \qquad \text{as desired.} \qquad \square$$

We reviewed Theorem 85.6.

THEOREM 92.5. *Let* $S, T \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$, $v \in S$. *Define* $L \in \mathcal{L}_{T \otimes S}^T$ *by:* $L_A \ = \ A \ *_{\mathbb{R}ST} \ v$. *Then* $\partial_v^{ST} f \ \supseteq \ L \circ (f')$.

Proof. Want: $\forall x \in S$, $(\partial_v^{ST} f)_x \ =^* \ (L \circ (f'))_x$.
Given $x \in S$. Want: $(\partial_v^{ST} f)_x \ =^* \ (L \circ (f'))_x$.
By HW#9-2, $(\partial_v^{ST} f)_x \ =^* \ (f'_x) \ *_{\mathbb{R}ST} \ v$.
Then $(\partial_v^{ST} f)_x \ =^* \ (f'_x) \ *_{\mathbb{R}ST} \ v \ = \ L(f'_x) \ = \ (L \circ (f'))_x$, as desired. $\square$

THEOREM 92.6. *Let* $S, T \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$, $v \in S$.
Then $\mathrm{dom}[f'] \subseteq \mathrm{dom}[\partial_v^{ST} f]$.

Proof. We have $\mathrm{im}[f'] \subseteq T \otimes S = \mathrm{dom}[L]$, so $\mathrm{dom}[L \circ (f')] = \mathrm{dom}[f']$.
By Theorem 92.5, $\partial_v^{ST} f \ \supseteq \ L \circ (f')$, so $\mathrm{dom}[\partial_v^{ST} f] \ \supseteq \ \mathrm{dom}[L \circ (f')]$.
Then $\mathrm{dom}[f'] \ = \ \mathrm{dom}[L \circ (f')] \ \subseteq \ \mathrm{dom}[\partial_v^{ST} f]$, as desired. $\square$

THEOREM 92.7. *Let* $V, W \in \mathrm{TNSR}^+$, $L \in \mathcal{L}_V^W$.
Then $V = \mathrm{dom}[L']$.

Proof. Since $L : V \to W$ we get $L : V \dashrightarrow W \otimes V$, and so $\mathrm{dom}[L'] \subseteq V$.
Want: $V \subseteq \mathrm{dom}[L']$. Want: $\forall x \in V$, $x \in \mathrm{dom}[L']$.
Given $x \in V$. Want $x \in \mathrm{dom}[L']$.
Since $L \in \mathcal{L}_V^W$, we get $\mathrm{s}\ell_L^{VW} \in W \otimes V$.
By HW#6-4, $D_x L = L$, and so $L'_x = \mathrm{s}\ell_L^{VW}$.
Then $L'_x = \mathrm{s}\ell_L^{VW} \in W \otimes V$, and so $L'_x \neq ☹$, and so $x \in \mathrm{dom}[L']$. $\square$

THEOREM 92.8. *Let* $S, T \in \mathrm{TNSR}^+$, $v \in S$, $f : S \dashrightarrow T$.
Then $\mathrm{dom}[f''] \subseteq \mathrm{dom}[(\partial_v^{ST} f)']$.

Proof. Define $L \in \mathcal{L}_{T \otimes S}^T$ by: $L_A \ = \ A \ *_{\mathbb{R}ST} \ v$.
By Theorem 92.5, $\partial_v^{ST} f \ \supseteq \ L \circ (f')$.
Let $g := f'$. Then $\partial_v^{ST} f \ \supseteq \ L \circ g$.
Then $(\partial_v^{ST} f)' \ \supseteq \ (L \circ g)'$, so $\mathrm{dom}[(\partial_v^{ST} f)'] \ \supseteq \ \mathrm{dom}[(L \circ g)']$.
Want: $\forall x \in \mathrm{dom}[f'']$, $x \in \mathrm{dom}[(\partial_v^{ST} f)']$.
Given $x \in \mathrm{dom}[f'']$. Want: $x \in \mathrm{dom}[(\partial_v^{ST} f)']$.
Since $\mathrm{dom}[(L \circ g)'] \ \subseteq \ \mathrm{dom}[(\partial_v^{ST} f)']$,

it suffices to show: $x \in \mathrm{dom}[(L \circ g)']$.
By Theorem 90.25, it suffices to show: ($x \in \mathrm{dom}[g']$) & ($g_x \in \mathrm{dom}[L']$).
We have $x \in \mathrm{dom}[f''] = \mathrm{dom}[g']$. Want: $g_x \in \mathrm{dom}[L']$.
Since $x \in \mathrm{dom}[g'] \subseteq \mathrm{dom}[g]$, we get $g_x \in \mathrm{im}[g]$.
Since $f : S \dashrightarrow T$, we see that $f' : S \dashrightarrow T \otimes S$, and so $\mathrm{im}[f'] \subseteq T \otimes S$.
Since $L \in \mathcal{L}^T_{T\otimes S}$, we conclude, by Theorem 92.7, that $\mathrm{dom}[L'] = T \otimes S$.
Then $g_x \in \mathrm{im}[g] = \mathrm{im}[f'] \subseteq T \otimes S = \mathrm{dom}[L']$, as desired. □

93. Some basic properties of directional derivatives

THEOREM 93.1. *Let $V, W \in \mathrm{TNSR}^+$, $f, g : V \dashrightarrow W$, $u \in V$. Then $\partial^{VW}_u(f + g) \supseteq (\partial^{VW}_u f) + (\partial^{VW}_u g)$.*

Proof. Want: $\forall x \in V$, $(\partial^{VW}_u(f + g))_x \;=^*\; ((\partial^{VW}_u f) + (\partial^{VW}_u g))_x$.
Given $x \in V$. Want: $(\partial^{VW}_u(f + g))_x \;=^*\; ((\partial^{VW}_u f) + (\partial^{VW}_u g))_x$.
Let $\phi := f \circ (i^u_x)$ and let $\psi := g \circ (i^u_x)$.
Then $\phi'_0 = (\partial^{VW}_u f)_x$ and $\psi'_0 = (\partial^{VW}_u g)_x$.
Also, since $\phi + \psi = (f \circ (i^u_x)) + (g \circ (i^u_x)) = (f + g) \circ (i^u_x)$,
we get $(\phi + \psi)'_0 = (\partial^{VW}_u(f + g))_x$.
By Theorem 81.3, we have $(\phi + \psi)'_0 \;=^*\; \phi'_0 + \psi'_0$.
Then
$$\begin{aligned}(\partial^{VW}_u(f + g))_x &= (\phi + \psi)'_0 \;=^*\; \phi'_0 + \psi'_0\\ &= ((\partial^{VW}_u f) + (\partial^{VW}_u g))_x,\end{aligned}$$
as desired. □

THEOREM 93.2. *Let $V, W \in \mathrm{TNSR}^+$, $c \in \mathbb{R}$, $f : V \dashrightarrow W$, $u \in V$. Then $\partial^{VW}_u(c \cdot f) \supseteq c \cdot (\partial^{VW}_u f)$.*

Proof. Unassigned HW. □

THEOREM 93.3. *Let $V := \mathbb{R}^2$, $W := \mathbb{R}$, $C \in \mathcal{C}^W_V$. Then: $\partial^{VW}_1 C \;=\; \partial^{VW}_2 C \;=\; \mathbf{0}^W_V$.*

Proof. Unassigned HW. □

THEOREM 93.4. *Let $V := \mathbb{R}^2$, $W := \mathbb{R}$, $\lambda, \mu \in \mathbb{R}$. Define $L \in \mathcal{L}^W_V$ by $L(x, y) = (\lambda, \mu) \bullet_V (x, y)$. Then: ($\partial^{VW}_1 L = C^\lambda_V$) & ($\partial^{VW}_2 L = C^\mu_V$).*

Proof. Unassigned HW. □

THEOREM 93.5. *Let $V := \mathbb{R}^2$, $W := \mathbb{R}$, $\alpha, \gamma, \delta \in \mathbb{R}$. Define $Q \in \mathcal{Q}^W_V$ by $Q(x, y) = \begin{bmatrix} \alpha & \gamma \\ \gamma & \delta \end{bmatrix} \bullet_{V\otimes V} ((x, y)^{\otimes 2})$. Then, $\forall x, y \in \mathbb{R}$, we have:*

$$(\partial_1^{VW} Q)(x,y) \;=\; 2\alpha x \,+\, 2\gamma y \qquad\qquad \textit{and}$$
$$(\partial_2^{VW} Q)(x,y) \;=\; 2\gamma x \,+\, 2\delta y.$$

Proof. Unassigned HW. ☐

DEFINITION 93.6. *Let* $V \in \text{TNSR}^+$, $f : \mathbb{R} \to V$.
Then $\partial_{\varnothing}^{\mathbb{R}V} f := f'$.

We paraphrase HW#9-2:
Let $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$, $u \in V$.
Then: $\qquad \partial_u^{VW} f \;\supseteq\; f' \;*_{\mathbb{R}VW}\; u,$
$\qquad$ *i.e.*, $\forall p \in V$, $\qquad (\partial_u^{VW} f)_p \;=^*\; f'_p \;*_{\mathbb{R}VW}\; u.$

THEOREM 93.7. *Let* $V, W, Z \in \text{TNSR}^+$.
Let $A \in Z \otimes V \otimes W$, $\quad x \in V$, $\quad y \in W$.
Then: $\qquad \text{Bilin}_A^{VWZ}(x,y) \;=\; A \;*_{\mathbb{R},V\otimes W,Z}\; (x \otimes y).$

Proof. Let $B := \text{Bilin}_A^{VWZ}$. $\qquad$ Then $B \in \mathcal{B}_{VW}^Z$ and $[B]_{VW}^Z = A$.
Let $* \;:=\; *_{\mathbb{R},V\otimes W,Z}$. $\qquad$ Then $* \in \mathcal{B}_{Z\otimes V\otimes W, V\otimes W}^Z$.
We have: $\forall i \in \mathcal{I}_V$, $\forall j \in \mathcal{I}_W$, $\forall k \in \mathcal{I}_Z$,
$$A_{k\|i\|j} = ([B]_{VW}^Z)_{k\|i\|j} = \pi_k^Z(B(\varepsilon_i^V, \varepsilon_j^W)) = (B(\varepsilon_i^V, \varepsilon_j^W))_k.$$
Want: $\qquad \text{Bilin}_A^{VWZ}(x,y) \;=\; A \;*\; (x \otimes y).$
Want: $\forall k \in \mathcal{I}_Z$, $\quad (\,\text{Bilin}_A^{VWZ}(x,y)\,)_k \;=\; (\,A \;*\; (x \otimes y)\,)_k.$
Given $k \in \mathcal{I}_Z$. $\quad$ Want: $(\,\text{Bilin}_A^{VWZ}(x,y)\,)_k \;=\; (\,A \;*_{\mathbb{R},V\otimes W,Z}\; (x \otimes y)\,)_k.$
We calculate: $\text{Bilin}_A^{VWZ}(x,y) \;=\; B(x,y)$
$$= \; B\left(\sum_{i\in\mathcal{I}_V} x_i\varepsilon_i \;,\; \sum_{i\in\mathcal{I}_V} x_i\varepsilon_i\right)$$
$$= \; \sum_{i\in\mathcal{I}_V}\sum_{i\in\mathcal{I}_V} x_i \;\cdot\; y_j \;\cdot\; (B(\varepsilon_i^V, \varepsilon_j^W))$$
Then: $(\,\text{Bilin}_A^{VWZ}(x,y)\,)_k \;=\; \sum_{i\in\mathcal{I}_V}\sum_{i\in\mathcal{I}_V} x_i \;\cdot\; y_j \; cdot \; (\,B(\varepsilon_i^V, \varepsilon_j^W)\,)_k.$
We compute: $(\,A \;*\; (x \otimes y)\,)_k \;=\; (\,A \;*\; x \otimes y)\,)_{k\|\varnothing}$
$$= \; \sum_{\ell\in\mathcal{I}_{V\otimes W}} A_{k\|\ell} \;*\; (x \otimes y)_{\ell\|\varnothing}$$
$$= \; \sum_{\ell\in\mathcal{I}_{V\otimes W}} A_{k\|\ell} \;\cdot\; (x \otimes y)_{\ell}$$
$$= \; \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} A_{k\|i\|j} \;\cdot\; (x \otimes y)_{i\|j}$$
$$= \; \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} A_{k\|i\|j} \;\cdot\; x_i \;\cdot\; y_j$$
Then $(\,\text{Bilin}_A^{VWZ}(x,y)\,)_k \;=\; \sum_{i\in\mathcal{I}_V}\sum_{i\in\mathcal{I}_V} x_i \;\cdot\; y_j \;\cdot\; (\,B(\varepsilon_i^V, \varepsilon_j^W)\,)_k$

$$= \sum_{i\in\mathcal{I}_V} \sum_{i\in\mathcal{I}_V} x_i \cdot y_j \cdot A_{k\|i\|j}$$
$$= \sum_{i\in\mathcal{I}_V} \sum_{i\in\mathcal{I}_V} A_{k\|i\|j} \cdot x_i \cdot y_j$$
$$= (A * (x \otimes y))_k, \qquad \text{as desired.} \qquad \square$$

THEOREM 93.8. *Let $h : \mathbb{R}^2 \dashrightarrow \mathbb{R}$ and let $p \in \mathrm{dom}[h']$.*
Then (1) $(h'_p)_1 = (\partial_1 h)_p$ and (2) $(h'_p)_2 = (\partial_2 h)_p$.

Proof. Proof of (1):
Let $V := \mathbb{R}^2$.
We have $(\partial_1 h)_p =^* h'_p *_{\mathbb{R}V\mathbb{R}} \varepsilon_1^V = h'_p \bullet_V \varepsilon_1^V = (h'_p)_1 \neq ☹$.
Then $(h'_p)_1 = (\partial_1 h)_p$, as desired.
End of proof of (1).

Proof of (2):
Unassigned HW.
End of proof of (2). $\square$

THEOREM 93.9. *Let $V := \mathbb{R}^2$ and let $W := \mathbb{R}$.*
Let $R : V \dashrightarrow W$ and let $z := 0_2$. Assume $z \in \mathrm{dom}[R'']$.
Assume $(\partial_1\partial_1 R)_z = (\partial_1\partial_2 R)_z = (\partial_2\partial_2 R)_z = (\partial_1 R)_z = (\partial_2 R)_z = R_z = 0$.
Then $R \in \mathcal{O}_2^{VW}$.

Proof. Since $z \in \mathrm{dom}[R'']$, it follows that R' is defined near z,
so choose $B \in \mathcal{B}_V(z)$ s.t. $B \subseteq \mathrm{dom}[R']$.
Since $B \in \mathcal{B}_V(z)$, choose $r > 0$ s.t. $B = B_V(z, r)$.
Let $\delta := r/\sqrt{2}$, $J := (-\delta, \delta)$. Then $J^2 \subseteq B_V(z, r)$.
Then $J^2 \subseteq B_V(z, r) = B \subseteq \mathrm{dom}[R']$.
Let $g := \partial_1 R$, $h := \partial_2 R$, $i := i_z^{\varepsilon_1^V}$.
By HW#9-3, choose $\sigma \in \widehat{\mathcal{O}}_1^{WW}$ and $\tau \in \widehat{\mathcal{O}}_1^{VV}$ s.t.
$$f = (g \circ i \circ \sigma \circ \pi_1^V) \cdot \pi_1^V + (h \circ \tau) \cdot \pi_2^V \qquad \text{near } z.$$
Define $* \in \mathcal{B}_{WW}^W$ by $a * b = ab$.
Then $f = (g \circ i \circ \sigma \circ \pi_1^V) * \pi_1^V + (h \circ \tau) * \pi_2^V$ near z.
By HW#10-5, we have: $g \circ i \in \mathcal{O}_1^{WW}$.
Then $g \circ i \circ \sigma \circ \pi_1^V \in \mathcal{O}_1^{WW} \circ \widehat{\mathcal{O}}_1^{WW} \circ \widehat{\mathcal{O}}_1^{VW} \subseteq \mathcal{O}_1^{VW}$.
Then $(g \circ i \circ \sigma \circ \pi_1^V) * \pi_1^V \in \mathcal{O}_1^{VW} * \widehat{\mathcal{O}}_1^{VW} \subseteq \mathcal{O}_2^{VW}$.
By Theorem 92.8, $\mathrm{dom}[R''] \subseteq \mathrm{dom}[(\partial_2 R)']$.
Then $z \in \mathrm{dom}[R''] \subseteq \mathrm{dom}[(\partial_2 R)'] = \mathrm{dom}[h']$.
By Theorem 93.8, $(h'_z)_1 = (\partial_1 h)_z$ and $(h'_z)_2 = (\partial_2 h)_z$.

So, since $(\partial_1 h)_z = (\partial_1\partial_2 R)_z = 0$ and $(\partial_2 h)_z = (\partial_2\partial_2 R)_z = 0$,
we conclude that $(h'_z)_1 = 0$ and $(h'_z)_2 = 0$.
Also, we have $W \otimes V = \mathbb{R} \otimes \mathbb{R}^2 = \mathbb{R}^2$.
Then $h'_z = (\,(h'_z)_1\,,\,(h'_z)_2\,) = (0,0) = 0_{\mathbb{R}^2} = 0_{W\otimes V}$.
So, since $h_z = (\partial_2 R)_z = 0 = 0_W$, by HW#8-1, we get: $h \in \mathcal{O}_1^{VW}$.
Then $h \circ \tau \in \mathcal{O}_1^{VW} \circ \widehat{\mathcal{O}}_1^{VV} \subseteq \mathcal{O}_1^{VW}$.
Also, $(h\circ\tau) * \pi_2^V \in \mathcal{O}_1^{VW} * \widehat{\mathcal{O}}_1^{VW} \subseteq \mathcal{O}_2^{VW}$.
So, since $(g \circ i \circ \sigma \circ \pi_1^V) * \pi_1^V \in \mathcal{O}_2^{VW}$,
we get: $(g \circ i \circ \sigma \circ \pi_1^V) * \pi_1^V + (h\circ\tau) * \pi_2^V \in \mathcal{O}_2^{VW}$.
So, since $f = (g \circ i \circ \sigma \circ \pi_1^V) * \pi_1^V + (h\circ\tau) * \pi_2^V$ near z,
we conclude that $f \in \mathcal{O}_2^{VW}$, as desired. □

THEOREM 93.10. *Let $V := \mathbb{R}^2$ and let $W := \mathbb{R}$.*
Let $S : V \dashrightarrow W$ and let $z := 0_2$. Assume $z \in \mathrm{dom}[S'']$.
Assume $(\partial_1\partial_1 S)_z = (\partial_2\partial_1 S)_z = (\partial_2\partial_2 S)_z = (\partial_1 S)_z = (\partial_2 S)_z = S_z = 0$.
Then $S \in \mathcal{O}_2^{VW}$.

Proof. Define $L \in \mathcal{L}_V^V$ by $L(x,y) = (y,x)$ Then $L \circ L = \mathrm{id}_V$.
Let $R := S \circ L$. Then, for all $x, y \in \mathbb{R}$, we have $R(x,y) = S(y,x)$.
Unassigned HW: Show:

$$(\partial_1\partial_1 R)_z = (\partial_1\partial_2 R)_z = (\partial_2\partial_2 R)_z = (\partial_1 R)_z = (\partial_2 R)_z = r_z = 0.$$

Then, by Theorem 93.9, we have $R \in \mathcal{O}_2^{VW}$.
Also, $R \circ L = S \circ L \circ L = S \circ \mathrm{id}_V = S$ and $L \in \mathcal{L}_V^V \subseteq \widehat{\mathcal{O}}_1^{VV}$.
Then $S = R \circ L \in \mathcal{O}_2^{VW} \circ \widehat{\mathcal{O}}_1^{VV} = \mathcal{O}_2^{VW}$, as desired. □

94. Directional derivatives commute with one another

THEOREM 94.1. *Let $V := \mathbb{R}^2$, $W := \mathbb{R}$, $f : V \dashrightarrow W$, $z := 0_2$.*
Assume $z \in \mathrm{dom}[f'']$. Then $(\partial_1\partial_2 f)_z = (\partial_2\partial_1 f)_z$.

Proof. Let $\alpha := (\partial_1\partial_1 f)_z$, $\beta := (\partial_1\partial_2 f)_z$.
Let $\gamma := (\partial_2\partial_1 f)_z$, $\delta := (\partial_2\partial_2 f)_z$.
Want: $\beta = \gamma$. Let $\lambda := (\partial_1 f)_z$, $\mu := (\partial_2 f)_z$, $\rho := f_z$.
Define $C \in \mathcal{C}_V^W$, $L \in \mathcal{L}_V^W$, $P, Q \in \mathcal{Q}_V^W$ by:

$$C(x,y) = \rho,$$
$$L(x,y) = (\lambda,\mu) \bullet_V (x,y),$$
$$P(x,y) = \begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix} \bullet_{V\otimes V} ((x,y)^{\otimes 2}).$$
$$Q(x,y) = \begin{bmatrix} \alpha & \gamma \\ \gamma & \delta \end{bmatrix} \bullet_{V\otimes V} ((x,y)^{\otimes 2}).$$

Let $\quad R := f - \left(C + L + \dfrac{P}{2!} \right) \quad$ and $\quad S := f - \left(C + L + \dfrac{Q}{2!} \right)$.
By HW#9-5, we have:
$\quad (\partial_1\partial_1 R)_z = (\partial_1\partial_2 R)_z = (\partial_2\partial_2 R)_z = (\partial_1 R)_z = (\partial_2 R)_z = R_z = 0.$
Then, by Theorem 93.9, $R \in \mathcal{O}_2^{VW}$.
Unassigned HW:
$\quad (\partial_1\partial_1 S)_z = (\partial_2\partial_1 S)_z = (\partial_2\partial_2 S)_z = (\partial_1 S)_z = (\partial_2 S)_z = S_z = 0.$
Then, by Theorem 93.10, $S \in \mathcal{O}_2^{VW}$.
Since $z \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[f']$, we conclude that f is defined near z.
Then $f - f = \mathbf{0}_V^W$ near z. $\qquad$ Then $R - S = -\dfrac{P}{2} + \dfrac{Q}{2}$ near z.
Then $2 \cdot (R - S) = Q - P$ near z.
Since $R, S \in \mathcal{O}_2^{VW}$, we conclude that $2 \cdot (R - S) \in \mathcal{O}_2^{VW}$.
Then $Q - P \in \mathcal{O}_2^{VW}$. $\qquad$ Since $Q, P \in \mathcal{Q}_V^W$, we get $Q - P \in \mathcal{Q}_V^W$.
Then $Q - P \in (\mathcal{O}_2^{VW}) \cap (\mathcal{Q}_V^W)$.
So, since $(\mathcal{O}_2^{VW}) \cap (\mathcal{Q}_V^W) = \{\mathbf{0}_V^W\}$, it follows that $Q - P = \mathbf{0}_V^W$.
Then $P = Q$, so $P(1,1) = Q(1,1)$.
Then $\alpha + 2\beta + \delta = P(1,1) = Q(1,1) = \alpha + 2\gamma + \delta$.
Then $2\beta = 2\gamma$, and so $\beta = \gamma$, as desired. ∎

THEOREM 94.2. *Let* $\quad X, Z \in \mathrm{TNSR}^+, \quad \phi : \mathbb{R} \dashrightarrow X$.
Let $L \in \mathcal{L}_X^Z$, $\quad t \in \mathrm{dom}[\phi']$. $\qquad$ *Then* $\quad (L \circ \phi)'_t = L(\phi'_t)$.

Proof. We have $(L \circ \phi)'_t \;=\; (D_t(L \circ \phi))(1) \;=^*\; ((D_{\phi_t} L) \circ (D_t \phi))(1)$
$\qquad = \; (L \circ (D_t \phi))(1) \;=\; L((D_t\phi)(1)) \;=\; L(\phi'_t) \;\neq\; ☹.$
Then $(L \circ \phi)'_t \;=\; L(\phi'_t)$, as desired. ∎

THEOREM 94.3. *Let* $V, X, Z \in \mathrm{TNSR}^+$.
Let $\quad f : V \dashrightarrow X, \quad L \in \mathcal{L}_X^Z, \quad u \in V$.
Then: $\qquad \partial_u(L \circ f) \;\supseteq\; L \circ (\partial_u f)$.

Proof. Want: $\forall q \in V$, $(\partial_u(L \circ f))_q \;=^*\; (L \circ (\partial_u f))_q$.
Given $q \in V$. $\qquad$ Want: $(\partial_u(L \circ f))_q \;=^*\; (L \circ (\partial_u f))_q$.
Want: ($(L \circ (\partial_u f))_q \neq ☹$) $\Rightarrow$ ($(\partial_u(L \circ f))_q \;=\; (L \circ (\partial_u f))_q$).
Assume $(L \circ (\partial_u f))_q \neq ☹$. $\qquad$ Want: $(\partial_u(L \circ f))_q \;=\; (L \circ (\partial_u f))_q$.
Let $\qquad i := i_q^u, \qquad \phi : f \circ i$.
Then $\quad (\partial_u(L \circ f))_q \;=\; (L \circ f \circ (i_q^u))'_0 \;=\; (L \circ f \circ i)'_0 \;=\; (L \circ \phi)'_0$.
Also, $\quad (\partial_u f)_q \;=\; (f \circ (i_q^u))'_0 \;=\; (f \circ i)'_0 \;=\; \phi'_0$. $\quad$ Then $\phi'_0 = (\partial_u f)_q$.
Because $(L \circ (\partial_u f))_q \neq ☹$, $\qquad$ it follows that $(\partial_u f)_q \neq ☹$.
Then $\quad \phi'_0 \;=\; (\partial_u f)_q \;\neq\; ☹, \qquad$ and so $\quad 0 \in \mathrm{dom}[\phi']$.
Then, by Theorem 94.2, we have : $\qquad (L \circ \phi)'_0 \;=\; L(\phi'_0)$.

Then $(\partial_u(L \circ f))_q \;=\; (L \circ \phi)'_0 \;=\; L(\phi'_0)$
$\;=\; L((\partial_u f)_q) \;=\; (L \circ (\partial_u f))_q$, as desired. □

THEOREM 94.4. *Let $S, T \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$, $u \in S$.*
Then *(1)* $\mathrm{dom}[f''] \subseteq \mathrm{dom}[(\partial_u f)']$ *and*
(2) $\mathrm{dom}[f'] \subseteq \mathrm{dom}[\partial_u f]$.

Proof. By Theorem 92.8, (1) holds. By Theorem 92.6, (2) holds.
Then both (1) and (2) hold, as desired. □

THEOREM 94.5. *Let $S, T \in \mathrm{TNSR}^+$, $f : S \dashrightarrow T$, $u, v \in S$.*
Then $\mathrm{dom}[f''] \subseteq \mathrm{dom}[(\partial_u \partial_v f)']$.

Proof. Let $g := \partial_v f$. By (2) of Theorem 94.4, $\mathrm{dom}[g'] \subseteq \mathrm{dom}[\partial_u g]$.
By (1) of Theorem 94.4, $\mathrm{dom}[f''] \subseteq \mathrm{dom}[(\partial_v f)']$.
Then $\mathrm{dom}[f''] \subseteq \mathrm{dom}[(\partial_v f)'] = \mathrm{dom}[g']$
$\subseteq \mathrm{dom}[\partial_u g] = \mathrm{dom}[\partial_u \partial_v f]$, as desired. □

THEOREM 94.6. *Let $V := \mathbb{R}^2$, $X \in \mathrm{TNSR}^+$, $f : V \dashrightarrow X$, $z := 0_2$.*
Assume $z \in \mathrm{dom}[f'']$. Then $(\partial_1 \partial_2 f)_z = (\partial_2 \partial_1 f)_z$.

Proof. Want: $\forall j \in \mathcal{I}_X$, $(\,(\partial_1 \partial_2 f)_z\,)_j \;=\; (\,(\partial_2 \partial_1 f)_z\,)_j$.
Given $j \in \mathcal{I}_X$. Want: $(\,(\partial_1 \partial_2 f)_z\,)_j \;=\; (\,(\partial_2 \partial_1 f)_z\,)_j$.
Want: $\pi^X_j((\partial_1 \partial_2 f)_z) = \pi^X_j((\partial_2 \partial_1 f)_z)$.
Let $L := \pi^X_j$. Want: $L((\partial_1 \partial_2 f)_z) = L((\partial_2 \partial_1 f)_z)$.

Claim 1: $(\partial_1 \partial_2 (L \circ f))_z = L((\partial_1 \partial_2 f)_z)$.
Proof of Claim 1:
By two applications of Theorem 94.3, we get: $\partial_1 \partial_2 (L \circ f) \supseteq L \circ (\partial_1 \partial_2 f)$.
Then $(\partial_1 \partial_2 (L \circ f))_z \;=^*\; (L \circ (\partial_1 \partial_2 f))_z$.
By Theorem 94.5, we get $\mathrm{dom}[f''] \subseteq \mathrm{dom}[\partial_1 \partial_2 f]$.
Then $z \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[\partial_1 \partial_2 f]$,
so $(\partial_1 \partial_2 f)_z \neq$ ☹, so $L((\partial_1 \partial_2 f)_z) \neq$ ☹.
Then $(\partial_1 \partial_2 (L \circ f))_z \;=^*\; (L \circ (\partial_1 \partial_2 f))_z \;=\; L((\partial_1 \partial_2 f)_z) \neq$ ☹.
Then $(\partial_1 \partial_2 (L \circ f))_z = L((\partial_1 \partial_2 f)_z)$.
End of proof of Claim 1.

Claim 2: $(\partial_2 \partial_1 (L \circ f))_z = L((\partial_2 \partial_1 f)_z)$.
Proof of Claim 2:
By two applications of Theorem 94.3, we get: $\partial_2 \partial_1 (L \circ f) \supseteq L \circ (\partial_2 \partial_1 f)$.
Then $(\partial_2 \partial_1 (L \circ f))_z \;=^*\; (L \circ (\partial_2 \partial_1 f))_z$.

By Theorem 94.5, we get $\mathrm{dom}[f''] \subseteq \mathrm{dom}[\partial_2\partial_1 f]$.
Then $z \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[\partial_2\partial_1 f]$,
so $(\partial_2\partial_1 f)_z \neq ☹$, so $L((\partial_2\partial_1 f)_z) \neq ☹$.
Then $(\partial_2\partial_1(L \circ f))_z \;\; =^* \;\; (L \circ (\partial_2\partial_1 f))_z \;\; = \;\; L((\partial_2\partial_1 f)_z) \neq ☹$.
Then $(\partial_2\partial_1(L \circ f))_z = L((\partial_2\partial_1 f)_z)$.
End of proof of Claim 2.

Since $L \in \mathcal{L}_Z^{\mathbb{R}}$, we see that $\mathrm{dom}[L''] = X$.
Since $z \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[f]$, we get $f_z \in \mathrm{im}[f]$.
Since $z \in \mathrm{dom}[f'']$ and since $f_z \in \mathrm{im}[f] \subseteq X = \mathrm{dom}[L'']$,
we conclude, by Theorem 90.25, that $z \in \mathrm{dom}[(L \circ f)'']$.
Then, by Theorem 94.1, we have $(\partial_1\partial_2(L \circ f))_z = (\partial_2\partial_1(L \circ f))_z$.
By Claim 1, $(\partial_1\partial_2(L \circ f))_z = L((\partial_1\partial_2 f)_z)$.
By Claim 2, $L((\partial_2\partial_1 f)_z) = (\partial_2\partial_1(L \circ f))_z$.
Then $L((\partial_1\partial_2 f)_z) = (\partial_1\partial_2(L \circ f))_z = (\partial_2\partial_1(L \circ f))_z = L((\partial_2\partial_1 f)_z)$. □

DEFINITION 94.7. *Let $S \in \mathrm{TNSR}^+$, $q, v, w \in S$.*
Then $j_q^{vw} : \mathbb{R}^2 \to S$ is defined by $j_q^{vw}(a, b) = q + av + bw$.

Unassigned HW: $S \in \mathrm{TNSR}^+$, $q, v, w \in S$.
Show: $\mathbb{R}^2 = \mathrm{dom}[j] = \mathrm{dom}[j'] = \mathrm{dom}[j'']$.

THEOREM 94.8. *Let $S, X \in \mathrm{TNSR}^+$, $f : S \dashrightarrow X$.*
Let $q, v, w \in S$, $j := j_q^{vw}$.
Then: (1) $\partial_1(f \circ j) = (\partial_v f) \circ j$ and
(2) $\partial_2(f \circ j) = (\partial_w f) \circ j$.

Proof. Proof of (1):
Unassigned HW.
End of proof of (1).

Proof of (2):
Want: $\forall y \in \mathbb{R}^2$, $(\partial_2(f \circ j))_y = ((\partial_w f) \circ j)_y$.
Given $y \in \mathbb{R}^2$. Want: $(\partial_2(f \circ j))_y = ((\partial_w f) \circ j)_y$.
Let $p := j(y)$ and $e := \varepsilon_2^V$. Want: $(\partial_e(f \circ j))_y = (\partial_w f)_p$.
Want: $(f \circ j \circ (i_y^e))'_0 = (f \circ (i_p^w))'_0$.
Want: $j \circ (i_y^e) \;\; = \;\; i_p^w$.
Want: $\forall t \in \mathbb{R}$, $(j \circ (i_y^e))(t) = (i_p^w)(t)$.
Given $t \in \mathbb{R}$. Want: $(j \circ (i_y^e))(t) = (i_p^w)(t)$.

We compute $i_y^e(t) = y + te = y + t\varepsilon_2^V = (y_1, y_2) + t \cdot (0,1) = (y_1, y_2 + t)$.
We compute $(i_p^w)(t) = p + tw$ and $p = j(y) = j_q^{vw}(y_1, y_2) = q + y_1 v + y_2 w$.

$$\begin{aligned} \text{Then } (j \circ (i_y^e))(t) &= j((i_y^e)(t)) = j(y_1, y_2 + t) \\ &= q + y_1 v + (y_2 + t) w \\ &= q + y_1 v + y_2 w + tw \\ &= p + tw = (i_p^w)(t), \qquad \text{as desired.} \end{aligned}$$

End of proof of (2). □

THEOREM 94.9. *Let* $S, X \in \text{TNSR}^+$, $f : S \dashrightarrow X$.
Let $q, v, w \in S$, $j := j_q^{vw}$.
Then: (1) $\partial_1 \partial_2 (f \circ j) = (\partial_v \partial_w f) \circ j$ *and*
(2) $\partial_2 \partial_1 (f \circ j) = (\partial_w \partial_v f) \circ j$.

Proof. Proof of (1):
By (2) of Theorem 94.8, $\partial_2 (f \circ j) = (\partial_w f) \circ j$.
Let $g := \partial_w f$. Then $\partial_2 (f \circ j) = g \circ j$.
By (1) of Theorem 94.8, $\partial_1 (g \circ j) = (\partial_v g) \circ j$.
Then $\partial_1 \partial_2 (f \circ j) = \partial_1 (g \circ j) = (\partial_v g) \circ j = (\partial_v \partial_w g) \circ j$.
End of proof of (1).

Proof of (2):
Unassigned HW.
End of proof of (2). □

THEOREM 94.10. *Let* $S, X \in \text{TNSR}^+$, $f : S \dashrightarrow X$.
Let $q \in \text{dom}[f'']$, $v, w \in S$.
Then $(\partial_v \partial_w f)_q = (\partial_w \partial_v f)_q$.

Proof. Let $j := j_q^{vw}$, $z := 0_2$.
We have $j_z = j_{0_2} = j_{(0,0)} = q + 0 \cdot v + 0 \cdot w = q$.
Want: $((\partial_v \partial_w f) \circ j)_z = ((\partial_w \partial_v f) \circ j)_z$.
Since $z \in \mathbb{R}^2 = \text{dom}[j'']$ and $j_z = q \in \text{dom}[f'']$,
it follows, from Theorem 90.25, that $z \in \text{dom}[(f \circ j)'']$.
Then, by Theorem 94.6, we get: $(\partial_1 \partial_2 (f \circ j))_z = (\partial_2 \partial_1 (f \circ j))_z$.
By Theorem 94.9, we have:
$\partial_1 \partial_2 (f \circ j) = (\partial_v \partial_w) \circ j$ and $\partial_2 \partial_1 (f \circ j) = (\partial_w \partial_v) \circ j$.

$$\begin{aligned} \text{Then } ((\partial_v \partial_w f) \circ j)_z &= (\partial_1 \partial_2 (f \circ j))_z \\ &= (\partial_2 \partial_1 (f \circ j))_z = ((\partial_w \partial_v f) \circ j)_z, \qquad \text{as desired.} \end{aligned}$$ □

THEOREM 94.11. *Let* $V := \mathbb{R}^2$, $W := \mathbb{R}$, $z := 0_2$.
Then there exists $f : V \to W$ *s.t.*

(1) $\mathrm{dom}[f'] = V$, *(2)* $\forall k \in \mathbb{N}_1^\times$, $\mathrm{dom}[f^{(k)}] = V_z^\times$,
(3) $(\partial_1\partial_2 f)_z = 0$ *and* *(4)* $(\partial_2\partial_1 f)_z = 1$.

Proof. Let $X := \pi_1^V$ and $Y := \pi_2^V$.
Let $S := \{ (x,y) \in \mathbb{R}^2 \text{ s.t. } 2 \cdot |x| \leqslant |y| \}$.
Let $T := \{ (x,y) \in \mathbb{R}^2 \text{ s.t. } 2 \cdot |y| \leqslant |x| \}$.
Choose $f : V \to W$ s.t.
(A) $\mathrm{dom}[f'] = V$, (B) $\forall k \in \mathbb{N}_1^\times$, $\mathrm{dom}[f^{(k)}] = V_z^\times$,
(C) $f = XY$ on S and (D) $f = \mathbf{0}_V^W$ on T.
By (A) and (B), we have (1) and (2). Want: (3) and (4).
Since $\partial_2 f = \mathbf{0}_V^W$ on $\mathbb{R} \times \{0\}$, we get $(\partial_1\partial_2 f)_z = 0$, verifying (3).
Since $\partial_1 f = Y$ on $\{0\} \times \mathbb{R}$, we get $(\partial_2\partial_1 f)_z = 1$, verifying (4).
Then (3) and (4) hold, as desired. □

95. A higher order Chain Rule

THEOREM 95.1. *Let* $f, g : \mathbb{R} \to \mathbb{R}$.
Then $(g \circ f)' \supseteq (g' \circ f) \cdot (f')$ *and*
$(g \circ f)'' \supseteq (g'' \circ f) \cdot ((f')^{\otimes 2}) + (g' \circ f) \cdot (f'')$.

Proof. Unassigned HW. Follows from: Chain and Product Rules. □

THEOREM 95.2. *Let* $V := \mathbb{R}^2$, $S := V \otimes V$, $f : \mathbb{R} \to V$, $g : V \to \mathbb{R}$.
Then $(g \circ f)' \supseteq (g' \circ f) \bullet_V (f')$ *and*
$(g \circ f)'' \supseteq (g'' \circ f) \bullet_S ((f')^{\otimes 2}) + (g' \circ f) \bullet_V (f'')$.

Proof. Unassigned HW. Follows from: Chain and Product Rules, together with some tensor algebra. □

Let $V := \mathbb{R}^2$, $S := V \otimes V$, $f : \mathbb{R} \to V$, $g : V \to \mathbb{R}$ and $p \in V$.
Assume that $p \in \mathrm{dom}[f'']$ and that $f_p \in \mathrm{dom}[g'']$.
Assume that g has a "critical point" at f_p, *i.e.*, assume that $g'_{f_p} = 0$.
Then, according to Theorem 95.2, $(g \circ f)''_p = (g''_{f_p}) \bullet_S ((f'_p)^{\otimes 2})$.
In this situation, $(g \circ f)''_p$ does not depend on f''_p, only on g''_{f_p} and f'_p.

96. Positive definite criterion

DEFINITION 96.1. *Let* $n \in \mathbb{N}$, $A \in \mathbb{R}^{n \times n}$, $m \in [1..n]$.
Then: $\mathrm{PS}_m(A) := A \,|\, ([1..m]^2)$.

In Definition 96.1, $\mathrm{PS}_m(A)$ is called the mth **principal submatrix** of A. Its determinant is called the the mth **principal minor** of A.

THEOREM 96.2. *Let* $A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Then $\mathrm{PS}_1(A) = [1]$ *and* $\mathrm{PS}_2(A) = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ *and* $\mathrm{PS}_3(A) = A$.

THEOREM 96.3. *Let* $n \in \mathbb{N}$, $V := \mathbb{R}^n$, $S := V \otimes V$, $z := 0_n$.
Let $A \in S$. *Assume that* A *is symmetric.*
Then: $(\; \forall u \in V_z^\times, \; \mathrm{Bilin}_A^{VV\mathbb{R}}(u,u) \; > \; 0 \;)$
$\Leftrightarrow \; (\; \forall u \in V_z^\times, \; A \bullet_S (u^{\otimes 2}) \; > \; 0 \;)$
$\Leftrightarrow \; (\; \forall m \in [1..n], \; \det(\mathrm{PS}_m(A)) \; > \; 0 \;)$.

That is: a symmetric matrix is positive definite iff all of its principal minors are positive.

There is a similar criterion for negative definite. However, for positive and negative semidefinite, the criteria are slightly more complicated.

97. The second total derivative is symmetric

THEOREM 97.1. *Let* $V, W \in \mathrm{TNSR}^+$, $A \in W \otimes V$, $i \in \mathcal{I}_V$, $j \in \mathcal{I}_W$.
Then $A_{j\|i} \;\; = \;\; (A \;*_{\mathbb{R}VW}\; \varepsilon_i^V)_j$.

Proof. Unassigned HW. □

THEOREM 97.2. *Let* $V, W \in \mathrm{TNSR}^+$ *and let* $f : V \dashrightarrow W$.
Let $u \in V$ *and let* $W' := W \otimes V$.
Then $\partial_u f : V \dashrightarrow W$ *and* $(\partial_u f)' : V \dashrightarrow W'$ *and* $f' : V \dashrightarrow W$ *and* $\partial_u(f') : V \dashrightarrow W'$.

Proof. Unassigned HW. □

THEOREM 97.3. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$.
Let $p \in \mathrm{dom}[f'']$, $u \in V$.
Then: $(\partial_u(f'))_p = (\partial_u f)'_p$.

Proof. Let $g := f'$, $h := \partial_u f$. Want: $(\partial_u g)_p = h'_p$.
Let $W' := W \otimes V$. We have: $g : V \dashrightarrow W'$ and $\partial_u g : V \dashrightarrow W'$.
Since $p \in \mathrm{dom}[f''] = \mathrm{dom}[g'] \subseteq \mathrm{dom}[\partial_u g]$, we get $(\partial_u g)_p \in \mathrm{im}[\partial_u g]$.
Then $(\partial_u g)_p \in \mathrm{im}[\partial_u g] \subseteq W'$.
We have: $h : V \dashrightarrow W$ and $h' : V \dashrightarrow W'$.
Since $p \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[(\partial_u f)'] = \mathrm{dom}[h']$, we get $h'_p \in \mathrm{im}[h']$.

Then $h'_p \in \mathrm{im}[h'] \subseteq W'$.
Want: $\forall k \in \mathcal{I}_{W'}$, $((\partial_u g)_p)_k = (h'_p)_k$.
Given $k \in \mathcal{I}_{W'}$. Want: $((\partial_u g)_p)_k = (h'_p)_k$.
Let $L := \pi_k^{W'}$. Want: $L((\partial_u g)_p) = L(h'_p)$.
Want: $(L \circ (\partial_u g))_p = (L \circ (h'))_p$.
Since $\mathrm{dom}[L] = W'$, we get:
$\mathrm{dom}[\partial_u g] = \mathrm{dom}[L \circ (\partial_u g)]$ and $\mathrm{dom}[h'] = \mathrm{dom}[L \circ (h')]$.
We have $p \in \mathrm{dom}[\partial_u g] = \mathrm{dom}[L \circ (\partial_u g)]$.
Also, $p \in \mathrm{dom}[h'] = \mathrm{dom}[L \circ (h')]$.
Then $(L \circ (\partial_u g))_p \neq ☹ \neq (L \circ (h'))_p$.
Since $k \in \mathcal{I}_{W'} = \mathcal{I}_{W \otimes V}$, choose $i \in \mathcal{I}_V$ and $j \in \mathcal{I}_W$ s.t. $k = j \| i$.
Then $L = \pi_{j\|i}^{W'}$. Let $M := \pi_j^W$.
By HW#10-5, both $L \circ (h') \subseteq M \circ (\partial_i h)$ and $L \circ (f') \subseteq M \circ (\partial_i f)$.
By Theorem 94.3, $L \circ (\partial_u g) \subseteq \partial_u(L \circ g)$.
By Theorem 94.3, $M \circ (\partial_i f) \subseteq \partial_i(M \circ f)$,
and so $\partial_u(M \circ (\partial_i f)) \subseteq \partial_u \partial_i (M \circ f)$.
Since $L \circ (f') \subseteq M \circ (\partial_i f)$, we get $\partial_u(L \circ (f')) \subseteq \partial_u(M \circ (\partial_i f))$.
Then $L \circ (\partial_u g) \subseteq \partial_u(L \circ g) = \partial_u(L \circ (f')) \subseteq \partial_u(M \circ (\partial_i f)) \subseteq \partial_u \partial_i (M \circ f)$,
and so $L \circ (\partial_u g) \subseteq \partial_u \partial_i (M \circ f)$,
and so $(L \circ (\partial_u g))_p \overset{*}{=} (\partial_u \partial_i (M \circ f))_p$.
So, since $☹ \neq (L \circ (\partial_u g))_p$,
we see that: $(L \circ (\partial_u g))_p = (\partial_u \partial_i (M \circ f))_p$.
By Theorem 94.3, $M \circ (\partial_i h) \subseteq \partial_i(M \circ h)$.
By Theorem 94.3, $M \circ (\partial_u f) \subseteq \partial_u(M \circ f)$,
and so $\partial_i(M \circ (\partial_u f)) \subseteq \partial_i \partial_u (M \circ f)$.
Then $L \circ (h') \subseteq M \circ (\partial_i h) \subseteq \partial_i(M \circ h) = \partial_i(M \circ (\partial_u f)) \subseteq \partial_u \partial_u (M \circ f)$,
and so $L \circ (h') \subseteq \partial_u \partial_u (M \circ f)$,
and so $(L \circ (h'))_p \overset{*}{=} (\partial_u \partial_i (M \circ f))_p$.
So, since $☹ \neq (L \circ (h'))_p$,
we see that: $(L \circ (h'))_p = (\partial_i \partial_u (M \circ f))_p$.
Sicne $M \in \mathcal{L}_W^{\mathbb{R}}$, we get $\mathrm{dom}[M''] = W$.
Since $p \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[f]$, we get $f_p \in \mathrm{im}[f]$.
Then $f_p \in \mathrm{im}[f] \subseteq W = \mathrm{dom}[M'']$.
Since $p \in \mathrm{dom}[f'']$ and $f_p \in \mathrm{dom}[M'']$,
it follows, by Theorem 90.25, that: $p \in \mathrm{dom}[(M \circ f)'']$.
Then, by Theorem 94.10, we get: $(\partial_u \partial_i (M \circ f))_p = (\partial_i \partial_u (M \circ f))_p$.
Then $(L \circ (\partial_u g))_p = (\partial_u \partial_i (M \circ f))_p = (\partial_i \partial_u (M \circ f))_p = (L \circ (h'))_p$. □

THEOREM 97.4. *Let $V := \mathbb{R}^2$, $W := \mathbb{R}$. Then $\exists f, g : V \to W$ s.t.*
$$\partial_q(f') \nsubseteq (\partial_1 f') \quad \text{and} \quad (\partial_1 g)' \nsubseteq \partial_1(g').$$

Proof. Define $f, g : V \to W$ by $f(x, y) = y^2 \cdot |x|$ and $g(x, y) = |y|$.
Want: $\partial_q(f') \nsubseteq (\partial_1 f)'$ and $(\partial_1 g)' \nsubseteq \partial_1(g')$.
We have: $\forall x > 0, \forall y \in \mathbb{R}, \quad f(x, y) = xy^2$.
Then: $\forall x > 0, \forall y \in \mathbb{R}, \quad (\,(\partial_1 f)(x, y)\,,\,(\partial_2 f)(x, y)\,) = (\,y^2\,,\,2xy\,)$.
Then: $\forall x > 0, \forall y \in \mathbb{R}, \quad f'(x, y) = (y^2, 2xy)$.
Then: $\forall x > 0, \quad f'(x, 0) = (0, 0)$.
We have: $\forall x < 0, \forall y \in \mathbb{R}, \quad f(x, y) = -xy^2$.
Then: $\forall x < 0, \forall y \in \mathbb{R}, \quad (\,(\partial_1 f)(x, y)\,,\,(\partial_2 f)(x, y)\,) = (\,-y^2\,,\,-2xy\,)$.
Then: $\forall x < 0, \forall y \in \mathbb{R}, \quad f'(x, y) = (-y^2, -2xy)$.
Then: $\forall x < 0, \quad f'(x, 0) = (0, 0)$.
We have $f = (\pi_2^V)^2 \cdot |\pi_1^V| \subseteq \widehat{\mathcal{O}}_2^{VW} \cdot \widehat{\mathcal{O}}_1^{VW} \subseteq \widehat{\mathcal{O}}_3^{VW} \subseteq o_2^{VW} \subseteq o_1^{VW}$,
and so $f'(0, 0) = (0, 0)$.
We conclude: $\forall x \in \mathbb{R}, \quad f'(x, 0) = (0, 0)$.
Then $(\partial_1(f'))(0, 0) = (0, 0)$, so $(0, 0) \in \text{dom}[\partial_1(f')]$.
We have: $\forall y \in \mathbb{R}_0^\times, \quad (\partial_1 f)(0, y) = ☹$,
and so $\neg(\ \partial_1 f$ is defined near $(0, 0)\ )$,
and so $(\partial_1 f)'(0, 0) = ☹$,
and so $(0, 0) \notin \text{dom}[(\partial_1 f)']$.
Since $(0, 0) \in \text{dom}[\partial_1(f')]$ and $(0, 0) \notin \text{dom}[(\partial_1 f)']$,
we conclude that $\text{dom}[\partial_1(f')] \nsubseteq \text{dom}[(\partial_1 f)']$.
Then $\partial_1(f') \nsubseteq (\partial_1 f)'$. Want: $(\partial_1 g)' \nsubseteq \partial_1(g')$.
We have: $\forall x, y \in \mathbb{R}, \quad (\partial_1 g)(x, y) = 0$.
Then: $\forall x, y \in \mathbb{R}, \quad (\ (\partial_1\partial_1 g)(x, y)\ ,\ (\partial_2\partial_1 g)(x, y)\ ) = (\ 0\ ,\ 0\ )$.
Then: $\forall x, y \in \mathbb{R}, \quad (\partial_1 g)'(x, y) = (0, 0)$.
Then: $(\partial_1 g)'(0, 0) = (0, 0)$, and so $(0, 0) \in \text{dom}[(\partial_1 g)']$.
We have: $\forall x \in \mathbb{R}, \quad (\partial_2 g)(x, 0) = ☹$.
Then: $(\partial_2 g)'(0, 0) = ☹$, and so $(0, 0) \notin \text{dom}[\partial_2 g]$.
So, since $\text{dom}[\partial_1(g')] \subseteq \text{dom}[g'] \subseteq \text{dom}[\partial_2 g]$,
we get: $(0, 0) \notin \text{dom}[\partial_1(g')]$.
Since $(0, 0) \in \text{dom}[(\partial_1 g)']$ and $(0, 0) \notin \text{dom}[\partial_1(g')]$,
we conclude that $\text{dom}[(\partial_1 g)'] \nsubseteq \text{dom}[\partial_1(g')]$.
Then $(\partial_1 g)' \nsubseteq \partial_1(g')$, as desired. □

You are not required to read through the proof of the next theorem. You should be aware, however, that it is true.

THEOREM 97.5. *Let* $V := \mathbb{R}^2$, $W := \mathbb{R}$, $z := 0_2$.
Then $\exists h : V \to W$ *s.t.* h *is infinitely differentiable on* $V_z^\times$,
and s.t. $(\partial_1(h'))_z = (0,0)$ *and* $(\partial_1 h)' = (0,1)$.

Proof. Choose $\psi : \mathbb{R} \to \mathbb{R}$ infinitely differentiable s.t.
$0 \leqslant \psi \leqslant 1$ on $\mathbb{R}$,
$\psi = 1$ on $[-1;1]$,
$\psi = 0$ on $\mathbb{R}\backslash(-2;2)$ and
$|\psi'| < 2$ on $\mathbb{R}$.
Define $h : V \to W$ by:

$$h(x,y) = \begin{cases} \left(\psi\left(x/\sqrt{y}\right)\right) \cdot xy, & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ \left(\psi\left(x/\sqrt{-y}\right)\right) \cdot xy, & \text{if } y < 0. \end{cases}$$

Want: h is infinitely differentiable on $V_z^\times$,
and $(\partial_1(h'))_z = (0,0)$ and $(\partial_1 h)'_z = (0,1)$.
Let $S := \{(x,y) \text{ s.t. } |y| < x^2\}$.
Then $h = 0$ on S, so h is infinitely differentiable on S.
Let $X := \pi_1^V$ and $Y := \pi_2^V$. $\forall x,y \in \mathbb{R}$, $X(x,y) = x$ and $Y(x,y) = y$.
Let $T := \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$.
Then $\sqrt{Y}$ is infinitely differentiable on T.
So, since $h = (\psi \circ (X/\sqrt{Y})) \cdot XY$ on T,
we see that h is infinitely differentiable on T.
Let $U := \{(x,y) \in \mathbb{R}^2 \,|\, y < 0\}$.
Then $\sqrt{-Y}$ is infinitely differentiable on U.
So, since $h = (\psi \circ (X/\sqrt{-Y})) \cdot XY$ on U,
we see that h is infinitely differentiable on U.
Since h is infinitely differentiable on S, on T and on U,
and since $S \cup T \cup U = V_z^\times$,
we conclude that h is infinitely differentiable on $V_z^\times$.
Want: $(\partial_1(h'))_z = (0,0)$ and $(\partial_1 h)'_z = (0,1)$.
Since $h \in \widehat{\mathcal{O}}_2^{VW} \subseteq \mathcal{o}_1^{VW}$, we see that $h'(z) = (0,0)$.
Let $R := \{(x,0) \,|\, x \in \mathbb{R}_0^\times\}$. Then $R \subseteq S$.
Since $h = 0$ on S, we conclude that $h' = (0,0)$ on S.
So, since $R \subseteq S$, we conclude that $h' = (0,0)$ on R.
So since $h'(0,0) = h'(z) = (0,0)$,
we conclude: $\forall x \in \mathbb{R}$, $h'(x,0) = (0,0)$.
Then $(\partial_1(h'))_z = (0,0)$. Want: $(\partial_1 h)'_z = (0,1)$.
Since $\mathrm{s}\ell_Y^{VW} = (0,1)$, it suffices to show: $(\partial_1 h)_z^T - Y \in \mathcal{o}_1^{VW}$.

We have: $\forall x \in \mathbb{R}, \quad h(x,0) = 0.$
Then: $\forall x \in \mathbb{R}, \quad (\partial_1 h)(x,0) = 0.$
Then $(\partial_1 h)_z = 0.$ It follows that $(\partial_1 h)_z^T = \partial_1 h.$
Want: $\partial_1 h - Y \in \mathcal{O}_1^{VW}.$
We have: $\forall x \in \mathbb{R}$, $\forall y > 0$,

$$(\partial_1 h)(x,y) = \left(\psi'\left(x/\sqrt{y}\right)\right) \cdot \left(1/\sqrt{y}\right) \cdot xy + \left(\psi\left(x/\sqrt{y}\right)\right) \cdot y.$$

Recall: $\forall x \in \mathbb{R}, \quad (\partial_1 h)(x,0) = 0.$
We have: $\forall x \in \mathbb{R}$, $\forall y < 0$,

$$(\partial_1 h)(x,y) = \left(\psi'\left(x/\sqrt{-y}\right)\right) \cdot \left(1/\sqrt{-y}\right) \cdot xy + \left(\psi\left(x/\sqrt{-y}\right)\right) \cdot y.$$

Let $\eta := \partial_1 h - Y$. Want: $\eta \in \mathcal{O}_1^{VW}$.
Want: $\forall x, y \in \mathbb{R}, \quad |\eta(x,y)| \leqslant 2 \cdot |x| \cdot \sqrt{|y|} + x^2.$
Recall: $\forall x \in \mathbb{R}$, $(\partial_1 h)(x,0) = 0$. Also, $\forall x \in \mathbb{R}$, $Y(x,0) = 0$.
Then: $\forall x \in \mathbb{R}$, $\eta(x,0) = 0 - 0 = 0$.
Then: $\forall x \in \mathbb{R}, \quad |\eta(x,0)| = 0 = 2 \cdot |x| \cdot \sqrt{|0|} + x^2.$
Want: $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}_0^\times, \quad |\eta(x,y)| \leqslant 2 \cdot |x| \cdot \sqrt{|y|} + x^2.$
Given $x \in \mathbb{R}$, $y \in \mathbb{R}_0^\times$. Want: $|\eta(x,y)| \leqslant 2 \cdot |x| \cdot \sqrt{|y|} + x^2.$
Let $\chi := \psi - C_V^1$.
Since $0 \leqslant \psi \leqslant 1$ on $\mathbb{R}$, we conclude: $-1 \leqslant \chi \leqslant 0$ on $\mathbb{R}$.
Then $|\chi| \leqslant 1$ on $\mathbb{R}$.
We have: If $y > 0$, then

$$\eta(x,y) = \left(\psi'\left(x/\sqrt{y}\right)\right) \cdot x \cdot \sqrt{y} + \left(\chi\left(x/\sqrt{y}\right)\right) \cdot y.$$

We have: If $y < 0$, then

$$\eta(x,y) = -\left(\psi'\left(x/\sqrt{-y}\right)\right) \cdot x \cdot \sqrt{-y} + \left(\chi\left(x/\sqrt{-y}\right)\right) \cdot y.$$

Then $|\eta(x,y)| \leqslant |\psi'(x,y)| \cdot |x| \cdot \sqrt{|y|} + |\chi(x/\sqrt{|y|})| \cdot |y|$.
Recall: $|\psi'| \leqslant 2$ on $\mathbb{R}$.
Then $|\eta(x,y)| \leqslant 2 \cdot |x| \cdot \sqrt{|y|} + |\chi(x/\sqrt{|y|})| \cdot |y|$.
Want: $|\chi(x/\sqrt{|y|})| \cdot |y| \leqslant x^2$.
Since $|\chi| \leqslant 1$ on $\mathbb{R}$, we see that $|\chi(x/\sqrt{|y|})| \leqslant 1$.
We have: If $|y| \leqslant x^2$, then $|\chi(x/\sqrt{|y|})| \cdot |y| \leqslant 1 \cdot |y| = |y| \leqslant x^2$.
Also, if $|y| > x^2$, then $\sqrt{|y|} > |x|$, so $|\, x/\sqrt{|y|}\,| < 1$, so $\psi(x/\sqrt{|y|}) = 1$,
so $\chi(x/\sqrt{|y|}) = 0$, so $|\chi(x/\sqrt{|y|})| \cdot |y| = 0 \leqslant x^2$.
Then $|\chi(x/\sqrt{|y|})| \cdot |y| \leqslant x^2$, as desired. □

THEOREM 97.6. *Let* $V, W \in \mathrm{TNSR}^+$.
Let $f : V \dashrightarrow W, \quad p \in \mathrm{dom}[f''], \quad u, v \in V$.
Then: $\mathrm{Bilin}_{f''_p}^{VVW}(u,v) \;=\; (\partial_u \partial_v f)_p$.

Proof. Since $p \in \mathrm{dom}[f'']$, we see that $f''_p \in W \otimes V \otimes V$.
Then $\mathrm{Bilin}_{f''_p} \in \mathcal{B}_{VV}^W$, so $\mathrm{Bilin}_{f''_p}(u,v) \in W$, so $\mathrm{Bilin}_{f''_p}(u,v) \neq ☹$.

By Theorem 97.3, $(\partial_v(f'))_p = (\partial_v f)'_p$.
Let $g := f'$, $h := \partial_v f$ and $A := f''_p$.
Then $A = g'_p$ and $(\partial_v g)_p = (\partial_v(f'))_p = (\partial_v f)'_p = h'_p$.
By Theorem 93.7, $\text{Bilin}_A^{VVW}(u,v) = A *_{\mathbb{R},V\otimes V,W} (u \otimes v)$.
By HW#10-3, $A *_{\mathbb{R},V\otimes V,W} (u \otimes v) = (A *_{\mathbb{R},V,W\otimes V} v) *_{\mathbb{R}VW} u$
By HW#9-2, $(g'_p) *_{\mathbb{R},V,W\otimes V} v \overset{*}{=} (\partial_v g)_p$.
By HW#9-2, $(h'_p) *_{\mathbb{R}VW} u \overset{*}{=} (\partial_u h)_p$.
We have:

$$\begin{aligned}\text{Bilin}_{f''_p}^{VVW}(u,v) &= \text{Bilin}_A^{VVW}(u,v)\\ &= A *_{\mathbb{R},V\otimes V,W} (u\otimes v)\\ &= (A *_{\mathbb{R},V,W\otimes V} v) *_{\mathbb{R}VW} u\\ &= ((g'_p) *_{\mathbb{R},V,W\otimes V} v) *_{\mathbb{R}VW} u\\ &\overset{*}{=} (\partial_v g)_p *_{\mathbb{R}VW} u\\ &= (h'_p) *_{\mathbb{R}VW} u\\ &\overset{*}{=} (\partial_u h)_p\\ &= (\partial_u\partial_v f)_p.\end{aligned}$$

So, since ☹ $\neq \text{Bilin}_{f''_p}^{VVW}(u,v)$, we conclude that

$$\text{Bilin}_{f''_p}^{VVW}(u,v) = (\partial_u\partial_v f)_p, \qquad \text{as desired.}$$ □

THEOREM 97.7. *Let $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$, $p \in \text{dom}[f'']$. Then* $\text{Bilin}_{f''_p}^{VVW} \in \mathcal{SB}_V^W$.

Proof. Since $p \in \text{dom}[f'']$, we get $f''_p \in W \otimes V \otimes V$, so $\text{Bilin}_{f''_p} \in \mathcal{B}_{VV}^W$.
Want: $\forall u, v \in V$, $\text{Bilin}_{f''_p}^{VVW}(u,v) = \text{Bilin}_{f''_p}^{VVW}(v,u)$.
Given $u, v \in V$. Want: $\text{Bilin}_{f''_p}^{VVW}(u,v) = \text{Bilin}_{f''_p}^{VVW}(v,u)$.
By Theorem 97.6, $\text{Bilin}_{f''_p}^{VVW}(u,v) = (\partial_u\partial_v f)_p$
and $\text{Bilin}_{f''_p}^{VVW}(v,u) = (\partial_v\partial_u f)_p$.
By Theorem 94.10, $(\partial_u\partial_v f)_p = (\partial_v\partial_u f)_p$.
Then

$$\begin{aligned}\text{Bilin}_{f''_p}^{VVW}(u,v) &= (\partial_u\partial_v f)_p\\ &= (\partial_v\partial_u f)_p = \text{Bilin}_{f''_p}^{VVW}(v,u), \text{ as desired.}\end{aligned}$$ □

98. Multivariable Choice Mean Value Theorem

THEOREM 98.1. *Let $m \in \mathbb{N}$, $U := \mathbb{R}^m$, $W := \mathbb{R}$, $\phi : U \dashrightarrow W$. Let $\delta := 0$, $J := (-\delta;\delta)$, $A := J^m$. Assume $\phi_{0_V} = 0_W$. Assume: $\forall i \in \mathcal{I}_U$, $A \subseteq \text{dom}[\partial_i\phi]$. Then: $\exists \alpha : \mathcal{I}_U \to \widehat{\mathcal{O}}_1^{UU}$ s.t.*

$$\phi = \sum_{i\in\mathcal{I}_U} ((\partial_i\phi)\circ\alpha_i)\cdot\pi_i^U \qquad \text{on } A.$$

Proof. Define $S : \mathcal{I}_U \times A \to 2^{\mathbb{R}}$ by:

$$S_{(i,x)} = \{\ c \in [0|x_i] \quad \text{s.t.}$$

$$\begin{aligned} &(\phi(x_1,\ldots,x_{i-1},\ \ x_i\ \ ,0,\cdots,0)) \\ -\ &(\phi(x_1,\ldots,x_{i-1},\ \ 0\ \ ,0,\cdots,0)) \\ =\quad &((\partial_i\phi)(x_1,\ldots,x_{i-1},\ \ c\ \ ,0,\cdots,0))\ \cdot\ x_i\ \}. \end{aligned}$$

By the one-variable Mean Value Theorem, we have:

$\forall i \in \mathcal{I}_U$, $\forall x \in A$, $\quad S_{(i,x)} \neq \varnothing$.

It follows that: $\quad \forall i \in \mathcal{I}_U$, $\forall x \in A$, $\quad \mathrm{CH}(S_{(i,x)}) \in S_{(i,x)}$.

Also, by definition of A, we have:

$\forall i \in \mathcal{I}_U$, $\forall x \in A$, $\quad x_i \in J$.

So, since J is an interval and since $0 \in J$, we get:

$\forall i \in \mathcal{I}_U$, $\forall x \in A$, $\quad [0|x_i] \subseteq J$.

By definition of S, we have: $\quad \forall i \in \mathcal{I}_U$, $\forall x \in A$, $\quad S_{(i,x)} \subseteq [0|x_i]$.

Then: $\forall i \in \mathcal{I}_U$, $\forall x \in A$, $\quad \mathrm{CH}(S_{(i,x)}) \in S_{(i,x)} \subseteq [0|x_i] \subseteq J$.

Define $\alpha : \mathcal{I}_U \to A^A$ by:

$\alpha_i(x) \ = \ (x_1,\ldots,x_{i-1},\ \mathrm{CH}(S_{(i,x)})\ ,0,\ldots,0)$.

We have: $\forall i \in \mathcal{I}_U$, $\forall x \in A$, $\quad (\alpha_i(x))_i = \mathrm{CH}(S_{(i,x)}) \in [0|x_i]$,

so $|(\alpha_i(x))_i| \leqslant |x_i|$, $\qquad$ so $((\alpha_i(x))_i)^2 \leqslant x_i^2$.

Then, $\forall i \in \mathcal{I}_U$, $\forall x \in A$, $\quad |\alpha_i(x)|_U^2 \ \leqslant\ |x|_U^2$.

Then, $\forall i \in \mathcal{I}_U$, $\forall x \in A$, $\quad |\alpha_i(x)|_U \ \leqslant\ |x|_U$.

Then, by HW#9-1, we have: $\forall i \in \mathcal{I}_U$, $\quad \alpha_i \in \widehat{\mathcal{O}}_1^{UU}$.

Then $\alpha : \mathcal{I}_U \to \widehat{\mathcal{O}}_1^{UU}$. $\qquad$ Want: $\phi \ = \ \sum_{i \in \mathcal{I}_U} ((\partial_i\phi) \circ \alpha_i) \cdot \pi_i^U \quad$ on A.

Want: $\forall x \in A$, $\qquad \phi_x \ = \ \left(\sum_{i \in \mathcal{I}_U} ((\partial_i\phi) \circ \alpha_i) \cdot \pi_i^U \right)_x$.

Given $x \in A$. $\qquad$ Want: $\phi_x \ = \ \left(\sum_{i \in \mathcal{I}_U} ((\partial_i\phi) \circ \alpha_i) \cdot \pi_i^U \right)_x$.

Want: $\phi_x \ = \ \sum_{i \in \mathcal{I}_U} ((\partial_i\phi)(\alpha_i(x))) \cdot x_i$.

We compute: $\phi_x \ = \ \phi_x - 0_W \ = \ \phi_x - \phi_{0_U} \ = \ (\phi(x)) - (\phi(0_U)$

$$\begin{aligned} &= (\phi(x_1,\ldots,x_m) - (\phi(0,\ldots,0)) \\ &= \sum_{i \in \mathcal{I}_U} \Bigg(\quad (\phi(x_1,\ldots,x_{i-1},\quad x_i \quad ,0,\ldots,0)) \\ &\qquad\qquad -\ (\phi(x_1,\ldots,x_{i-1},\quad 0 \quad ,0,\ldots,0)) \quad \Bigg) \\ &= \sum_{i \in \mathcal{I}_U} \Big(((\partial_i\phi)(x_1,\ldots,x_{i-1},\mathrm{CH}(S_{(i,x)}),0,\ldots,0)) \cdot x_i \Big) \\ &= \sum_{i \in \mathcal{I}_U} ((\partial_i\phi)(\alpha_i(x))) \cdot x_i, \qquad \text{as desired.} \qquad \square \end{aligned}$$

DEFINITION 98.2. *Let* $U, V \in \mathrm{TNSR}^+$, $\lambda : \mathcal{I}_U \to \mathcal{I}_V$.
Then $\widehat{\lambda} \in \mathcal{L}_U^V$ *is defined by* $\widehat{\lambda}(x) = \sum_{i \in \mathcal{I}_V} x_i \varepsilon_i^V$.

THEOREM 98.3. *Let* $U := \mathbb{R}^6$ *and* $V := \mathbb{R}^{2\times 3}$.

$$\textit{Let} \qquad \lambda := \begin{pmatrix} 1 \mapsto (1,1) \\ 2 \mapsto (1,2) \\ 3 \mapsto (1,3) \\ 4 \mapsto (2,1) \\ 5 \mapsto (2,2) \\ 6 \mapsto (2,3) \end{pmatrix}.$$

Then, $\forall u, v, w, x, y, z \in \mathbb{R}$, *we have:* $\widehat{\lambda}(u, v, w, x, y, z) \;=\; \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$.

THEOREM 98.4. *Let* $U := \mathbb{R}^6$ *and* $V := \mathbb{R}^{2\times 3}$.

$$\textit{Let} \qquad \lambda := \begin{pmatrix} 1 \mapsto (1,1) \\ 2 \mapsto (2,1) \\ 3 \mapsto (1,2) \\ 4 \mapsto (2,2) \\ 5 \mapsto (1,3) \\ 6 \mapsto (2,3) \end{pmatrix}.$$

Then, $\forall u, v, w, x, y, z \in \mathbb{R}$, *we have:* $\widehat{\lambda}(u, v, w, x, y, z) \;=\; \begin{bmatrix} u & w & y \\ v & x & z \end{bmatrix}$.

THEOREM 98.5. *Let* $U, V, W \in \mathrm{TNSR}^+$, $\lambda : \mathcal{I}_u \to \mathcal{I}_V$, $\mu : \mathcal{I}_V \to \mathcal{I}_W$.
Then $\widehat{\mu \circ \lambda} = \widehat{\mu} \circ \widehat{\lambda}$.

Proof. Unassigned HW. □

THEOREM 98.6. *Let* $U \in \mathrm{TNSR}^+$, $\lambda := \mathrm{id}_{\mathcal{I}_U}$. *Then* $\widehat{\lambda} = \mathrm{id}_U$.

Proof. Unassigned HW. □

THEOREM 98.7. *Let* $U, V \in \mathrm{TNSR}^+$, $\lambda : \mathcal{I}_U \to \mathcal{I}_V$.
Assume that $\lambda : \mathcal{I}_U \hookrightarrow\!\!> \mathcal{I}_V$.
Then $\widehat{\lambda} : U \hookrightarrow\!\!> V$ *and* $(\widehat{\lambda})^{-1} = \widehat{(\lambda^{-1})}$.

Proof. Unassigned HW. □

THEOREM 98.8. *Let* $U, V \in \mathrm{TNSR}^+$, $\lambda : \mathcal{I}_U \to \mathcal{I}_V$, $i \in \mathcal{I}_U$.
Then $\widehat{\lambda}(\varepsilon_i^U) = \varepsilon_{\lambda(i)}^V$. *Also,* $\forall x \in U$, $\pi_{\lambda(i)}^V(\widehat{\lambda}(x)) \;=\; \pi_i^U(x)$.

Proof. Unassigned HW. □

THEOREM 98.9. *Let* $U, V, W \in \mathrm{TNSR}^+$.
$\quad$ *Let* $L \in \mathcal{L}_U^V$, $f : V \dashrightarrow W$, $x \in U$.
$\quad$ *Then* $\partial_x(f \circ L) \;=\; (\partial_{L(x)} f) \circ L$.

Proof. Want: $\forall p \in U$, $(\partial_x(f \circ L))_p \;=\; ((\partial_{L(x)} f) \circ L)_p$.
Given $p \in U$. Want: $(\partial_x(f \circ L))_p \;=\; ((\partial_{L(x)} f) \circ L)_p$.
We have: $(\partial_x(f \circ L))_p \;=\; (f \circ L \circ (i_p^x))'_0$.
Let $y := L(x)$ and let $q := L(p)$.
Then $((\partial_{L(x)} f) \circ L)_p \;=\; ((\partial_y f) \circ L)_p \;=\; (\partial_y f)_{L(p)}$
$\qquad = \; (\partial_y f)_q \;=\; (f \circ (i_q^y))'_0$.
Want: $(f \circ L \circ (i_p^x))'_0 \;=\; (f \circ (i_q^y))'_0$.
Want: $L \circ (i_p^x) \;=\; i_q^y$.
Want: $\forall t \in \mathbb{R}$, $(L \circ (i_p^x))(t) \;=\; (i_q^y)(t)$.
Given $t \in \mathbb{R}$. Want: $(L \circ (i_p^x))(t) \;=\; (i_q^y)(t)$.
We compute: $(L \circ (i_p^x))(t) \;=\; L((i_p^x)(t)) \;=\; L(p + tx)$
$\qquad = \; (L(p)) \,+\, t \cdot (L(x))$
$\qquad = \; q + ty \;=\; (i_q^y)(t)$, as desired. $\square$

The following is another **Choice MVT**:

THEOREM 98.10. *Let* $V \in \mathrm{TNSR}^+$, $W := \mathbb{R}$, $f : V \dashrightarrow W$.
Assume $f_{0_V} = 0_W$.
Assume: $\forall j \in \mathcal{I}_V$, $\partial_j f$ *is defined near* 0_V.
Then: $\exists \beta : \mathcal{I}_V \to \mathcal{O}_1^{VV}$ *s.t.* $\displaystyle f \;=\; \sum_{j \in \mathcal{I}_V} ((\partial_j f) \circ \beta_j) \cdot \pi_j^V$ *near* 0_V.

Proof. Let $m := \#\mathcal{I}_V$, $U := \mathbb{R}^m$. Then $\mathcal{I}_U = [1..m]$, so $\#\mathcal{I}_U = m$.
Since $\#\mathcal{I}_U = m = \#\mathcal{I}_V$, choose $\lambda : \mathcal{I}_U \hookrightarrow\!\!> \mathcal{I}_V$. Let $\phi := f \circ \widehat{\lambda}$.
We have: $\forall i \in \mathcal{I}_U$, $\widehat{\lambda}(\varepsilon_i^U) = \varepsilon_{\lambda(i)}$.
Then: $\forall i \in \mathcal{I}_U$, $\partial_i \phi \;=\; \partial_{\varepsilon_i^U}(f \circ \widehat{\lambda}) \;=\; (\partial_{\widehat{\lambda}(\varepsilon_i^U)} f) \circ \widehat{\lambda}$
$\qquad = \; (\partial_{\varepsilon^U_{\lambda(i)}} f) \circ \widehat{\lambda} \;=\; (\partial_{\lambda(i)} f) \circ \widehat{\lambda}$.
Then: $\forall i \in \mathcal{I}_U$, $\partial_i \phi \;=\; (\partial_{\lambda_i} f) \circ \lambda$.
Choose $D \in \mathcal{B}_V(0_V)$, s.t., $\forall j \in \mathcal{I}_V$, $D \subseteq \mathrm{dom}[\partial_j f]$.
Choose $C \in \mathcal{B}_U(0_U)$ s.t. $\widehat{\lambda}_*(C) \subseteq D$.
Then, $\forall i \in \mathcal{I}_U$, $C \subseteq \mathrm{dom}[(\partial_{\lambda_i} f) \circ \lambda] = \mathrm{dom}[\partial_i \phi]$.
Choose $r > 0$ s.t. $C = B_U(0_U, r)$.
Let $\delta := r/\sqrt{m}$. Then $B_U(0_U, \delta) \subseteq (\delta; \delta)^m \subseteq B_U(0_U, r)$.
Let $J := (-\delta; \delta)$, $A := J^m$. Then $B_U(0_U, \delta) \subseteq A \subseteq B_U(0_U, r)$.
Then $A \subseteq B_U(0_U, r) = A \subseteq C$.
Then, $\forall i \in \mathcal{I}_U$, $A \subseteq C \subseteq \mathrm{dom}[\partial_i \phi]$.

By Theorem 98.1, choose $\alpha : \mathcal{I}_U \to \widehat{\mathcal{O}}_1^{UU}$ s.t.
$$\phi \quad = \quad \sum_{i \in \mathcal{I}_U} ((\partial_i \phi) \circ \alpha_i) \cdot \pi_i^U \qquad \text{on } A.$$
Since $\lambda : \mathcal{I}_U \hookrightarrow\!\!> \mathcal{I}_V$, we see that $\widehat{\lambda} : U \hookrightarrow\!\!> V$. Let $\kappa := \lambda^{-1}$.
Then $\kappa : \mathcal{I}_V \hookrightarrow\!\!> \mathcal{I}_U$ and $\widehat{\kappa} : V \hookrightarrow\!\!> U$ and $\widehat{\lambda}^{-1} = \widehat{\kappa}$.
We have $\widehat{\kappa} \in \mathcal{L}_V^U \subseteq \widehat{\mathcal{O}}_1^{VU}$ and $\widehat{\lambda} \in \mathcal{L}_U^V \subseteq \widehat{\mathcal{O}}_1^{UV}$.
Also, $\forall i \in \mathcal{I}_U$, we have: $\alpha_i \in \widehat{\mathcal{O}}_1^{UU}$.
Then, $\forall i \in \mathcal{I}_U$, we have: $\widehat{\lambda} \circ \alpha_i \circ \widehat{\kappa} \in \widehat{\mathcal{O}}_1^{VV}$.
Define $\beta : \mathcal{I}_V \to \mathcal{O}_1^{VV}$ by: $\beta_j = \widehat{\lambda} \circ \alpha_{\kappa_j} \circ \widehat{\kappa}$.
Then, $\forall i \in \mathcal{I}_U$, we have: $\beta_{\lambda_i} = \widehat{\lambda} \circ \alpha_i \circ \widehat{\kappa}$.
Want: $f = \sum_{j \in \mathcal{I}_V} ((\partial_j f) \circ \beta_j) \cdot \pi_j^V$ near 0_V.
Choose $B \in \mathcal{B}_V(0_V)$ s.t. $\widehat{\kappa}_*(B) \subseteq B_U(0_U, \delta)$.
Want: $f = \sum_{j \in \mathcal{I}_V} ((\partial_j f) \circ \beta_j) \cdot \pi_j^V$ on B.
Want: $\forall y \in B$, $f_y = \left(\sum_{j \in \mathcal{I}_V} ((\partial_j f) \circ \beta_j) \cdot \pi_j^V \right)_y$.
Given $y \in B$. Want: $f_y = \left(\sum_{j \in \mathcal{I}_V} ((\partial_j f) \circ \beta_j) \cdot \pi_j^V \right)_y$.
Let $x := \widehat{\kappa}_y$. Then $\widehat{\lambda}_x = y$. Then $f_y = f(\widehat{\lambda}_x) = (f \circ \lambda)_x = \phi_x$.
Want: $\phi_x = \left(\sum_{j \in \mathcal{I}_V} ((\partial_j f) \circ \beta_j) \cdot \pi_j^V \right)_{\widehat{\lambda}_x}$.
Recall that $\lambda : \mathcal{I}_U \hookrightarrow\!\!> \mathcal{I}_V$.
Want: $\phi_x = \left(\sum_{i \in \mathcal{I}_U} ((\partial_{\lambda_i} f) \circ \beta_{\lambda_i}) \cdot \pi_{\lambda_i}^V \right)_{\widehat{\lambda}_x}$.
Recall: $\forall i \in \mathcal{I}_U$, $\beta_{\lambda_i} = \widehat{\lambda} \circ \alpha_i \circ \widehat{\kappa}$.
Want: $\phi_x = \left(\sum_{i \in \mathcal{I}_U} ((\partial_{\lambda_i} f) \circ \widehat{\lambda} \circ \alpha_i \circ \widehat{\kappa}) \cdot \pi_{\lambda_i}^V \right)_{\widehat{\lambda}_x}$.
Recall: $\forall i \in \mathcal{I}_U$, $\partial_i \phi = (\partial_{\lambda_i} f) \circ \lambda$.
Want: $\phi_x = \left(\sum_{i \in \mathcal{I}_U} ((\partial_i \phi) \circ \alpha_i \circ \widehat{\kappa}) \cdot \pi_{\lambda_i}^V \right)_{\widehat{\lambda}_x}$.
We have $\widehat{\kappa}(\widehat{\lambda}_x) = x$. Also, $\forall i \in \mathcal{I}_U$, $\pi_{\lambda_i}^V(\widehat{\lambda}_x) = \pi_i^U(x) = (\pi_i^U)_x$.
Want: $\phi_x = \sum_{i \in \mathcal{I}_U} ((\partial_i \phi) \circ \alpha_i)_x \cdot (\pi_i^U)_x$.
Since $y \in B \subseteq V = \mathrm{dom}[\widehat{\kappa}]$, we get: $y \in B \cap (\mathrm{dom}[\widehat{\kappa}])$, so $\widehat{\kappa}_y \in \widehat{\kappa}_*(B)$.

Recall: $x = \widehat{\kappa}_y$ and $B_U(0_U, \delta) \subseteq A$, and $\widehat{\kappa}_*(B) \subseteq B_U(0_U, \delta)$.
Then $x = \widehat{\kappa}_y \in \widehat{\kappa}_*(B) \subseteq B_U(0_U, \delta) \subseteq A$.

Then, by choice of α, $\phi_x = \left(\sum_{i \in \mathcal{I}_U} ((\partial_i \phi) \circ \alpha_i) \cdot \pi_i^U \right)_x$.

Then: $\phi_x = \sum_{i \in \mathcal{I}_U} ((\partial_i \phi \circ \alpha_i)_x \cdot (\pi_i^U)_x$, as desired. □

99. Functions with small derivatives are small

THEOREM 99.1. *Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.
Then: (($\forall m \in \mathcal{I}_W$, $\pi_m^W \circ f$ is continuous at p)
$\Leftrightarrow$ (f is continuous at p))*

Proof. Unassigned HW. □

THEOREM 99.2. *Let $V, W \in \mathrm{TNSR}^+$, $\alpha : V \dashrightarrow W$.
Then: ($\forall q \in V$, $|\alpha_q - \alpha_{0_V}|_W^2 = \sum_{m \in \mathcal{I}_W} |(\pi_m^W \circ \alpha)_q - (\pi_m^W \circ \alpha)_{0_V}|^2$)
and (($\forall m \in \mathcal{I}_W$, $\pi_m^W \circ \alpha$ is continuous at 0_V)
$\Leftrightarrow$ (α is continuous at 0_V))
and ($\forall m \in \mathcal{I}_W$, $\mathrm{dom}[\pi_m^W \circ \alpha] = \mathrm{dom}[\alpha]$)
and (($\forall m \in \mathcal{I}_W$, $\pi_m^W \circ \alpha \in \mathrm{DNZ}_V^{\mathbb{R}}$) $\Leftrightarrow$ ($\alpha \in \mathrm{DNZ}_V^W$))
and (($\forall m \in \mathcal{I}_W$, $(\pi_m^W \circ \alpha)_{0_V} = 0$) $\Leftrightarrow$ ($\alpha_{0_V} = 0_W$))
and (($\forall m \in \mathcal{I}_W$, $\pi_m^W \circ \alpha \in \mathrm{CVZ}_V^{\mathbb{R}}$) ($\alpha \in \mathrm{CVZ}_V^W$))
and (($\forall m \in \mathcal{I}_W$, $\pi_m^W \circ \alpha \in \mathrm{BNZ}_V^{\mathbb{R}}$) ($\alpha \in \mathrm{BNZ}_V^W$)).*

Proof. Unassigned HW. □

THEOREM 99.3. *Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $k \in \mathbb{N}_0$.
Then: (($\forall m \in \mathcal{I}_W$, $\pi_m^W \circ f \in \mathcal{O}_k^{V\mathbb{R}}$) $\Leftrightarrow$ ($f \in \mathcal{O}_k^{VW}$))
and (($\forall m \in \mathcal{I}_W$, $\pi_m^W \circ f \in \widehat{\mathcal{O}}_k^{V\mathbb{R}}$) $\Leftrightarrow$ ($f \in \widehat{\mathcal{O}}_k^{VW}$)).*

Proof. Unassigned HW. □

DEFINITION 99.4. *Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $S \subseteq V$.
Then f is* **infinitely differentiable** *on S means:
$\forall k \in \mathbb{N}$, $S \subseteq \mathrm{dom}[f^{(k)}]$.*

THEOREM 99.5. *$\exists f : \mathbb{R} \to \mathbb{R}^2$ s.t.
(f is infinitely differentiable on $\mathbb{R}$)
& ($f_{-1} = f_1$) & ($\nexists c \in (-1; 1)$ s.t. $f'_c = 0_2$).*

Proof. Define $g, h : \mathbb{R} \to \mathbb{R}$ by $g_t = t^2$, $h_t = t^3 - t$.
Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f_t = (g_t, h_t)$.
Want: (f is infinitely differentiable on $\mathbb{R}$)
& ($f_{-1} = f_1$) & ($\nexists c \in (-1;1)$ s.t. $f'_c = 0_2$).
We compute $f_{-1} = (g_{-1}, h_{-1}) = (1,0) = (g_1, h_1) = f_1$.
Also, $\forall t \in \mathbb{R}$, $f'_t = (2t, 3t^2 - 1)$.
Also, $\forall t \in \mathbb{R}$, $f''_t = (2, 6t)$.
Also, $\forall t \in \mathbb{R}$, $f'''_t = (0, 6)$.
Also, $\forall t \in \mathbb{R}$, $\forall k \in [4..\infty)$, $f^{(k)}_t = (0,0)$.
Then f is infinitely differentiable.
Want: $\nexists c \in (-1;1)$ s.t. $f'_c = 0_2$.
Assume: $\exists c \in (-1;1)$ s.t. $f'_c = 0_2$. Want: Contradiction.
Choose $c \in (-1;1)$ s.t. $f'_c = 0_2$.
Then $(2c, 3c^2 - 1) = f'_c = 0_2 = (0,0)$, so $2c = 0$ and $3c^2 - 1 = 0$.
Since $2c = 0$, we conclude that $c = 0$ and so $3c^2 - 1 = 3 \cdot 0^2 - 1 = -1$.
Then $0 = 3c^2 - 1 = -1$, so $0 = -1$.
However, $0 \neq -1$. Contradiction. □

THEOREM 99.6. *Let $V, W, X \in \mathrm{TNSR}^+$.*
Let $f : V \dashrightarrow W$, $L \in \mathcal{L}_W^X$, $j \in \mathcal{I}_V$.
Then: $\partial_j(L \circ f) \supseteq L \circ (\partial_j f)$.

Proof. Let $u := \varepsilon_j^V$. Then $\partial_j(L \circ f) = \partial_u(L \circ f)$ and $\partial_j f = \partial_u f$.
By Theorem 94.3, we have $\partial_u(L \circ f) \supseteq L \circ (\partial_u f)$.
Then $\partial_j(L \circ f) = \partial_u(L \circ f) \supseteq L \circ (\partial_u f) = L \circ (\partial_j f)$. □

THEOREM 99.7. *Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $k \in \mathbb{N}_0$.*
Assume $f_{0_V} = 0_W$. Assume: $\forall j \in \mathcal{I}_V$, $\partial_j f \in \mathcal{O}_k^{VW}$. Then $f \in \mathcal{O}_{k+1}^{VW}$.

Proof. By Theorem 99.3, it suffices to show: $\forall m \in \mathcal{I}_W$, $\pi_m^W \circ f \in \mathcal{O}_{k+1}^{V\mathbb{R}}$.
Given $m \in \mathcal{I}_W$. Want: $\pi_m^W \circ f \in \mathcal{O}_{k+1}^{V\mathbb{R}}$.
Let $g := \pi_m^W \circ f$. Want: $g \in \mathcal{O}_{k+1}^{V\mathbb{R}}$.
By Theorem 98.10, choose $\beta : \mathcal{I}_V \to \widehat{\mathcal{O}}_1^{VV}$ s.t.

$$g = \sum_{j \in \mathcal{I}_V} ((\partial_j g) \circ \beta_j) \cdot \pi_j^V \quad \text{near } 0_V.$$

Want: $\forall j \in \mathcal{I}_V$, $((\partial_j g) \circ \beta_j) \cdot \pi_j^V \in \mathcal{O}_{k+1}^{V\mathbb{R}}$.
Given $j \in \mathcal{I}_V$. Want: $((\partial_j g) \circ \beta_j) \cdot \pi_j^V \in \mathcal{O}_{k+1}^{V\mathbb{R}}$.
Since $\pi_j^V \in \mathcal{L}_V^W \subseteq \widehat{\mathcal{O}}_1^{V\mathbb{R}}$, it suffices to show: $(\partial_j g) \circ \beta_j \in \mathcal{O}_k^{V\mathbb{R}}$.
So, since $\beta_j \in \widehat{\mathcal{O}}_1^{VV}$, it suffices to show: $\partial_j g \in \mathcal{O}_k^{V\mathbb{R}}$.
So, since $\partial_j g = \partial_j((\pi_m^W) \circ f) \supseteq \pi_m^W \circ (\partial_j f)$,

it suffices to show: $\pi_m^W \circ (\partial_j f) \in \mathcal{O}_k^{V\mathbb{R}}$.
By assumption, $\partial_j f \in \mathcal{O}_k^{VW}$.
So, by Theorem 99.3, we have $\pi_m^W \circ (\partial_j f) \in \mathcal{O}_k^{V\mathbb{R}}$, as desired. □

THEOREM 99.8. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $k \in \mathbb{N}_0$.
Assume $f_{0_V} = 0_W$. *Assume:* $\forall j \in \mathcal{I}_V$, $(\partial_j f)_{0_V} = 0_W$.
Assume: $\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f \in \mathcal{O}_k^{VW}$. *Then* $f \in \mathcal{O}_{k+2}^{VW}$.

Proof. Since $f_{0_V} = 0_W$, by Theorem 99.7,
it suffices to show: $\forall j \in \mathcal{I}_V$, $\partial_j f \in \mathcal{O}_{k+1}^{VW}$.
Given $j \in \mathcal{I}_V$. Want: $\partial_j f \in \mathcal{O}_{k+1}^{VW}$.
Let $g := \partial_j f$. Want: $g \in \mathcal{O}_{k+1}^{VW}$.
Since $g_{0_V} = (\partial_j f)_{0_V} = 0_W$, by Theorem 99.7,
it suffices to show: $\forall i \in \mathcal{I}_V$, $\partial_i g \in \mathcal{O}_k^{VW}$.
Given $i \in \mathcal{I}_V$. Want: $\partial_i g \in \mathcal{O}_k^{VW}$.
We have $\partial_i g = \partial_i \partial_j f \in \mathcal{O}_k^{VW}$, as desired. □

100. Properties of partials imply properties of function

THEOREM 100.1. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.
Then: (1) $\partial_j(f(p + \bullet)) = (\partial_j f)(p + \bullet)$
and (2) ($p \in \mathrm{dom}[f]$) $\Rightarrow$
(($(\partial_j(f_p^T) = (\partial_j f)(p + \bullet)$) & ($(f_p^T)' = f'(p + \bullet)$))
and (3) (f *is defined near* p) $\Rightarrow$ ($f(p + \bullet)$ *is defined near* 0_V)
and (4) (f *is continuous at* p) $\Rightarrow$ ($f(p{+}\bullet)$ *is continuous at* 0_V).

Proof. Unassigned HW. □

THEOREM 100.2. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.
Assume: $\forall j \in \mathcal{I}_V$, $\partial_j f$ *is defined near* p *and continuous at* p.
Then $p \in \mathrm{dom}[f']$.

Proof. Choose $i \in \mathcal{I}_V$. Then $p \in \mathrm{dom}[\partial_i f] \subseteq \mathrm{dom}[f]$. Let $g := f_p^T$.
Then $g_{0_V} = 0_W$ and $g' = f'(p + \bullet)$. Then $\mathrm{dom}[f'] = (\mathrm{dom}[g']) + p$.
It suffices to show: $0_V \in \mathrm{dom}[g']$.
By HW#12-5, it suffices to show:
$\forall j \in \mathcal{I}_V$, $\partial_j g$ is defined near 0_V and continuous at 0_V.
Given $j \in \mathcal{I}_V$. Want: $\partial_j g$ is defined near 0_V and continuous at 0_V.
Since $p \in \mathrm{dom}[f]$ and $g = f_p^T$, we get $\partial_j g = (\partial_j f)(p + \bullet)$.
Then $\mathrm{dom}[\partial_j g] = (\mathrm{dom}[\partial_j f]) - p$.
So, since $\partial_j f$ is defined near p,
we see that $\partial_j g$ is defined near 0_V.

Want: $\partial_j g$ is continuous at 0_V.
Since $\partial_j g = (\partial_j f)(p + \bullet)$ and since $\partial_j f$ is continuous at p,
we see that $\partial_j g$ is continuous at 0_V, as desired. $\square$

DEFINITION 100.3. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$ and let $p \in X$.
Then f is **continuous near** *p (from X to Y) means:*
$\exists B \in \mathcal{B}_X(p)$ s.t. f is continuous on B from X to Y.

THEOREM 100.4. *Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.*
Assume: f' is continuous near p.
Then: $\forall j \in \mathcal{I}_V$, $\partial_j f$ is continuous near p.

Proof. Given $j \in \mathcal{I}_V$. Want: $\partial_j f$ is continuous near p.
Since f' is continuous near p,
choose $B \in \mathcal{B}_V(p)$ s.t. f' is continuous on B.
Want: $\partial_j f$ is continuous on B.
Want: $\forall q \in B$, $\partial_j f$ is continuous at q.
Given $q \in B$. Want: $\partial_j f$ is continuous at q.
Define $L \in \mathcal{L}_{W \otimes V}^{W}$ by $L(x) = x *_{\mathbb{R}VW} \varepsilon_j^V$. Then L is continuous.
Also, $\mathrm{im}[f'] \subseteq W \otimes V = \mathrm{dom}[L]$, so $\mathrm{im}[f'] \subseteq \mathrm{dom}[L]$.
We have $\partial_j f \supseteq f' *_{\mathbb{R}VW} \varepsilon_j^V = L \circ (f')$, so $\partial_j f \supseteq L \circ (f')$.
Since f' is continuous on B, we get: $B \subseteq \mathrm{dom}[f']$.
So, since $\mathrm{im}[f'] \subseteq \mathrm{dom}[L]$, we conclude: $B \subseteq \mathrm{dom}[L \circ (f')]$.
So, since $\partial_j f \supseteq L \circ (f')$, we conclude: $\partial_j f = L \circ (f')$ on B.
So, since $q \in B$, by the Recentering Lemma,
we conclude that: $\partial_j f = L \circ (f')$ near q.
It therefore suffices to show: $L \circ (f')$ is continuous at q.
Since $q \in B$ and since f' is continuous on B,
it follows that: f' is continuous at q.
We have $q \in B \subseteq \mathrm{dom}[f']$, so $f'_q \in \mathrm{im}[f']$.
Since $f'_q \in \mathrm{im}[f'] \subseteq \mathrm{dom}[L]$ and since L is continuous,
we see that: L is continuous at f'_q.
So, since f' is continuous at q,
we get: $L \circ (f')$ is continuous at q, as desired. $\square$

101. Defined at total derivatives of each partial derivative

You are not expected to know the result or the proof in this section.

It shows that definedness of the total derivative of each partial does not guarantee definedness of the second total derivative.

THEOREM 101.1. *Let* $V := \mathbb{R}^2$, $W := \mathbb{R}$, $z := 0_2$.
Then: $\exists f : V \to W$ *s.t.* $z \in \mathrm{dom}[(\partial_1 f)']$, $z \in \mathrm{dom}[(\partial_2 f)']$, $z \notin \mathrm{dom}[f'']$.

Proof. Choose an infinitely differentiable function $\lambda : \mathbb{R} \to \mathbb{R}$ s.t.

$0 \leqslant \lambda \leqslant 1$ on $\mathbb{R}$ and $|\lambda'| \leqslant 100$ on $\mathbb{R}$ and
$\lambda = 0$ on $(-\infty; 1/2] \cup [3/2; \infty)$ and $\lambda_1 = 1$.

Choose an infinitely differentiable function $\rho : \mathbb{R} \to \mathbb{R}$ s.t.

$0 \leqslant \rho \leqslant 1$ on $\mathbb{R}$ and $|\rho'| \leqslant 100$ on $\mathbb{R}$ and
$\rho = 1$ on $(-\infty; 2]$ and $\rho = 0$ on $[4; \infty)$.

Define $\eta : \mathbb{R}^2 \to \mathbb{R}$ by

$$\eta(x, y) = \begin{cases} ((\lambda(y/x)) \cdot (\rho(x^2 + y^2)) \cdot x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Then: $\forall x, y \in \mathbb{R}$, $(\, x^2 + y^2 \geqslant 4\,) \Rightarrow (\, \eta(x, y) = 0\,)$.
Also: $\forall x \in [-1; 1]$, $\eta(x, x) = x$.
Then: $(\partial_{(1,1)}\eta)(0, 0) = 1$.
Also: $\forall x, y \in \mathbb{R}$, $(\, y \notin [\, x/2 \,|\, 3x/2\,]\,) \Rightarrow (\, \eta(x, y) = 0\,)$.
Also: $\forall y \in \mathbb{R}_0^\times$, because $\eta = 0$ near $(0, y)$,
we conclude: $(\partial_1\eta)(x, y) = 0$ and $(\partial_2\eta)(x, y) = 0$.
Also, because $\eta = 0$ on $\mathbb{R} \times \{0\}$, we conclude: $(\partial_1\eta)(0, 0) = 0$.
Also, because $\eta = 0$ on $\{0\} \times \mathbb{R}$, we conclude: $(\partial_2\eta)(0, 0) = 0$.
Since $(\partial_1\eta)(0, 0) = 0$ and $(\partial_2\eta)(0, 0) = 0$ and $(\partial_{(1,1)}\eta)(0, 0) = 1$,
it follows that $\eta'(z) = ☹$, *i.e.*, that $z \neq \mathrm{dom}[\eta']$.
We compute: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$,

$$\begin{aligned} (\partial_1\eta)(x, y) &= (\lambda'(y/x)) \cdot (\rho(x^2 + y^2)) \cdot (-y/x) \\ &\quad + (\lambda(y/x)) \cdot (\rho'(x^2 + y^2)) \cdot (2x^2) \\ &\quad + (\lambda(y/x)) \cdot (\rho(x^2 + y^2)) \cdot (1) \qquad \text{and} \\ (\partial_2\eta)(x, y) &= (\lambda'(y/x)) \cdot (\rho(x^2 + y^2)) \cdot (1) \\ &\quad + (\lambda(y/x)) \cdot (\rho'(x^2 + y^2)) \cdot (2xy). \end{aligned}$$

Also: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$, $(\, |y/x| > 3/2\,) \Rightarrow (\, \lambda(y/x) = 0\,)$.
Then: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$, $(\, |y/x| > 3/2\,) \Rightarrow (\, \eta(x, y) = 0\,)$.
Then: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$,
$(\, |y/x| > 3/2\,) \Rightarrow (\, (\partial_1\eta)(x, y) = (\partial_2\eta)(x, y) = 0\,)$.
Also: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$, $(\, x > 2\,) \Rightarrow (\, \rho(x^2 + y^2) = 0\,)$.
Then: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$, $(\, x > 2\,) \Rightarrow (\, \eta(x, y) = 0\,)$.
Then: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$,
$(\, x > 2\,) \Rightarrow (\, (\partial_1\eta)(x, y) = (\partial_2\eta)(x, y) = 0\,)$.

Also: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$, $(\, y > 2\,) \Rightarrow (\, \rho(x^2 + y^2) = 0\,)$.
Then: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$, $(\, y > 2\,) \Rightarrow (\, \eta(x,y) = 0\,)$.
Then: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$,

$$(\, y > 2\,) \Rightarrow (\, (\partial_1\eta)(x,y) = (\partial_2\eta)(x,y) = 0\,).$$

It follows that: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$,

$$|(\partial_1\eta)(x,y)| \quad \leqslant \quad 100\cdot 1\cdot (3/2) + 1\cdot 100\cdot 2\cdot 2^2 + 1\cdot 1\cdot 1 \quad = \quad 953.$$

It also follows that: $\forall x \in \mathbb{R}_0^\times$, $\forall y \in \mathbb{R}$,

$$|(\partial_2\eta)(x,y)| \quad \leqslant \quad 100\cdot 1\cdot 1 \,+\, 1\cdot 100\cdot 2\cdot 2\cdot 2 \quad = \quad 900.$$

So, since $\partial_1\eta = \partial_2\eta = 0$ on $\{0\} \times \mathbb{R}$, we get:

$$\forall x, y \in \mathbb{R}, \qquad |(\partial_1\eta)(x,y)| \leqslant 1000 \quad \text{and} \quad |(\partial_2\eta)(x,y)| \leqslant 1000.$$

Define $q, c \in \mathbb{R}^\mathbb{N}$ by $q_i = (1/2)^i$ and $c_i = (1/100)^i$.
$\forall i \in \mathbb{N}$, define $g_i : V \to W$ by $g_i(x,y) = c_i^2 \cdot \left(\eta\left(\dfrac{x - q_i}{c_i}, \dfrac{y}{c_i}\right)\right)$.
Then: $\forall i \in \mathbb{N}$, $(q_i, 0) \notin \mathrm{dom}[g_i']$.
Also: $\forall i \in \mathbb{N}$, $\forall x, y \in \mathbb{R}$,

$$|(\partial_1 g)(x,y)| \;\leqslant\; (c_i^2)\cdot\left(\frac{1}{c_i}\right)\cdot 1000 \;=\; 1000\cdot c_i \qquad \text{and}$$

$$|(\partial_2 g)(x,y)| \;\leqslant\; (c_i^2)\cdot\left(\frac{1}{c_i}\right)\cdot 1000 \;=\; 1000\cdot c_i.$$

We have: $\forall x, y \in \mathbb{R}$,

$$(\, x < -2\,) \Rightarrow (\, \eta(x,y) = 0\,).$$

Then: $\forall i \in \mathbb{N}$, $\forall x, y \in \mathbb{R}$,

$$(\, x < q_i - 2c_i\,) \Rightarrow (\, g_i(x,y) = 0\,).$$

Then: $\forall i \in \mathbb{N}$, $\forall x, y \in \mathbb{R}$,

$$(\, x < q_i - 2c_i\,) \Rightarrow (\, (\partial_1 g_i)(x,y) = (\partial_2 g_i)(x,y) = 0\,).$$

We have: $\forall i \in \mathbb{N}$, $q_{i+1} = 1/(2^{i+1}) < (1/2)^i - 2\cdot(1/100)^i = q_i - 2c_i$.
Then: $\forall i \in \mathbb{N}$, $\forall x, y \in \mathbb{R}$,

$$\begin{aligned}
&(\, |x| < q_{i+1}\,) \Rightarrow (\, x < q_{i+1}\,) \Rightarrow (\, x < q_i - 2c_i\,)\\
&\quad \Rightarrow (\quad (\partial_1 g_i)(x,y) = (\partial_2 g_i)(x,y) = 0 \quad)\\
&\quad \Rightarrow (\quad (\, |(\partial_1 g_i)(x,y)| \;\leqslant\; 1000\cdot c_i\cdot (x/q_{i+1})^2\,) \quad \& \\
&\qquad\qquad (\, |(\partial_2 g_i)(x,y)| \;\leqslant\; 1000\cdot c_i\cdot (x/q_{i+1})^2\,) \quad).
\end{aligned}$$

Also: $\forall i \in \mathbb{N}$, $\forall x, y \in \mathbb{R}$,

$$\begin{aligned}
&(\, |x| \geqslant q_{i+1}\,) \Rightarrow (\, (x/q_{i+1})^2 \geqslant 1\,)\\
&\quad \Rightarrow (\quad (\, |(\partial_1 g_i)(x,y)| \;\leqslant\; 1000\cdot c_i\cdot (x/q_{i+1})^2\,) \quad \& \\
&\qquad\qquad (\, |(\partial_2 g_i)(x,y)| \;\leqslant\; 1000\cdot c_i\cdot (x/q_{i+1})^2\,) \quad).
\end{aligned}$$

We conclude: $\forall i \in \mathbb{N}$, $\forall x, y \in \mathbb{R}$,

$$\begin{aligned}
&(\quad (\, |(\partial_1 g_i)(x,y)| \;\leqslant\; 1000\cdot c_i\cdot (x/q_{i+1})^2\,) \quad \& \\
&\qquad (\, |(\partial_2 g_i)(x,y)| \;\leqslant\; 1000\cdot c_i\cdot (x/q_{i+1})^2\,) \quad).
\end{aligned}$$

We have: $\forall i \in \mathbb{N}$, $c_i \cdot (1/q_{i+1})^2 \;=\; (1/100)^i \cdot (2^i)^2 \;=\; (4/100)^i \;<\; 1$.
We conclude: $\forall i \in \mathbb{N}$, $\forall x, y \in \mathbb{R}$,

$$(\quad (\,|(\partial_1 g_i)(x,y)| \;\leqslant\; 1000x^2\,) \quad \& \quad (\,|(\partial_2 g_i)(x,y)| \;\leqslant\; 1000x^2\,) \quad).$$

We have: $\forall i \in \mathbb{N}$, $q_{i+1} + 2c_{i+1} = (1/2)^{i+1} + 2 \cdot (1/100)^{i+1}$
$< (1/2)^i - 2 \cdot (1/100)^i = q_i - 2c_i$.
$\forall i \in \mathbb{N}$, let $T_i := \{(x,y) \,|\, q_i - 2c_i \leqslant x \leqslant q_i + 2c_i\}$.
Then: $\forall i, j \in \mathbb{N}$, $(\, i \neq j\,) \;\Rightarrow\; (\, T_i \cap T_j = \varnothing\,)$.
We have: $\forall x, y \in \mathbb{R}$,

$$(\,|x| > 2\,) \;\Rightarrow\; (\,\eta(x,y) = 0\,).$$

Then: $\forall i \in \mathbb{N}$, $\forall x, y \in \mathbb{R}$,

$$(\,|x - q_i| > 2c_i\,) \;\Rightarrow\; (\,g_i(x,y) = 0\,).$$

Then: $\forall i \in \mathbb{N}$, $\forall x, y \in \mathbb{R}$,

$$(\,g_i(x,y) \neq 0\,) \;\Rightarrow\; (\,|x - q_i| \leqslant 2c_i\,) \;\Rightarrow\; (\,x \in T_i\,).$$

$\forall i \in \mathbb{N}$, let $S_i := \{(x,y) \,|\, g_i(x,y) \neq 0\}$.
Then: $\forall i \in \mathbb{N}$, $S_i \subseteq T_i$.
Recall: $\forall i, j \in \mathbb{N}$, $(\, i \neq j\,) \;\Rightarrow\; (\, T_i \cap T_j = \varnothing\,)$.
Let $f := g_1 + g_2 + \cdots$.
Then, $\forall p \in \mathbb{R}^2$, $\exists i \in \mathbb{N}$ s.t.: $\qquad f = g_i$ near p.
Then, $\forall p \in \mathbb{R}^2$, $\exists i \in \mathbb{N}$ s.t.

$$(\,f'(p) = g_i'(p)\,) \,\&\, (\,(\partial_1 f)(p) = (\partial_1 g_i)(p)\,) \,\&\, (\,(\partial_2 f)(p) = (\partial_2 g_i)(p)\,).$$

Then: $\forall x, y \in \mathbb{R}$,

$$(\quad (\,|(\partial_1 f)(x,y)| \;\leqslant\; 1000x^2\,) \quad \& \quad (\,|(\partial_2 f)(x,y)| \;\leqslant\; 1000x^2\,) \quad).$$

Then $\partial_1 f, \partial_2 f \in \widehat{\mathcal{O}}_2^{VW}$.
Since $\partial_1 f, \partial_2 f \in \widehat{\mathcal{O}}_2^{VW} \subseteq \mathcal{o}_1^{VW}$, we get: $(\partial_1 f)' = z = (\partial_2 f)'$.
Then $z \in \operatorname{dom}[(\partial_1 f)']$ and $z \in \operatorname{dom}[(\partial_2 f)']$. $\qquad$ Want: $z \notin \operatorname{dom}[f'']$.
Recall: $\qquad \forall i \in \mathbb{N}, \quad (q_i, 0) \notin \operatorname{dom}[g_i']$.
Also: $\qquad \forall i \in \mathbb{N}, \quad f = g_i$ near $(q_i, 0)$.
Then: $\qquad \forall i \in \mathbb{N}, \quad (q_i, 0) \notin \operatorname{dom}[f']$.
So, since $q_\bullet \to 0$ in $\mathbb{R}$, we see that f' is not defined near z.
Then $z \notin \operatorname{dom}[f'']$, as desired. $\square$

102. Partial derivative properties to total derivative properties

THEOREM 102.1. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$.
Let $p \in V$, $j \in \mathcal{I}_V$, $k \in \mathcal{I}_W$.
Then: $\pi_k \circ (\partial_j f) \supseteq \pi_{k\|j} \circ (f')$.

Proof. Want: $\forall p \in V$, $(\pi_k \circ (\partial_j f))\ \ =^*\ \ (\pi_{k\|j} \circ (f'))_p$.
Given $p \in V$. Want: $(\pi_k \circ (\partial_j f))\ \ =^*\ \ (\pi_{k\|j} \circ (f'))_p$.
Want: $(\ (\pi_{k\|j} \circ (f'))_p \neq ☹\ ) \ \Rightarrow\ (\ (\pi_k \circ (\partial_j f)) = (\pi_{k\|j} \circ (f'))_p\ )$.
Assume: $(\pi_{k\|j} \circ (f'))_p \neq ☹$. Want: $(\pi_k \circ (\partial_j f)) = (\pi_{k\|j} \circ (f'))_p$.
Let $* := *_{\mathbb{R}VW}$. By HW#9-2, $(\partial_j f)_p\ \ =^*\ \ (f'_p) * (\varepsilon_j^V)$.
As $\pi_{k\|j}(f'_p) = (\pi_{k\|j} \circ (f'))_p \neq ☹$, we conclude that $f'_p \neq ☹$.
Then $f'_p \in \mathrm{im}[f'] \subseteq W \otimes V$, so $(f'_p) * (\varepsilon_j^V) \in W$, so $(f'_p) * (\varepsilon_j^V) \neq ☹$.
Then $(\partial_j f)_p\ \ =^*\ \ (f'_p) * (\varepsilon_j^V)\ \ \neq\ \ ☹$, so $(\partial_j f)_p = (f'_p) * (\varepsilon_j^V)$.
Then

$$\begin{aligned}
(\pi_k \circ (\partial_j f)) &= \partial_k((\partial_j f)_p) = \pi_k((f'_p) * (\varepsilon_j^V)) = ((f'_p) * (\varepsilon_j^V))_k \\
&= ((f'_p) * (\varepsilon_j^V))_{k\|\varnothing} = \sum_{i \in \mathcal{I}_V} ((f'_p)_{k\|i}) \cdot ((\varepsilon_j^V)_{i\|\varnothing}) \\
&= \sum_{i \in \mathcal{I}_V} ((f'_p)_{k\|i}) \cdot ((\varepsilon_j^V)_i) \\
&= \big(((f'_p)_{k\|j}) \cdot ((\varepsilon_j^V)_j) \big) + \left(\sum_{i \in \mathcal{I}_V \setminus \{j\}} ((f'_p)_{k\|i}) \cdot ((\varepsilon_j^V)_i) \right) \\
&= \big(((f'_p)_{k\|j}) \cdot 1 \big) + \left(\sum_{i \in \mathcal{I}_V \setminus \{j\}} ((f'_p)_{k\|i}) \cdot 0 \right) \\
&= (f'_p)_{k\|j} = \pi_{k\|j}(f'_p) = (\pi_{k\|j} \circ (f'))_p.
\end{aligned}$$

□

DEFINITION 102.2. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$, $p \in X$.
Then f is **bounded near** *p (from X to Y) means:*
$\exists B \in \mathcal{B}_X(p)$ s.t. $f_*(B)$ is bounded in Y.

THEOREM 102.3. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$, $p \in X$.
Assume: f is continuous at p from X to Y.
Then: f is bounded near p from X to Y.

Proof. Want: $\exists B \in \mathcal{B}_X(p)$ s.t. $f_*(B)$ is bounded in Y.
Let $C := B_Y(f_p, 1)$. Then $C \in \mathcal{B}_Y(f_p)$.
So, since f is continuous at p from X to Y,
choose $B \in \mathcal{B}_X(p)$ s.t. $f_*(B) \subseteq C$.
Want: $f_*(B)$ is bounded in Y.
Since $C = B_Y(f_p, 1) \in \mathcal{B}_Y$ and since $f_*(B) \subseteq C$,
it follows that $f_*(B)$ is bounded in Y, as desired. □

THEOREM 102.4. . *Let* $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.
Then: (f' *is continuous near* p)
$\Leftrightarrow$ ($\forall j \in \mathcal{I}_V$, $\partial_j f$ *is continuous near* p).

Proof. Proof of $\Rightarrow$*:* This is Theorem 100.4. *End of proof of* $\Rightarrow$.

Proof of $\Leftarrow$*:*
Assume: $\forall j \in \mathcal{I}_V$, $\partial_j f$ is continuous near p.
Want: f' is continuous near p.
Choose $B \in \mathcal{B}_V(p)$ s.t. $\forall j \in \mathcal{I}_V$, $\partial_j f$ is continuous on B.
Want: f' is continuous on B. Want: $\forall q \in B$, f' is continuous at q.
Given $q \in B$. Want: f' is continuous at q.
By Theorem 99.1,
we wish to show: $\forall \ell \in \mathcal{I}_{W \otimes V}$, $\pi_\ell \circ (f')$ is continuous at q.
Given $\ell \in \mathcal{I}_{W \otimes V}$. Want: $\pi_\ell \circ (f')$ is continuous at q.
Since $\ell \in \mathcal{I}_{W \otimes V}$, choose $j \in \mathcal{I}_V$, $k \in \mathcal{I}_W$ s.t. $\ell = k\|j$.
Want: $\pi_{k\|j} \circ (f')$ is continuous at q.
Since $\partial_j f$ is continuous on B and $q \in B$,
we conclude that $\partial_j f$ is continuous at q.
Then, by Theorem 99.1, $\pi_k \circ (\partial_j f)$ is continuous at q.
Also, by Theorem 102.1, we have: $\pi_k \circ (\partial_j f) \subseteq \pi_{k\|j} \circ (f')$.
So, by Theorem 44.13,
we wish to show: $q \in \text{dom}[\pi_{k\|j} \circ (f')]$.
Since $\text{im}[f'] \subseteq W \otimes V = \text{dom}[\pi_{k\|j}]$, we get: $\text{dom}[f'] = \text{dom}[\pi_{k\|j} \circ (f')]$.
We therefore wish to show: $q \in \text{dom}[f']$.
By Theorem 100.2, it suffices to show:
$\forall i \in \mathcal{I}_V$, $\partial_i f$ is defined near q and continuous at q.
Given $i \in \mathcal{I}_V$. Want: $\partial_i f$ is defined near q and continuous at q.
Since $\partial_i f$ is continuous on B and $q \in B$,
we see that $\partial_i f$ is continuous at q.
Want: $\partial_i f$ is defined near q.
Since $q \in B \in \mathcal{B}_V$, by the Recentering Lemma (Theorem 38.16),
choose $A \in \mathcal{B}_V(q)$ s.t. $A \subseteq B$.
By the choice of B, we know that $\partial_i f$ is continuous on B,
so $B \subseteq \text{dom}[\partial_i f]$.
Since $A \in \mathcal{B}_V(q)$ and $A \subseteq B \subseteq \text{dom}[\partial_i f]$,
it follows that $\partial_i f$ is defined near q, as desired.
End of proof of $\Leftarrow$. ∎

THEOREM 102.5. *Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.*
Assume: $\forall j \in \mathcal{I}_V$, *$\partial_j f$ is defined near p and bounded near p.*
Then f is continuous near p.

Proof. Choose $B \in \mathcal{B}_V(p)$ s.t., $\forall j \in \mathcal{I}_V$,
$B \subseteq \mathrm{dom}[\partial_j f]$ and $(\partial_j f)_*(B)$ is bounded in W.
Want: f is continuous on B. Want: $\forall y \in B$, f is continuous at y.
Given $y \in B$. Want: f is continuous at y. Let $g := f_y^T$.
Want: g is continuous at 0_V. Want: $g \in \mathrm{CVZ}_V^W$.
Since $\widehat{\mathcal{O}}_1^{VW} \subseteq \mathcal{O}_0^{VW} = \mathrm{CVZ}_V^W$, it suffices to show: $g \in \widehat{\mathcal{O}}_1^{VW}$.
Then, by HW#12-3, we wish to prove: $\forall j \in \mathcal{I}_V$, $\partial_j g \in \widehat{\mathcal{O}}_0^{VW}$.
Given $j \in \mathcal{I}_V$. Want: $\partial_j g \in \widehat{\mathcal{O}}_0^{VW}$.
We have $\partial_j g = \partial_j(f_y^T) = (\partial_j f)(y + \bullet)$.
Then $\mathrm{dom}[\partial_j g] = (\mathrm{dom}[\partial_j f]) - y$. Recall: $B \subseteq \mathrm{dom}[\partial_j f]$.
Then $B - y \subseteq (\mathrm{dom}[\partial_j f]) - y = \mathrm{dom}[\partial_j g]$.
Because $\partial_j g = (\partial_j f)(y + \bullet)$,
we conclude that $(\partial_j g)_*(B - y) = (\partial_j f)_*(B)$,
so, since $(\partial_j f)_*(B)$ is bounded in W,
it follows that $(\partial_j g)_*(B - y)$ is bounded in W.
Since $B \in \mathcal{B}_V$, we get: $B - y \in \mathcal{B}_V$. Also, $0_V = y - y \in B - y$.
As $0_V \in B - y \in \mathcal{B}_V$, by the Recentering Lemma (Theorem 38.16),
choose $A \in \mathcal{B}_V(0_V)$ s.t. $A \subseteq B$.
Then $A \subseteq B - y \subseteq \mathrm{dom}[\partial_j g]$.
So, as $A \in \mathcal{B}_V(0_V)$, we see that $\partial_j g$ is defined near 0_V.
That is, $\partial_j g \in \mathrm{DNZ}_V^W$.
Also, $(\partial_j g)_*(A) \subseteq (\partial_j g)_*(B - y)$,
so, since $(\partial_j g)_*(B - y)$ is bounded in W,
we conclude that $(\partial_j g)_*(A)$ is bounded in W.
So, since $A \in \mathcal{B}_V(0_V)$ and since $\partial_j g \in \mathrm{DNZ}_V^W$,
we see that $\partial_j g \in \mathrm{BNZ}_V^W$.
Then $\partial_j g \in \mathrm{BNZ}_V^W = \widehat{\mathcal{O}}_0^{VW}$, as desired. □

THEOREM 102.6. *Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p, u \in V$.*
Assume: $\forall k \in \mathcal{I}_W$, $\partial_u(\pi_k^W \circ f)$ is defined near p.
Then $\partial_u f$ is defined near p.

Proof. Choose $B \in \mathcal{B}_V(p)$ s.t., $\forall k \in \mathcal{I}_W$, $B \subseteq \mathrm{dom}[\partial_u(\pi_k^W \circ f)]$.
Want: $B \subseteq \mathrm{dom}[\partial_u f]$. Want: $\forall q \in B$, $q \in \mathrm{dom}[\partial_u f]$.
Given $q \in B$. Want: $q \in \mathrm{dom}[\partial_u f]$.

We have: $\forall k \in \mathcal{I}_W$, $q \in B \subseteq \text{dom}[\partial_u(\pi_k^W \circ f)]$.
Then, by HW#13-2, we get: $q \in \text{dom}[\partial_u f]$, as desired. □

Let X and Y be metric spaces, $f : X \dashrightarrow Y$, $p \in X$.
Recall: If f is continuous at p, the $p \in \text{dom}[f]$.

Let X and Y be metric spaces, $f : X \dashrightarrow Y$, $S \subseteq X$.
Assue f is continuous on S. Then $S \subseteq \text{dom}[f]$.

THEOREM 102.7. *Let $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.*
Assume: $\forall j \in \mathcal{I}_V$, $\partial_j f$ is defined near p and continuous at p.
Then f' is continuous at p.

Proof. By Theorem 100.2, we get: $p \in \text{dom}[f']$.
Since $f : V \dashrightarrow W$, we get $f' : V \dashrightarrow W \otimes V$.
By Theorem 99.1, we wish to show:
$\forall \ell \in \mathcal{I}_W \otimes V$, $\pi_\ell \circ (f')$ is continuous at p.
Given $\ell \in \mathcal{I}_W \otimes V$. Want: $\pi_\ell \circ (f')$ is continuous at p.
Since $\ell \in \mathcal{I}_W \otimes V$, choose $j \in \mathcal{I}_V$, $k \in \mathcal{I}_W$ s.t. $\ell = k \| j$.
Since $\partial_j f$ is continuous at p, by Theorem 99.1,
we see that $\pi_k \circ (\partial_j f)$ is continuous at p.
Also, $p \in \text{dom}[f'] \subseteq \text{dom}[\pi_\ell \circ (f')]$.
By Theorem 102.1, we get: $\pi_\ell \circ (f') \subseteq \pi_k \circ (\partial_j f)$.
So, since $p \in \text{dom}[\pi_\ell \circ (f')]$ and since $\pi_k \circ (\partial_j f)$ is continuous at p,
by Theorem 44.13, we get: $\pi_\ell \circ (f')$ is continuous at p. □

THEOREM 102.8. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$, $B \in \mathcal{B}_X$, $p \in B$. Assume: f is continuous on B.
Then f is defined near p and continuous at p.

Proof. Since f is continuous on B and $p \in B$,
we conclude that f is continuous at p.
Want: f is defined near p.
By the Recentering Lemma (Theorem 38.16),
choose $A \in \mathcal{B}_V(p)$ s.t. $A \subseteq B$. Want: $A \subseteq \text{dom}[f]$.
Since f is continuous on B, we get: $B \subseteq \text{dom}[f]$.
Then $A \subseteq B \subseteq \text{dom}[f]$, as desired. □

See Theorem 100.4 for the converse to the next result, Theorem 102.9:

THEOREM 102.9. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $q \in V$.
Assume: $\forall j \in \mathcal{I}_V$, $\partial_j f$ *is continuous near* q.
Then: f' *is continuous near* q.

Proof. Choose $B \in \mathcal{B}_V(q)$ s.t., $\forall j \in \mathcal{I}_V$, $\partial_j f$ is continuous on B.
Want: f' is continuous on B. Want: $\forall p \in B$, f' is continuous at p.
Given $p \in B$. Want: f' is continuous at p.
By Theorem 102.8, it suffices to show:
$\forall j \in \mathcal{I}_V$, $\partial_j f$ is defined near p and continuous at p.
Given $j \in \mathcal{I}_V$. Want: $\partial_j f$ is defined near p and continuous at p.
By the choice of B, we know that $\partial_j f$ is continuous on B.
Then, by Theorem 102.8, $\partial_j f$ is defined near p and continuous at p. □

The next result is the order two version of Theorem 102.5.

THEOREM 102.10. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.
Assume: $\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f$ *is defined near* p *and bounded near* p.
Then f' *is continuous near* p.

Proof. By Theorem 102.9, it suffices to show:
$\forall j \in \mathcal{I}_V$, $\partial_j f$ is continuous near p.
Given $j \in \mathcal{I}_V$. Want: $\partial_j f$ is continuous near p.
Let $g := \partial_j f$. Want: g is continuous near p.
By Theorem 102.5, it suffices to show:
$\forall i \in \mathcal{I}_V$, $\partial_i g$ is defined near p and bounded near p.
Given $i \in \mathcal{I}_V$. Want: $\partial_i g$ is defined near p and bounded near p.
By assumption, $\partial_i \partial_j f$ is defined near p and bounded near p.
So since $\partial_i g = \partial_i \partial_j f$, we get:
$\partial_i g$ is defined near p and bounded near p, as desired. □

Recall Theorem 99.6.

THEOREM 102.11. *Let* X *be a metric space,* Y *a set.*
Let $f, g : X \dashrightarrow Y$, $p \in X$.
Assume that $f \subseteq g$ *and that* f *is defined near* p.
Then $f = g$ *near* p.

Proof. Unassigned HW. □

THEOREM 102.12. *Let* X *be a metric space,* Y *a set.*
Let $f, g : X \dashrightarrow Y$, $p \in X$.
Assume that $f = g$ *near* p *and that* f *is defined near* p.
Then g *is defined near* p.

Proof. Unassigned HW. □

THEOREM 102.13. *Let X and Y be metric spaces.*
Let $f, g : X \dashrightarrow Y$, $p \in X$.
Assume that $f = g$ *near* p *and that* f *is continuous at* p.
Then g is continuous at p.

Proof. Unassigned HW. □

THEOREM 102.14. *Let $V, W \in$ TNSR$^+$.*
Let $f, g : V \dashrightarrow W$, $p \in V$, $j \in \mathcal{I}_V$.
Assume that $f = g$ *near* p. *Then* $\partial_j f = \partial_j g$ *near* p.

Proof. Unassigned HW. □

THEOREM 102.15. *Let $V, W \in$ TNSR$^+$.*
Let $f, g : V \dashrightarrow W$, $p \in V$.
Assume that $f = g$ *near* p. *Then* $f' = g'$ *near* p.

Proof. Unassigned HW. □

THEOREM 102.16. *Let $V, W \in$ TNSR$^+$, $f, g : V \dashrightarrow W$, $p \in V$.*
Assume: $\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f$ is defined near p and continuous at p.
Then f'' is continuous at p.

Proof. By Theorem 102.3, $\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f$ is bounded near p.
Also, by assumption, $\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f$ is defined near p.
Then, by Theorem 102.10, f' is continuous near p.
Then: f' is continuous at p and f' is defined near p.

Claim: Let $i, j \in \mathcal{I}_V$, $k \in \mathcal{I}_W$.
Then: $\partial_i(\pi_{k\|j} \circ (f')) = \pi_k \circ (\partial_i \partial_j f)$ near p.
Proof of Claim:
Since f' is defined near p, and since $\mathrm{im}[f'] \subseteq W \otimes V = \mathrm{dom}[\pi_{k\|j}]$, we see that $\pi_{k\|j} \circ (f')$ is defined near p.
By Theorem 102.1, $\pi_k \circ (\partial_j f) \supseteq \pi_{k\|j} \circ (f')$.
Then: $\pi_{k\|j} \circ (f') \subseteq \pi_k \circ (\partial_j f)$.
So, since $\pi_{k\|j} \circ (f')$ is defined near p, it follows that:
$\pi_{k\|j} \circ (f') = \pi_k \circ (\partial_j f)$ near p.
Then: $\partial_i(\pi_{k\|j} \circ (f')) = \partial_i(\pi_k \circ (\partial_j f))$ near p.
Want: $\partial_i(\pi_k \circ (\partial_j f)) = \pi_k \circ (\partial_i \partial_j f)$ near p.
By assumption, $\partial_i \partial_j f$ is defined near p.

So, since $\mathrm{im}[\partial_i\partial_j f] \subseteq W = \mathrm{dom}[\pi_k]$,

we see that $\pi_k \circ (\partial_i\partial_j f)$ is defined near p.

By Theorem 94.3, $\partial_i(\pi_k \circ (\partial_j f)) \supseteq \pi_k \circ (\partial_i\partial_j f)$.

So, since $\pi_k \circ (\partial_i\partial_j f)$ is defined near p, it follows that:

$\partial_i(\pi_k \circ (\partial_j f)) \;=\; \pi_k \circ (\partial_i\partial_j f)$ near p, as desired.

End of proof of Claim.

Let $g := f'$. Want: g' is continuous at p.

Since $f : V \dashrightarrow W$ and $g = f'$, we get $g : V \dashrightarrow W \otimes V$.

By Theorem 102.7, it suffices to show:

$\forall i \in \mathcal{I}_V$, $\partial_i g$ is defined near p and continuous at p.

Given $i \in \mathcal{I}_V$. Want: (A) $\partial_i g$ is defined near p and

(B) $\partial_i g$ is continuous at p.

Proof of (A):

By HW#13-2, it suffices to show:

$\forall \ell \in \mathcal{I}_{W\otimes V}$, $\partial_i(\pi_\ell \circ g)$ is defined near p.

Given $\ell \in \mathcal{I}_{W\otimes V}$. Want: $\partial_i(\pi_\ell \circ g)$ is defined near p.

Since $\ell \in \mathcal{I}_{W\otimes V}$, choose $j \in \mathcal{I}_V$ and $k \in \mathcal{I}_W$ s.t. $\ell = k\|j$.

By assumption, $\partial_i\partial_j f$ is defined near p.

So, since $\mathrm{im}[\partial_i\partial_j f] \subseteq W = \mathrm{dom}[\pi_k]$,

we see that $\pi_k \circ (\partial_i\partial_j f)$ is defined near p.

By the claim: $\partial_i(\pi_{k\|j} \circ (f')) \;=\; \pi_k \circ (\partial_i\partial_j f)$ near p.

So, since $\ell = k\|j$ and $g = f'$, we see that:

$\partial_i(\pi_\ell \circ g) \;=\; \pi_k \circ (\partial_i\partial_j f)$ near p.

So, since $\pi_k \circ (\partial_i\partial_j f)$ is defined near p, we conclude:

$\partial_i(\pi_\ell \circ g)$ is defined near p, as desired.

End of proof of (A).

Proof of (B):

By Theorem 99.1, it suffices to show:

$\forall \ell \in \mathcal{I}_{W\otimes V}$, $\pi_\ell \circ (\partial_i g)$ is continuous at p.

Given $\ell \in \mathcal{I}_{W\otimes V}$. Want: $\pi_\ell \circ (\partial_i g)$ is continuous at p.

By (A), $\partial_i g$ is defined near p. Then $p \in \mathrm{dom}[\partial_i g]$.

So, since $\mathrm{im}[\partial_i g] \subseteq W \otimes V = \mathrm{dom}[\pi_\ell]$,

we see that $p \in \mathrm{dom}[\pi_\ell \circ (\partial_i g)]$.

By Theorem 94.3, $\partial_i(\pi_\ell \circ g) \supseteq \pi_\ell \circ (\partial_i g)$.

Then, by Theorem 44.13, it suffices to show:

$\partial_i(\pi_\ell \circ g)$ is continuous at p.
Since $\ell \in \mathcal{I}_{W \otimes V}$, choose $j \in \mathcal{I}_V$ and $k \in \mathcal{I}_W$ s.t. $\ell = k \| j$.
By assumption, $\partial_i \partial_j f$ is continuous at p.
Then, by Theorem 99.1,
we see that $\pi_k \circ (\partial_i \partial_j f)$ is continuous at p.
By the claim, $\partial_i(\pi_{k\|j} \circ (f')) \;=\; \pi_k \circ (\partial_i \partial_j f)$ near p.
So, since $\ell = k\|j$ and $g = f'$, we see that:
$\partial_i(\pi_\ell \circ g) \;=\; \pi_k \circ (\partial_i \partial_j f)$ near p.
So, since $\pi_k \circ (\partial_i \partial_j f)$ is continuous at p, by Theorem 65.7, we have:
$\partial_i(\pi_\ell \circ g)$ is continuous at p, as desired.
End of proof of (B). □

THEOREM 102.17. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$.
Let $i, j \in \mathcal{I}_V$, $k \in \mathcal{I}_W$. *Then* $\pi_k \circ (\partial_i \partial_j f) \;\supseteq\; \pi_{k\|i\|j} \circ (f'')$.

Proof. Want: $\forall p \in V$, $(\pi_k \circ (\partial_i \partial_j f))_p \;=^*\; (\pi_{k\|i\|j} \circ (f''))_p$.
Given $p \in V$. Want: $(\pi_k \circ (\partial_i \partial_j f))_p \;=^*\; (\pi_{k\|i\|j} \circ (f''))_p$.
Want: $(\, (\pi_{k\|i\|j} \circ (f''))_p \neq ☹ \,) \Rightarrow (\, (\pi_k \circ (\partial_i \partial_j f))_p \;=\; (\pi_{k\|i\|j} \circ (f''))_p \,)$.
Assume: $(\pi_{k\|i\|j} \circ (f''))_p \neq ☹$. Want: $(\pi_k \circ (\partial_i \partial_j f))_p \;=\; (\pi_{k\|i\|j} \circ (f''))_p$.
Since $(\pi_{k\|i\|j} \circ (f''))_p \neq ☹$, we get $f''_p \neq ☹$, and so $p \in \mathrm{dom}[f'']$.
By Theorem 97.6, we have: $\mathrm{Bilin}^{VVW}_{f''_p}(\varepsilon_i, \varepsilon_j) \;=\; (\partial_i \partial_j f)_p$.
Let $A := f''_p$ and $B := \mathrm{Bilin}^{VVW}_A$. Then $B(\varepsilon_i, \varepsilon_j) = (\partial_i \partial_j f)_p$.
Since $\mathrm{Bilin}^{VVW}_A = B$, get $A = [B]^W_{VV}$, and so $A_{k\|i\|j} = \pi_k(B(\varepsilon_i, \varepsilon_j))$.
We have $(\pi_{k\|i\|j} \circ (f''))_p = \pi_{k\|i\|j}(f''_p) = \pi_{k\|i\|j}(A) = A_{k\|i\|j}$.
Then $(\pi_k \circ (\partial_i \partial_j f))_p \;=\; \pi_k((\partial_i \partial_j f)_p) \;=\; \pi_k(B(\varepsilon_i, \varepsilon_j))$
$= \; A_{k\|i\|j} \;=\; (\pi_{k\|i\|j} \circ (f''))_p$, as desired. □

THEOREM 102.18. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $q \in V$.
Assume: f'' *is continuous near* q.
Then: $\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f$ *is continuous near* q.

Proof. This is HW#13-3. □

THEOREM 102.19. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$.
Let $h, i, j \in \mathcal{I}_V$, $k \in \mathcal{I}_W$. *Then* $\pi_k \circ (\partial_h \partial_i \partial_j f) \;\supseteq\; \pi_{k\|h\|i\|j} \circ (f''')$.

Proof. Unassigned HW. □

THEOREM 102.20. *Let* $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $q \in V$.
Assume: f''' *is continuous near* q.
Then: $\forall h, i, j \in \mathcal{I}_V$, $\partial_h \partial_i \partial_j f$ *is continuous near* q.

Proof. Unassigned HW. □

THEOREM 102.21. *Let $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$, $q \in V$.*
Assume: $\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f$ is continuous near q.
Then: f'' is continuous near q.

Proof. Choose $B \in \mathcal{B}_V(q)$ s.t., $\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f$ is continuous on B.
Want: f'' is continuous on B. Want: $\forall p \in B$, f'' is continuous at p.
Given $p \in B$. Want: f'' is continuous at p.
By Theorem 102.16, it suffices to show:
$\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f$ is defined near p and continuous at p.
Given $i, j \in \mathcal{I}_V$. Want: $\partial_i \partial_j f$ is defined near p and continuous at p.
By the choice of B, we know that $\partial_i \partial_j f$ is continuous on B.
Then, by Theorem 102.8,
$\partial_i \partial_j f$ is defined near p and continuous at p, as desired. □

THEOREM 102.22. *Let $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$, $q \in V$.*
Assume: $\forall h, i, j \in \mathcal{I}_V$, $\partial_h \partial_i \partial_j f$ is continuous near q.
Then: f''' is continuous near q.

Proof. Choose $B \in \mathcal{B}_V(q)$ s.t., $\forall h, i, j \in \mathcal{I}_V$, $\partial_h \partial_i \partial_j f$ is continuous on B.
Want: f''' is continuous on B. Want: $\forall p \in B$, f''' is continuous at p.
Given $p \in B$. Want: f''' is continuous at p.
By HW#13-5, it suffices to show:
$\forall h, i, j \in \mathcal{I}_V$, $\partial_h \partial_i \partial_j f$ is defined near p and continuous at p.
Given $h, i, j \in \mathcal{I}_V$. Want: $\partial_h \partial_i \partial_j f$ is defined near p and continuous at p.
By the choice of B, we know that $\partial_h \partial_i \partial_j f$ is continuous on B.
Then, by Theorem 102.8,
$\partial_h \partial_i \partial_j f$ is defined near p and continuous at p, as desired. □

Recall Theorem 99.7.

THEOREM 102.23. *Let $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$.*
Assume $f_{0_V} = 0_W$ and $f'_{0_V} = 0_{W \otimes V}$. Then $f \in \mathcal{O}_1^{VW}$.

Proof. Let $L := D_{0_V} f$. Since $f'_{0_V} = 0_{W \otimes V}$, we get: $L = \mathbf{0}_V^W$.
Then $L \neq ☹$, and so $f_{0_V}^T - L \in \mathcal{O}_1^{VW}$.
Since $f_{0_V} = 0_W$, we get $f_{0_V}^T = f$.
Then $f = f - \mathbf{0}_V^W = f_{0_V}^T - L \in \mathcal{O}_1^{VW}$, as desired. □

THEOREM 102.24. *Let $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$, $q \in \text{dom}[f']$.*
Then: ($f'_q = 0_{W \otimes V}$) $\Leftrightarrow$ ($\forall j \in \mathcal{I}_V$, $(\partial_j f)_q = 0$).

Proof. Since $q \in \mathrm{dom}[f']$, we get $f'_q \in \mathrm{im}[f']$.
Then $f'_q \in \mathrm{im}[f'] \subseteq W \otimes V = \mathrm{dom}[\pi_{k\|j}]$, so $\pi_{k\|j}(f'_q) \neq ☹$.
Since $(\pi_{k\|j} \circ (f'))_q = \pi_{k\|j}(f'_q) \neq ☹$,
by Theorem 102.1, we get: $(\pi_k \circ (\partial_j f))_q = (\pi_{k\|j} \circ (f'))_q$.
It follows that: $(f'_q = 0_{W\otimes V})$
$\Leftrightarrow$ $(\forall j \in \mathcal{I}_V,\ \forall k \in \mathcal{I}_W,\ (\pi_{k\|j} \circ (f'))_q = 0)$
$\Leftrightarrow$ $(\forall j \in \mathcal{I}_V,\ \forall k \in \mathcal{I}_W,\ (\pi_k \circ (\partial_j f))_q = 0)$
$\Leftrightarrow$ $(\forall j \in \mathcal{I}_V,\ (\partial_j f)_q = 0_W)$, as desired. □

THEOREM 102.25. *Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $q \in \mathrm{dom}[f'']$. Then: $(f''_q = 0_{W\otimes V\otimes V}) \Leftrightarrow (\forall i, j \in \mathcal{I}_V,\ (\partial_i \partial_j f)_q = 0)$.*

Proof. Unassigned HW. □

103. Second order Taylor theorem

THEOREM 103.1. *Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$. Assume $f_{0_V} = 0_W$, $f'_{0_V} = 0_{W\otimes V}$ and $f''_{0_V} = 0_{W\otimes V\otimes V}$. Then $f \in \mathcal{O}_2^{VW}$.*

Proof. By Theorem 99.7, it suffices to show:
$\forall j \in \mathcal{I}_V, \quad \partial_j f \in \mathcal{O}_1^{VW}$.
Given $j \in \mathcal{I}_V$. Want: $\partial_j f \in \mathcal{O}_1^{VW}$.
Let $g := \partial_j f$. Want: $g \in \mathcal{O}_1^{VW}$.
Since $f : V \dashrightarrow W$ and $g = \partial_j f$, we get $g : V \dashrightarrow W$.
Since $f'_{0_V} = 0_{W\otimes V}$, by Theorem 102.24, we get $(\partial_j f)_{0_V} = 0_W$.
Then $g_{0_V} = (\partial_j f)_{0_V} = 0_W$.
Then, by Theorem 102.23, it suffices to show: $g'_{0_V} = 0_{W\otimes V}$.
By Theorem 94.5, we have $\mathrm{dom}[f''] \subseteq \mathrm{dom}[(\partial_j f)']$.
Then $0_V \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[(\partial_j f)'] = \mathrm{dom}[g']$.
Then, by Theorem 102.24, we wish to show: $\forall i \in \mathcal{I}_V$, $(\partial_i g)_{0_V} = 0_W$.
Given $i \in \mathcal{I}_V$. Want: $(\partial_i g)_{0_V} = 0_W$.
By assumption, we have: $f''_{0_V} = 0_{W\otimes V\otimes V}$.
Then, by Theorem 102.25, we get: $(\partial_i \partial_j f)_{0_V} = 0_W$.
Then: $(\partial_i g)_{0_V} = (\partial_i \partial_j f)_{0_V} = 0_W$, as desired. □

THEOREM 103.2. *Let $V, W \in \mathrm{TNSR}^+$, $L \in \mathcal{L}_V^W$, $x, u \in V$. Then $(\partial_u L)_x = L(u)$.*

Proof. Let $i := i_x^u$. Then $(\partial_u L)_x = (L \circ i)'_0$.
We calculate: $\forall t \in \mathbb{R}$,
$(L \circ i)_t = L(i_t) = L(x + tu) = (L(x)) + t \cdot (L(u))$.
Define $M \in \mathcal{L}_{\mathbb{R}}^W$ by $M(h) = h \cdot (L(u))$.

Then $\mathrm{s}\ell_M^{\mathbb{R}W} = M(1) = 1 \cdot (L(u)) = L(u)$.
We calculate: $\forall h \in \mathbb{R}$,

$$\begin{aligned}(L \circ i)_h^T &= (L \circ i)_{0+h} - (L \circ i)_0 \\ &= (L \circ i)_h - (L \circ i)_0 \\ &= (\ (L(x)) + h \cdot (L(u))\ ) \\ &\quad - (\ (L(x)) + 0 \cdot (L(u))\ ) \\ &= h \cdot (L(u)) = M_h.\end{aligned}$$

Then $(L \circ i)_0^T = M$. Then $(L \circ i)_0^T - M = \mathbf{0}_{\mathbb{R}}^W \in \mathcal{o}_1^{\mathbb{R}W}$.
Then $D_0(L \circ i) = M$, so $(L \circ i)'_0 = \mathrm{s}\ell_M^{\mathbb{R}W}$.
Then $(\partial_u L)_x = (L \circ i)'_0 = \mathrm{s}\ell_M^{\mathbb{R}W} = L(u)$, as desired. □

THEOREM 103.3. *Let $V, W \in \mathrm{TNSR}^+$, $L \in \mathcal{L}_V^W$, $x, u, v \in V$. Then $(\partial_u \partial_v L)_x = 0_W$.*

Proof. Let $C := C_V^{L(v)}$. Then $(\partial_u C)_x = 0_W$.
By Theorem 103.2, we have: $\forall y \in V$, $(\partial_v L)_y = L(v)$.
Then: $\forall y \in V$, $(\partial_v L)_y = L(v) = C_y$.
Then $\partial_v L = C$. Then $(\partial_u \partial_v L)_x = (\partial_u C)_x = 0_W$, as desired. □

THEOREM 103.4. *Let $V, W \in \mathrm{TNSR}^+$, $B \in \mathcal{SB}_V^W$, $Q := B(\bullet, \bullet)$. Let $x, u \in V$. Then $(\partial_u Q)_x = 2 \cdot (B(x, u))$.*

Proof. Let $i := i_x^u$. Then $(\partial_u Q)_x = (Q \circ i)'_0$.
We calculate: $\forall t \in \mathbb{R}$,

$$\begin{aligned}(Q \circ i)_t = Q(i_t) &= B(i_t, i_t). \\ &= B(x + tu, x + tu). \\ &= (B(x, x)) + (2t) \cdot (B(x, u)) + (t^2) \cdot (B(u, u)). \\ &= (Q(x)) + (2t) \cdot (B(x, u)) + (t^2) \cdot (Q(u)).\end{aligned}$$

Define $M \in \mathcal{L}_{\mathbb{R}}^W$ by $M(h) = (2h) \cdot (B(x, u))$.
Then $\mathrm{s}\ell_M^{\mathbb{R}W} = M(1) = (2 \cdot 1) \cdot (B(x, u)) = 2 \cdot (B(x, u))$.
Define $P \in \mathcal{Q}_{\mathbb{R}}^W$ by $P(h) = (h^2) \cdot (Q(u))$.
We calculate: $\forall h \in \mathbb{R}$,

$$\begin{aligned}(Q \circ i)_h^T &= (Q \circ i)_{0+h} - (Q \circ i)_0 \\ &= (Q \circ i)_h - (Q \circ i)_0 \\ &= (\ (Q(x)) + (2h) \cdot (B(x, u)) + (h^2) \cdot (Q(u))\ ) \\ &\quad - (\ (Q(x)) + (2 \cdot 0) \cdot (B(x, u)) + (0^2) \cdot (Q(u))\ ) \\ &= (2h) \cdot (B(x, u)) + (h^2) \cdot (Q(u)) \\ &= M_h + P_h.\end{aligned}$$

Then $(Q \circ i)_0^T = M + P$.
Then $(Q \circ i)_0^T - M = P \in \mathcal{Q}_{\mathbb{R}}^W \subseteq \widehat{\mathcal{O}}_2^{\mathbb{R}W} \subseteq \mathcal{o}_1^{\mathbb{R}W}$.

Then $D_0(Q \circ i) = M$, so $(Q \circ i)'_0 = \mathrm{s}\ell_M^{\mathbb{R}W}$.
Then $(\partial_u Q)_x = (Q \circ i)'_0 = \mathrm{s}\ell_M^{\mathbb{R}W} = 2 \cdot (B(x,u))$, as desired. ☐

THEOREM 103.5. *Let $V, W \in \mathrm{TNSR}^+$, $B \in \mathcal{SB}_V^W$, $Q := B(\bullet, \bullet)$.*
Let $x, u, v \in V$. Then $(\partial_u \partial_v Q)_x = 2 \cdot (B(u,v))$.

Proof. Let $L := 2 \cdot (B(\bullet, v))$. Then $L \in \mathcal{L}_V^W$.
By Theorem 103.4, we have: $\forall y \in V$, $(\partial_v Q)_y = 2 \cdot (B(y,v))$.
Then: $\forall y \in V$, $(\partial_v Q)_y = 2 \cdot (B(y,v)) = L_y$. Then $\partial_v Q = L$.
By Theorem 103.2, $(\partial_u L)_x = L(u)$.
Then $(\partial_u \partial_v Q)_x = (\partial_u L)_x = L(u) = 2 \cdot (B(u,v))$, as desired. ☐

The following theorem is the **Second Order Taylor Theorem**:

THEOREM 103.6. *Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $x \in \mathrm{dom}[f'']$.*
Let $L := \mathrm{Lin}_{f'_x}$ and $Q := ((1/2) \cdot (\mathrm{Bilin}_{f''_x}))(\bullet, \bullet)$.
Let $R := f_x^T - L - Q$. Then $R \in \mathcal{O}_2^{VW}$.

Proof. Since $x \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[f'] \subseteq \mathrm{dom}[f]$, we get $f_x^T(0_V) = 0_W$.
Let $B := (1/2) \cdot (\mathrm{Bilin}_{f''_x})$. Then $Q = B(\bullet, \bullet)$.
Since $L \in \mathcal{L}_V^W$, we get $\mathrm{dom}[L''] = V$ and $L_{0_V} = 0_W$.
Since $Q \in \mathcal{Q}_V^W$, we get $\mathrm{dom}[Q''] = V$ and $Q_{0_V} = 0_W$.
We have $(f_x^T)' = f'(x + \bullet)$ and $(f_x^T)'' = (f'(x + \bullet))' = f''(x + \bullet)$.
Also, $R' = ((f_x^T) - L - Q)' \supseteq (f_x^T)' - L' - Q'$.
Then $R'' \supseteq ((f_x^T)' - L' - Q')' \supseteq (f_x^T)'' - L'' - Q''$.
Then $R' \supseteq (f'(x + \bullet)) - L' - Q'$ and $R'' \supseteq (f''(x + \bullet)) - L'' - Q''$.
Then $\mathrm{dom}[R''] \supseteq \mathrm{dom}[(f''(x + \bullet)) - L'' - Q'']$.
Since $x \in \mathrm{dom}[f'']$, we get $0_V \in \mathrm{dom}[f''(x + \bullet)]$.
So, since $0_V \in V = \mathrm{dom}[L'']$ and $0_V \in V = \mathrm{dom}[Q'']$,
we conclude: $0_V \in \mathrm{dom}[(f''(x + \bullet)) - L'' - Q'']$.
Then $0_V \in \mathrm{dom}[(f''(x + \bullet)) - L'' - Q''] \subseteq \mathrm{dom}[R'']$.
Also, $R_{0_V} = (f_x^T - L - Q)_{0_V} = 0_W - 0_W - 0_W = 0_W$.
Then, by Theorem 103.1, it suffices to show:
both $R'_{0_V} = 0_{W \otimes V}$ and $R''_{0_V} = 0_{W \otimes V \otimes V}$.

Proof of (1):
We have $0_V \in \mathrm{dom}[R''] \subseteq \mathrm{dom}[R']$.
By Theorem 102.24, it suffices to show: $\forall j \in \mathcal{I}_V$, $(\partial_j R)_{0_V} = 0_W$.
Given $j \in \mathcal{I}_V$. Want: $(\partial_j R)_{0_V} = 0_W$.
Let $u := \varepsilon_j^V$. Then $\partial_j R = \partial_u R$.
By Theorem 103.2, $(\partial_u L)_{0_V} = L(u)$.

By Theorem 103.4, $(\partial_u Q)_{0_V} = B(0_V, u)$.
Since $B \in \mathcal{B}^W_{VV}$, we get $B(0_V, u) = 0_W$.
Since $L = \mathrm{Lin}_{f'_x}$, we get $L(u) = \mathrm{Lin}_{f'_x}(u)$.
By HW#8-4, $(\partial_u f)(x) =^* (D_x f)(u)$.
By Theorem 90.1, $(D_x f)(u) = \mathrm{Lin}_{f'_x}(u)$.
Then $((\partial_u f)(x + \bullet))_{0_V} = (\partial_u f)(x) =^* (D_x f)(u) = \mathrm{Lin}_{f'_x}(u)$.
Since $x \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[f']$, we get $f'_x \in \mathrm{im}[f'] \subseteq W \otimes V$.
Then $\mathrm{Lin}_{f'_x} \in \mathcal{L}^W_V$. Then $\mathrm{Lin}_{f'_x}(u) \in W$.
Then $(\mathrm{Lin}_{f'_x}(u)) - (\mathrm{Lin}_{f'_x}(u)) - (0_W) = 0_W$.

$$\begin{aligned}\text{Then } (\partial_j R)_{0_V} &= (\partial_u R)_{0_V} \\ &= (\partial_u (f^T_x - L - Q))_{0_V} \\ &=^* (\partial_u (f^T_x))_{0_V} - (\partial_u L)_{0_V} - (\partial_u Q)_{0_V} \\ &= ((\partial_u f)(x + \bullet))_{0_V} - (L(u)) - (B(0_V, u)) \\ &=^* (\mathrm{Lin}_{f'_x}(u)) - (\mathrm{Lin}_{f'_x}(u)) - (0_W). \\ &= 0_W \quad \neq \quad ☹.\end{aligned}$$

Then $(\partial_j R)_{0_V} = 0_W$, as desired.
End of proof of (1).

Proof of (2):
We have $0_V \in \mathrm{dom}[R'']$.
By Theorem 102.25, it suffices to show: $\forall i, j \in \mathcal{I}_V$, $(\partial_i \partial_j R)_{0_V} = 0_W$.
Given $i, j \in \mathcal{I}_V$. Want: $(\partial_i \partial_j R)_{0_V} = 0_W$.
Let $u := \varepsilon^V_i$ and let $v := \varepsilon^V_j$. Then $\partial_i \partial_j R = \partial_u \partial_v R$.
By Theorem 103.3, $(\partial_u \partial_v L)_{0_V} = 0_W$.
By Theorem 103.5, $(\partial_u \partial_v Q)_{0_V} = 2 \cdot (B(u, v))$.
Since $x \in \mathrm{dom}[f'']$, by Theorem 97.6,
we get: $\mathrm{Bilin}_{f''_x}(u, v) = (\partial_u \partial_v f)_x$.
Then $((\partial_u \partial_v f)(x + \bullet))_{0_V} = (\partial_u \partial_v f)(x) = \mathrm{Bilin}_{f''_x}(u, v)$.
Since $B := (1/2) \cdot (\mathrm{Bilin}_{f''_x})$, we get $2 \cdot (B(u, v)) = \mathrm{Bilin}_{f''_x}(u, v)$.
Since $x \in \mathrm{dom}[f'']$, we get $f''_x \in \mathrm{im}[f''] \subseteq W \otimes V \otimes V$.
Then $\mathrm{Bilin}_{f''_x} \in \mathcal{B}^W_{VV}$. Then $\mathrm{Bilin}_{f''_x}(u, v) \in W$.
Then $(\mathrm{Bilin}_{f''_x}(u, v)) - (0_W) - (\mathrm{Bilin}_{f''_x}(u, v)) = 0_W$.

$$\begin{aligned}\text{Then } (\partial_i \partial_j R)_{0_V} &= (\partial_u \partial_v R)_{0_V} \\ &= (\partial_u \partial_v (f^T_x - L - Q))_{0_V} \\ &=^* (\partial_u \partial_v (f^T_x))_{0_V} - (\partial_u \partial_v L)_{0_V} - (\partial_u \partial_v Q)_{0_V} \\ &= ((\partial_u \partial_v f)(x + \bullet))_{0_V} - (0_W) - 2 \cdot (B(u, v)) \\ &= (\mathrm{Bilin}_{f'_x}(u)) - (0_W) - (\mathrm{Bilin}_{f''_x}(u, v)). \\ &= 0_W \quad \neq \quad ☹.\end{aligned}$$

Then $(\partial_i\partial_j R)_{0_V} \;=\; 0_W$, as desired.
End of proof of (2). $\square$

104. Second Derivative Theorem

THEOREM 104.1. *Let* $V \in \mathrm{TNSR}^+$.
Then (V, d_V) *is a proper metric space.*

Proof. Let $W := \mathbb{R}^m$.
By Theorem 60.10, (W, d_W) is a proper metric space.
Let $m := \#\mathcal{I}_V$. Then $\#\mathcal{I}_V = m = \#[1..m]$.
Choose $\lambda : \mathcal{I}_V \hookrightarrow\!\!> [1..m]$.
Then $\widehat{\lambda} : V \hookrightarrow\!\!> W$ is an isometry from (V, d_V) to (W, d_W).
Then (V, d_V) is a proper metric space. $\square$

THEOREM 104.2. *Let* $V \in \mathrm{TNSR}^+$, $Q \in \mathcal{Q}_V^{\mathbb{R}}$.
Assume: $Q > 0$ *on* $V_{0_V}^{\times}$. *Then:* $\exists \varepsilon > 0$ *s.t.* $Q \geqslant \varepsilon \cdot (|\bullet|_V^2)$ *on* V.

Proof. Let $S := \{x \in V \text{ s.t. } |x|_V = 1\}$.
Since $Q \in \mathcal{Q}_V^{\mathbb{R}}$, it follows that Q is continuous from V to $\mathbb{R}$.
Then $Q|S$ is continuous from S to $\mathbb{R}$.
Since $S = (|\bullet|_V)^*(\{1\})$ and since $|\bullet|_V : V \to \mathbb{R}$ is continuous
and since $\{1\}$ is closed in $\mathbb{R}$, we get: S is closed in V.
Moreover, as $S \subseteq B_V(0_V, 2)$, we see that S is bounded in V.
Since S is closed and bounded in V,
and since (by Theorem 104.1), (V, d_V) is proper,
we see that S is compact.
So, since $Q|S$ is continuous from S to $\mathbb{R}$,
it follows, from the Extreme Value Theorem (Theorem 61.2),
that $\min(\mathrm{im}[Q|S]) \neq ☹$.
Let $\varepsilon := \min(\mathrm{im}[Q|S])$. Then $\varepsilon \in \mathrm{im}[Q|S]$ and $\mathrm{im}[Q|S] \geqslant \varepsilon$.
Since $S \subseteq V_{0_V}^{\times}$ and since $Q > 0$ on $V_{0_V}^{\times}$, we get: $\mathrm{im}[Q|S] > 0$.
Then $\varepsilon \in \mathrm{im}[Q|S] > 0$, so $\varepsilon > 0$.
Want: $Q \geqslant \varepsilon \cdot (|\bullet|_V^2)$ on V.
Since $Q \in \mathcal{Q}_V^{\mathbb{R}}$, we see that $Q_{0_V} = 0$.
Then $Q_{0_V} = 0 = \varepsilon \cdot 0^2 = \varepsilon \cdot (|0_V|_V^2) = (\varepsilon \cdot (|\bullet|_V^2))_{0_V}$.
Want: $Q \geqslant \varepsilon \cdot (|\bullet|_V^2)$ on $V_{0_V}^{\times}$.
Want: $\forall x \in V_{0_V}^{\times}$, $Q_x \geqslant (\varepsilon \cdot (|\bullet|_V^2))_x$.
Given $x \in V_{0_V}^{\times}$. Want: $Q_x \geqslant (\varepsilon \cdot (|\bullet|_V^2))_x$.
Let $a := |x|_V$. As $x \in V_{0_V}^{\times}$, we get: $a > 0$.

Let $u := x/a$. Then $x = au$.
Also, $|u|_V = |x|_V/a = a/a = 1$, so $u \in S$.
Then $Q_u \in \mathrm{im}[Q|S] \geqslant \varepsilon$, so $Q_u \geqslant \varepsilon$.
Since $Q_u \geqslant \varepsilon$ and since $a^2 \geqslant 0$, we get $a^2 \cdot Q_u \geqslant a^2 \cdot \varepsilon$.
Since $Q \in \mathcal{Q}_V^{\mathbb{R}}$, we get $Q_{au} = a^2 \cdot Q_u$. Recall: $a = |x|_V$.
Then $Q_x = Q_{au} = a^2 \cdot Q_u \geqslant a^2 \cdot \varepsilon = \varepsilon \cdot (|x|_V^2) = (\varepsilon \cdot (|\bullet|_V^2))_x$. □

THEOREM 104.3. *Let $V, W \in \mathrm{TNSR}^+$, $\alpha \in \mathrm{CVZ}_V^W$, $\varepsilon > 0$. Then $\exists B \in \mathcal{B}_V(0_V)$ s.t. $|\alpha|_W \leqslant \varepsilon$ on B.*

Proof. Since $\alpha \in \mathrm{CVZ}_V^W$, we get: $\alpha_{0_V} = 0_W$.
Since $\alpha \in \mathrm{CVZ}_V^W \subseteq \mathrm{DNZ}_V^W$, choose $C \in \mathcal{B}_V(0_V)$ s.t. $C \subseteq \mathrm{dom}[\alpha]$.
Since $\alpha \in \mathrm{CVZ}_V^W$, we know that α is continuous at 0_V from V to W.
Choose $D \in \mathcal{B}_V(0_V)$ s.t. $\alpha_*(D) \subseteq B_W(\alpha_{0_V}, \varepsilon)$.
Let $B := C \cap D$. Then $B \in \mathcal{B}_V(0_V)$.
Want: $|\alpha|_W \leqslant \varepsilon$ on B.
Want: $\forall x \in B$, $(|\alpha|_W)_x \leqslant \varepsilon$.
Given $x \in B$. Want: $(|\alpha|_W)_x \leqslant \varepsilon$.
Since $x \in B \subseteq C \subseteq \mathrm{dom}[\alpha]$ and since $x \in B \subseteq D$,
it follows that $\alpha_x \in \alpha_*(D)$.
Since $\alpha_x \in \alpha_*(D) \subseteq B_W(\alpha_{0_V}, \varepsilon)$, we get: $|\alpha_x - \alpha_{0_V}| < \varepsilon$.
Since $\alpha_{0_V} = 0_W$, we get: $\alpha_x - \alpha_{0_V} = \alpha_x$.
Since $(|\alpha|_W)_x = |\alpha_x|_W = |\alpha_x - \alpha_{0_V}|_W < \varepsilon$,
we get $(|\alpha|_W)_x < \varepsilon$, and so $(|\alpha|_W)_x \leqslant \varepsilon$, as desired. □

THEOREM 104.4. *Let $V \in \mathrm{TNSR}^+$, $f : V \dashrightarrow \mathbb{R}$, $x \in \mathrm{dom}[f'']$. Let $L := \mathrm{Lin}_{f'_x}$ and $B := \mathrm{Bilin}_{f''_x}$. Assume that $L = 0$ on V and that $B(\bullet, \bullet) > 0$ on $V_{0_V}^\times$. Then f has a strict local minimum at x.*

Proof. Since $x \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[f]$, we get: $(f_x^T)_{0_V} = 0$.
Want: f_x^T has a strict local minimum at 0_V.
Let $g := f_x^T$. Then $g_{0_V} = 0$.
Want: g has a strict local minimum at 0_V.
Want: $\exists U \in \mathcal{B}_V(0_V)$ s.t. $g > g_{0_V}$ on $U_{0_V}^\times$.
Let $Q := ((1/2) \cdot B)(\bullet, \bullet)$. Then $Q = (1/2) \cdot (B(\bullet, \bullet))$.
So, since $B(\bullet, \bullet) > 0$ on $V_{0_V}^\times$, we see that $Q > 0$ on $V_{0_V}^\times$.
Also, $Q = ((1/2) \cdot \mathrm{Bilin}_{f''_x})(\bullet, \bullet)$.
Let $R := f_x^T - L - Q$. By Theorem 103.6, $R \in \mathcal{O}_2^{V\mathbb{R}}$.
Since $R \in \mathcal{O}_2^{V\mathbb{R}} = (\mathrm{CVZ}_V^{\mathbb{R}}) \cdot (|\bullet|_V^2)$,

choose $\alpha \in \mathrm{CVZ}_V^{\mathbb{R}}$ s.t. $R = \alpha \cdot (|\bullet|_V^2)$.
Since $L = 0$ on V, we conclude that $L = \mathbf{0}_V^{\mathbb{R}}$.
Then $g - Q = g - \mathbf{0}_V^{\mathbb{R}} - Q = f_x^T - L - Q = R$.
By Theorem 104.2, choose $\varepsilon > 0$ s.t. $Q \geqslant \varepsilon \cdot (|\bullet|_V^2)$ on V.
By Theorem 104.3, choose $U \in \mathcal{B}_V(0_V)$ s.t. $|\alpha| \leqslant \varepsilon/2$ on U.
Want: $g > g_{0_V}$ on $U_{0_V}^\times$. Want: $\forall y \in U_{0_V}^\times$, $g_y > g_{0_V}$.
Given $y \in U_{0_V}^\times$. Want: $g_y > g_{0_V}$.
Since $y \in U_{0_V}^\times \subseteq V_{0_V}^\times$ and since $Q > 0$ on $V_{0_V}^\times$,
we conclude that $Q_y > 0$. Then $Q_y/2 > 0$.
Since $y \in U_{0_V}^\times \subseteq U$ and since $Q \geqslant \varepsilon \cdot (|\bullet|_V^2)$ on U,
we conclude that $Q_y \geqslant \varepsilon \cdot |y|_V^2$. Then $\varepsilon \cdot |y|_V^2 \leqslant Q_y$.
Since $y \in U_{0_V}^\times \subseteq U$ and since $|\alpha| \leqslant \varepsilon/2$ on U,
we conclude that $|\alpha_y| \leqslant \varepsilon/2$.
We have $g - Q = R = \alpha \cdot (|\bullet|_V^2)$, so $(g - Q)_y = (\alpha \cdot (|\bullet|_V^2))_y$.
Then $g_y - Q_y = (g - Q)_y = (\alpha \cdot (|\bullet|_V^2))_y = \alpha_y \cdot (|y|_V^2)$,
and so $g_y - Q_y = \alpha_y \cdot (|y|_V^2)$, and so $|g_y - Q_y| = |\alpha_y| \cdot |y|_V^2$.
Recall: $|\alpha_y| \leqslant \varepsilon/2$ and $\varepsilon \cdot |y|_V^2 \leqslant Q_y$.
Then $|g_y - Q_y| = |\alpha_y| \cdot |y|_V^2 \leqslant (\varepsilon/2) \cdot |y|_V^2 = (\varepsilon \cdot |y|_V^2)/2 \leqslant Q_y/2$.
Since $|g_y - Q_y| \leqslant Q_y/2$, we get: $Q_y - (Q_y/2) \leqslant g_y \leqslant Q_y + (Q_y/2)$.
Then $g_y \geqslant Q_y - (Q_y/2)$. Recall: $Q_y/2 > 0$ and $g_{0_V} = 0$.
Then $g_y \geqslant Q_y - (Q_y/2) = Q_y(1 - (1/2)) = Q_y/2 > 0 = g_{0_V}$. $\square$

105. Partitions

DEFINITION 105.1. *Let $\mathcal{P}$ be a set of sets.*
Then $\mathcal{P}$ is a **partition** *means:*
$$\forall P, Q \in \mathcal{P}, \qquad (\ (P = Q)\ \vee\ (P \cap Q = \varnothing)\ ).$$

DEFINITION 105.2. *Let X be a set and let $\mathcal{P}$ be a set of sets.*
Then $\mathcal{P}$ is a **partition of** X *means:*
$$(\ (\mathcal{P} \text{ is a partition})\ \&\ (\bigcup \mathcal{P} = X)\ ).$$

DEFINITION 105.3. *We define:* $\forall j \in \mathbb{N}_0$,
$$\mathcal{I}_j := \left\{ \left[\frac{m}{2^j}, \frac{m+1}{2^j}\right) \,\middle|\, m \in \mathbb{Z} \right\} \qquad \text{and}$$
$$\mathcal{S}_j := \{\ I \times J \mid I, J \in \mathcal{I}_j\ \}.$$

We often refer to elements of a partition $\mathcal{P}$ as "legos" or "$\mathcal{P}$-legos".

Let $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2})$ and let $j := 10^{100}$.

As you can assemble a Star Wars Death Star out of legos,
so, too, you can build a good approximation to D out of $\mathcal{S}_j$-legos.

THEOREM 105.4. *We have:* $\forall j \in \mathbb{N}_0$,
$\mathcal{I}_j$ *is a partition of* $\mathbb{R}$ *and* $\mathcal{S}_j$ *is a partition of* $\mathbb{R}^2$.

DEFINITION 105.5. *Let* $\mathcal{P}$ *be a partition.*
Then $\widehat{\mathcal{P}} := \{ \bigcup \mathcal{A} \mid \mathcal{A} \subseteq \mathcal{P} \}$.

Note: $\varnothing, \bigcup \mathcal{P} \in \widehat{\mathcal{P}}$ and $\mathcal{P} \subseteq \widehat{\mathcal{P}}$.

We often refer to elements of $\widehat{\mathcal{P}}$ as "assemblies" or "$\mathcal{P}$-assemblies".

Let $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2})$ and let $j := 10^{100}$.
As you can assemble a Star Wars Death Star out of legos,
so, too, you can find sets in $\widehat{\mathcal{S}}_j$ that approximate D.

DEFINITION 105.6. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be partitions.*
Then $\mathcal{P} << \mathcal{Q}$ *means: both* $\mathcal{P} \subseteq \widehat{\mathcal{Q}}$ *and* $\bigcup \mathcal{P} = \bigcup \mathcal{Q}$.

THEOREM 105.7. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be partitions.*
Assume $\mathcal{P} << \mathcal{Q}$. *Then* $\widehat{\mathcal{P}} \subseteq \widehat{\mathcal{Q}}$.

THEOREM 105.8. *We have:*
both $\mathcal{I}_0 << \mathcal{I}_1 << \mathcal{I}_2 << \cdots$ *and* $\mathcal{S}_0 << \mathcal{S}_1 << \mathcal{S}_2 << \cdots$.

106. Partition measures

DEFINITION 106.1. *Let* μ *be a function.*
Then μ *is a* **partition measure** *means:*
both $\mathrm{dom}[\mu]$ *is a partition and* $\mathrm{im}[\mu] \subseteq [0; \infty]$.

DEFINITION 106.2. *Let* μ *be a partition measure.*
Then we define: $\mathcal{P}_\mu := \mathrm{dom}[\mu]$ *and* $X_\mu := \bigcup \mathcal{P}_\mu$.

DEFINITION 106.3. $\forall j \in \mathbb{N}_0$, *define* $\alpha^j : \mathcal{S}_j \to [0; \infty]$ *by* $\alpha^j_S = 4^{-j}$.

THEOREM 106.4. $\forall j \in \mathbb{N}_0$, *we have:*
α^j *is a partition measure and* $\mathcal{P}_{\alpha^j} = \mathcal{S}_j$ *and* $X_{\alpha^j} = \mathbb{R}^2$.

DEFINITION 106.5. *Let* X *be a set and let* μ *be a function.*
Then μ *is a* **partition measure on** X *means:*
both μ *is a partition measure and* $X_\mu = X$.

THEOREM 106.6. $\forall j \in \mathbb{N}_0$, α^j *is a partition measure on* $\mathbb{R}^2$.

DEFINITION 106.7. *Let μ be a partition measure.*
Then we define $\widehat{\mu} : \widehat{\mathcal{P}_\mu} \to [0;\infty]$ by: $\quad \widehat{\mu}_B \;=\; \displaystyle\sum_{P \in \mathcal{P}_\mu \cap 2^B} \mu_P$.

We call $\widehat{\mu}$ the **assembly measure** of μ.

Note: $\varnothing, X_\mu \in \mathrm{dom}[\widehat{\mu}]$ and $\mu \subseteq \widehat{\mu}$.

THEOREM 106.8. *Let μ be a partition measure and let $S, T \in \widehat{\mathcal{P}_\mu}$.*
Then $\widehat{\mu}_{S \cup T} \leqslant \widehat{\mu}_S + \widehat{\mu}_T$.

DEFINITION 106.9. *Let μ and ν be partition measures.*
Then $\mu << \nu$ means: both $\mu \subseteq \widehat{\nu}$ and $X_\mu = X_\nu$.

THEOREM 106.10. *Let μ and ν be partition measures.*
Assume $\mu << \nu$. Then $\widehat{\mu} \subseteq \widehat{\nu}$.

THEOREM 106.11. *We have: $\alpha^0 << \alpha^1 << \alpha^2 << \cdots$.*

107. Approximating sets

DEFINITION 107.1. *Let $\mathcal{P}$ be a partition, $B \subseteq \bigcup \mathcal{P}$.*
Then $B^-_{\mathcal{P}} := \bigcup\{P \in \mathcal{P} \,|\, P \subseteq B\}$ and $B^+_{\mathcal{P}} := \bigcup\{P \in \mathcal{P} \,|\, P \cap B \neq \varnothing\}$.

We call $B^-_{\mathcal{P}}$ the **inner assembly** or **inner $\mathcal{P}$-assembly** of B.
We call $B^+_{\mathcal{P}}$ the **outer assembly** or **outer $\mathcal{P}$-assembly** of B.

Let $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2})$.
We drew a picture of D.
We added each of the four squares in $\{S \in \mathcal{S}_0 \,|\, S \subseteq D\}$.
Note that $D^-_{\mathcal{S}_0}$ is the union of those four squares.
Adding five more squares, we displayed
all of the 13 squares in $\{S \in \mathcal{S}_0 \,|\, S \cap D \neq \varnothing\}$.
Note that $D^+_{\mathcal{S}_0}$ is the union of those 13 squares. squares.

Let $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2})$, $A := D \cap \mathbb{Q}^2$.
We discussed how to picture A.
We noted that $\{S \in \mathcal{S}_0 \,|\, S \subseteq A\} \;=\; \varnothing$.
Therefore, $A^-_{\mathcal{S}_0} \;=\; \varnothing$.
We noted that $\{S \in \mathcal{S}_0 \,|\, S \cap A \neq \varnothing\} \;=\; \{S \in \mathcal{S}_0 \,|\, S \cap D \neq \varnothing\}$.
Those nine squares from before are $\{S \in \mathcal{S}_0 \,|\, S \cap A \neq \varnothing\}$.
Note that $A^+_{\mathcal{S}_0}$ is the union of those nine squares.

Let $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2}), \quad A := D \cap \mathbb{Q}^2, \quad j = 10^{100}$.
We discussed why
$D^-_{\mathcal{S}_j}$ and $D^+_{\mathcal{S}_j}$ are both indistinguishable from D on the board.
We discussed why
$A^-_{\mathcal{S}_j} = \varnothing$, while $A^+_{\mathcal{S}_j}$ is indistinguishable from D on the board.

THEOREM 107.2. *Let $\mathcal{P}$ be a partition, $B \subseteq \bigcup \mathcal{P}$.*
Then: $B^-_{\mathcal{P}} \subseteq B \subseteq B^+_{\mathcal{P}}$.

THEOREM 107.3. *Let X be a set, $B \subseteq X$.*
Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of X. *Assume:* $\mathcal{P} << \mathcal{Q}$.
Then: $B^-_{\mathcal{P}} \subseteq B^-_{\mathcal{Q}} \subseteq B \subseteq B^+_{\mathcal{Q}} \subseteq B^+_{\mathcal{P}}$.

108. Approximating measures of sets

DEFINITION 108.1. *Let μ be a partition measure.*
Then $\mu^-, \mu^+ : 2^{X_\mu} \to [0; \infty]$ are defined by:
$$\mu^-(B) = \widehat{\mu}(B^-_{\mathcal{P}_\mu}) \qquad \text{and} \qquad \mu^+(B) = \widehat{\mu}(B^+_{\mathcal{P}_\mu}).$$

We call μ^- the **inner measure** of μ.
We call μ^+ the **outer measure** of μ.

THEOREM 108.2. *Let $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2})$.*
Then $(\alpha^0)^-(D) = 4$ and $(\alpha^0)^+(D) = 13$.

It has been known since antiquity that the area of D is 2π.
Note that $(\alpha^0)^-(D) = 4 < 2\pi < 9 = (\alpha^0)^+(D)$.
If we approximate D by $\mathcal{S}_0$-assemblies (from the inside and outside),
we don't get such a good approximation to the correct area.

THEOREM 108.3. *Let* $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2}), \quad j := 10^{100}$.
Then: $2\pi - 0.1 < (\alpha^j)^-(D) < 2\pi < (\alpha^j)^+(D) < 2\pi + 0.1$.

If we approximate D by $\mathcal{S}_j$-assemblies (from the inside and outside),
we get a good approximation to the correct area.

THEOREM 108.4. *Let μ be a partition measure and let $S, T \subseteq X_\mu$.*
Then both $\mu^-_{S \cup T} \geqslant \mu^-_S + \mu^-_T$ and $\mu^+_{S \cup T} \leqslant \mu^+_S + \mu^+_T$.

THEOREM 108.5. *Let μ be a partition measure.*
Then: $\mu^- = \widehat{\mu} = \mu^+$ *on* $\widehat{\mathcal{P}_\mu}$
and $\mu^- \leqslant \mu^+$ *on* 2^X.

THEOREM 108.6. *Let X be a set.*
Let μ and ν be a partition measures on X.
Then: $\mu^- \leqslant \nu^- \leqslant \nu^+ \leqslant \mu^+$ *on* 2^X.

THEOREM 108.7.
$$(\alpha^0)^- \leqslant (\alpha^1)^- \leqslant (\alpha^2)^- \leqslant \cdots \leqslant \cdots \leqslant (\alpha^2)^+ \leqslant (\alpha^1)^+ \leqslant (\alpha^0)^+$$
on $2^{\mathbb{R}^2}$.

109. The definition of area

DEFINITION 109.1. *Define* $\alpha^+, \alpha^- : 2^{\mathbb{R}^2} \to [0;\infty]$ *by*
$$\alpha^-(B) = \sup\{(\alpha^j)^-(B) \,|\, j \in \mathbb{N}_0\}$$
and
$$\alpha^+(B) = \inf\{(\alpha^j)^+(B) \,|\, j \in \mathbb{N}_0\}.$$

We call $\alpha^-(B)$ the **inner area** of B.
We call $\alpha^+(B)$ the **outer area** of B.

THEOREM 109.2. *Let $S, T \in \mathcal{M}$.*
Then both $\alpha^-_{S\cup T} \geqslant \alpha^-_S + \alpha^-_T$ *and* $\alpha^+_{S\cup T} \leqslant \alpha^+_S + \alpha^+_T$.

THEOREM 109.3. *Define* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $F(x, y) = (y, x)$. *Let* $S \in \mathcal{M}$.
Then both $\alpha^-(F_*(S)) = \alpha^-(S)$ *and* $\alpha^+(F_*(S)) = \alpha^+(S)$.

THEOREM 109.4. *Let* $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2})$.
Then $\alpha^-(D) = 2\pi = \alpha^+(D)$.
Also, $\alpha^-(D \cap \mathbb{Q}^2) = 0 \neq 2\pi = \alpha^+(D \cap \mathbb{Q}^2)$.

DEFINITION 109.5. *We define:*
$$\mathcal{M} \quad := \quad \{\ B \subseteq \mathbb{R}^2 \ |\ (\ B \text{ is bounded in } \mathbb{R}^2\ )\ \&\ (\ \alpha^-_B = \alpha^+_B\ )\ \}.$$

That is, $\mathcal{M}$ is the collection of
bounded subsets of $\mathbb{R}^2$ whose inner and outer areas agree.
A subset B of $\mathbb{R}^2$ is **Jordan measurable** if $B \in \mathcal{M}$.

Let $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2})$. Then $D \in \mathcal{M}$ and $D \cap \mathbb{Q}^2 \notin \mathcal{M}$.

THEOREM 109.6. *Let $S, T \in \mathcal{M}$. Then $S \cup T \in \mathcal{M}$.*

THEOREM 109.7. *Define* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $F(x, y) = (y, x)$.
Let $S \in \mathcal{M}$. *Then* $F_*(S) \in \mathcal{M}$.

Note that $\alpha^-|\mathcal{M} = \alpha^+|\mathcal{M}$.

DEFINITION 109.8. $\alpha := \alpha^-|\mathcal{M}$.

Note: $\forall B \subseteq \mathbb{R}^2$, $(\,(\,B$ is bounded in $\mathbb{R}^2\,) \Rightarrow (\,\alpha_B^- \leqslant \alpha_B^+ < \infty\,)\,)$.
Also, $\forall B \in \mathcal{M}$, B is bounded in $\mathbb{R}^2$.
It follows that $\mathrm{im}[\alpha] \subseteq [0;\infty)$. Then $\alpha : \mathcal{M} \to [0;\infty)$.

The function $\alpha : \mathcal{M} \to [0;\infty)$ is called **Jordan area**.

Note: $\forall B \subseteq \mathbb{R}^2$, $(\,B$ is Jordan measurable$\,) \Leftrightarrow (\,\alpha_B \neq ☹\,)$.

THEOREM 109.9. *Let $S, T \in \mathcal{M}$. Then $\alpha_{S\cup T} \leqslant \alpha_S + \alpha_T$.*

THEOREM 109.10. *Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x,y) = (y,x)$. Let $S \in \mathcal{M}$. Then $\alpha(F_*(S)) = \alpha(S)$.*

110. Jordan/Riemann Integration

In this section, we develop an integration theory that is often called **Riemann integration**. It is closely tied to Jordan area, and so might also be called **Jordan integration**.

DEFINITION 110.1. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
We define $G_f := \{(x,y) \in \mathbb{R}^2 \,|\, y = f(x)\}$,
and $O_f := \{(x,y) \in \mathbb{R}^2 \,|\, 0 < y < f(x)\}$.

In Definition 110.1, G_f is called the **graph** of f,
and O_f is called the **ordinate set** of f.

DEFINITION 110.2. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
We define $\displaystyle\int_{\mathbb{R}} f \quad := \quad (\alpha(\mathcal{O}_f)) - (\alpha(\mathcal{O}_{-f}))$.

Note that $\alpha(\mathcal{O}_f)$ is the Jordan area of the region under f.
Note that $\alpha(\mathcal{O}_{-f})$ is equal to the Jordan area of the region over f.
Thus, the integral of f is the area under f minus the area over f.

DEFINITION 110.3. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Then f is **Jordan integrable** *means:* $\displaystyle\int_{\mathbb{R}} f \neq ☹$.

THEOREM 110.4. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Then: $(\,f$ is Jordan integrable$\,) \Leftrightarrow (\,\mathcal{O}_f, \mathcal{O}_{-f} \in \mathcal{M}\,)$.

That is, f is Jordan integrable iff the regions under and over f are both Jordan measurable.

Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(x) = 1/(1 + x^2)$.
Then $\mathcal{O}_\phi$ is not bounded in $\mathbb{R}^2$, so $\mathcal{O}_\phi \notin \mathcal{M}$.
Then ϕ is not Jordan integrable.

Also, the characteristic function $\chi_{[0;1]\cap\mathbb{Q}}^{[0;1]}$ is not Jordan integrable.

DEFINITION 110.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $a, b \in \mathbb{R}$.
Assume that $[a|b] \subseteq \mathrm{dom}[f]$. *Let* $I := \int_{\mathbb{R}} (f|[a|b])$.

Then we define: $$\int_a^b f \;:=\; \begin{cases} I, & \text{if } a < b \\ 0, & \text{if } a = b \\ -I, & \text{if } a > b. \end{cases}$$

Also, we define: $$⨍_a^b f \;:=\; \left(\frac{1}{b-a}\right) \cdot \left(\int_a^b f\right).$$

Note: $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall a \in \mathrm{dom}[f]$, both $\int_a^a f = 0$ and $⨍_a^a f = ☹$.

111. Continuity on a compact implies uniform continuity

THEOREM 111.1. *Let* X *be a compact metric space. Let* $p, q \in X^{\mathbb{N}}$.
Then $\exists$*strictly increasing* $m \in \mathbb{N}^{\mathbb{N}}$ *s.t.*
$p \circ m$ *and* $q \circ m$ *are both convergent in* X.

Proof. Since X is compact, p is subconvergent in X.
Choose a strictly increasing $k \in \mathbb{N}^{\mathbb{N}}$ s.t. $p \circ k$ is convergent in X.
Since X is compact, $q \circ k$ is subconvergent in X.
Choose a strictly increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $q \circ k \circ \ell$ is convergent in X.
Let $m := k \circ \ell$. Then $q \circ m$ is convergent in X.
Want: $p \circ m$ is convergent in X.
Since $p \circ m = p \circ k \circ \ell$, it follows that $p \circ m$ is a subsequence of $p \circ k$.
Since $p \circ k$ is convergent in X and since $p \circ m$ is a subsequence of X,
we conclude that $p \circ m$ is convergent in X, as desired. □

THEOREM 111.2. *Let* X *and* Y *be metric spaces.*
Let $f : X \to Y$ *be continuous and let* $\varepsilon > 0$.
Assume that X *is compact.*

Then: $\exists \delta > 0$ *s.t.,* $\forall p, q \in X$,
$$(\, d_X(p,q) < \delta \,) \;\Rightarrow\; (\, d_Y(f_p, f_q) < \varepsilon \,).$$

Proof. Assume: $\forall \delta > 0$, $\exists p, q \in X$ s.t.
$(\, d_X(p,q) < \delta \,) \;\&\; (\, d_Y(f_p, f_q) \geqslant \varepsilon \,)$.
Want: Contradiction.
By the Axiom of Choice, choose $p, q \in X^{\mathbb{N}}$ s.t., $\forall j \in \mathbb{N}$,
$(\, d_X(p_j, q_j) < 1/j \,) \;\&\; (\, d_Y(f_{p_j}, f_{q_j}) \geqslant \varepsilon \,)$.
By Theorem 111.1,choose a strictly increasing $m \in \mathbb{N}^{\mathbb{N}}$ s.t.
$p \circ m$ and $q \circ m$ are both convergent in X.
Let $a := p \circ m$, $b := q \circ m$. Then a and b are both convergent in X.
Choose $\alpha, \beta \in X$ s.t. both $a_\bullet \to \alpha$ and $b_\bullet \to \beta$ in X.
Then $(d_X(a,b))_\bullet \to d_X(\alpha, \beta)$ in $\mathbb{R}$.
By our choice of p and q, we have: $\forall j \in \mathbb{N}$, $d_X(p_j, q_j) \leqslant 1/j$.
Then: $\forall j \in \mathbb{N}$, $0 \leqslant (d_X(p,q))_j \leqslant 1/j$.
Then, by the Squeeze Theorem, $(d_X(p,q))_\bullet \to 0$ in $\mathbb{R}$.
Also, $d_X(a,b) = (d_X(p,q)) \circ m$, so $d_X(a,b)$ is a subsequence of $d_X(p,q)$.
Then $(d_X(a,b))_\bullet \to 0$ in $\mathbb{R}$.
So, since $(d_X(a,b))_\bullet \to d_X(\alpha,\beta)$ in $\mathbb{R}$,
we conclude that $0 = d_X(\alpha, \beta)$. Then $\alpha = \beta$.
Since f is continuous, since $a_\bullet \to \alpha$ in X and since $b_\bullet \to \beta \in X$,
it follows that both $(f \circ a)_\bullet \to f_\alpha$ and $(f \circ b)_\bullet \to f_\beta$ in Y.
Then $(d_Y(f \circ a, f \circ b))_\bullet \to d_Y(f_\alpha, f_\beta)$ in $\mathbb{R}$.
Since $\alpha = \beta$, it follows that $d_Y(f_\alpha, f_\beta) = 0$.
Then $(d_Y(f \circ a, f \circ b))_\bullet \to 0$ in $\mathbb{R}$.
Choose $K \in \mathbb{N}$ s.t., $\forall i \in \mathbb{N}$,
$(\, i \geqslant K \,) \;\Rightarrow\; (\, (d_Y(f \circ a, f \circ b))_i < \varepsilon \,)$.
Then we have: $(d_Y(f \circ a, f \circ b))_K < \varepsilon$.
Let $j := m_K$. Then $p_j = (p \circ m)_K = a_K$ and $q_j = (q \circ m)_K = b_K$.
By our choice of p and q, we have $d_Y(f_{p_j}, f_{q_j}) \geqslant \varepsilon$, so $\varepsilon \leqslant d_Y(f_{p_j}, f_{q_j})$.
Then $\varepsilon \leqslant d_Y(f_{p_j}, f_{q_j}) = d_Y(f_{a_K}, f_{b_K}) = (d_Y(f \circ a, f \circ b))_K < \varepsilon$.
Then $\varepsilon < \varepsilon$. Contradiction. ☐

THEOREM 111.3. *Let* $a, b \in \mathbb{R}$. *Assume* $a < b$.
Let $f : [a; b] \to \mathbb{R}$ *be continuous.* *Then* $\alpha(G_f) = 0$.

Proof. Unassigned HW. ☐

THEOREM 111.4. *Let* $C := S_{\mathbb{R}^2}(0_2, \sqrt{2})$. *Then* $\alpha(C) = 0$.

Proof. Since $\alpha(C) \geqslant 0$, it suffices to show: $\alpha(C) \leqslant 0$.
Define $f : [-\sqrt{2}, \sqrt{2}] \to \mathbb{R}$ by $f(x) = \sqrt{2 - x^2}$.
Since $C = G_f \cup G_{-f}$, we get: $\alpha(C) \leqslant (\alpha(G_f)) + (\alpha(G_{-f}))$.
By Theorem 111.3, we have both $\alpha(G_f) = 0$ and $\alpha(G_{-f}) = 0$.
Then $\alpha(C) \leqslant (\alpha(G_f)) + (\alpha(G_{-f})) = 0 + 0 = 0$, as desired. □

DEFINITION 111.5. *Let* $p, q \in \mathbb{R}^2$.
Then: $[p|q] \quad := \quad \{\ (1-t)p + tq \mid t \in [0;1]\ \}$.

The set $[p|q]$ is called the **closed line segment** from p to q.

THEOREM 111.6. *Let* $p, q \in \mathbb{R}^2$. *Then* $\alpha([p|q]) = 0$.

Proof. We have $p = (p_1, p_2)$ and $q = (q_1, q_2)$
Exactly one of the following is true:
(1) $p_1 = q_1$ or (2) $p_1 \neq q_1$.

Case (1):
Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x, y) = (y, x)$.
Define $h : [p_2|q_2] \to \mathbb{R}$ by $h(t) = p_1$. Then $[p|q] = F_*(G_h)$.
By Theorem 111.3, $\alpha(G_h) = 0$.
By Theorem 109.10, $\alpha(F_*(G_h)) = \alpha(G_h)$.
Then $\alpha([p|q]) = \alpha(F_*(G_f)) = \alpha(G_f) = 0$, as desired.

End of Case (1).

Case (2):
Define $f : [p_1|q_1] \to \mathbb{R}$ by $f(t) = (1-t)p_2 + tq_2$. Then $[p|q] = G_f$.
By Theorem 111.3, $\alpha(G_f) = 0$.
Then $\alpha([p|q]) = \alpha(G_f) = 0$, as desired.
End of Case (2). □

112. Bounding sets and Jordan measurability

DEFINITION 112.1. *Let* $\mathcal{P}$ *be a partition and let* $B, V \subseteq \bigcup \mathcal{P}$.
Then B $\mathcal{P}$-**bounds** V *means:* $\forall P \in \mathcal{P}$,
$(\ P \cap V \neq \varnothing \neq P \backslash V\ ) \Rightarrow (\ P \cap B \neq \varnothing\ )$.

THEOREM 112.2. *Let* $C := S_{\mathbb{R}^2}(0_2, \sqrt{2})$, $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2})$, $j \in \mathbb{N}_0$.
Then C $\mathcal{S}_j$*-bounds* D.

Proof. Unassigned HW. □

THEOREM 112.3. *Let $\mathcal{P}$ be a partition,* $B, V \subseteq \bigcup \mathcal{P}$.
Assume: B $\mathcal{P}$-bounds V. *Then:* $V_{\mathcal{P}}^{+} \backslash V_{\mathcal{P}}^{-} \subseteq B_{\mathcal{P}}^{+}$.

Proof. Unassigned HW. □

THEOREM 112.4. *Let μ be a partition measure,* $B, V \subseteq X_{\mu}$.
Assume: B $\mathcal{P}_{\mu}$-bounds V. *Then:* $\mu_V^+ - \mu_V^- \leqslant \mu_B^+$.

Proof. Unassigned HW. □

THEOREM 112.5. *Let $B, V \subseteq \mathbb{R}^2$.* *Assume:* $\alpha_B = 0$.
Assume: $\forall j \in \mathbb{N}_0$, B $\mathcal{S}_j$-bounds V. *Then: $V \in \mathcal{M}$.*

Proof. Want: $\alpha_V^- = \alpha_V^+$.
Since $\alpha_V^- \leqslant \alpha_V^+$, it suffices to show: $\alpha_V^+ \leqslant \alpha_V^-$.
Want: $\alpha_V^+ - \alpha_V^- \leqslant 0$. Want: $\forall \varepsilon > 0$, $\alpha_V^+ - \alpha_V^- < \varepsilon$.
Given $\varepsilon > 0$. Want: $\alpha_V^+ - \alpha_V^- < \varepsilon$.
Let $S := \{(\alpha^j)_B^+ \,|\, j \in \mathbb{N}_0\}$. Then $\alpha_B^+ := \inf S$.
By assumption, we have: $\alpha_B^+ = 0$.
As $\inf S = \alpha_B^+ = 0 < \varepsilon$, we get $\neg(\varepsilon \leqslant S)$. Choose $t \in S$ s.t. $t < \varepsilon$.
Since $t \in S = \{(\alpha^j)_B^+ \,|\, j \in \mathbb{N}_0\}$, choose $k \in \mathbb{N}_0$ s.t. $t = (\alpha^k)_B^+$.
We have $\alpha_V^+ = \inf\{(\alpha^j)_V^+ \,|\, j \in \mathbb{N}_0\}$, so $\alpha_V^+ \leqslant \{(\alpha^j)_V^+ \,|\, j \in \mathbb{N}_0\}$,
and so $\alpha_V^+ \leqslant (\alpha^k)_V^+$.
Also, $\alpha_V^- = \sup\{(\alpha^j)_V^- \,|\, j \in \mathbb{N}_0\}$, so $\alpha_V^- \geqslant \{(\alpha^j)_V^- \,|\, j \in \mathbb{N}_0\}$,
and so $\alpha_V^- \geqslant (\alpha^k)_V^-$.
Since $\alpha_V^+ \leqslant (\alpha^k)_V^+$ and $\alpha_V^- \geqslant (\alpha^k)_V^-$,
we get: $\alpha_V^+ - \alpha_V^- \leqslant (\alpha^k)_V^+ - (\alpha^k)_V^-$.
By Theorem 112.4, we have: $(\alpha^k)_V^+ - (\alpha^k)_V^- \leqslant (\alpha^k)_B^+$.
Then $\alpha_V^+ - \alpha_V^- \leqslant (\alpha^k)_V^+ - (\alpha^k)_V^- \leqslant (\alpha^k)_B^+ = t < \varepsilon$, as desired. □

THEOREM 112.6. *Let $D := \overline{B}_{\mathbb{R}^2}(0_2, \sqrt{2})$.* *Then $D \in \mathcal{M}$.*

Proof. Let $C := S_{\mathbb{R}^2}(0_2, \sqrt{2})$. By Theorem 111.4, $\alpha_C = 0$.
By Theorem 112.2, we have: $\forall j \in \mathbb{N}_0$, C $\mathcal{S}_j$-bounds D.
Then, by Theorem 112.5, $D \in \mathcal{M}$, as desired. □

THEOREM 112.7. *Let $a, b \in \mathbb{R}$.* *Assume $a \leqslant b$.*
Let $f : [a; b] \to [0; \infty)$ be continuous. *Then $\mathcal{O}_f \in \mathcal{M}$.*

Proof. Let $V := \mathcal{O}_f$. Want: $V \in \mathcal{M}$.
Let $p := (a, 0)$, $q := (a, f_a)$, $r := (b, 0)$, $s := (b, f_b)$.
Let $C := [p|q]$, $D := [r|s]$, $E := [p|r]$.
Let $B := C \cup D \cup E \cup G_f$.

Then $\alpha(B) \leqslant (\alpha(C)) + (\alpha(D)) + (\alpha(E)) + (\alpha(G_f))$.
Unassigned HW: Show, $\forall j \in \mathbb{N}_0$, B $\mathcal{S}_j$-bounds V.
By Theorem 111.6, we have: $\alpha(C) = \alpha(D) = \alpha(E) = 0$.
By Theorem 111.3, we have: $\alpha(G_f) = 0$.
Then $\alpha(B) \leqslant (\alpha(C)) + (\alpha(D)) + (\alpha(E)) + (\alpha(G_f)) = 0+0+0+0 = 0$.
So, since $\alpha(B) \geqslant 0$, we get: $\alpha(B) = 0$.
Then, by Theorem 112.5, we have: $V \in \mathcal{M}$, as desired. □

THEOREM 112.8. *Let $a, b \in \mathbb{R}$.*
Let $f : [a|b] \to \mathbb{R}$ be continuous. Then $\mathcal{O}_f \in \mathcal{M}$.

Proof. Let $\alpha := \min\{a, b\}$ and $\beta := \max\{a, b\}$.
Then $\alpha \leqslant \beta$ and $f : [\alpha; \beta] \to \mathbb{R}$.
Define $g := (f + |f|)/2$. Then g is continuous.
Also, $\forall x \in [\alpha; \beta]$, we have $g_x = (f_x + |f_x|)/2 = \max\{f_x, 0\}$.
Then $\mathcal{O}_f = \mathcal{O}_g$. Also, $g : [\alpha; \beta] \to [0; \infty)$.
By Theorem 112.7, we have: $\mathcal{O}_g \in \mathcal{M}$.
Then $\mathcal{O}_f = \mathcal{O}_g \in \mathcal{M}$, as desired. □

113. Basic properties of integration

THEOREM 113.1. *Let $f : \mathbb{R} \dashrightarrow [0; \infty]$, $a \geqslant 0$.*
Then $\displaystyle\int_{\mathbb{R}} (a \cdot f) =^* a \cdot \left(\int_{\mathbb{R}} f\right)$.

THEOREM 113.2. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$.*
Then $\displaystyle\int_{\mathbb{R}} (a \cdot f) =^* a \cdot \left(\int_{\mathbb{R}} f\right)$.

THEOREM 113.3. *Let $D \subseteq \mathbb{R}$, $f, g : D \to [0; \infty]$.*
Then $\displaystyle\int_{\mathbb{R}} (f + g) =^* \left(\int_{\mathbb{R}} f\right) + \left(\int_{\mathbb{R}} g\right)$.

THEOREM 113.4. *Let $D \subseteq \mathbb{R}$, $f, g : D \to \mathbb{R}$.*
Then $\displaystyle\int_{\mathbb{R}} (f + g) =^* \left(\int_{\mathbb{R}} f\right) + \left(\int_{\mathbb{R}} g\right)$.

THEOREM 113.5. *Let $D \subseteq \mathbb{R}$. Let $f, g : D \to \mathbb{R}$.*
Assume that f and g are both Jordan integrable.
Assume that $f \leqslant g$ on D. Then $\displaystyle\int_{\mathbb{R}} f \leqslant \int_{\mathbb{R}} g$.

THEOREM 113.6. *Let $a, b, u \in \mathbb{R}$. Assume $a < b$. Let $D := [a, b]$.*
Then $\displaystyle\int_{\mathbb{R}} C_D^u = u \cdot (b - a)$.

114. The Fundamental Theorem of Calculus

THEOREM 114.1. *Let $a, b \in \mathbb{R}$. Assume $a \neq b$. Let $D := [a|b]$. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume that f is continuous on D.*

Then: $$\min(f_*(D)) \leqslant ⨍_a^b f \leqslant \max(f_*(D)).$$

Proof. Let $\alpha := \min\{a, b\}$ and let $\beta := \max\{a, b\}$.

Then: $\alpha < \beta$ and $D = [\alpha; \beta]$ and $⨍_a^b f = ⨍_\alpha^\beta f$.

Let $y := \min(f_*(D))$ and let $z := \max(f_*(D))$.

Since f is continuous on D, it follows that $D \subseteq \mathrm{dom}[f]$.

Then $y \leqslant f \leqslant z$ on D. Let $\phi := f|D$. Then $C_D^y \leqslant \phi \leqslant C_D^z$.

Since $\phi = f|D = f|[\alpha; \beta]$, we get: $\int_\alpha^\beta f = \int_{\mathbb{R}} \phi$.

Want: $$y \leqslant \frac{1}{\beta - \alpha} \int_\alpha^\beta f \leqslant z.$$

Want: $$y \leqslant \frac{1}{\beta - \alpha} \int_{\mathbb{R}} \phi \leqslant z.$$

Want: $$y \cdot (\beta - \alpha) \leqslant \int_{\mathbb{R}} \phi \leqslant z \cdot (\beta - \alpha).$$

By Theorem 113.6, we get:

both $\int_{\mathbb{R}} C_D^y = y \cdot (\beta - \alpha)$ and $\int_{\mathbb{R}} C_D^z = z \cdot (\beta - \alpha)$.

Since $C_D^y \leqslant \phi \leqslant C_D^z$, by Theorem 113.5,

we see that $\int_{\mathbb{R}} C_D^y \leqslant \int_{\mathbb{R}} \phi \leqslant \int_{\mathbb{R}} C_D^z$.

Then: $y \cdot (\beta - \alpha) \leqslant \int_{\mathbb{R}} \phi \leqslant z \cdot (\beta - \alpha)$, as desired. □

The following theorem is the **Fundamental Theorem of Calculus**:

THEOREM 114.2. *Let $a, b \in \mathbb{R}$. Assume that $a < b$.*

Let $f : [a; b] \to \mathbb{R}$ be continuous.

Define $g : [a; b] \to \mathbb{R}$ by: $g(x) = \int_a^x f$.

Let $p \in (a; b)$. Then $g'_p = f_p$.

Proof. Let $m := f_p$. Want: $g'_p = m$.

Define $L \in \mathcal{L}_{\mathbb{R}}^{\mathbb{R}}$ by $L_h = mh$. Then $\mathrm{s}\ell_L^{\mathbb{R}\mathbb{R}} = m$.

Want: $g'_p = \mathrm{s}\ell_L^{\mathbb{R}\mathbb{R}}$. Want: $D_p g = L$. Want $g_p^T - L \in \mathcal{O}_1^{\mathbb{R}\mathbb{R}}$.

Let $\alpha := \mathrm{adj}_0^0 \left(\frac{g_p^T - L}{\mathrm{id}_{\mathbb{R}}} \right)$. Then $g_p^T - L = \alpha \cdot \mathrm{id}_{\mathbb{R}}$.

Since $\mathrm{id}_{\mathbb{R}} \in \widehat{\mathcal{O}}_1^{\mathbb{R}\mathbb{R}}$, it suffices to prove: $\alpha \in \mathcal{O}_0^{\mathbb{R}\mathbb{R}}$.
Since $\mathcal{O}_0^{\mathbb{R}\mathbb{R}} = \mathrm{CVZ}_{\mathbb{R}}^{\mathbb{R}}$, it suffices to prove: $\alpha \in \mathrm{CVZ}_{\mathbb{R}}^{\mathbb{R}}$.
We have $\alpha \in \mathrm{DNZ}_{\mathbb{R}}^{\mathbb{R}}$ and $\alpha_0 = 0$.
It remains to show: α is continuous at 0.
Want: $\forall \varepsilon > 0$, $\exists U \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\alpha_*(U) \subseteq (-\varepsilon; \varepsilon)$.
Given $\varepsilon > 0$. Want: $\exists U \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\alpha_*(U) \subseteq (-\varepsilon; \varepsilon)$.
Since f is continuous, f is continuous at p. Recall: $f_p = m$.
Since $p \in (a; b)$, choose $V \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $p + V \subseteq (a; b)$.
Choose $U \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $U \subseteq V$ and $f_*(p + U) \subseteq (m - \varepsilon; m + \varepsilon)$.
Want: $\alpha_*(U) \subseteq (-\varepsilon; \varepsilon)$. We have $\alpha_0 = 0 \in (-\varepsilon; \varepsilon)$.
Want: $\alpha_*(U_0^\times) \subseteq (-\varepsilon; \varepsilon)$. Want: $\forall y \in \alpha_*(U_0^\times)$, $y \in (-\varepsilon, \varepsilon)$.
Given $y \in \alpha_*(U_0^\times)$. Want: $y \in (-\varepsilon, \varepsilon)$. Want: $-\varepsilon < y < \varepsilon$.
Since $y \in \alpha_*(U_0^\times)$, choose $h \in U_0^\times \cap (\mathrm{dom}[\alpha])$ s.t. $y = \alpha(h)$.
Since $U \in \mathcal{B}_{\mathbb{R}}(0)$, it follows that U is an interval, so $p+U$ is an interval.
Since $0, h \in U$, we get $p, p + h \in p + U$.
So, since $p + U$ is an interval, we get $[p|p + h] \subseteq p + U$.
We have $U \subseteq V$, so $p + U \subseteq p + V$.
Then $p + U \subseteq p + V \subseteq (a; b) \subseteq [a; b] = \mathrm{dom}[f]$.
So, since $f_*(p + U) \subseteq (m - \varepsilon; m + \varepsilon)$,
we conclude that: $m - \varepsilon < f < m + \varepsilon$ on $p + U$.
Then: $-\varepsilon < f - C_{\mathbb{R}}^m < \varepsilon$ on $p + U$.
So, since $[p|p + h] \subseteq p + U$, we get: $-\varepsilon < f - C_{\mathbb{R}}^m < \varepsilon$ on $[p|p + h]$.

Then, by Theorem 114.1, $$-\varepsilon \ < \ ⨍_p^{p+h} (f - C_{\mathbb{R}}^m) \ < \ \varepsilon.$$

It therefore suffices to show: $$y = ⨍_p^{p+h} (f - C_{\mathbb{R}}^m).$$

We have: both $$g_p^T(h) = \int_p^{p+h} f \quad \text{and} \quad L(h) = \int_p^{p+h} C_{\mathbb{R}}^m.$$

Then: $$(g_p^T - L)(h) = \int_p^{p+h} (f - C_{\mathbb{R}}^m).$$

Since $h \in U_0^\times \subseteq \mathbb{R}_0^\times$ and $\alpha = \mathrm{adj}_0^0 \left(\dfrac{g_p^T - L}{\mathrm{id}_{\mathbb{R}}} \right)$,

we get: $$\alpha(h) = \frac{(g_p^T - L)(h)}{h}.$$

Then:
$$\begin{aligned} y \;&=\; \alpha(h) \;=\; \frac{(g_p^T - L)(h)}{h} \;=\frac{1}{h}\cdot((g_p^T - L)(h)) \\ &=\; \frac{1}{h}\cdot\int_p^{p+h}(f - C_{\mathbb{R}}^m) \;=\; \fint_p^{p+h}(f - C_{\mathbb{R}}^m). \end{aligned}$$
□

Index of terms

FOR OFFICE USE ONLY: Theorem 94.4