MATH 4604 Spring 2019, Final exam
Handout date: Monday 13 May 2019
Instructor: Scot Adams

PRINT YOUR NAME:

SOLUTIONS

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Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let X be a metric space. Then X is **proper** means ...

$\forall s \in X^{\mathbb{N}},$

$(s \text{ is bounded in } X) \Rightarrow (s \text{ is subconvergent in } X)$

B. (5 pts) Let X be a metric space. Then X is **complete** means ...

$\forall s \in X^{\mathbb{N}},$

$(s \text{ is Cauchy in } X) \Rightarrow (s \text{ is convergent in } X)$

C. (5 pts) Let $V, W \in$ TNSR and let $L \in \mathcal{L}_V^W$. Then $[L]_V^W \in W \otimes V$ is defined by: $\forall i \in \mathcal{I}_V$, $\forall j \in \mathcal{I}_W$, $([L]_V^W)_{j\|i} = \cdots$

$\pi_j^W (L(\varepsilon_i^V))$

D. (5 pts) Let $V, W \in \mathrm{TNSR}$, $f : V \dashrightarrow W$, $p \in V$.
Then $\mathrm{LINS}_p^{VW} f := \cdots$.

$$\{ L \in \mathcal{L}_V^W \mid f_p^T - L \in \mathcal{O}_1^{VW} \}$$

E. (5 pts) Let $V, W \in \mathrm{TNSR}$, $f : V \dashrightarrow W$, $p \in V$.
Then $D_p^{VW} f := \cdots$.

$$UE\,(LINS_p^{VW} f)$$

F. (5 pts) Let μ be a function.
Then μ is a **partition measure** means ...

$$(\mathrm{dom}[\mu] \text{ is a partition})$$
$$\& \, (\mathrm{im}[\mu] \subseteq [0; \infty])$$

II. True or false (no partial credit):

a. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Assume: f is one-to-one. Then: $\forall x \in \mathbb{R}$, $f'(x) \neq 0$.

False

$id_{\mathbb{R}}^3$

b. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}^2$. Assume: $\forall x \in \mathbb{R}$, $f'(x) \neq (0,0)$. Then f is one-to-one.

False

$(\cos, \sin)$

c. (5 pts) Let $f : [1,2] \cup (3;4] \hookrightarrow \mathbb{R}$. Assume that f is continuous. Then f^{-1} is continuous.

False

$$f(x) = \begin{cases} x+4, & \text{if } 1 \leq x \leq 2 \\ x+3, & \text{if } 3 < x \leq 4 \end{cases}$$

d. (5 pts) $\forall f, g \in \widehat{\mathcal{O}}_0, \quad g \circ f \in \widehat{\mathcal{O}}_0$.

False

$f = g = C^2_{(-1;1)}$

e. (5 pts) Let $\mathcal{P}$ be a partition and let $B \subseteq \bigcup \mathcal{P}$. Then $B_{\mathcal{P}}^- \subseteq B \subseteq B_{\mathcal{P}}^+$.

True

Th 107.2, p. 275

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PLEASE DO NOT WRITE BELOW THE LINE

I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$(\partial_1 f)(0_2) = 0 \quad \text{and} \quad (\partial_2 f)(0_2) = 0 \quad \text{and} \quad f'(0_2) = ☹.$$

$$S := (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$$

$$f := \chi_S^{\mathbb{R}^2}$$

2. (10 pts) Let $V, Z \in \text{TNSR}$, $K \in \mathcal{K}_V^Z$.
Show: $\exists C \geqslant 0$ s.t., $\forall p \in V$, $|K(p)|_Z \leqslant C \cdot |p|_V^3$.

Pf: Choose $T \in \mathcal{ST}_V^Z$ s.t. $K = T(\bullet, \bullet, \bullet)$

Choose $C \geq 0$ s.t., $\forall p, q, r \in V$,

$$|T(p,q,r)| \leq C \cdot |p|_V \cdot |q|_V \cdot |r|_V$$

Want: $\forall p \in V$ $\quad |K(p)|_Z \leq C \cdot |p|_V^3$

Given $p \in V$. Want: $|K(p)|_Z \leq C \cdot |p|_V^3$

$$\begin{aligned} |K(p)|_Z &= |(T(\bullet,\bullet,\bullet))(p)|_Z \\ &= |T(p,p,p)|_Z \\ &\leq C \cdot |p|_V \cdot |p|_V \cdot |p|_V \\ &= C \cdot |p|_V^3 \end{aligned}$$

$\blacksquare$

3. (10 pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$, $u, v \in \mathbb{R}^2$.
Assume: $\forall p \in \mathbb{R}^2$, $(\partial_u f)(p) = 0$. Show: $f(v) = f(v+u)$.

Pf: Define $g : \mathbb{R} \to \mathbb{R}$ by $g_t = f(v + tu)$

Want: $g_0 = g_1$. Want: g is constant on $\mathbb{R}$

Want: $\forall t \in \mathbb{R}$, $g'_t = 0$. Given $t \in \mathbb{R}$ Want: $g'_t = 0$

$(g(t+\cdot))' = g'(t+\cdot)$ $\therefore$ $(g(t+\cdot))'_0 = g'_t$

Want: $(g(t+\cdot))'_0 = 0$

$p := v + tu$, $i := i^u_p$

$(f \circ i)'_0 = (\partial_u f)(p) = 0$

Want: $g(t+\cdot) = f \circ i$

Want: $\forall h \in \mathbb{R}$, $(g(t+\cdot))_h = (f \circ i)_h$

$$
\begin{aligned}
(g(t+\cdot))_h &= g(t+h) = f(v + (t+h)u) \\
&= f(v + tu + hu) = f(p + hu) \\
&= f(i_h) = (f \circ i)_h .
\end{aligned}
$$

$\blacksquare$

4. (15 pts.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and let $v, w \in \mathbb{R}^2$.
Assume: $\forall p \in \mathbb{R}^2$, $f'(p) = 0_2$. Show: $f(v) = f(w)$.

Pf: $u := w - v$. Want: $f(v) = f(v+u)$

By #3, Want: $\forall p \in \mathbb{R}^2$, $(\partial_u f)(p) = 0$

Given $p \in \mathbb{R}^2$. Want: $(\partial_u f)(p) = 0$

$$\begin{aligned}
(\partial_u f)(p) &\overset{*}{=} (D_p f)(u) \\
&= f'_p \underset{\mathbb{R}, \mathbb{R}^2, \mathbb{R}}{*} u \\
&= 0_2 \underset{\mathbb{R}, \mathbb{R}^2, \mathbb{R}}{*} u \\
&= 0 \neq 0
\end{aligned}$$

$\therefore (\partial_u f)(p) = 0$

∎