MATH 4604 Spring 2019, Midterm #1
Handout date: Thursday 28 February 2019
Instructor: Scot Adams

PRINT YOUR NAME:

SOLUTIONS

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One hour exam.

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathrm{dom}[f]$.
Then f **is strictly decreasing** on S means ...

$\forall x, y \in S,$

$(x < y) \Rightarrow (f_x > f_y)$

B. (5 pts) Let Z be a metric space and let $f : Z \dashrightarrow \mathbb{R}$.
Then f has a **strict local minimum** at p in Z means: ...

$\exists B \in \mathcal{B}_Z(p)$ s.t.

$f > f_p$ on $B_p^\times$

C. (5 pts) CVZ := ⋯.

$\{ \alpha \in DNZ \mid (\alpha_0 = 0)$

$\& \, (\alpha$ continuous at 0 from $\mathbb{R}$ to $\mathbb{R}) \}$

D. (5 pts) Let $k \in \mathbb{N}_0$. Then $\hat{\mathcal{O}}_k := \cdots$.

$$(\mathrm{BNZ}) \cdot (|\bullet|^k)$$

E. (5 pts) Let X be a metric space. Then X is **compact** means: ...

$\forall s \in X^{\mathbb{N}}$,

s is subconvergent in X

F. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$. Then $f'(p) := \cdots$.

$$\mathrm{UE}\{ s\ell_L \mid L \in \mathrm{LINS}_p f \}$$

II. True or false (no partial credit):

a. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Assume $\mathrm{dom}[f'] = \mathbb{R}$.
Assume that f is strictly increasing on $\mathbb{R}$. Then $f' > 0$ on $\mathbb{R}$.

False

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$

b. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Assume $\mathrm{dom}[f'] = \mathbb{R}$.
Assume $f(0) = \max(\mathrm{im}[f])$. Then $f'(0) = 0$.

True

Extreme Value Theorem

c. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$.
Assum both $f(0) = 0$ and $f' = C_{\mathbb{R}}^0$. Then $f = C_{\mathbb{R}}^0$.

True

Th 72.23 (4), p. 180

d. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$. Then $\#(\mathrm{LINS}_p f) \leqslant 1$.

True

Th 66.8, p. 146

e. (5 pts) We have $(\hat{\mathcal{O}}_3) \cdot (o_3) \subseteq o_9$.

False

$\boxed{id_{\mathbb{R}}^3 \in \hat{\mathcal{O}}_3 \text{ \& } id_{\mathbb{R}}^4 \in o_3 \text{ \& } (id_{\mathbb{R}}^3) \cdot (id_{\mathbb{R}}^4) = id_{\mathbb{R}}^7 \notin o_9}$

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PLEASE DO NOT WRITE BELOW THE LINE

I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Give an example of two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that:
$$(\ 0 \notin \mathrm{dom}[f'] \ ) \ \& \ (\ 0 \notin \mathrm{dom}[g'] \ ) \ \& \ (\ (f+g)'(0) = 1 \ ).$$

Define $f, g : \mathbb{R} \longrightarrow \mathbb{R}$

by $f(x) = x - |x|$

& $g(x) = |x|$.

2. (10 pts) Let $f, \gamma, h : \mathbb{R} \dashrightarrow \mathbb{R}$, let $p \in \mathbb{R}$ and let $q := f_p$.
Show: $((\gamma \circ f) \cdot h)'_p =^* (\gamma'_q) \cdot (f'_p) \cdot (h_p) + (\gamma_q) \cdot (h'_p)$.

Pf: By the Product Rule,

$$((\gamma \circ f) \cdot h)'_p =^* (\gamma \circ f)'_p \cdot (h_p) + (\gamma \circ f)_p \cdot (h'_p).$$

By the Chain Rule,

$$(\gamma \circ f)'_p =^* (\gamma'_{f_p}) \cdot (f'_p).$$

Also, $(\gamma \circ f)_p = \gamma_{f_p}$.

Then

$$\begin{aligned} ((\gamma \circ f) \cdot h)'_p &=^* (\gamma'_{f_p}) \cdot (f'_p) \cdot (h_p) + (\gamma_{f_p}) \cdot (h'_p) \\ &= (\gamma'_q) \cdot (f'_p) \cdot (h_p) + (\gamma_q) \cdot (h'_p) \end{aligned}$$

∎

Pf: Let $\phi := (g\circ f)'$, $\gamma := g'$, $h := f'$, $\psi := (\gamma\circ f)\cdot h$

Let $R := (g''_q)\cdot(f'_p)^2 + (g'_q)\cdot(f''_p)$. Want: $\phi'_p = R$

Since $R = (g''_q)\cdot(f'_p)\cdot(h_p) + (\gamma_q)\cdot(h'_q)$, by Problem 2, $\psi'_p =^* R$

3. (10 pts) Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, let $p \in \mathrm{dom}[f'']$ and let $q := f_p$.
Assume: $q \in \mathrm{dom}[g'']$. Prove: $(g\circ f)''_p = (g''_q)\cdot(f'_p)^2 + (g'_q)\cdot(f''_p)$.

Hint: Let $\phi := (g\circ f)'$ and $\psi := (g'\circ f)\cdot f'$. Show: $\phi = \psi$ near p.
You may then conclude, without proof, that $\phi'_p = \psi'_p$.
To compute ψ'_p, use Problem 2 (replacing γ by g' and h by f').

Since $p \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[f']$ & $q \in \mathrm{dom}[g''] \subseteq \mathrm{dom}[g']$, $R \neq ⊙$.

So, since $\psi'_p =^* R$, we get $\psi'_p = R$. Want: $\phi'_p = \psi'_p$.

Want $\phi = \psi$ near p. Want: $\exists C \in \mathcal{B}_{\mathbb{R}}(p)$ s.t. $\phi = \psi$ on C

$p \in \mathrm{dom}[f''] = \mathrm{dom}[h']$. Choose $A \in \mathcal{B}_{\mathbb{R}}(p)$ s.t. $A \subseteq \mathrm{dom}[h]$

$q \in \mathrm{dom}[g''] = \mathrm{dom}[\gamma']$. Choose $U \in \mathcal{B}_{\mathbb{R}}(q)$ s.t. $U \subseteq \mathrm{dom}[\gamma]$

$p \in \mathrm{dom}[f''] \subseteq \mathrm{dom}[f']$ $\therefore$ f contin at p. Choose $B \in \mathcal{B}_{\mathbb{R}}(p)$ s.t. $f_*(B) \subseteq U$

$C := A \cap B$. Then $C \in \{A, B\} \subseteq \mathcal{B}_{\mathbb{R}}(p)$. Want: $\phi = \psi$ on C

Want: $\forall x \in C$, $\phi_x = \psi_x$. Given $x \in C$. Want: $\phi_x = \psi_x$.

$x \in C \subseteq A \subseteq \mathrm{dom}[h] = \mathrm{dom}[f']$. Then $x \in \mathrm{dom}[f'] \subseteq \mathrm{dom}[f]$.

So, since $x \in C \subseteq B$, we have: $f_x \in f_*(B)$

So, since $f_*(B) \subseteq U \subseteq \mathrm{dom}[\gamma] = \mathrm{dom}[g']$, we have: $f_x \in \mathrm{dom}[g']$

So, since $x \in \mathrm{dom}[f']$, by the Chain Rule $(g\circ f)'_x = (g'_{f_x})\cdot(f'_x)$

Then $\phi_x = (g\circ f)'_x = (g'_{f_x})\cdot(f'_x) = (\gamma_{f_x})\cdot(h_x) = ((\gamma\circ f)\cdot h)_x = \psi_x$.

4. (15 pts.) Let $g : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume g' is c/d on $[0;3]$.
Assume $g_0 = g'_0 = 0 = g_3$. Show: $\exists c \in (0;3)$ s.t. $g''_c = 0$.

Pf: Since g' c/d on $[0;3]$, we get:

g' contin on $[0;3]$ & $(0;3) \subseteq \text{dom}[g'']$

Since g' contin on $[0;3]$, we get: $[0;3] \subseteq \text{dom}[g']$.

Since $[0;3] \subseteq \text{dom}[g']$, we get: g contin on $[0;3]$.

So, since $(0;3) \subseteq [0;3] \subseteq \text{dom}[g']$, we get: g c/d on $[0;3]$

So, since $g_0 = 0 = g_3$, by Rolle's thm,

choose $b \in (0;3)$ s.t. $g'_b = 0$.

Since $b \in (0;3)$, we get: $(0;b) \subseteq (0;3)$

Want: $\exists c \in (0;b)$ s.t. $g''_c = 0$

Since $b \in (0;3) \subseteq [0;3]$, we get: $[0;b] \subseteq [0;3]$.

So, since g' contin on $[0;3]$, we get: g' contin on $[0;b]$

So, since $(0;b) \subseteq (0;3) \subseteq \text{dom}[g'']$, we get: g' c/d on $[0;b]$

So, since $g'_0 = 0 = g'_b$, by Rolle's Thm

$\exists c \in (0;b)$ s.t. $g''_c = 0$.