

Quantified equivalences

- (1) Axiom of Extensionality / Set-equality equivalency: Let A and B be sets. Then “ $A = B$ ” is equivalent to the quantified statement “ $\forall x, [(x \in A) \Leftrightarrow (x \in B)]$ ”.
- (2) Subset equivalency: Let A and B be sets. Then “ $A \subseteq B$ ” is equivalent to the quantified statement “ $\forall x \in A, [x \in B]$ ”.
- (3) Union equivalency / Intersection equivalency: Let \mathcal{S} be a set of sets. Then, $\forall z$,

$$\left(\left[z \in \bigcup \mathcal{S} \right] \Leftrightarrow \left[\exists X \in \mathcal{S} \text{ s.t. } z \in X \right] \right) \quad \&$$

$$\left(\left[z \in \bigcap \mathcal{S} \right] \Leftrightarrow \left[\forall X \in \mathcal{S}, z \in X \right] \right).$$

- (4) Function-equality equivalency: Let f and g be functions and let S be a set. Assume that $\text{dom}[f] \subseteq S$ and that $\text{dom}[g] \subseteq S$. Then the statement “ $f = g$ ” is equivalent to the quantified statement “ $\forall x \in S, [f(x) = g(x)]$ ”.
- (5) Union-closure equivalency: Let \mathcal{S} be a set of sets, X a set. Then “ $X \in \langle \mathcal{S} \rangle_{\cup}$ ” is equivalent to the quantified statement “ $\forall z \in X, \exists A \in \mathcal{S} \text{ s.t. } z \in A \subseteq X$ ”. In other words, “some of the sets in \mathcal{S} will cover X exactly” is equivalent to “every point of X can be covered by a set from \mathcal{S} without going outside X ”.
- (6) Base equivalency: Let \mathcal{B} be any set of sets. Then the statement “ \mathcal{B} is a base for a topology” is equivalent to “ $\langle \mathcal{B} \rangle_{\cup}$ is a topology”, which is equivalent to the quantified statement “ $\forall U, V \in \mathcal{B}, U \cap V \in \langle \mathcal{B} \rangle_{\cup}$ ”. In other words, a set of sets is a base for a topology iff “each pairwise-intersection is a union”.
- (7) Forward-image containmentment equivalency: Let f be a function. Let S, T be sets. Then “ $f_*(S) \subseteq T$ ” is equivalent to the quantified statement “ $\forall x \in \text{dom}[f], ([x \in S] \Rightarrow [f(x) \in T])$ ”.
- (8) Open equivalency: Let X be a topological space, $W \subseteq X$. Then “ W is open in X ” is equivalent to the quantified statement “ $\forall q \in W, \exists V \in \mathcal{N}_X(q) \text{ s.t. } V \subseteq W$ ”. In other words, “the set is open” is equivalent to “every point of the set is covered by a neighborhood that stays inside the set”.
- (9) Closure point equivalency: Let Z be a topological space, $S \subseteq Z$, $q \in Z$. Then the statement “ $q \in \text{Cl}_Z(S)$ ” is equivalent to the quantified statement “ $\forall U \in \mathcal{N}_Z(q) \text{ s.t. } U \cap S \neq \emptyset$ ”.

- (10) Interior point equivalency: Let Z be a topological space, $S \subseteq Z$, $q \in Z$. Then the statement “ $q \in \text{Int}_Z(S)$ ” is equivalent to the quantified statement “ $\exists U \in \mathcal{N}_Z(q)$ s.t. $U \subseteq S$ ”.
- (11) Boundary point equivalency: Let Z be a topological space, let $S \subseteq Z$ and let $q \in Z$. Then “ $q \in \partial_Z(S)$ ” is equivalent to the quantified statement “ $\forall U \in \mathcal{N}_Z(q)$, $U \cap S \neq \emptyset \neq U \setminus S$ ”.
- (12) Limit point equivalency: Let Z be a topological space, let $S \subseteq Z$ and let $q \in Z$. Then “ $q \in \text{LP}_Z(S)$ ” is equivalent to the quantified statement “ $\forall P \in \mathcal{N}_Z^\times(q)$, $P \cap S \neq \emptyset$ ”.
- (13) Topological space function-limit equivalency: Let Y and Z be topological spaces, $f : Y \dashrightarrow Z$, $a \in Y$, $b \in Z$. Then: “ $f \rightarrow b$ in Z near a in Y ” is equivalent to the quantified statement “ $\forall V \in \mathcal{N}_Z(b)$, $\exists U \in \mathcal{N}_Y^\times(a)$ s.t., $\forall x \in \text{dom}[f]$,

$$[x \in U] \quad \Rightarrow \quad [f(x) \in V]”.$$

- (14) Metric space function-limit equivalency: Let Y and Z be metric spaces and let $f : Y \dashrightarrow Z$. Let $a \in Y$, $b \in Z$. Then: “ $f \rightarrow b$ in Z near a in Y ” is equivalent to the quantified statement “ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \text{dom}[f]$,

$$[0 < d_Y(x, a) < \delta] \quad \Rightarrow \quad [d_Z(f(x), b) < \varepsilon]”.$$

- (15) Normed vector space function-limit equivalency: Let Y and Z be normed vector spaces, $f : Y \dashrightarrow Z$, $a \in Y$, $b \in Z$. Then: “ $f \rightarrow b$ in Z near a in Y ” is equivalent to the quantified statement “ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \text{dom}[f]$,

$$[0 < \|x - a\|_Y < \delta] \quad \Rightarrow \quad [\|f(x) - b\|_Z < \varepsilon]”.$$

- (16) Topological space sequence-limit equivalency: Let Z be a topological space. Let $s \in Z^\mathbb{N}$, $b \in Z$. Then: “ $s_\bullet \rightarrow b$ in Z ” is equivalent to “ $\forall V \in \mathcal{N}_Z(b)$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[j \geq K] \quad \Rightarrow \quad [s_j \in V]”.$$

- (17) Metric space sequence-limit equivalency: Let Z be a metric space. Let $s \in Z^\mathbb{N}$, $b \in Z$. Then “ $s_\bullet \rightarrow b$ in Z ” is equivalent to the quantified statement “ $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[j \geq K] \quad \Rightarrow \quad [d_Z(s_j, b) < \varepsilon]”.$$

- (18) Normed vector space sequence-limit equivalency: Let Z be a normed vector space, $s \in Z^{\mathbb{N}}$, $b \in Z$. Then: “ $s_{\bullet} \rightarrow b$ in Z ” is equivalent to “ $\forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$[j \geq K] \quad \Rightarrow \quad [|s_j - b|_Z < \varepsilon]”.$$

- (19) Topological space continuity equivalency: Let Y and Z be topological spaces, let $f : Y \dashrightarrow Z$ and let $a \in \text{dom}[f]$. Then the statement “ f is (Y, Z) -continuous at a ” is equivalent to the statement “ $\forall V \in \mathcal{N}_Z(f(a)), \exists U \in \mathcal{N}_Y(a)$ s.t., $\forall x \in \text{dom}[f]$,

$$[x \in U] \quad \Rightarrow \quad [f(x) \in V]”.$$

- (20) Metric space continuity equivalency: Let Y and Z be metric spaces, let $f : Y \dashrightarrow Z$ and let $a \in \text{dom}[f]$. Then the statement “ f is (Y, Z) -continuous at a ” is equivalent to the quantified statement “ $\forall \varepsilon > 0, \exists \delta > 0$ s.t., $\forall x \in \text{dom}[f]$,

$$[d_Y(x, a) < \delta] \quad \Rightarrow \quad [d_Z(f(x), f(a)) < \varepsilon]”.$$

- (21) Normed vector space continuity equivalency: Let Y and Z be normed vector spaces, let $f : Y \dashrightarrow Z$ and let $a \in \text{dom}[f]$. Then the statement “ f is (Y, Z) -continuous at a ” is equivalent to the quantified statement “ $\forall \varepsilon > 0, \exists \delta > 0$ s.t., $\forall x \in \text{dom}[f]$,

$$[|x - a|_Y < \delta] \quad \Rightarrow \quad [|[f(x)] - [f(a)]|_Z < \varepsilon]”.$$

- (22) K -bounded quantified equivalence: Let V, W be normed vector spaces, $T \in L(V, W)$, $K \geq 0$. Then:

$$[T \text{ is } K\text{-bounded}] \quad \Leftrightarrow \quad [\forall u \in S_V, |T(u)|_W \leq K].$$

- (23) Another K -bounded quantified equivalence: Let V and W be normed vector spaces, $T \in L(V, W)$ and $K \geq 0$. Then:

$$[T \text{ is } K\text{-bounded}] \quad \Leftrightarrow \quad [\forall x \in V, |T(x)|_W \leq K \cdot |x|_V].$$

- (24) δ -bounded below quantified equivalence: Let V, W be normed vector spaces, $T \in L(V, W)$, $\delta > 0$. Then:

$$[T \text{ is } \delta\text{-bdd below}] \quad \Leftrightarrow \quad [\forall u \in S_V, |T(u)|_W \geq \delta].$$

- (25) Another δ -bounded below quantified equivalence: Let V and W be normed vector spaces, $T \in L(V, W)$ and $\delta > 0$. Then:

$$[T \text{ is } \delta\text{-bdd below}] \quad \Leftrightarrow \quad [\forall x \in V, |T(x)|_W \geq \delta \cdot |x|_V].$$

Other useful facts

- (A) For any set \mathcal{S} of sets, for any $X \in \mathcal{S}$, we have: $X \subseteq \bigcup \mathcal{S}$.

- (B) For any vector space V , for any $k \in \mathbb{N}$, for any $x_1, \dots, x_k \in V$,
 $\langle \{x_1, \dots, x_k\} \rangle_{\text{lin}} = \{ c_1x_1 + \dots + c_kx_k \mid c_1, \dots, c_k \in \mathbb{R} \}$.
- (C) For any function f , for any set S , for any x , if [$(x \in \text{dom}[f])$
and $(x \in S)$], then [$f(x) \in f_*(S)$].