Homework for MATH 4604 (Advanced Calculus II)
Spring 2019

Homework 13: Due on Tuesday 30 April

13-1. Let $W \in \mathrm{TNSR}^+$, $\phi : \mathbb{R} \to W$.
Assume: $\forall k \in \mathcal{I}_W, \quad 0 \in \mathrm{dom}\,[((\pi_k^W) \circ \phi)']$.
Show: $0 \in \mathrm{dom}\,[\phi']$.

13-2. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $q, u \in V$.
Assume: $\forall k \in \mathcal{I}_W, \quad q \in \mathrm{dom}\,[\partial_u((\pi_k^W) \circ f)]$.
Show: $q \in \mathrm{dom}\,[\partial_u f]$.

13-3. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.
Assume: f'' is continuous near p.
Show: $\forall i, j \in \mathcal{I}_V, \quad \partial_i \partial_j f$ is continuous near p.

13-4. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.
Assume: $\forall h, i, j \in \mathcal{I}_V$,
$\partial_h \partial_i \partial_j f$ is both defined near p and bounded near p.
Show: f'' is continuous near p.

13-5. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.
Assume: $\forall h, i, j \in \mathcal{I}_V$,
$\partial_h \partial_i \partial_j f$ is both defined near p and continuous at p.
Show: f''' is continuous at p.

Homework 12: Due on Tuesday 23 April

12-1. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $k \in \mathbb{N}_0$.
Assume: $\forall m \in \mathcal{I}_W, \quad (\pi_m^W) \circ f \in \mathcal{O}_k^{V\mathbb{R}}$.
Show: $f \in \mathcal{O}_k^{VW}$.

12-2. Let $V, W, X \in \mathrm{TNSR}^+$, $k \in \mathbb{N}_0$.
Show:
$$(\mathcal{O}_k^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW}) \subseteq \mathcal{O}_k^{VX} \quad \text{and}$$
$$(\widehat{\mathcal{O}}_1^{WX}) \circ (\mathcal{O}_k^{VW}) \subseteq \mathcal{O}_k^{VX} \quad \text{and}$$
$$(\widehat{\mathcal{O}}_k^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW}) \subseteq \widehat{\mathcal{O}}_k^{VX}.$$

12-3. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $k \in \mathbb{N}_0$.
Assume: $f_{0_V} = 0_W$.
Assume: $\forall j \in \mathcal{I}_V$, $\partial_j f \in \widehat{\mathcal{O}}_k^{VW}$. Show: $f \in \widehat{\mathcal{O}}_{k+1}^{VW}$.

12-4. Let $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$, $k \in \mathbb{N}_0$.
Assume: $f_{0_V} = 0_W$. Assume: $\forall j \in \mathcal{I}_V$, $(\partial_j f)_{0_V} = 0_W$.
Assume: $\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f \in \widehat{\mathcal{O}}_k^{VW}$. Show: $f \in \widehat{\mathcal{O}}_{k+2}^{VW}$.

12-5. Let $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$.
Assume: $f_{0_V} = 0_W$.
Assume: $\forall j \in \mathcal{I}_V$, $\partial_j f$ is both defined near 0_V and continuous at 0_V.
Show: $0_V \in \text{dom}\,[f']$.

Homework 11: Due on Tuesday 16 April

11-1. Let $V := \mathbb{R}^2$, $S := V \otimes V$, $f : \mathbb{R} \dashrightarrow V$, $g : V \dashrightarrow \mathbb{R}$.
Let $p \in \mathbb{R}$. Assume $f'_p = (1,2)$ and $f''_p = (3,4)$.
Let $q := f_p$. Assume $g'_q = (5,6)$ and $g''_q = \begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix}$.
Compute: $(g \circ f)'_p$ and $(g \circ f)''_p$.

11-2. Let $V := \mathbb{R}^2$, $S := V \otimes V$, $A := \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
Show: $\forall q \in V$, $A \bullet_S (q^{\otimes 2}) \geqslant 0$.

11-3. Let $V := \mathbb{R}^2$, $S := V \otimes V$, $a, b, c \in \mathbb{R}$, $A := \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $z := 0_2$.
Assume: $a > 0$ and $ac - b^2 > 0$.
Show: $\forall q \in V_z^\times$, $A \bullet_S (q^{\otimes 2}) > 0$.

11-4. Let $V := \mathbb{R}^2$, $S := V \otimes V$, $f : \mathbb{R} \dashrightarrow V$, $g : V \dashrightarrow \mathbb{R}$.
Let $p \in \mathbb{R}$. Assume $f'_p \neq (0,0)$ and $f''_p \neq ☹$.
Let $q := f_p$. Assume $g'_q = (0,0)$ and $g''_q = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.
Show: $(g \circ f)''_p > 0$.

11-5. Let $V, W \in \text{TNSR}^+$, $f : V \dashrightarrow W$, $i \in \mathcal{I}_V$, $j \in \mathcal{I}_W$, $W' := W \otimes V$.
Show: $(\pi_{j\|i}^{W'}) \circ (f') \subseteq (\pi_j^W) \circ (\partial_i f)$.

Homework 10: Due on Tuesday 9 April

10-1. Let $V, W, X \in \text{TNSR}^+$, $f : V \dashrightarrow W$, $g : W \dashrightarrow X$, $u \in V$.
Show: $\forall p \in V$, $(\partial_u^{VX}(g \circ f))_p =^* (g'_{f_p}) *_{\mathbb{R}VW} ((\partial_u^{VW} f)_p)$.

10-2. Let $S, V, W, Z \in \mathrm{TNSR}^+$, $\quad * \in \mathcal{B}_{VW}^{Z}$.
Let $f : S \dashrightarrow V$, $\quad g : S \dashrightarrow W$, $\quad u \in S$.
Show: $\quad \forall p \in S, \;\; (\partial_u^{SZ}(f*g))_p \;\; =^* \;\; (\,((\partial_u^{SV} f)_p)*g_p\,) \;+\; (\,f_p*((\partial_u^{SW} g)_p)\,)$.

10-3. Let $V, W, X \in \mathrm{TNSR}^+$, $\quad A \in X \otimes V \otimes W$, $\quad y \in V$, $\quad z \in W$.
Show: $\qquad (A \;\; *_{\mathbb{R},W,X\otimes V} \;\; z) \;\; *_{\mathbb{R}VX} \;\; y \quad = \quad A \;\; *_{\mathbb{R},V\otimes W,X} \;\; (y \otimes z)$.

10-4. Let $V, W, X \in \mathrm{TNSR}^+$, $\quad f : V \dashrightarrow W$, $\quad g : W \dashrightarrow X$, $\quad u, v \in V$.
Show:
$$\partial_v^{VX}\partial_u^{VX}(g \circ f) \;\;\supseteq\;\; (g'' \circ f) \;\; *_{\mathbb{R},W\otimes W,X} \;\; ((\partial_u^{VW} f) \otimes (\partial_v^{VW} f)) \;\; + \;\; (g' \circ f) \;\; *_{\mathbb{R}WX} \;\; (\partial_v^{VW}\partial_u^{VW} f).$$

10-5. Let $V := \mathbb{R}^2$, $\quad W := \mathbb{R}$, $\quad g : V \dashrightarrow W$, $\quad z := 0_2$, $\quad i := i_z^{\varepsilon_1^V}$.
Assume: $(g_z = 0)$ & $((\partial_1^{VW} g)_z = 0)$. $\qquad$ Show: $g \circ i \in o_1^{WW}$.

Homework 9: Due on Tuesday 2 April

9-1. Let $V, W \in \mathrm{TNSR}^+$, $\quad \alpha : V \dashrightarrow W$, $\quad \delta > 0$, $\quad B := B_V(0_V, \delta)$.
Assume: $B \subseteq \mathrm{dom}\,[\alpha]$. $\qquad$ Assume: $\forall q \in B,\; |\alpha_q|_W \leqslant |q|_V$.
Show: $\alpha \in \widehat{\mathcal{O}}_1^{VW}$.

9-2. Let $S, T \in \mathrm{TNSR}^+$, $\quad f : S \dashrightarrow T$, $\quad x, v \in S$.
Show: $\qquad (\partial_v^{ST} f)_x \quad =^* \quad (f'_x) \;\; *_{\mathbb{R}ST} \;\; v$.

9-3. Let $V := \mathbb{R}^2$, $\quad W := \mathbb{R}$, $\quad f : V \dashrightarrow W$, $\quad z := 0_2$.
Assume: $\qquad (f'$ is defined near $z)$ $\;\;$ & $\;\;$ $(f_z = 0)$.
Let $g := \partial_1^{VW} f$, $\quad h := \partial_2^{VW} f$, $\quad i := i_z^{\varepsilon_1^V}$.
Show: $\exists \sigma \in \widehat{\mathcal{O}}_1^{WW}$, $\exists \tau \in \widehat{\mathcal{O}}_1^{VV}$ s.t.
$$f \;\; = \;\; (g \circ i \circ \sigma \circ \pi_1^V) \cdot \pi_1^V \;\; + \;\; (h \circ \tau) \cdot \pi_2^V \qquad \text{near } z.$$

9-4. Let $V := \mathbb{R}^2$, $\quad W := \mathbb{R}$, $\quad h : V \dashrightarrow W$, $\quad z := 0_2$.
Assume: $z \in \mathrm{dom}\,[h']$. $\quad$ Assume: $h_z = (\partial_1^{VW} h)_z = (\partial_2^{VW} h)_z = 0$.
Show: $\qquad h \in o_1^{VW}$.

9-5. Let $V := \mathbb{R}^2$, $\quad W := \mathbb{R}$, $\quad f : V \dashrightarrow W$, $\quad z := 0_2$.
Let $\alpha := (\partial_1\partial_1 f)_z$, $\qquad \beta := (\partial_1\partial_2 f)_z$, $\qquad \delta := (\partial_2\partial_2 f)_z$,
$\lambda := (\partial_1 f)_z$, $\qquad \mu := (\partial_2 f)_z$, $\qquad \rho := f_z$.
Assume $\alpha \neq ☹$, $\quad \beta \neq ☹$, $\quad \delta \neq ☹$, $\quad \lambda \neq ☹$, $\quad \mu \neq ☹$, $\quad \rho \neq ☹$.
Define $\quad C \in \mathcal{C}_V^W$, $\quad L \in \mathcal{L}_V^W$, $\quad Q \in \mathcal{Q}_V^W$ $\quad$ by
$$C(x,y) \;=\; \rho, \qquad L(x,y) \;=\; (\lambda, \mu) \;\bullet_V\; (x,y),$$
$$P(x,y) \;=\; \begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix} \bullet_{V\otimes V} \;((x,y)^{\otimes 2}).$$

Let $R := f - \left(C + L + \frac{P}{2!} \right)$.
Show: $(\partial_1 \partial_1 R)_z = (\partial_1 \partial_2 R)_z = (\partial_2 \partial_2 R)_z = (\partial_1 R)_z = (\partial_2 R)_z = R_z = 0$.

Homework 8: Due on Tuesday 26 March

8-1. Let $S, T \in \text{TNSR}^+$, $R : S \dashrightarrow T$.
Assume ($R_{0_S} = 0_T$) & ($R'_{0_S} = 0_{T \otimes S}$). Show: $R \in \mathcal{O}_1^{ST}$.

8-2. Let $S, T \in \text{TNSR}^+$, $f : S \dashrightarrow T$.
Let $p \in \text{dom}\,[f']$, $L := \text{Lin}^{ST}_{f'_p}$, $R := f^T_p - L$.
Show: ($R_{0_S} = 0_T$) & ($R'_{0_S} = 0_{T \otimes S}$).

8-3. Let $S \in \text{TNSR}^+$, $p, v \in S$, $i := i^v_p$. Show: $i' = C^v_{\mathbb{R}}$.

8-4. Let $S, T \in \text{TNSR}^+$, $f : S \dashrightarrow T$, $p, v \in S$.
Show: $(\partial^{ST}_v f)_p \;=^*\; (D^{ST}_p f)_v$.

8-5. Let $V := \mathbb{R}^2$, $W \in \mathbb{R}$, $R : V \dashrightarrow W$,
$\delta > 0$, $J := (-\delta; \delta)$, $x \in J$.
Assume: $J^2 \subseteq \text{dom}\,[R']$.
Show: $\exists \alpha \in [0|x]$ s.t. $R|^{(x,0)}_{(0,0)} = ((\partial^{VW}_1 R)(\alpha, 0)) \cdot x$.

Homework 7: Due on Tuesday 12 March

7-1. Let $V, W \in$ TNSR and let $L, M \in \mathcal{L}^W_V$.
Assume: $[L]^W_V = [M]^W_V$. Show: $L = M$.

7-2. Let $V, W \in$ TNSR and let $A \in W \otimes V$.
Show: $\exists L \in \mathcal{L}^W_V$ s.t. $[L]^W_V = A$.

7-3. Let I be a finite set and let $z \in \mathbb{R}^I$.
Show: $$\sum_{j \in I} |z_j| \;\leqslant\; \sqrt{\#\, I} \cdot \sqrt{\sum_{j \in I} z_j^2}.$$

7-4. Let $U, V, W \in$ TNSR and let $B \in \mathcal{B}^W_{UV}$.
Show: $\exists C \geqslant 0$ s.t., $\forall p \in U$, $\forall q \in V$,
$$|B(p,q)|_W \;\leqslant\; C \cdot |p|_U \cdot |q|_V.$$

7-5. Let $T, U, V, W \in$ TNSR and let $* \in \mathcal{B}^W_{UV}$.
Show: $\text{BNZ}_{TU} \; * \; \text{BNZ}_{TV} \subseteq \text{BNZ}_{TW}$.

Homework 6: Due on Tuesday 5 March

6-1. Let $T \in \mathrm{TNSR}$, $\quad v \in T$. $\qquad$ Show: $v = \sum_{j \in \mathcal{I}_T} v_j \varepsilon_j^T$.

6-2. Let $S, T \in \mathrm{TNSR}$, $\quad A \in T \otimes S$, $\quad L := \mathrm{Lin}_A^{ST}$, $\quad i \in \mathcal{I}_S$.
Show: $L(\varepsilon_i^S) = \sum_{j \in \mathcal{I}_T} A_{j\|i} \cdot \varepsilon_j^T$.

6-3. Let $S, T \in \mathrm{TNSR}^+$, $\quad C \in \mathcal{C}_S^T$, $\quad p \in S$. $\qquad$ Show: $D_p^{ST} C = \mathbf{0}_S^T$.

6-4. Let $S, T \in \mathrm{TNSR}^+$, $\quad L \in \mathcal{L}_S^T$, $\quad p \in S$. $\qquad$ Show: $D_p^{ST} L = L$.

6-5. Let $V, W \in \mathrm{TNSR}^+$, $\quad Q \in \mathcal{Q}_V^W$, $\quad B \in \mathcal{SB}_V^W$.
Assume: $Q = B(\bullet, \bullet)$. $\qquad$ Show: $\forall x \in V$, $D_x^{VW} Q = 2 \cdot (B(x, \bullet))$.

Homework 5: Due on Tuesday 26 February

5-1. Let $S := \mathbb{R}^2$ and let $T := \mathbb{R}$. $\quad$ Show: $\mathcal{Q}_{ST} \subseteq \widehat{\mathcal{O}}_2^{ST}$.

5-2. Let $S, T \in \mathrm{TNSR}^+$. $\quad$ Show: $(\mathrm{BNZ}_{ST}) \cdot (\mathrm{CVZ}_{S\mathbb{R}}) \subseteq \mathrm{CVZ}_{ST}$.

5-3. Let $S \in \mathrm{TNSR}^+$, $\quad f, g : \mathbb{R} \dashrightarrow S$, $\quad p \in \mathbb{R}$.
Show: $(f \bullet g)_p^T = (f_p^T) \bullet (g_p) + (f_p) \bullet (g_p^T) + (f_p^T) \bullet (g_p^T)$.

5-4. Let $S, T, U \in \mathrm{TNSR}^+$, $\quad f : S \dashrightarrow T$, $\quad g : T \dashrightarrow U$.
Let $p \in S$, $\quad q := f_p$. $\qquad$ Show: $(f \circ g)_p^T = (g_q^T) \circ (f_p^T)$

5-5. Let $S, T \in \mathrm{TNSR}$, $\quad i \in \mathcal{I}_S$, $\quad j \in \mathcal{I}_T$. $\qquad$ Show: $i\|j \in \mathcal{I}_{S \otimes T}$.

Homework 4: Due on Tuesday 19 February

4-1. Let $k \in \mathbb{N}_0$, $\alpha \in \widehat{\mathcal{O}}_k$ and $\lambda := \mathrm{adj}_0^0 \left(\frac{\alpha}{|\bullet|^k} \right)$. $\qquad$ Show: $\lambda \in \mathrm{BNZ}$.

4-2. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $U \in \mathbb{R}$.
Assume that U is open in $\mathbb{R}$ and that $U \subseteq \mathrm{dom}\,[f']$.
Assume that f is semi-increasing on U.
Let $T := f'_*(U)$. $\qquad$ Show: $T \geqslant 0$.

4-3. Let $f : \mathbb{R} \to \mathbb{R}$ and let J be an interval.
Assume that f is c/d on J. $\qquad$ Let $I := \mathrm{Int}_{\mathbb{R}} J$ and $T := f'_*(I)$.
Assume $T \geqslant 0$. $\qquad$ Show: f is semi-increasing on J.

4-4. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.
Show: $(f(p + \bullet))' = f'(p + \bullet)$.

4-5. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \operatorname{dom}[f]$.
Show: $[\,(\, f$ has a strict local minimum at p in $\mathbb{R}\,)$
$\Leftrightarrow$ $(\, f_p^T$ has a strict local minimum at 0 in $\mathbb{R}\,)\,]$.

Homework 3: Due on Tuesday 12 February

3-1. Define $\alpha : \mathbb{R} \dashrightarrow \mathbb{R}$ by $\alpha_h = \dfrac{h^2}{9 \cdot (3+h)}$. Show: $\alpha \in \widehat{\mathcal{O}}_2$.

3-2. Define $r : \mathbb{R} \dashrightarrow \mathbb{R}$ by $r_x = 1/x$. Show: $r'_3 = -1/9$.

3-3. Define $L \in \mathcal{L}$ by $L_x = 7x$. Show: $L' = C_{\mathbb{R}}^7$.

3-4. Let $a, b \in \mathbb{R}$. Assume $a < b$. Let $I := (a;b)$, $J := [a;b]$.
Show: $\operatorname{Int}_{\mathbb{R}} J = I$ and $\operatorname{Cl}_{\mathbb{R}} I = J$.

3-5. Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that
$[\ f(-2) = f(2)\ ]\ \&\ [\ \forall x \in (-2;2),\ f'_x \neq 0\ ]$.

Homework 2: Due on Tuesday 5 February

2-1. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$, $p \in \operatorname{dom}[f]$, $S \subseteq \mathbb{R}$ and $q := f_p$.
Show: $(\, f > a$ on $S\,) \Leftrightarrow (\, f_p^T > a - q$ on $S - p\,)$.

2-2. Show: (BNZ) $\circ$ (CVZ) $\subseteq$ BNZ.

2-3. Let $j, k \in \mathbb{N}$. Show: $\widehat{\mathcal{O}}_j \circ \mathcal{o}_k \subseteq \mathcal{o}_{jk}$.

2-4. Let $\phi \in$ CVZ, $\varepsilon > 0$. Show: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $|\phi| < \varepsilon$ on B.

2-5. Let $\alpha \in \mathcal{o}_1$, $L \in \mathcal{L} \backslash \{\mathbf{0}\}$. Show: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $|\alpha| \leqslant |L|$ on B.

Homework 1: Due on Tuesday 29 January

1-1. Let $j \in \mathbb{N}_0$. Show: $\mathcal{o}_j \supseteq$ (CVZ) $\cdot$ $(|\bullet|^j)$.

1-2. Show: CVZ $\subseteq$ BNZ.

1-3. Show: (BNZ) $\cdot$ (BNZ) $\subseteq$ BNZ.

1-4. Show: (BNZ) $\cdot$ (CVZ) $\subseteq$ CVZ.

1-5. Let $j \in \mathbb{N}_0$. Show: $\mathcal{o}_j \subseteq \widehat{\mathcal{O}}_j$.
