MATH 4604 Spring 2019, Midterm #2
Handout date: Thursday 28 March 2019
Instructor: Scot Adams

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SOLUTIONS

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Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) Let $V, W \in \mathrm{TNSR}^+$. Let $f : V \dashrightarrow W$, $S \subseteq V$. Then f is **c/d** on S means: ...

f is continuous on S

&

$\mathrm{Int}_V S \subseteq \mathrm{dom}[f']$

B. (5 pts) Let $V, W, X \in \mathrm{TNSR}^+$. Then $\mathcal{B}^X_{VW} := \cdots$.

$$\{B : V \times W \longrightarrow X \mid (\forall p \in V,\ B(p, \bullet) \in \mathcal{L}^X_W) \ \&\ (\forall q \in W,\ B(\bullet, q) \in \mathcal{L}^X_V)\}$$

C. (5 pts) Let Z be a metric space, let $p \in Z$, and let $f : Z \dashrightarrow \mathbb{R}$. Then f has a **strict local maximum** at p in Z means: ...

$\exists B \in \mathcal{B}_Z(p)$

s.t. $f < f_p$ on $B_p^\times$

D. (5 pts) Let $V, W, X \in \mathrm{TNSR}^+$, $A \in W \otimes V$, $B \in X \otimes W$.
Then $B *_{VWX} A$:= $\cdots$

$$\left[\left(\mathrm{Lin}_B^{WX}\right) \circ \left(\mathrm{Lin}_A^{VW}\right)\right]_V^X$$

E. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathbb{R}$.
Then f is **semi-decreasing** on S means: ...

$$\forall x, y \in S$$

$$(x \leq y) \Longrightarrow (f_x \geq f_y)$$

F. (5 pts) Let $\sigma \in \mathbb{N}$. Then TNSR_σ := $\cdots$.

$$\{\mathbb{R}^m \mid m \in \mathbb{N}^\sigma\}$$

II. True or false (no partial credit):

a. (5 pts) Let $f : [0,1] \hookrightarrow\!\!> [3,7]$. Assume f is continuous.
Then f^{-1} is continuous.

True

Thm 62.9, p.138

b. (5 pts) Let $f, g : \mathbb{R} \to \mathbb{R}$. Assume that $0 \in \mathrm{dom}[(g \circ f)']$.
Then $0 \in \mathrm{dom}[f']$ and $f(0) \in \mathrm{dom}[g']$.

False

$f := |\bullet|$, $g := C_{\mathbb{R}}^4$

c. (5 pts) Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p, q \in V$.
Then $(f(p + \bullet))'(q) = f'(p + q)$.

True

HW 4-4

d. (5 pts) We have $\mathcal{O}_2 \circ \widehat{\mathcal{O}}_4 \subseteq \mathcal{O}_8$.

True

Th 69.16, p.165

e. (5 pts) We have $\mathcal{O}_2 \cdot \widehat{\mathcal{O}}_4 \subseteq \mathcal{O}_8$.

False

$\mathrm{id}_{\mathbb{R}}^3 \in \mathcal{O}_2$, $\mathrm{id}_{\mathbb{R}}^4 \in \widehat{\mathcal{O}}_4$, $\mathrm{id}_{\mathbb{R}}^7 \notin \mathcal{O}_8$

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I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Find $f \in \mathcal{O}_0$ and $g \in \widehat{\mathcal{O}}_0$ s.t. $g \circ f \notin \mathcal{O}_0$.

Hint: $\mathcal{O}_0 =$ CVZ and $\widehat{\mathcal{O}}_0 =$ BNZ.

$$f := C^0_{\mathbb{R}}$$

$$g := C^1_{\mathbb{R}}$$

$$g \circ f = C^1_{\mathbb{R}}$$

2. (10 pts) Let $V := \mathbb{R}^3$, $W := \mathbb{R}$.
Define $K \in \mathcal{K}_V^W$ by $K(x, y, z) = x^3 + 6y^2 z$.
Let $p := (1, 2, 3)$. Compute K'_p.

$\forall x, y, z \in \mathbb{R}$,

$$(\partial_1^{VW} K)(x, y, z) = 3x^2$$

$$(\partial_2^{VW} K)(x, y, z) = 12yz$$

$$(\partial_3^{VW} K)(x, y, z) = 6y^2$$

$$(\partial_1^{VW} K)(p) = 3 \cdot 1^2 = 3$$

$$(\partial_2^{VW} K)(p) = 12 \cdot 2 \cdot 3 = 72$$

$$(\partial_3^{VW} K)(p) = 6 \cdot 2^2 = 24$$

$$K'_p = (3, 72, 24)$$

3. (10 pts) Let $V, W, X \in \mathrm{TNSR}$, $B \in X \otimes W$, $A \in W \otimes V$.
Show: $|B *_{VWX} A|_{X\otimes V} \leqslant |B|_{X\otimes W} \cdot |A|_{W\otimes V}$.

Hint: $\forall k \in \mathcal{I}_X$, $\forall i \in \mathcal{I}_V$, $(B *_{VWX} A)_{k\|i} = \sum_{j\in\mathcal{I}_W} B_{k\|j} \cdot A_{j\|i}$.
Use Cauchy-Schwarz.

$$|B \underset{VWX}{*} A|^2_{X\otimes V} = \sum_{\ell \in \mathcal{I}_{X\otimes V}} \left(B \underset{VWX}{*} A\right)^2_\ell = \sum_{i\in\mathcal{I}_V} \sum_{k\in\mathcal{I}_X} \left(B \underset{VWX}{*} A\right)^2_{k\|i}$$

$$= \sum_{i\in\mathcal{I}_V} \sum_{k\in\mathcal{I}_X} \left(\sum_{j\in\mathcal{I}_W} B_{k\|j} \cdot A_{j\|i} \right)^2$$

$$\leqslant \sum_{i\in\mathcal{I}_V} \sum_{k\in\mathcal{I}_X} \sum_{m\in\mathcal{I}_W} \sum_{n\in\mathcal{I}_W} B^2_{k\|m} \cdot A^2_{n\|i}$$

$$= \left(\sum_{k\in\mathcal{I}_X} \sum_{m\in\mathcal{I}_W} B^2_{k\|m} \right) \cdot \left(\sum_{i\in\mathcal{I}_V} \sum_{n\in\mathcal{I}_W} A^2_{n\|i} \right)$$

$$= \left(\sum_{p\in\mathcal{I}_{X\otimes W}} B^2_p \right) \cdot \left(\sum_{q\in\mathcal{I}_{W\otimes V}} A^2_q \right)$$

$$= |B|^2_{X\otimes W} \cdot |A|^2_{W\otimes V}$$

$$0 \leqslant |B \underset{VWX}{*} A|^2_{X\otimes V} \leqslant |B|^2_{X\otimes W} \cdot |A|^2_{W\otimes V}$$

$$\therefore |B \underset{VWX}{*} A|_{X\otimes V} \leqslant |B|_{X\otimes W} \cdot |A|_{W\otimes V} \quad \blacksquare$$

4. (15 pts.) Let $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$.
Assume: $f_0 = (1,2)$ and $g'_{(1,2)} = (0,0)$ and $g''_{(1,2)} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
Assume: $0 \in \mathrm{dom}[f'']$. Show that $(g \circ f)''_0 \geqslant 0$.

Hint: $$(g \circ f)''_0 =^* (g''_{f_0}) \bullet_{\mathbb{R}^{2\times 2}} ((f'_0)^{\otimes 2}) + (g'_{f_0}) \bullet_{\mathbb{R}^2} (f''_0).$$

$a := (f'_0)_1$, $b := (f'_0)_2$. Then $a \neq \odot \neq b$.

Also, $f'_0 = (a,b)$, so $(f'_0)^{\otimes 2} = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$.

$$(g \circ f)''_0 =^* (g''_{f_0}) \bullet_{\mathbb{R}^{2\times 2}} ((f'_0)^{\otimes 2}) + (g'_{f_0}) \bullet_{\mathbb{R}^2} (f''_0)$$
$$= g''_{(1,2)} \bullet_{\mathbb{R}^{2\times 2}} ((f'_0)^{\otimes 2}) + (g'_{(1,2)}) \bullet_{\mathbb{R}^2} (f''_0)$$
$$= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \bullet_{\mathbb{R}^{2\times 2}} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} + (0,0) \bullet_{\mathbb{R}^2} (f''_0)$$
$$= a^2 + 2ab + 2ab + 4b^2 + 0 = (a+2b)^2 \neq \odot$$

$$(g \circ f)''_0 = (a+2b)^2 \geqslant 0$$