Solutions for MATH 4604 (Advanced Calculus II)
Spring 2019

Homework 13: Due on Tuesday 30 April

13-1. Let $W \in \mathrm{TNSR}^+$, $\phi : \mathbb{R} \to W$.
Assume: $\forall k \in \mathcal{I}_W$, $0 \in \mathrm{dom}\,[((\pi_k^W) \circ \phi)']$.
Show: $0 \in \mathrm{dom}\,[\phi']$.

Proof: Let $x := \displaystyle\sum_{k \in \mathcal{I}_W} (((\pi_k^W) \circ \phi)'_0) \cdot \varepsilon_k^W$. Want: $\phi'_0 = x$.
We have: $\forall k \in \mathcal{I}_W$, $x_k = ((\pi_k^W) \circ \phi)'_0$.
Define $L \in \mathcal{L}_{\mathbb{R}}^W$ by $L(t) = tx$. Then $\mathrm{s}\ell_L^{\mathbb{R}W} = L(1) = x$.
Also, $\phi'_0 = \mathrm{s}\ell_{D_0\phi}^{\mathbb{R}W}$. Want: $\mathrm{s}\ell_{D_0\phi}^{\mathbb{R}W} = \mathrm{s}\ell_L^{\mathbb{R}W}$.
Want: $D_0\phi = L$. Want: $\phi_0^T - L \in \mathcal{o}_1^{\mathbb{R}W}$.
Let $\psi := \phi_0^T - L$. Want: $\psi \in \mathcal{o}_1^{\mathbb{R}W}$.
By HW#12-1, it suffices to show: $\forall k \in \mathcal{I}_W$, $(\pi_k^W) \circ \psi \in \mathcal{o}_1^{\mathbb{R}\mathbb{R}}$.
Given $k \in \mathcal{I}_W$. Want: $(\pi_k^W) \circ \psi \in \mathcal{o}_1^{\mathbb{R}\mathbb{R}}$.
Let $\chi := (\pi_k^W) \circ \phi$. Define $M \in \mathcal{L}_{\mathbb{R}}^{\mathbb{R}}$ by $M(t) = tx_k$.
We have $\mathrm{s}\ell_M^{\mathbb{R}\mathbb{R}} = M(1) = 1 \cdot x_k = x_k = ((\pi_k^W) \circ \phi)'_0 = \chi'_0 = \mathrm{s}\ell_{D_0\chi}^{\mathbb{R}\mathbb{R}}$.
Then $M = [\mathrm{s}\ell_M^{\mathbb{R}\mathbb{R}}]_{\mathbb{R}}^{\mathbb{R}} = [\mathrm{s}\ell_{D_0\chi}^{\mathbb{R}\mathbb{R}}]_{\mathbb{R}}^{\mathbb{R}} = D_0\chi$.
Since $D_0\chi = M$, we get $\chi_0^T - M \in \mathcal{o}_1^{\mathbb{R}\mathbb{R}}$.
Want: $\chi_0^T - M = (\pi_k^W) \circ \psi$. Want: $\forall h \in \mathbb{R}$, $(\chi_0^T - M)_h = ((\pi_k^W) \circ \psi)_h$.
Given $h \in \mathbb{R}$. Want: $(\chi_0^T - M)_h = ((\pi_k^W) \circ \psi)_h$.
We have $\psi_h = (\phi_0^T - L)_h = \phi_{0+h} - \phi_0 - L_h = \phi_h - \phi_0 - hx$.
Then $\pi_k^W(\psi_h) = (\pi_k^W(\phi_h)) - (\pi_k^W(\phi_0)) - hx_k$.
Then:

$$\begin{aligned}(\chi_0^T - M)_h &= \chi_{0+h} - \chi_0 - M_h = \chi_h - \chi_0 - M_h \\ &= ((\pi_k^W) \circ \phi)_h - ((\pi_k^W) \circ \phi)_0 - hx_k \\ &= (\pi_k^W(\phi_h)) - (\pi_k^W(\phi_0)) - hx_k \\ &= \pi_k^W(\psi_h) = ((\pi_k^W) \circ \psi)_h, \quad \text{as desired.}\end{aligned}$$

QED

13-2. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $q, u \in V$.
Assume: $\forall k \in \mathcal{I}_W$, $q \in \mathrm{dom}\,[\partial_u((\pi_k^W) \circ f)]$.
Show: $q \in \mathrm{dom}\,[\partial_u f]$.

Proof: Let $\phi := f \circ (i_q^u)$. Then $(\partial_u f)_q = \phi'_0$. Want: $0 \in \mathrm{dom}\,[\phi']$.
By HW#13-1, it suffices to show: $\forall k \in \mathcal{I}_W$, $0 \in \mathrm{dom}\,[((\pi_k^W) \circ \phi)']$.
Given $k \in \mathcal{I}_W$. Want: $0 \in \mathrm{dom}\,[((\pi_k^W) \circ \phi)']$.
By assumption, $q \in \mathrm{dom}\,[\partial_u((\pi_k^W) \circ f)]$.

So, since $(\partial_u((\pi_k^W)\circ f))_q = ((\pi_k^W)\circ f\circ(i_q^u))'_0$,

we get: $0 \in \mathrm{dom}\,[((\pi_k^W)\circ f\circ(i_q^u))']$.

Then $0 \in \mathrm{dom}\,[((\pi_k^W)\circ f\circ(i_q^u))'] = \mathrm{dom}\,[((\pi_k^W)\circ\phi)']$, as desired. QED

13-3. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.

Assume: f'' is continuous near p.

Show: $\forall i, j \in \mathcal{I}_V$, $\partial_i\partial_j f$ is continuous near p.

Proof: Given $i, j \in \mathcal{I}_V$. Want: $\partial_i\partial_j f$ is continuous near p.

Choose $B \in \mathcal{B}_V(p)$ s.t. f'' is continuous on B.

Want: $\partial_i\partial_j f$ is continuous on B.

Want: $\forall p \in B$, $\partial_i\partial_j f$ is continuous at p.

Given $p \in B$. Want: $\partial_i\partial_j f$ is continuous at p.

By Theorem 99.1, it suffices to show:

$\forall k \in \mathcal{I}_W$, $\pi_k \circ (\partial_i\partial_j f)$ is continuous at p.

Given $\forall k \in \mathcal{I}_W$. Want: $\pi_k \circ (\partial_i\partial_j f)$ is continuous at p.

Since f'' is continuous on B, we conclude: $B \subseteq \mathrm{dom}\,[f'']$.

So, since $\mathrm{im}\,[f''] \subseteq W \otimes V \otimes V = \mathrm{dom}\,[\pi_{k\|i\|j}]$,

we conclude that: $B \subseteq \mathrm{dom}\,[\pi_{k\|i\|j} \circ (f'')]$.

By Thoerem 102.17, we have: $\pi_k \circ (\partial_i\partial_j f) \supseteq \pi_{k\|i\|j} \circ (f'')$.

So, since $B \subseteq \mathrm{dom}\,[\pi_{k\|i\|j} \circ (f'')]$, we get:

$\pi_k \circ (\partial_i\partial_j f) = \pi_{k\|i\|j} \circ (f'')$ on B.

So, since $p \in B$, by the Recentering Lemma (Theorem 38.16), we get:

$\pi_k \circ (\partial_i\partial_j f) = \pi_{k\|i\|j} \circ (f'')$ near p.

Since f'' is continuous on B and $p \in B$,

we see that f'' is continuous at p.

Then, by Theorem 99.1, $\pi_{k\|i\|j} \circ (f'')$ is continuous at p.

So, as $\pi_k \circ (\partial_i\partial_j f) = \pi_{k\|i\|j} \circ (f'')$ near p,

it follows, from Theorem 65.7,

that $\pi_k \circ (\partial_i\partial_j f)$ is continuous at p, as desired. QED

13-4. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.

Assume: $\forall h, i, j \in \mathcal{I}_V$,

$\partial_h\partial_i\partial_j f$ is both defined near p and bounded near p.

Show: f'' is continuous near p.

Proof: By Theorem 102.21, it suffices to show:

$\forall i, j \in \mathcal{I}_V$, $\partial_i\partial_j f$ is continuous near p.

Given $i, j \in \mathcal{I}_V$. Want: $\partial_i\partial_j f$ is continuous near p.

Let $g := \partial_i \partial_j f$. Want: g is continuous near p.
By Theorem 102.5, it suffices to show:
$\forall h \in \mathcal{I}_V$, $\partial_h g$ is defined near p and bounded near p.
Given $h \in \mathcal{I}_V$. Want: $\partial_h g$ is defined near p and bounded near p.
By assumption, $\partial_h \partial_i \partial_j f$ is defined near p and bounded near p.
So since $\partial_h g = \partial_h \partial_i \partial_j f$, we get:
$\partial_h g$ is defined near p and bounded near p, as desired. QED

13-5. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $p \in V$.
Assume: $\forall h, i, j \in \mathcal{I}_V$,
$\partial_h \partial_i \partial_j f$ is both defined near p and continuous at p.
Show: f''' is continuous at p.

Proof: By Theorem 102.3, $\forall h, i, j \in \mathcal{I}_V$, $\partial_h \partial_i \partial_j f$ is bounded near p.
Also, by assumption, $\forall h, i, j \in \mathcal{I}_V$, $\partial_h \partial_i \partial_j f$ is defined near p.
Then, by HW#13-4, f'' is continuous near p.
Then: f'' is continuous at p and f'' is defined near p.

Claim: Let $h, i, j \in \mathcal{I}_V$, $k \in \mathcal{I}_W$.
Then: $\partial_h(\pi_{k\|i\|j} \circ (f'')) = \pi_k \circ (\partial_h \partial_i \partial_j f)$ near p.
Proof of Claim:
Since f'' is defined near p, by Theorem 97.3,
we see that $\partial_j(f') = (\partial_j f)'$ near p.
Since f'' is defined near p and since $\mathrm{im}\,[f''] \subseteq W \otimes V \otimes V = \mathrm{dom}\,[\pi_{k\|i\|j}]$,
we conclude that $\pi_{k\|i\|j} \circ (f'')$ is defined near p.
Let $\alpha := f'$ and $\ell := k\|i$. By Theorem 102.1, $\pi_\ell \circ (\partial_j \alpha) \supseteq \pi_{\ell\|j} \circ (\alpha')$.
That is, $\pi_{k\|i} \circ (\partial_j(f')) \supseteq \pi_{k\|i\|j} \circ (f'')$.
So, since $\pi_{k\|i\|j} \circ (f'')$ is defined near p,
it follows that $\pi_{k\|i} \circ (\partial_j(f')) = \pi_{k\|i\|j} \circ (f'')$ near p.
Since f'' is defined near p, by Theorem 94.4,
we see that $(\partial_j f)'$ is defined near p.
Then, since $\mathrm{im}\,[(\partial_j f)'] \subseteq W \otimes V = \mathrm{dom}\,[\pi_{k\|i}]$,
we conclude that $\pi_{k\|i} \circ ((\partial_j f)')$ is defined near p.
Let $\beta := \partial_j f$. By Theorem 102.1, $\pi_k \circ (\partial_i \beta) \supseteq \pi_{k\|i} \circ (\beta')$.
That is, $\pi_k \circ (\partial_i \partial_j f) \supseteq \pi_{k\|i} \circ ((\partial_j f)')$.
So, since $\pi_{k\|i} \circ ((\partial_j f)')$ is defined near p,
it follows that $\pi_k \circ (\partial_i \partial_j f) = \pi_{k\|i} \circ ((\partial_j f)')$ near p.
So, since $\partial_j(f') = (\partial_j f)'$ near p,
it follows that $\pi_k \circ (\partial_i \partial_j f) = \pi_{k\|i} \circ (\partial_j(f'))$ near p.

So, since $\pi_{k\|i} \circ (\partial_j(f')) = \pi_{k\|i\|j} \circ (f'')$ near p.

it follows that $\pi_k \circ (\partial_i \partial_j f) = \pi_{k\|i\|j} \circ (f'')$ near p.

Then: $\partial_h(\pi_{k\|i\|j} \circ (f'')) = \partial_h(\pi_k \circ (\partial_i \partial_j f))$ near p.

Want: $\partial_h(\pi_k \circ (\partial_i \partial_j f)) = \pi_k \circ (\partial_h \partial_i \partial_j f)$ near p.

By assumption, $\partial_h \partial_i \partial_j f$ is defined near p.

So, since $\operatorname{im}[\partial_h \partial_i \partial_j f] \subseteq W = \operatorname{dom}[\pi_k]$,

we see that $\pi_k \circ (\partial_h \partial_i \partial_j f)$ is defined near p.

By Theorem 94.3, $\partial_h(\pi_k \circ (\partial_i \partial_j f)) \supseteq \pi_k \circ (\partial_h \partial_i \partial_j f)$.

So, since $\pi_k \circ (\partial_h \partial_i \partial_j f)$ is defined near p, it follows that:

$\partial_h(\pi_k \circ (\partial_i \partial_j f)) = \pi_k \circ (\partial_h \partial_i \partial_j f)$ near p, as desired.

End of proof of Claim.

Let $g := f''$. Want: g' is continuous at p.

Since $f : V \dashrightarrow W$ and $g = f''$, we get $g : V \dashrightarrow W \otimes V \otimes V$.

By Theorem 102.7, it suffices to show:

$\forall h \in \mathcal{I}_V$, $\partial_h g$ is defined near p and continuous at p.

Given $h \in \mathcal{I}_V$. Want: (A) $\partial_h g$ is defined near p and

(B) $\partial_h g$ is continuous at p.

Proof of (A):

By HW#13-2, it suffices to show:

$\forall \ell \in \mathcal{I}_{W \otimes V}$, $\partial_h(\pi_\ell \circ g)$ is defined near p.

Given $\ell \in \mathcal{I}_{W \otimes V \otimes V}$. Want: $\partial_h(\pi_\ell \circ g)$ is defined near p.

Since $\ell \in \mathcal{I}_{W \otimes V \otimes V}$, choose $i, j \in \mathcal{I}_V$ and $k \in \mathcal{I}_W$ s.t. $\ell = k\|i\|j$.

By assumption, $\partial_h \partial_i \partial_j f$ is defined near p.

So, since $\operatorname{im}[\partial_h \partial_i \partial_j f] \subseteq W = \operatorname{dom}[\pi_k]$,

we see that $\pi_k \circ (\partial_h \partial_i \partial_j f)$ is defined near p.

By the claim: $\partial_h(\pi_{k\|i\|j} \circ (f'')) = \pi_k \circ (\partial_h \partial_i \partial_j f)$ near p.

So, since $\ell = k\|i\|j$ and $g = f''$, we see that:

$\partial_h(\pi_\ell \circ g) = \pi_k \circ (\partial_h \partial_i \partial_j f)$ near p.

So, since $\pi_k \circ (\partial_h \partial_i \partial_j f)$ is defined near p, we conclude:

$\partial_h(\pi_\ell \circ g)$ is defined near p, as desired.

End of proof of (A).

Proof of (B):

By Theorem 99.1, it suffices to show:

$\forall \ell \in \mathcal{I}_{W \otimes V \otimes V}$, $\pi_\ell \circ (\partial_h g)$ is continuous at p.

Given $\ell \in \mathcal{I}_{W \otimes V \otimes V}$. Want: $\pi_\ell \circ (\partial_h g)$ is continuous at p.

By (A), $\partial_h g$ is defined near p. Then $p \in \mathrm{dom}\,[\partial_h g]$.
So, since $\mathrm{im}\,[\partial_h g] \subseteq W \otimes V \otimes V = \mathrm{dom}\,[\pi_\ell]$,
we see that $p \in \mathrm{dom}\,[\pi_\ell \circ (\partial_h g)]$.
By Theorem 94.3, $\partial_h(\pi_\ell \circ g) \supseteq \pi_\ell \circ (\partial_h g)$.
Then, by Theorem 44.13, it suffices to show:
$\partial_h(\pi_\ell \circ g)$ is continuous at p.
Since $\ell \in \mathcal{I}_{W\otimes V\otimes V}$, choose $i, j \in \mathcal{I}_V$ and $k \in \mathcal{I}_W$ s.t. $\ell = k\|i\|j$.
By assumption, $\partial_h\partial_i\partial_j f$ is continuous at p.
Then, by Theorem 99.1,
we see that $\pi_k \circ (\partial_h\partial_i\partial_j f)$ is continuous at p.
By the claim, $\partial_h(\pi_{k\|i\|j} \circ (f')) = \pi_k \circ (\partial_h\partial_i\partial_j f)$ near p.
So, since $\ell = k\|i\|j$ and $g = f''$, we see that:
$\partial_h(\pi_\ell \circ g) = \pi_k \circ (\partial_h\partial_i\partial_j f)$ near p.
So, since $\pi_k \circ (\partial_h\partial_i\partial_j f)$ is continuous at p, by Theorem 65.7, we have:
$\partial_h(\pi_\ell \circ g)$ is continuous at p, as desired.
End of proof of (B). QED

Homework 12: Due on Tuesday 23 April

12-1. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $k \in \mathbb{N}_0$.
Assume: $\forall m \in \mathcal{I}_W$, $(\pi_m^W) \circ f \in \mathcal{O}_k^{V\mathbb{R}}$.
Show: $f \in \mathcal{O}_k^{VW}$.

Proof:
Claim: $f_{0_V} = 0_W$.
Proof of Claim:
Want: $\forall m \in \mathcal{I}_W$, $\pi_m^W(f_{0_V}) = 0$.
Given $m \in \mathcal{I}_W$. Want: $\pi_m^W(f_{0_V}) = 0$.
Since $(\pi_m^W) \circ f \in \mathcal{O}_k^{V\mathbb{R}} \subseteq \mathcal{O}_0^{V\mathbb{R}} = \mathrm{CVZ}_V^{\mathbb{R}}$, we conclude: $((\pi_m^W) \circ f)_{0_V} = 0$.
Then $\pi_m^W(f_{0_V}) = ((\pi_m^W) \circ f)_{0_V} = 0$, as desired.
End of proof of Claim.

Let $\phi := \mathrm{adj}_{0_V}^{0_W}\left(\dfrac{f}{|\bullet|_V^k}\right)$. Then $\phi_{0_V} = 0_W$.
By the claim, $f_{0_V} = 0_W$, and it follows that: $\phi \cdot (|\bullet|_V^k) = f$.
Want: $f \in (\mathrm{CVZ}_V^W) \cdot (|\bullet|_V^k)$. Want: $\phi \in \mathrm{CVZ}_V^W$.
Want: $\forall m \in \mathcal{I}_W$, $(\pi_m^W) \circ \phi \in \mathrm{CVZ}_V^{\mathbb{R}}$.
Given $m \in \mathcal{I}_W$. Want: $(\pi_m^W) \circ \phi \in \mathrm{CVZ}_V^{\mathbb{R}}$.
By assumption, $(\pi_m^W) \circ f \in \mathcal{O}_k^{V\mathbb{R}}$. Then $(\pi_m^W) \circ f \in (\mathrm{CVZ}_V^{\mathbb{R}}) \cdot (|\bullet|_V^k)$.

Choose $\psi \in \mathrm{CVZ}_V^{\mathbb{R}}$ s.t. $(\pi_m^W) \circ f = \psi \cdot (|\bullet|_V^k)$.
Want: $(\pi_m^W) \circ \phi = \psi$. Want: $(\pi_m^W) \circ \phi = \psi$ on V.
Since $\psi \in \mathrm{CVZ}_V^{\mathbb{R}}$, it follows that $\psi_{0_V} = 0$.
We have $((\pi_m^W) \circ \phi)_{0_V} = (\pi_m^W)(\phi_{0_V}) = \pi_m^W(0_W) = 0 = \psi_0$.
Want: $(\pi_m^W) \circ \phi = \psi$ on $V_{0_V}^\times$. Want: $\forall q \in V_{0_V}^\times$, $((\pi_m^W) \circ \phi)_q = \psi_q$.
Given $q \in V_{0_V}^\times$. Want: $((\pi_m^W) \circ \phi)_q = \psi_q$. Want: $(\pi_m^W)(\phi_q) = \psi_q$.
Since $q \in V_{0_V}^\times$, we get $|q|_V \neq 0$, and so $|q|_V^k \neq 0$.
Recall: $\phi \cdot (|\bullet|_V^k) = f$ and $(\pi_m^W) \circ f = \psi \cdot (|\bullet|_V^k)$.
Then $(\pi_m^W(\phi_q)) \cdot (|q|_V^k) = \pi_m^W(\phi_q \cdot (|q|_V^k)) = \pi_m^W((\phi \cdot (|\bullet|_V^k))_q) = \pi_m^W(f_q)$
$= ((\pi_m^W) \circ f)_q = (\psi \cdot (|\bullet|_V^k))_q = \psi_q \cdot (|q|_V^k)$.
So, since $|q|_V^k \neq 0$, dividing by $|q|_V^k$, we get: $(\pi_m^W(\phi_q)) = \psi_q$. QED

12-2. Let $V, W, X \in \mathrm{TNSR}^+$, $k \in \mathbb{N}_0$.
Show:
$$(\mathcal{O}_k^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW}) \subseteq \mathcal{O}_k^{VX} \quad \text{and}$$
$$(\widehat{\mathcal{O}}_1^{WX}) \circ (\mathcal{O}_k^{VW}) \subseteq \mathcal{O}_k^{VX} \quad \text{and}$$
$$(\widehat{\mathcal{O}}_k^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW}) \subseteq \widehat{\mathcal{O}}_k^{VX}.$$

Proof: Exactly one of the following is true:
(1) $k \neq 0$ or (2) $k = 0$.

Case (1):
Since $k \neq 0$, $(\mathcal{O}_k^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW}) \subseteq \mathcal{O}_{k\cdot 1}^{VX} = \mathcal{O}_k^{VX}$ and
$(\widehat{\mathcal{O}}_1^{WX}) \circ (\mathcal{O}_k^{VW}) \subseteq \mathcal{O}_{1\cdot k}^{VX} = \mathcal{O}_k^{VX}$.
Want: $(\widehat{\mathcal{O}}_k^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW}) \subseteq \widehat{\mathcal{O}}_k^{VX}$.
Since $k \neq 0$, $(\widehat{\mathcal{O}}_k^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW}) \subseteq \widehat{\mathcal{O}}_{k\cdot 1}^{VX} = \widehat{\mathcal{O}}_k^{VX}$.
End of Case (1).

Case (2):
We have
$$\begin{aligned}(\mathcal{O}_k^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW}) &= \mathcal{O}_0^{WX} \circ (\widehat{\mathcal{O}}_1^{VW}) \\ &\subseteq (\mathcal{O}_0^{WX}) \circ (\mathcal{O}_0^{VW}) \\ &\subseteq (\mathrm{CVZ}_W^X) \circ (\mathrm{CVZ}_V^W) \\ &\subseteq \mathrm{CVZ}_V^X = \mathcal{O}_0^{VX} \subseteq \mathcal{O}_k^{VX}.\end{aligned}$$
Also,
$$\begin{aligned}(\widehat{\mathcal{O}}_1^{WX}) \circ (\mathcal{O}_k^{VW}) &= (\widehat{\mathcal{O}}_1^{WX}) \circ (\mathcal{O}_0^{VW}) \\ &\subseteq (\mathcal{O}_0^{WX}) \circ (\mathcal{O}_0^{VW}) \\ &\subseteq (\mathrm{CVZ}_W^X) \circ (\mathrm{CVZ}_V^W) \\ &\subseteq \mathrm{CVZ}_V^X = \mathcal{O}_0^{VX} \subseteq \mathcal{O}_k^{VX}.\end{aligned}$$
Want: $(\widehat{\mathcal{O}}_k^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW}) \subseteq \widehat{\mathcal{O}}_k^{VX}$.
We have $(\widehat{\mathcal{O}}_k^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW}) = (\widehat{\mathcal{O}}_0^{WX}) \circ (\widehat{\mathcal{O}}_1^{VW})$

$$\begin{aligned} &= (\widehat{\mathcal{O}}_0^{WX}) \circ (\mathcal{O}_0^{VW}) \\ &= (\mathrm{BNZ}_W^X) \circ (\mathrm{CVZ}_V^W) \\ &\subseteq \mathrm{BNZ}_V^X = \widehat{\mathcal{O}}_0^{VX} \subseteq \widehat{\mathcal{O}}_k^{VX}. \end{aligned}$$

End of Case (2). QED

12-3. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $k \in \mathbb{N}_0$.
Assume: $f_{0_V} = 0_W$.
Assume: $\forall j \in \mathcal{I}_V$, $\partial_j f \in \widehat{\mathcal{O}}_k^{VW}$. Show: $f \in \widehat{\mathcal{O}}_{k+1}^{VW}$.

Proof: By Theorem 99.2, it suffices to show: $\forall m \in \mathcal{I}_W$, $\pi_m^W \circ f \in \widehat{\mathcal{O}}_{k+1}^{V\mathbb{R}}$.
Given $m \in \mathcal{I}_W$. Want: $\pi_m^W \circ f \in \widehat{\mathcal{O}}_{k+1}^{V\mathbb{R}}$.
Let $g := \pi_m^W \circ f$. Want: $g \in \widehat{\mathcal{O}}_{k+1}^{V\mathbb{R}}$.
By Theorem 98.10, choose $\beta : \mathcal{I}_V \to \widehat{\mathcal{O}}_1^{VV}$ s.t.

$$g = \sum_{j \in \mathcal{I}_V} ((\partial_j g) \circ \beta_j) \cdot \pi_j^V \quad \text{near } 0_V.$$

Want: $\forall j \in \mathcal{I}_V$, $((\partial_j g) \circ \beta_j) \cdot \pi_j^V \in \widehat{\mathcal{O}}_{k+1}^{V\mathbb{R}}$.
Given $j \in \mathcal{I}_V$. Want: $((\partial_j g) \circ \beta_j) \cdot \pi_j^V \in \widehat{\mathcal{O}}_{k+1}^{V\mathbb{R}}$.
Since $\pi_j^V \in \mathcal{L}_V^W \subseteq \widehat{\mathcal{O}}_1^{V\mathbb{R}}$, it suffices to show: $(\partial_j g) \circ \beta_j \in \widehat{\mathcal{O}}_k^{V\mathbb{R}}$.
So, since $\beta_j \in \widehat{\mathcal{O}}_1^{VV}$, it suffices to show: $\partial_j g \in \widehat{\mathcal{O}}_k^{V\mathbb{R}}$.
So, since $\partial_j g = \partial_j((\pi_m^W) \circ f) \supseteq \pi_m^W \circ (\partial_j f)$,
it suffices to show: $\pi_m^W \circ (\partial_j f) \in \widehat{\mathcal{O}}_k^{V\mathbb{R}}$.
By assumption, $\partial_j f \in \widehat{\mathcal{O}}_k^{VW}$.
So, by Theorem 99.2, we have $\pi_m^W \circ (\partial_j f) \in \widehat{\mathcal{O}}_k^{V\mathbb{R}}$, as desired. QED

12-4. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $k \in \mathbb{N}_0$.
Assume: $f_{0_V} = 0_W$. Assume: $\forall j \in \mathcal{I}_V$, $(\partial_j f)_{0_V} = 0_W$.
Assume: $\forall i, j \in \mathcal{I}_V$, $\partial_i \partial_j f \in \widehat{\mathcal{O}}_k^{VW}$. Show: $f \in \widehat{\mathcal{O}}_{k+2}^{VW}$.

Proof: Since $f_{0_V} = 0_W$, by HW#12-3,
it suffices to show: $\forall j \in \mathcal{I}_V$, $\partial_j f \in \widehat{\mathcal{O}}_{k+1}^{VW}$.
Given $j \in \mathcal{I}_V$. Want: $\partial_j f \in \widehat{\mathcal{O}}_{k+1}^{VW}$.
Let $g := \partial_j f$. Want: $g \in \widehat{\mathcal{O}}_{k+1}^{VW}$.
Since $g_{0_V} = (\partial_j f)_{0_V} = 0_W$, by HW#12-3,
it suffices to show: $\forall i \in \mathcal{I}_V$, $\partial_i g \in \widehat{\mathcal{O}}_k^{VW}$.
Given $i \in \mathcal{I}_V$. Want: $\partial_i g \in \widehat{\mathcal{O}}_k^{VW}$.
We have $\partial_i g = \partial_i \partial_j f \in \widehat{\mathcal{O}}_k^{VW}$, as desired. QED

12-5. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$.
Assume: $f_{0_V} = 0_W$.
Assume: $\forall j \in \mathcal{I}_V$, $\partial_j f$ is both defined near 0_V and continuous at 0_V.
Show: $0_V \in \mathrm{dom}\,[f']$.

Proof: Define $q : \mathcal{I}_V \to W$ by $q_j = (\partial_j f)_{0_V}$.
Define $L \in \mathcal{L}_V^W$ by $L(x) = \displaystyle\sum_{j \in \mathcal{I}_V} x_j q_j$. Want: $f'_{0_V} = \mathrm{s}\ell_L^{VW}$.
Want: $D_{0_V}^{VW} f = L$. Want: $f_{0_V}^T - L \in \mathcal{O}_1^{VW}$.
Since $f_{0_V} = 0_W$, we see that $f_{0_V}^T = f$. Want: $f - L \in \mathcal{O}_1^{VW}$.
By Theorem 99.6, it suffices to show: $\forall j \in \mathcal{I}_V$, $\partial_j(f - L) \in \mathcal{O}_0^{VW}$.
Given $j \in \mathcal{I}_V$. Want: $\partial_j(f - L) \in \mathcal{O}_0^{VW}$.

Claim: $\partial_j L = C_V^{q_j}$.
Proof of Claim:
Want: $\forall x \in V$, $(\partial_j L)_x = (C_V^{q_j})_x$.
Given $x \in V$. Want: $(\partial_j L)_x = (C_V^{q_j})_x$. Want: $(\partial_j L)_x = q_j$.
Let $u := \varepsilon_j^V$. Then $\partial_u L = \partial_j L$. Want: $(\partial_u L)_x = q_j$.
Let $i := i_x^u$. Want: $(L \circ i)'_0 = q_j$.
We have
$$L(u) = \sum_{i \in \mathcal{I}_V} u_i q_i = \sum_{i \in \mathcal{I}_V} (\varepsilon_j^V)_i \cdot q_i$$
$$= \left((\varepsilon_j^V)_j \cdot q_j\right) + \left(\sum_{i \in \mathcal{I}_V \setminus \{j\}} (\varepsilon_j^V)_i \cdot q_i\right)$$
$$= (1 \cdot q_j) + \left(\sum_{i \in \mathcal{I}_V \setminus \{j\}} 0 \cdot q_i\right) = q_j + 0 = q_j$$
Then $L(u) = q_j$. Let $y := L(x)$.
Then, $\forall t \in \mathbb{R}$, we have:
$$(L \circ i)_t = L(i_t) = L(x + tu) = (L(x)) + t \cdot (L(u)) = y + tq_j.$$
Then: $\forall t \in \mathbb{R}$, $(L \circ i)_t = y + tq_j$.
Then: $\forall h \in \mathbb{R}$, $(L \circ i)_0^T(h) = (y + (0 + h) \cdot q_j) - (y + 0 \cdot q_j) = hq_j$.
Then $(L \circ i)_0^T \in \mathcal{L}_{\mathbb{R}}^W$. Then: $D_0(L \circ i) = L \circ i$.
Then $(L \circ i)'_0 = \mathrm{s}\ell_{L \circ i}^{\mathbb{R}W} = (L \circ i)(1) = 1 \cdot q_j = q_j$, as desired.
End of proof of Claim.

We have: $\partial_j(f - L) \supseteq (\partial_j f) - (\partial_j L)$.
So, by the claim, $\partial_j(f - L) \supseteq (\partial_j f) - (C_V^{q_j})$.
Want: $(\partial_j f) - (C_V^{q_j}) \in \mathcal{O}_0^{VW}$. Want: $(\partial_j f) - (C_V^{q_j}) \in \mathrm{CVZ}_V^W$.

By assumption, $\partial_j f$ is both defined near 0_V and continuous at 0_V.
Also, $C_V^{q_j}$ is both defined near 0_V and continuous at 0_V.
Then $(\partial_j f) - (C_V^{q_j})$ is both defined near 0_V and continuous at 0_V.
Want: $((\partial_j f) - (C_V^{q_j}))_{0_V} = 0_W$.
By definition of q, we have $q_j = (\partial_j f)_{0_V}$.
We compute $((\partial_j f) - (C_V^{q_j}))_{0_V} = ((\partial_j f)_{0_V}) - ((C_V^{q_j})_{0_V})$
$= q_j - q_j = 0_W$, as desired. QED

Homework 11: Due on Tuesday 16 April

11-1. Let $V := \mathbb{R}^2$, $S := V \otimes V$, $f : \mathbb{R} \dashrightarrow V$, $g : V \dashrightarrow \mathbb{R}$.
Let $p \in \mathbb{R}$. Assume $f'_p = (1,2)$ and $f''_p = (3,4)$.
Let $q := f_p$. Assume $g'_q = (5,6)$ and $g''_q = \begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix}$.
Compute: $(g \circ f)'_p$ and $(g \circ f)''_p$.

Solution:
We have $(g \circ f)'_p \;=^*\; g'_q \bullet_V f'_p$
$= (5,6) \bullet_V (1,2) = 5 \cdot 1 + 6 \cdot 2 = 17 \neq ☹$.
Then $(g \circ f)'_p = 17$.
We have: $(f'_p)^{\otimes 2} = \begin{bmatrix} 1\cdot 1 & 1\cdot 2 \\ 2\cdot 1 & 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
Also, $(g \circ f)''_p \;=^*\; g''_q \bullet_S ((f'_p)^{\otimes 2}) + g'_q \bullet_V f''_p$
$= \begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} \bullet_S \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + (5,6) \bullet_V (3,4)$
$= 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 2 + 0 \cdot 4 + 5 \cdot 3 + 6 \cdot 4$
$= 7 + 16 + 18 + 0 + 15 + 24 = 80 \neq ☹$.
Then $(g \circ f)''_p = 80$.

11-2. Let $V := \mathbb{R}^2$, $S := V \otimes V$, $A := \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
Show: $\forall q \in V$, $A \bullet_S (q^{\otimes 2}) \geqslant 0$.

Proof: Given $q \in V$. Want: $A \bullet_S (q^{\otimes 2}) \geqslant 0$.
Let $x := q_1$ and $y := q_2$. Then $q = (x,y)$.
Then $q^{\otimes 2} = \begin{bmatrix} x \cdot x & x \cdot y \\ y \cdot x & y \cdot y \end{bmatrix} = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}$.

Then $A \bullet_S (q^{\otimes 2})$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \bullet_S \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}$$
$$= 1 \cdot x^2 + 2 \cdot xy + 2 \cdot xy + 4 \cdot y^2$$
$$= x^2 + 4xy + 4y^2$$
$$= (x+2y)^2 \geqslant 0,$$ as desired. QED

11-3. Let $V := \mathbb{R}^2$, $S := V \otimes V$, $a, b, c \in \mathbb{R}$, $A := \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $z := 0_2$.

Assume: $a > 0$ and $ac - b^2 > 0$.

Show: $\forall q \in V_z^\times$, $A \bullet_S (q^{\otimes 2}) > 0$.

Proof: Given $q \in V_z^\times$. Want: $A \bullet_S (q^{\otimes 2}) > 0$.

Let $x := q_1$ and $y := q_2$. Then $q = (x, y)$.

Then $q^{\otimes 2} = \begin{bmatrix} x \cdot x & x \cdot y \\ y \cdot x & y \cdot y \end{bmatrix} = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}$.

We have $A \bullet_S (q^{\otimes 2})$

$$= \begin{bmatrix} a & b \\ b & c \end{bmatrix} \bullet_S \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}$$
$$= a \cdot x^2 + b \cdot xy + b \cdot xy + c \cdot y^2$$
$$= ax^2 + 2bxy + cy^2$$
$$= a \cdot \left(x^2 + \frac{2by}{a} x \right) + cy^2$$
$$= a \cdot \left(x^2 + \frac{2by}{a} x + \frac{b^2 y^2}{a^2} \right) + \frac{acy^2}{a} - \frac{b^2 y^2}{a}$$
$$= a \cdot \left(x + \frac{by}{a} \right)^2 + \frac{ac - b^2}{a} \cdot y^2.$$

Since $a > 0$ and $\left(x + \dfrac{by}{a} \right)^2 \geqslant 0$,

we see that $a \cdot \left(x + \dfrac{by}{a} \right)^2 \geqslant 0$.

Since $a > 0$ and $ac - b^2 > 0$ and $y^2 \geqslant 0$,

we see that $\dfrac{ac - b^2}{a} \cdot y^2 \geqslant 0$.

Then $A \bullet_S (q^{\otimes 2}) = a \cdot \left(x + \dfrac{by}{a} \right)^2 + \dfrac{ac - b^2}{a} \cdot y^2 \geqslant 0$.

Want: $A \bullet_S (q^{\otimes 2}) \neq 0$.

Assume: $A \bullet_S (q^{\otimes 2}) = 0$. Want: Contradiction.

We have $a \cdot \left(x + \dfrac{by}{a} \right)^2 + \dfrac{ac - b^2}{a} \cdot y^2 \quad = \quad A \bullet_S (q^{\otimes 2}) \quad = \quad 0.$

Then $\dfrac{ac - b^2}{a} \cdot y^2 \quad = \quad - \left(a \cdot \left(x + \dfrac{by}{a} \right)^2 \right) \quad \leqslant 0.$

So, since $\dfrac{ac - b^2}{a} \cdot y^2 \quad \geqslant \quad 0$, we see that $\dfrac{ac - b^2}{a} \cdot y^2 \quad = \quad 0.$

So, since $ac - b^2 > 0$ and $a > 0$, we see that $y^2 = 0$, and so $y = 0$.

Then $A \bullet_S (q^{\otimes 2}) = a \cdot \left(x + \dfrac{b \cdot 0}{a} \right)^2 + \dfrac{ac - b^2}{a} \cdot 0^2 = ax^2.$

Then $ax^2 = A \bullet_S (q^{\otimes 2}) = 0$, so $ax^2 = 0$.

So, since $a > 0$, we get $x^2 = 0$, and so $x = 0$.

Since $q \in V_z^\times$, we conclude that $q \neq z$.

Then $0_2 = z \neq q = (x, y) = (0, 0) = 0_2$, so $0_2 \neq 0_2$.

Contradiction. QED

11-4. Let $V := \mathbb{R}^2$, $S := V \otimes V$, $f : \mathbb{R} \dashrightarrow V$, $g : V \dashrightarrow \mathbb{R}$.

Let $p \in \mathbb{R}$. Assume $f'_p \neq (0, 0)$ and $f''_p \neq ☹$.

Let $q := f_p$. Assume $g'_q = (0, 0)$ and $g''_q = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.

Show: $(g \circ f)''_p > 0$.

Proof: We have: $(g \circ f)''_p \ =^* \ g''_q \bullet_S ((f'_p)^{\otimes 2}) + g'_q \bullet_V f''_p \neq ☹$,

so $(g \circ f)''_p = g''_q \bullet_S ((f'_p)^{\otimes 2}) + g'_q \bullet_V f''_p$.

So, since $g'_q = (0, 0)$, we get $(g \circ f)''_p = g''_q \bullet_S ((f'_p)^{\otimes 2})$.

Let $a := 1$, $b := 2$, $c := 5$, $A := \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $z := 0_2$.

We have $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = g''_q$ and $(g \circ f)''_p = g''_q \bullet_S ((f'_p)^{\otimes 2})$.

Then $(g \circ f)''_p = A \bullet_S ((f'_p)^{\otimes 2})$.

We have $a > 0$ and $ac - b^2 = 1 \cdot 5 - 2^2 = 1 > 0$.

Since $f''_p \neq ☹$, we get $p \in \mathrm{dom}\,[f''] \subseteq \mathrm{dom}\,[f']$, and so $f'_p \in \mathrm{im}\,[f']$.

Then $f'_p \in \mathrm{im}\,[f'] \subseteq V \otimes \mathbb{R} = V$.

So, since $f'_p \neq (0, 0) = 0_2 = z$, we conclude that $f'_p \in V_z^\times$.

Then, by HW#11-3, we have $A \bullet_S ((f'_p)^{\otimes 2}) > 0$.

Then: $(g \circ f)''_p = A \bullet_S ((f'_p)^{\otimes 2}) > 0$, as desired. QED

11-5. Let $V, W \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $i \in \mathcal{I}_V$, $j \in \mathcal{I}_W$, $W' := W \otimes V$.

Show: $(\pi^{W'}_{j\|i}) \circ (f') \quad \subseteq \quad (\pi^W_j) \circ (\partial_i f)$.

Proof: Want: $\forall x \in V$, $((\pi^{W'}_{j\|i}) \circ (f'))_x \ {}^*\!\!= \ ((\pi^W_j) \circ (\partial_i f))_x$.
Given $x \in V$. Want: $((\pi^{W'}_{j\|i}) \circ (f'))_x \ {}^*\!\!= \ ((\pi^W_j) \circ (\partial_i f))_x$.
Want: $(\pi^{W'}_{j\|i})(f'_x) \ {}^*\!\!= \ (\pi^W_j)((\partial_i f)_x)$.
Want: $[\ (\pi^{W'}_{j\|i})(f'_x) \ \neq ☹\ ] \Rightarrow [\ (\pi^{W'}_{j\|i})(f'_x) \ = \ (\pi^W_j)((\partial_i f)_x)\ ]$.
Assume $(\pi^{W'}_{j\|i})(f'_x) \ \neq$ ☹. Want: $(\pi^{W'}_{j\|i})(f'_x) \ = \ (\pi^W_j)((\partial_i f)_x)$.
Let $A := f'_x$. Want: $(\pi^{W'}_{j\|i})(A) \ = \ (\pi^W_j)((\partial_i f)_x)$.
Since $(\pi^{W'}_{j\|i})(f'_x) \ \neq$ ☹, we conclude that $f'_x \neq$ ☹, and so $f'_x \in \mathrm{im}\,[f']$.
Then $A = f'_x \in \mathrm{im}\,[f'] \subseteq W \otimes V$.
By HW#9-2, $(f'_x) \ *_{\mathbb{R}VW} \ \varepsilon^V_i \ \ {}^*\!\!= \ (\partial_{\varepsilon_i} f)_x$.
So, since $A = f'_x$ and $\partial_{\varepsilon_i} f = \partial_i f$, we get $A \ *_{\mathbb{R}VW} \ \varepsilon^V_i \ \ {}^*\!\!= \ (\partial_i f)_x$.
Since $A \in W \otimes V$ and $\varepsilon^V_i \in V$, we get $A \ *_{\mathbb{R}VW} \ \varepsilon^V_i \ \in W$.
In particular, $A \ *_{\mathbb{R}VW} \ \varepsilon^V_i \ \neq$ ☹.
Then ☹ $\neq \ A \ *_{\mathbb{R}VW} \ \varepsilon^V_i \ \ {}^*\!\!= \ (\partial_i f)_x$, so $A \ *_{\mathbb{R}VW} \ \varepsilon^V_i \ = \ (\partial_i f)_x$.
Then $(\pi^W_j)(A \ *_{\mathbb{R}VW} \ \varepsilon^V_i) \ = \ (\pi^W_j)((\partial_i f)_x)$.
We have $(A \ *_{\mathbb{R}VW} \ \varepsilon^V_i)_j \ = \ (A \ *_{\mathbb{R}VW} \ \varepsilon^V_i)_{j\|\varnothing}$

$$= \sum_{h \in \mathcal{I}_V} (A_{j\|h}) \cdot ((\varepsilon^V_i)_{h\|\varnothing})$$

$$= \sum_{h \in \mathcal{I}_V} (A_{j\|h}) \cdot ((\varepsilon^V_i)_h)$$

$$= \left(\sum_{h \in \mathcal{I}_V \setminus \{i\}} (A_{j\|h}) \cdot ((\varepsilon^V_i)_h) \right) + \left((A_{j\|i}) \cdot ((\varepsilon^V_i)_i \right)$$

$$= \left(\sum_{h \in \mathcal{I}_V \setminus \{i\}} (A_{j\|h}) \cdot 0 \right) + \left((A_{j\|i}) \cdot 1 \right)$$

$$= 0 + A_{j\|i} = A_{j\|i}.$$

Then $(\pi^{W'}_{j\|i})(A) \ = \ A_{j\|i} \ = \ (A \ *_{\mathbb{R}VW} \ \varepsilon^V_i)_j$
$= \ (\pi^W_j)(A \ *_{\mathbb{R}VW} \ \varepsilon^V_i) \ = \ (\pi^W_j)((\partial_i f)_x)$, as desired. QED

Homework 10: Due on Tuesday 9 April

10-1. Let $V, W, X \in \mathrm{TNSR}^+$, $f : V \dashrightarrow W$, $g : W \dashrightarrow X$, $u \in V$.
Show: $\forall p \in V$, $(\partial^{VX}_u (g \circ f))_p \ =^* \ (g'_{f_p}) \ *_{\mathbb{R}VW} \ ((\partial^{VW}_u f)_p)$.

Proof: Given $p \in V$. Want: $(\partial^{VX}_u (g \circ f))_p \ =^* \ (g'_{f_p}) \ *_{\mathbb{R}VW} \ ((\partial^{VW}_u f)_p)$.
Let $i := i^v_p$. Then $i_0 = p + 0 \cdot v = p$. Also, $(\partial^{VW}_u f)_p = (f \circ i)'_0$.
Let $\phi := f \circ i$. Then $\phi_0 = (f \circ i)_0 = f_{i_0} = f_p$. Also, $(\partial^{VW}_u f)_p = \phi'_0$.
By the Chain Rule, $(g \circ \phi)'_0 \ =^* \ (g'_{\phi_0}) \ *_{\mathbb{R}VW} \ (\phi'_0)$.

Then $(\partial_u^{VX}(g \circ f))_p \;=\; (g \circ f \circ i)'_0 \;=\; (g \circ \phi)'_0$
$\quad =^* \;\; (g'_{\phi_0}) \; *_{\mathbb{R}VW} \; (\phi'_0) \;=\; (g'_{f_p}) \; *_{\mathbb{R}VW} \; ((\partial_u^{VW} f)_p).$ QED

10-2. Let $S, V, W, Z \in \mathrm{TNSR}^+$, $\quad * \in \mathcal{B}^Z_{VW}$.
Let $f : S \dashrightarrow V$, $\quad g : S \dashrightarrow W$, $\quad u \in S$.
Show: $\forall p \in S,\;\; (\partial_u^{SZ}(f*g))_p \;=^*\; (\,((\partial_u^{SV} f)_p)*g_p\,) + (\,f_p*((\partial_u^{SW} g)_p)\,)$.

Proof: Given $p \in V$.
Want: $(\partial_u^{SZ}(f * g))_p \;=^*\; (\,((\partial_u^{SV} f)_p) * g_p\,) \;+\; (\,f_p * ((\partial_u^{SW} g)_p)\,)$.
Let $i := i^u_p$. Then $i_0 = p + 0 \cdot u = p$.
Also, $(\partial_u^{SV} f)_p = (f \circ i)'_0$ and $(\partial_u^{SW} g)_p = (g \circ i)'_0$.
Let $\phi := f \circ i$ and $\psi := g \circ i$ and $\chi := (f * g) \circ i$.
Then $(\partial_u^{SV} f)_p = \phi'_0$ and $(\partial_u^{SW} g)_p = \psi'_0$ and $(\partial_u^{SZ}(f * g))_p = \chi'_0$.
Also, $\phi_0 = f_{i_0} = f_p$ and $\psi_0 = g_{i_0} = g_p$.
Want: $\chi'_0 \;=^*\; \phi'_0 \,*\, \chi_0 \;+\; \phi_0 \,*\, \psi'_0$.

Claim: $\chi \;=\; \phi * \psi$.
Proof of Claim:
Want: $\forall t \in \mathbb{R}$, $\chi_t \;=\; (\phi * \psi)_t$. Given $t \in \mathbb{R}$. Want: $\chi_t \;=\; (\phi * \psi)_t$.
Let $q := i_t$. Then $\phi_t = (f \circ i)_t = f_q$, $\psi_t = (g \circ i)_t = g_q$ and
$\chi_t = ((f * g) \circ i)_t = (f * g)_q$.
Then $\chi_t \;=\; (f * g)_q \;=\; f_q * g_q \;=\; \phi_t * \psi_t \;=\; (\phi * \psi)_t$, as desired.
End of proof of Claim.

By the Claim and by the Product Rule, we have:
$$D_0^{\mathbb{R}Z}\chi \;=^*\; (D_0^{\mathbb{R}V}\phi) \,*\, \psi_0 \;+\; \phi_0 \,*\, (D_0^{\mathbb{R}W}\psi).$$
Then: $(D_0^{\mathbb{R}Z}\chi)(1) \;=^*\; ((D_0^{\mathbb{R}V}\phi)(1)) \,*\, \psi_0 + \phi_0 \,*\, ((D_0^{\mathbb{R}Z}\psi)(1))$.
So, since $\chi'_0 = \mathrm{s}\ell_\bullet^{\mathbb{R}Z}(D_0^{\mathbb{R}Z}\chi) = (D_0^{\mathbb{R}Z}\chi)(1)$,
and since $\phi'_0 = \mathrm{s}\ell_\bullet^{\mathbb{R}V}(D_0^{\mathbb{R}V}\phi) = (D_0^{\mathbb{R}V}\phi)(1)$,
and since $\psi'_0 = \mathrm{s}\ell_\bullet^{\mathbb{R}W}(D_0^{\mathbb{R}W}\psi) = (D_0^{\mathbb{R}W}\psi)(1)$,
we see that: $\chi'_0 \;=^*\; \phi'_0 \,*\, \chi_0 \;+\; \phi_0 \,*\, \psi'_0$, as desired. QED

10-3. Let $V, W, X \in \mathrm{TNSR}^+$, $\quad A \in X \otimes V \otimes W$, $\quad y \in V$, $\quad z \in W$.
Show: $(A \;*_{\mathbb{R},W,X\otimes V}\; z) \;*_{\mathbb{R}VX}\; y \;=\; A \;*_{\mathbb{R},V\otimes W,X}\; (y \otimes z)$.

Proof: Want: $\forall k \in \mathcal{I}_X$,
$$((A \;*_{\mathbb{R},W,X\otimes V}\; z) \;*_{\mathbb{R}VX}\; y)_k \;=\; (A \;*_{\mathbb{R},V\otimes W,X}\; (y \otimes z))_k.$$
Given $k \in \mathcal{I}_X$.
Want: $((A \;*_{\mathbb{R},W,X\otimes V}\; z) \;*_{\mathbb{R}VX}\; y)_k \;=\; (A \;*_{\mathbb{R},V\otimes W,X}\; (y \otimes z))_k$.

We compute:

$$
\begin{aligned}
(A \;\; *_{\mathbb{R},V\otimes W,X} \;\; (y\otimes z))_k \quad &= \quad (A \;\; *_{\mathbb{R},V\otimes W,X} \;\; (y\otimes z))_{k\|\varnothing} \\
&= \sum_{\ell\in\mathcal{I}_{V\otimes W}} A_{k\|\ell} \;\cdot\; (y\otimes z)_{\ell\|\varnothing} \\
&= \sum_{\ell\in\mathcal{I}_{V\otimes W}} A_{k\|\ell} \;\cdot\; (y\otimes z)_{\ell} \\
&= \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} \left(\; A_{k\|i\|j} \;\cdot\; (y\otimes z)_{i\|j} \;\right) \\
&= \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} \left(\; A_{k\|i\|j} \;\cdot\; (y \;\; *_{V\mathbb{R}W} \;\; z)_{i\|j} \;\right) \\
&= \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} \left(\; A_{k\|i\|j} \;\cdot\; \left(\sum_{m\in\mathcal{I}_{\mathbb{R}}} y_{i\|m} \;\cdot\; z_{m\|j}\right)\;\right) \\
&= \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} \left(\; A_{k\|i\|j} \;\cdot\; y_{i\|\varnothing} \;\cdot\; z_{\varnothing\|j} \;\right) \\
&= \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} \left(\; A_{k\|i\|j} \;\cdot\; y_i \;\cdot\; z_j \;\right).
\end{aligned}
$$

Then:

$$
\begin{aligned}
((A \;\; *_{\mathbb{R},W,X\otimes V} \;\; z) \;\; *_{\mathbb{R}VX} \;\; y)_k \quad &= \quad ((A \;\; *_{\mathbb{R},W,X\otimes V} \;\; z) \;\; *_{\mathbb{R}VX} \;\; y)_{k\|\varnothing} \\
&= \sum_{i\in\mathcal{I}_V} (A \;\; *_{\mathbb{R},W,X\otimes V} \;\; z)_{k\|i} \;\cdot\; y_{i\|\varnothing} \\
&= \sum_{i\in\mathcal{I}_V} (A \;\; *_{\mathbb{R},W,X\otimes V} \;\; z)_{k\|i} \;\cdot\; y_i \\
&= \sum_{i\in\mathcal{I}_V} (A \;\; *_{\mathbb{R},W,X\otimes V} \;\; z)_{k\|i\|\varnothing} \;\cdot\; y_i \\
&= \sum_{i\in\mathcal{I}_V} \left(\sum_{j\in\mathcal{I}_W} A_{k\|i\|j} \;\cdot\; z_{j\|\varnothing}\right) \;\cdot\; y_i \\
&= \sum_{i\in\mathcal{I}_V} \left(\sum_{j\in\mathcal{I}_W} A_{k\|i\|j} \;\cdot\; z_j\right) \;\cdot\; y_i \\
&= \sum_{i\in\mathcal{I}_V}\sum_{j\in\mathcal{I}_W} \left(\; A_{k\|i\|j} \;\cdot\; y_i \;\cdot\; z_j \;\right) \\
&= (A \;\; *_{\mathbb{R},V\otimes W,X} \;\; (y\otimes z))_k, \qquad \text{as desired.} \qquad \text{QED}
\end{aligned}
$$

10-4. Let $V,W,X\in\mathrm{TNSR}^+$, $\quad f:V \dashrightarrow W, \quad g:W\dashrightarrow X, \quad u,v\in V.$

Show: $\quad \partial_v^{VX}\partial_u^{VX}(g\circ f) \;\;\supseteq\;\; (g''\circ f) \;\; *_{\mathbb{R},W\otimes W,X} \;\; ((\partial_u^{VW}f)\otimes(\partial_v^{VW}f))$
$\qquad + \quad (g'\circ f) \;\; *_{\mathbb{R}WX} \;\; (\partial_v^{VW}\partial_u^{VW}f).$

Proof: Let $\chi := \partial_v^{VX}\partial_u^{VX}(g\circ f)$.
Let $\phi := (g''\circ f) \;\; *_{\mathbb{R},W\otimes W,X} \;\; ((\partial_u^{VW}f)\otimes(\partial_v^{VW}f))$.

Let $\psi := (g' \circ f) \quad *_{\mathbb{R}WX} \quad (\partial_v^{VW} \partial_u^{VW} f)$.
Want: $\chi \supseteq \phi + \psi$.
By HW#10-1, $\quad \partial_u^{VX}(g \circ f) \;\supseteq\; (g' \circ f) \; *_{\mathbb{R}WX} \; (\partial_u^{VW} f)$.
Then $\quad \partial_v^{VX} \partial_u^{VX}(g \circ f) \;\supseteq\; \partial_v^{VX}(\,(g' \circ f) \; *_{\mathbb{R}WX} \; (\partial_u^{VW} f)\,)$.
Recall: $\chi = \partial_v^{VX} \partial_u^{VX}(g \circ f)$. $\quad$ Let $\alpha := g' \circ f$ and $\kappa := \partial_u^{VW} f$.
Then $\quad \chi \;\supseteq\; \partial_v^{VX}(\,\alpha \; *_{\mathbb{R}WX} \; \kappa\,)$.
Let $* := *_{\mathbb{R}WX}$. $\quad$ Then $\quad \chi \;\supseteq\; \partial_v^{VX}(\,\alpha * \kappa\,)$.
Want: $\quad \partial_v^{VX}(\,\alpha * \kappa\,) \;\supseteq\; \phi + \psi$.
Let $Y := X \otimes W$. $\quad$ Then $* \in \mathcal{B}_{YW}^X$.
By HW#10-2, $\quad \partial_v^{VX}(\,\alpha * \kappa\,) \;\supseteq\; (\,(\partial_v^{VY}\alpha) * \kappa\,) + (\,\alpha * (\partial_v^{VW}\kappa)\,)$.
Want: $\quad (\partial_v^{VY}\alpha) * \kappa \;\supseteq\; \phi \quad$ and $\quad \alpha * (\partial_v^{VW}\kappa) \;=\; \psi$.
We have $\quad \alpha * (\partial_v^{VW}\kappa) \;=\; (g' \circ f) \; *_{\mathbb{R}WX} \; (\partial_v^{VW}\partial_u^{VW} f) \;=\; \psi$.
Want: $\quad (\partial_v^{VY}\alpha) * \kappa \;\supseteq\; \phi$.
Want: $\quad \forall q \in V, \quad ((\partial_v^{VY}\alpha) * \kappa)_q \;=^*\; \phi_q$.
Given $q \in V$. $\quad$ Want: $((\partial_v^{VY}\alpha) * \kappa)_q \;=^*\; \phi_q$.
By HW#10-1, $\quad \partial_u^{VY}(g' \circ f) \;\supseteq\; (g'' \circ f) \; *_{\mathbb{R}WY} \; (\partial_u^{VW} f)$.
Then $(\partial_v^{VY}\alpha)_q \;=\; (\partial_v^{VY}(g' \circ f))_q \;=^*\; ((g'' \circ f) \; *_{\mathbb{R}WY} \; (\partial_u^{VW} f))_q$.
Let $\quad A := (g'' \circ f)_q, \quad y := (\partial_u^{VW} f)_q, \quad z := (\partial_v^{VW} f)_q$.
Then $y = (\partial_u^{VW} f)_q = \kappa_q$.
Also, $\phi_q \;=\; ((g'' \circ f) \quad *_{\mathbb{R},W\otimes W,X} \quad ((\partial_u^{VW} f) \otimes (\partial_v^{VW} f)))_q$
$\qquad\qquad = \quad A \quad *_{\mathbb{R},W\otimes W,X} \quad (y \otimes z)$
By HW#10-3 (with V replaced by W),

$$(A \;\; *_{\mathbb{R},W,X\otimes W} \;\; z) \;\; *_{\mathbb{R}VW} \;\; y \quad = \quad A \;\; *_{\mathbb{R},W\otimes W,X} \;\; (y \otimes z).$$

Then $((\partial_v^{VY}\alpha) * \kappa)_q \;=\; (\partial_v^{VY}\alpha)_q \; * \; \kappa_q \;=\; (\partial_v^{VY}\alpha)_q \; * \; y$

$$\begin{aligned}
&=^* \;\; ((g'' \circ f) \;\; *_{\mathbb{R}WY} \;\; (\partial_u^{VW} f))_q \;\; * \;\; y \\
&= \;\; ((g'' \circ f)_q \;\; *_{\mathbb{R}WY} \;\; (\partial_u^{VW} f)_q) \;\; *_{\mathbb{R}WX} \;\; y. \\
&= \;\; (A \;\; *_{\mathbb{R},W,X\otimes W} \;\; z) \;\; *_{\mathbb{R}WX} \;\; y. \\
&= \;\; A \;\; *_{\mathbb{R},W\otimes W,X} \;\; (y \otimes z) \;\; = \;\; \phi_q.
\end{aligned}$$

QED

10-5. Let $V := \mathbb{R}^2, \quad W := \mathbb{R}, \quad g : V \dashrightarrow W, \quad z := 0_2, \quad i := i_z^{\varepsilon_1^V}$.
Assume: $(g_z = 0)$ & $((\partial_1^{VW} g)_z = 0)$. $\quad$ Show: $g \circ i \in \mathcal{O}_1^{WW}$.

Proof: We have $i_0 = z + 0 \cdot \varepsilon_1^V = z$ and $(\partial_1^{VW} g)_z = (\partial_{\varepsilon_1^V}^{VW} g)_z = (g \circ i)'_0$.
Then $\quad (g \circ i)_0 = g_{i_0} = g_z = 0 \quad$ and $\quad (g \circ i)'_0 = (\partial_1^{VW} g)_z = 0$.
Since $\quad (g \circ i)_0 = 0 \quad$ and $\quad (g \circ i)'_0 = 0, \quad$ it follows,
from HW#8-1 (with S and T replaced by W, and R by $g \circ i$),
that: $\quad g \circ i \in \mathcal{O}_1^{WW}, \quad$ as desired. $\quad$ QED

Homework 9: Due on Tuesday 2 April

9-1. Let $V, W \in \mathrm{TNSR}^+$, $\quad \alpha : V \dashrightarrow W$, $\quad \delta > 0$, $\quad B := B_V(0_V, \delta)$.
Assume: $B \subseteq \mathrm{dom}\,[\alpha]$. Assume: $\forall q \in B$, $|\alpha_q|_W \leqslant |q|_V$.
Show: $\alpha \in \widehat{\mathcal{O}}_1^{VW}$.

Proof: Let $z := 0_V$.
Since $z \in B_V(z, \delta) = B_V(0_V, \delta) = B$, by assumption, $|\alpha_z|_W \leqslant |z|_V$.
Then $|\alpha_z|_W \leqslant |z|_V = |0_V|_V = 0$, so $|\alpha_z|_W \leqslant 0$.
Then $0 \leqslant |\alpha_z|_W \leqslant 0$, so $|\alpha_z|_W = 0$, so $\alpha_z = 0_W$.
Then $\alpha_{0_V} = \alpha_z = 0_W$, so $\alpha_{0_V} = 0_W$. Let $\sigma := \mathrm{adj}_{0_V}^{0_W}\left(\dfrac{\alpha}{|\bullet|_V}\right)$.
As $\alpha_{0_V} = 0_W$, we get: ($\mathrm{dom}\,[\alpha] = \mathrm{dom}\,[\sigma]$) & ($\alpha = \sigma \cdot (|\bullet|_V)$).
Want: $\alpha \in (\mathrm{BNZ}_V^W) \cdot (|\bullet|_V)$. Want: $\sigma \in \mathrm{BNZ}_V^W$.
We have $B \in \mathrm{dom}\,[\alpha] = \mathrm{dom}\,[\sigma]$ and $B = B(0_V, \delta) \in \mathcal{B}_V(0_V)$.
Then $\sigma \in \mathrm{DNZ}_V^W$. By definition of σ, we have $\sigma_{0_V} = 0_W$.
Want: $\sigma_*(B)$ is bounded in W. Let $C := B_W(0_W, 2)$.
Since $C \in \mathcal{B}_W$, it suffices to show: $\sigma_*(B) \subseteq C$.
Want: $\forall q \in B \cap (\mathrm{dom}\,[\sigma])$, $\sigma_q \in C$.
Given $q \in B \cap (\mathrm{dom}\,[\sigma])$. Want: $\sigma_q \in C$.
Exactly one of the following is true:
(1) $q = 0_V$ or (2) $q \neq 0_V$.

Case (1):
We have $\sigma_q = \sigma_{0_V} = 0_W \in B_W(0, 2) = C$, as desired.
End of Case (1).

Case (2):
By definition of σ, $\sigma_q = \dfrac{\alpha_q}{|q|_V}$. Then $|\sigma_q|_W = \dfrac{|\alpha_q|_W}{|q|_V}$.
Since $q \in B \cap (\mathrm{dom}\,[\sigma]) \subseteq B$, by assumption, $|\alpha_q|_W \leqslant |q|_V$.
Then $|\sigma_q - 0_W|_W = |\sigma_q|_W = \dfrac{|\alpha_q|_W}{|q|_V} \leqslant 1 < 2$, so $\sigma_q \in B(0_W, 2)$.
Then $\sigma_q \in B(0_W, 1) = C$, as desired. *End of Case (2).* QED

9-2. Let $S, T \in \mathrm{TNSR}^+$, $\quad f : S \dashrightarrow T$, $\quad x, v \in S$.
Show: $(\partial_v^{ST} f)_x \;=^*\; (f'_x) \;{*_{\mathbb{R}ST}}\; v$.

Proof: Want: ($(f'_x) \;{*_{\mathbb{R}ST}}\; v \neq ☹$) $\Rightarrow$ ($(\partial_v^{ST} f)_x = (f'_x) \;{*_{\mathbb{R}ST}}\; v$).
Assume: $(f'_x) \;{*_{\mathbb{R}ST}}\; v \neq ☹$. Want: $(\partial_v^{ST} f)_x = (f'_x) \;{*_{\mathbb{R}ST}}\; v$.

Since $(f'_x) \;\; {}^*\mathbb{R}ST \;\; v \neq ☹$, we see that $f'_x \neq ☹$.
Then $\mathrm{s}\ell_\bullet^{ST}(D_x^{ST} f) = f'_x \neq ☹$, so $D_x^{ST} f \neq ☹$, so $D_x^{ST} f \in \mathcal{L}_S^T$.
Let $L := D_x^{ST} f$. Then $L \in \mathcal{L}_S^T$.
We have $\mathrm{s}\ell_L^{ST} = \mathrm{s}\ell_\bullet^{ST}(L) = \mathrm{s}\ell_\bullet^{ST}(D_x^{ST} f) = f'_x$.
Since $\mathrm{s}\ell_L^{ST} = f'_x$, by Theorem 85.6, we get $L_v \;=\; (f'_x) \;\; {}^*\mathbb{R}ST \;\; v$.
By HW#8-4, $(\partial_v^{ST} f)_x \;=^*\; (D_x^{ST} f)_v$.
Then $(\partial_v^{ST} f)_x \;=^*\; (D_x^{ST} f)_v \;=\; L_v \;=\; (f'_x) \;\; {}^*\mathbb{R}ST \;\; v \;\neq\; ☹$.
Then $(\partial_v^{ST} f)_x \;=\; (f'_x) \;\; {}^*\mathbb{R}ST \;\; v$, as desired. QED

9-3. Let $V := \mathbb{R}^2$, $W := \mathbb{R}$, $f : V \dashrightarrow W$, $z := 0_2$.
Assume: (f' is defined near z) & ($f_z = 0$).
Let $g := \partial_1^{VW} f$, $h := \partial_2^{VW} f$, $i := i_z^{\varepsilon_1^V}$.
Show: $\exists \sigma \in \widehat{\mathcal{O}}_1^{WW}$, $\exists \tau \in \widehat{\mathcal{O}}_1^{VV}$ s.t.
$$f \;=\; (g \circ i \circ \sigma \circ \pi_1^V) \cdot \pi_1^V \;+\; (h \circ \tau) \cdot \pi_2^V \qquad \text{near } z.$$

Proof: Since f' is defined near z, choose $C \in \mathcal{B}_V(z)$ s.t. $C \subseteq \mathrm{dom}\,[f']$.
Since $C \in \mathcal{B}_V(z)$, choose $r > 0$ s.t. $C = B_V(z, r)$.
Let $\delta := r/\sqrt{2}$ and $J := (-\delta; \delta)$. Then $J^2 \subseteq B_V(z, r)$.
Then $J^2 \subseteq B_V(z, r) = C \subseteq \mathrm{dom}\,[f]$.
Define $S : J \to 2^{\mathbb{R}}$ and $T : J^2 \to 2^{\mathbb{R}}$ by
$$S_x = \{\alpha \in [0|x] \text{ s.t. } f|_{(0,0)}^{(x,0)} \;=\; ((\partial_1^{VW} f)(\alpha, 0)) \cdot x\} \qquad \text{and}$$
$$T_x = \{\beta \in [0|y] \text{ s.t. } f|_{(x,0)}^{(x,y)} \;=\; ((\partial_2^{VW} f)(x, \beta)) \cdot y\}.$$

Claim 1: $\forall x \in J$, we have $\varnothing \neq S_x \subseteq J$.
Proof of Claim 1:
Given $x \in J$. Want: $\varnothing \neq S_x \subseteq J$.
By HW#8-5, we have $\varnothing \neq S_x$. Want: $S_x \subseteq J$.
By definition of S_x, we have $S_x \subseteq [0|x]$.
We have $0 \in (-\delta; \delta) = J$ and $x \in J$.
So, since J is an interval, we get $[0|x] \subseteq J$.
Then $S_x \subseteq [0|x] \subseteq J$, as desired.
End of proof of Claim 1.

Claim 2: $\forall x, y \in J$, we have $\varnothing \neq T_{(x,y)} \subseteq J$.
Proof of Claim 2:
Given $x, y \in J$. Want: $\varnothing \neq T_{(x,y)} \subseteq J$.
By Theorem 90.13, p. 218, we have $\varnothing \neq T_{(x,y)}$. Want: $T_{(x,y)} \subseteq J$.
By definition of $T_{(x,y)}$, we have $T_{(x,y)} \subseteq [0|y]$.

We have $0 \in (-\delta;\delta) = J$ and $y \in J$.
So, since J is an interval, we get $[0|y] \subseteq J$.
Then $T_{(x,y)} \subseteq [0|y] \subseteq J$, as desired.
End of proof of Claim 2.

Define $\sigma : J \dashrightarrow \mathbb{R}$ by $\sigma_x = \mathrm{CH}(S_x)$.
By Claim 1, $\quad \forall x \in J$, we have $\varnothing \neq S_x \subseteq J$.
Then $\quad \sigma : J \to J \quad$ and, $\quad \forall x \in J$, $\sigma_x \in S_x \subseteq J$.
Define $\mu : J^2 \dashrightarrow \mathbb{R}$ by $\mu_{(x,y)} = \mathrm{CH}(T_{(x,y)})$.
By Claim 2, $\quad \forall x, y \in J$, we have $\varnothing \neq T_{(x,y)} \subseteq J$.
Then $\quad \mu : J^2 \to J \quad$ and, $\quad \forall x, y \in J$, $\mu_{(x,y)} \in T_{(x,y)} \subseteq J$.
Define $\tau : J^2 \to J^2$ by $\tau_{(x,y)} = (x, \mu_{(x,y)})$.

Claim 3: $\sigma \in \widehat{\mathcal{O}}_1^{WW}$.
Proof of Claim 3:
Since $J = B_{\mathbb{R}}(0,\delta)$ and $J = \mathrm{dom}\,[\sigma]$, by HW#9-1,
it suffices to prove: $\quad \forall x \in J, \quad |\sigma_x|_W \leqslant |x|_W$.
Given $x \in J$. $\quad$ Want: $|\sigma_x|_W \leqslant |x|_W$.
By definition of S_x, we have $S_x \subseteq [0|x]$.
Then $\sigma_x \in S_x \subseteq [0|x]$.
Then $\sigma_x \in [0|x]$, so $|\sigma_x| \leqslant |x|$.
Then $|\sigma_x|_W = |\sigma_x| \leqslant |x| = |x|_W$, as desired.
End of proof of Claim 3.

Claim 4: $\tau \in \widehat{\mathcal{O}}_1^{VV}$.
Proof of Claim 4:
Let $B := B_V(z,\delta)$. $\quad$ Then $B \subseteq J^2 = \mathrm{dom}\,[\tau]$.
By HW#9-1, it suffices to prove: $\quad \forall q \in B, \quad |\tau_q|_V \leqslant |q|_V$.
Given $q \in B$. $\quad$ Want: $|\tau_q|_V \leqslant |q|_V$.
Let $x := q_1$, $y := q_2$. $\quad$ Then $q = (x,y)$ and $|q|_V^2 = x^2 + y^2$.
Since $(x,y) = q \in B \subseteq J^2$, we see that $x, y \in J$.
By definition of $T_{(x,y)}$, we have $T_{(x,y)} \subseteq [0|y]$.
Then $\mu_{(x,y)} \in T_{(x,y)} \subseteq [0|y]$.
Then $\mu_{(x,y)} \in [0|y]$, so $|\mu_{(x,y)}| \leqslant |y|$, so $(\mu_{(x,y)})^2 \leqslant y^2$.
Then $|\tau_q| = |\tau_{(x,y)}|_V^2 = |(x, \mu_{(x,y)})|_V^2 = x^2 + (\mu_{(x,y)})^2 \leqslant x^2 + y^2 = |q|_V^2$.
Since $0 \leqslant |\tau_q|_V^2 \leqslant |q|_V^2$, we get $|\tau_q|_V \leqslant |q|_V$, as desired.
End of proof of Claim 4.

By Claim 3 and Claim 4, we have $\sigma \in \widehat{\mathcal{O}}_1^{WW}$ and $\sigma \in \widehat{\mathcal{O}}_1^{WW}$.
Want: $f \;=\; (g \circ i \circ \sigma \circ \pi_1^V) \cdot \pi_1^V \;+\; (h \circ \tau) \cdot \pi_2^V$ near z.
Since $B_V(0,\delta) \in \mathcal{B}_V(z)$, it suffices to prove:

$$f \;=\; (g \circ i \circ \sigma \circ \pi_1^V) \cdot \pi_1^V \;+\; (h \circ \tau) \cdot \pi_2^V \qquad \text{on } B_V(0,\delta).$$

Want: $\forall q \in B_V(0,\delta), \quad f_q \;=\; ((g \circ i \circ \sigma \circ \pi_1^V) \cdot \pi_1^V \;+\; (h \circ \tau) \cdot \pi_2^V)_q$.
Given $q \in B_V(0,\delta)$. Want: $f_q \;=\; ((g \circ i \circ \sigma \circ \pi_1^V) \cdot \pi_1^V \;+\; (h \circ \tau) \cdot \pi_2^V)_q$.
Let $B := B_V(z,\delta)$. Let $x := q_1$, $y := q_2$. Then $q = (x,y)$.
Then $\pi_1^V(q) = x$ and $\pi_2^V(q) = y$.
Since $(x,y) = q \in B \subseteq J^2$, we see that $x, y \in J$.
Since $i = i_z^{\varepsilon_1^V}$, we have: $\forall t \in \mathbb{R}$, $i(t) = z + t\varepsilon_1^V = (0,0) + t(1,0) = (t,0)$.
Then $(i \circ \sigma \circ \pi_1^V)_q = (i \circ \sigma)_x = i(\sigma_x) = (\sigma_x, 0)$.
Want: $f_{(x,y)} \;=\; (g(\sigma_x,0)) \cdot x \;+\; (h(\tau_{(x,y)})) \cdot y$.
Since $\sigma_x \in S_x$, by definition of S_x,
we get $f|_{(0,0)}^{(x,0)} \;=\; ((\partial_1^{VW} f)(\sigma_x,0)) \cdot x$.
Then $f|_{(0,0)}^{(x,0)} \;=\; ((\partial_1^{VW} f)(\sigma_x,0)) \cdot x \;=\; (g(\sigma_x,0)) \cdot x$.
Since $\mu_{(x,y)} \in T_{(x,y)}$, by definition of $T_{(x,y)}$,
we get $f|_{(x,0)}^{(x,y)} \;=\; ((\partial_1^{VW} f)(x,\mu_{(x,y)})) \cdot y$.
Then $f|_{(x,0)}^{(x,y)} \;=\; ((\partial_1^{VW} f)(x,\mu_{(x,y)})) \cdot y \;=\; (h(x,\mu_{x,y})) \cdot y \;=\; (h(\tau_{x,y})) \cdot y$.
Since $f_{(0,0)} = f_z = 0$, we get $f_{(x,y)} = f|_{(0,0)}^{(x,y)}$.
Then $f_{(x,y)} \;=\; f|_{((0,0)}^{(x,y)} \;=\; (\, f|_{(0,0)}^{(x,0)}\,) \;+\; (\, f|_{(x,0)}^{(x,y)}\,)$
$\;=\; (g(\sigma_x,0)) \cdot x \;+\; (h(\tau_{(x,y)})) \cdot y$, as desired. QED

9-4. Let $V := \mathbb{R}^2$, $W := \mathbb{R}$, $h : V \dashrightarrow W$, $z := 0_2$.
Assume: $z \in \operatorname{dom}[h']$. Assume: $h_z = (\partial_1^{VW} h)_z = (\partial_2^{VW} h)_z = 0$.
Show: $h \in \mathcal{o}_1^{VW}$.

Proof: We have $h' : V \dashrightarrow W \otimes V$, so $\operatorname{dom}[h'] \subseteq V$ and $\operatorname{im}[h'] \subseteq W \otimes V$.
Since $z \in \operatorname{dom}[h']$, we get $h'_z \in \operatorname{im}[h']$.
Then $h'_z \in \operatorname{im}[h'] \subseteq W \otimes V = \mathbb{R} \otimes \mathbb{R}^2 = \mathbb{R}^2$.
Then $(h'_z)_1 \in \mathbb{R}$ and $(h'_z)_2 \in \mathbb{R}$. Then $(h'_z)_1 \neq ☹ \neq (h'_z)_2$.
We have $(\partial_1^{VW} h)_z \;=^*\; (h'_z)_1 \;\neq ☹$, so $(\partial_1^{VW} h)_z \;=\; (h'_z)_1$.
Also, $(\partial_2^{VW} h)_z \;=^*\; (h'_z)_2 \;\neq ☹$, so $(\partial_2^{VW} h)_z \;=\; (h'_z)_2$.
Then $(h'_z)_1 = (\partial_1^{VW} h)_z = 0$ and $(h'_z)_2 = (\partial_2^{VW} h)_z = 0$.
Then $h'_{0_V} = h'_z = (0,0) = 0_{W \otimes V}$. Also, $h_{0_V} = h_z = 0 = 0_W$.
Then, by HW#8-1, we get $h \in \mathcal{o}_1^{VW}$, as desired. QED

9-5. Let $V := \mathbb{R}^2$, $\quad W := \mathbb{R}$, $\quad f : V \dashrightarrow W$, $\quad z := 0_2$.

Let $\alpha := (\partial_1\partial_1 f)_z$, $\qquad \beta := (\partial_1\partial_2 f)_z$, $\qquad \delta := (\partial_2\partial_2 f)_z$,

$\lambda := (\partial_1 f)_z$, $\qquad \mu := (\partial_2 f)_z$, $\qquad \rho := f_z$.

Assume $\alpha \neq ☹$, $\quad \beta \neq ☹$, $\quad \delta \neq ☹$, $\quad \lambda \neq ☹$, $\quad \mu \neq ☹$, $\quad \rho \neq ☹$.

Define $\quad C \in \mathcal{C}_V^W$, $\quad L \in \mathcal{L}_V^W$, $\quad Q \in \mathcal{Q}_V^W \quad$ by

$$C(x,y) \;=\; \rho, \qquad L(x,y) \;=\; (\lambda,\mu) \;\bullet_V\; (x,y),$$

$$P(x,y) \;=\; \begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix} \bullet_{V\otimes V} \; ((x,y)^{\otimes 2}).$$

Let $R := f - \left(C + L + \dfrac{P}{2!} \right)$.

Show: $(\partial_1\partial_1 R)_z = (\partial_1\partial_2 R)_z = (\partial_2\partial_2 R)_z = (\partial_1 R)_z = (\partial_2 R)_z = R_z = 0$.

Proof: We have: $\forall x, y \in \mathbb{R}$,

$C_{(x,y)} = \rho$, $\quad (\partial_1 C)_{(x,y)} = 0$, $\quad (\partial_2 C)_{(x,y)} = 0$,

$(\partial_1\partial_1 C)_{(x,y)} = 0$, $\quad (\partial_1\partial_2 C)_{(x,y)} = 0$, $\quad (\partial_2\partial_2 C)_{(x,y)} = 0$,

$L_{(x,y)} = \lambda x + \mu y$, $\quad (\partial_1 L)_{(x,y)} = \lambda$, $\quad (\partial_2 L)_{(x,y)} = \mu$,

$(\partial_1\partial_1 L)_{(x,y)} = 0$, $\quad (\partial_1\partial_2 L)_{(x,y)} = 0$, $\quad (\partial_2\partial_2 L)_{(x,y)} = 0$,

$P_{(x,y)} = \alpha x^2 + 2\beta xy + \delta y^2$,

$(\partial_1 P)_{(x,y)} = 2\alpha x + 2\beta y$, $\quad (\partial_2 P)_{(x,y)} = 2\beta x + 2\delta y$,

$(\partial_1\partial_1 P)_{(x,y)} = 2\alpha$, $\quad (\partial_1\partial_2 P)_{(x,y)} = 2\beta$, $\quad (\partial_2\partial_2 P)_{(x,y)} = 2\delta$.

Then, since $z = 0_2 = (0,0)$, we get

$C_z = \rho$, $\quad (\partial_1 C)_z = 0$, $\quad (\partial_2 C)_z = 0$,

$(\partial_1\partial_1 C)_z = 0$, $\quad (\partial_1\partial_2 C)_z = 0$, $\quad (\partial_2\partial_2 C)_z = 0$,

$L_z = 0$, $\quad (\partial_1 L)_z = \lambda$, $\quad (\partial_2 L)_z = \mu$,

$(\partial_1\partial_1 L)_z = 0$, $\quad (\partial_1\partial_2 L)_z = 0$, $\quad (\partial_2\partial_2 L)_z = 0$,

$P_z = 0$,

$(\partial_1 P)_z = 0$, $\quad (\partial_2 P)_z = z$,

$(\partial_1\partial_1 P)_z = 2\alpha$, $\quad (\partial_1\partial_2 P)_z = 2\beta$, $\quad (\partial_2\partial_2 P)_z = 2\delta$.

Then

$$
\begin{aligned}
R_z \;&=^*\; f_z - (\rho + 0 + 0) \;=\; \rho - \rho \;=\; 0 \;\neq\; ☹,\\
(\partial_1 R)_z \;&=^*\; (\partial_1 f)_z - (0 + \lambda + 0) \;=\; \lambda - \lambda \;=\; 0 \;\neq\; ☹,\\
(\partial_2 R)_z \;&=^*\; (\partial_2 f)_z - (0 + \mu + 0) \;=\; \mu - \mu \;=\; 0 \;\neq\; ☹,\\
(\partial_1\partial_1 R)_z \;&=^*\; (\partial_1\partial_1 f)_z - (0 + 0 + (2\alpha/2)) \;=\; \alpha - \alpha \;=\; 0 \;\neq\; ☹,\\
(\partial_1\partial_2 R)_z \;&=^*\; (\partial_1\partial_2 f)_z - (0 + 0 + (2\beta/2)) \;=\; \beta - \beta \;=\; 0 \;\neq\; ☹,\\
(\partial_2\partial_2 R)_z \;&=^*\; (\partial_2\partial_2 f)_z - (0 + 0 + (2\delta/2)) \;=\; \delta - \delta \;=\; 0 \;\neq\; ☹.
\end{aligned}
$$

Then $(\partial_1\partial_1 R)_z = (\partial_1\partial_2 R)_z = (\partial_2\partial_2 R)_z = (\partial_1 R)_z = (\partial_2 R)_z = R_z = 0$,

as desired. QED

Homework 8: Due on Tuesday 26 March

8-1. Let $S, T \in \text{TNSR}^+$, $\quad R : S \dashrightarrow T$.
Assume ($R_{0_S} = 0_T$) & ($R'_{0_S} = 0_{T\otimes S}$). $\qquad$ Show: $R \in \mathcal{O}_1^{ST}$.

Proof: Let $L := D_{0_S} R$ and $A := R'_{0_S}$. Then $L = \text{Lin}_A^{ST}$ and $A = 0_{T\otimes S}$.
Then $L = \text{Lin}_{0_{T\otimes S}}^{ST} = \mathbf{0}_S^T$. $\quad$ Then $L \neq ☹$.
Since $D_{0_S} R = L \neq ☹$, it follows that $R_{0_S}^T - L \in \mathcal{O}_1^{ST}$.
Since $R_{0_S} = 0_T$, it follows that $R_{0_S}^T = R$.
Then $R = R - \mathbf{0}_S^T = R_{0_S}^T - \mathbf{0}_S^T = R_{0_S}^T - L \in \mathcal{O}_1^{ST}$, as desired. QED

8-2. Let $S, T \in \text{TNSR}^+$, $\quad f : S \dashrightarrow T$.
Let $p \in \text{dom}\,[f']$, $\quad L := \text{Lin}_{f'_p}^{ST}$, $\quad R := f_p^T - L$.
Show: ($R_{0_S} = 0_T$) & ($R'_{0_S} = 0_{T\otimes S}$).

Proof: We have $R_{0_S} = (f_p^T - L)(0_S) = [f_p^T(0_S)] - [L(0_S)]$.
Since $p \in \text{dom}\,[f'] \subseteq \text{dom}\,[f]$, we get $f_p^T(0_S) = 0_T$.
Since $L \in \mathcal{L}_S^T$, we get $L(0_S) = 0_T$.
Then $R_{0_S} = R(0_S) = (f_p^T - L)(0_S) = [f_p^T(0_S)] - [L(0_S)] = 0_T - 0_T = 0_T$.
Want: $R'_{0_S} = 0_{T\otimes S}$.
We have $D_{0_S}^{ST}(f_p^T) = D_{p+0_S}^{ST} f = D_p^{ST} f = \text{Lin}_{f'_p}^{ST} = L$.
Also, since $L \in \mathcal{L}_S^T$, we get $D_{0_S}^{ST} L = L$.
Then $D_{0_S}^{ST} R \;=\; D_{0_S}^{ST}(f_p^T - L) \;=^*\; [D_{0_S}^{ST}(f_p^T)] - [D_{0_S}^{ST}(L)]$
$\qquad\qquad\qquad\qquad = \; L - L \; = \; \mathbf{0}_S^T$.
Since $D_{0_S} R \;=^*\; \mathbf{0}_S^T \;\neq\; ☹$, it follows that $D_{0_S} R \;=\; \mathbf{0}_S^T$.
Then $R - 0_S = \text{s}\ell_\bullet^{ST}(D_{0_S} R) = \text{s}\ell_\bullet^{ST}(\mathbf{0}_S^T) = 0_{T\otimes S}$. QED

8-3. Let $S \in \text{TNSR}^+$, $\quad p, v \in S$, $\quad i := i_p^v$. $\qquad$ Show: $i' = C_{\mathbb{R}}^v$.

Proof: We have: $\forall t \in \mathbb{R}$, $i(t) = i_p^v(t) = p + tv$.
Want: $\forall t \in \mathbb{R}$, $\quad i'(t) = C_{\mathbb{R}}^v(t)$.
Given $t \in \mathbb{R}$. $\qquad$ Want: $i'(t) = C_{\mathbb{R}}^v(t)$. $\qquad$ Want $i'_t = v$.
Define $L \in \mathcal{L}_{\mathbb{R}}^S$ by $L(h) = hv$. $\qquad$ Then $\text{s}\ell_\bullet^{\mathbb{R}S}(L) = \text{s}\ell_L^{\mathbb{R}S} = L(1) = v$.
We have: $\forall h \in \mathbb{R}$, $\quad i_t^T(h) = [i(t+h)] - [i(t)]$
$\qquad\qquad = [p + (t+h)v] - [p + tv] = hv = L(h)$.
Then $i_t^T = L$, so $i_t^T - L = L - L = \mathbf{0}_{\mathbb{R}}^S \in \mathcal{O}_1^{\mathbb{R}S}$.
Since $i_t^T - L \in \mathcal{O}_1^{\mathbb{R}S}$, it follows that $L = D_t^{\mathbb{R}S} i$.
Then $i'_t = \text{s}\ell_\bullet^{\mathbb{R}S}(D_t^{\mathbb{R}S} i) = \text{s}\ell_\bullet^{\mathbb{R}S}(L) = v$, as desired. QED

8-4. Let $S, T \in \text{TNSR}^+$, $\quad f : S \dashrightarrow T$, $\quad p, v \in S$.

Show: $(\partial_v^{ST} f)_p \;=^*\; (D_p^{ST} f)_v$.

Proof: Let $i := i_p^v$ and let $g := f \circ i$.
Then $(\partial_v^{ST} f)_p = (f \circ i)'_0 = g'_0$.
Also, we have: $i_0 = i(0) = i_p^v(0) = p + 0 \cdot v = p$.
By the Linearization Chain Rule,
we have $D_0^{\mathbb{R}T}(f \circ i) \;=^*\; (D_{i_0}^{ST} f) \circ (D_0^{\mathbb{R}S} i)$.
Let $L := D_0^{\mathbb{R}S} i$ and let $M := D_0^{\mathbb{R}T} g$.
Then $M \;=\; D_0^{\mathbb{R}T}(f \circ i) \;=^*\; (D_{i_0}^{ST} f) \circ (D_0^{\mathbb{R}S} i) \;=\; (D_p^{ST} f) \circ L$.
Then $M(1) \;=^*\; ((D_p^{ST} f) \circ L)(1) \;=\; (D_p^{ST} f)(L(1))$.
We have $g'_0 = \text{s}\ell_\bullet^{\mathbb{R}T}(D_0^{\mathbb{R}T} g) = \text{s}\ell_\bullet^{\mathbb{R}T}(M) = \text{s}\ell_M^{\mathbb{R}T} = M(1)$.
By HW#8-3, $i' = C_{\mathbb{R}}^v$. $\qquad$ Then $i'_0 = i'(0) = C_{\mathbb{R}}^v(0) = v$.
We have $i'_0 = \text{s}\ell_\bullet^{\mathbb{R}S}(D_0^{\mathbb{R}S} i) = \text{s}\ell_\bullet^{\mathbb{R}S}(L) = \text{s}\ell_L^{\mathbb{R}S} = L(1)$.
Then $(\partial_v^{ST} f)_p \;=\; g'_0 \;=\; M(1) \;=^*\; (D_p^{ST})(L(1))$
$\;=\; D_p^{ST}(i'_0) \;=\; D_p^{ST}(v)$, as desired. QED

8-5. Let $V := \mathbb{R}^2$, $\quad W \in \mathbb{R}$, $\quad R : V \dashrightarrow W$,
$\delta > 0$, $\quad J := (-\delta; \delta)$, $\quad x \in J$.

Assume: $\quad J^2 \subseteq \text{dom}\,[R']$.

Show: $\quad \exists \alpha \in [0|x] \quad$ s.t. $\quad R|_{(0,0)}^{(x,0)} \;=\; ((\partial_1^{VW} R)(\alpha, 0)) \cdot x$.

Proof: Let $i := i_{(0,0)}^{(1,0)}$. $\qquad$ By HW#8-3, we have $i' = C_{\mathbb{R}}^{(1,0)}$.
Also: $\forall t \in \mathbb{R}$, $i_t = i(t) = (0,0) + t \cdot (1,0) = (t, 0)$.
Let $f := R \circ i$. $\quad$ Then $f_x = R(i_x) = R(x,0)$ and $f_0 = R(i_0) = R(0,0)$.

Claim: $J \subseteq \text{dom}\,[f']$.
Proof of Claim:
Want: $\forall t \in J$, $t \in \text{dom}\,[f']$.
Given $t \in J$. $\qquad$ Want: $t \in \text{dom}\,[f']$.
Since $t \in \mathbb{R} = \text{dom}\,[C_{\mathbb{R}}^{(0,1)} = i'$ and $i_t = (t, 0) \in J^2 \subseteq \text{dom}\,[R']$,
we conclude that $t \in \text{dom}\,[(R \circ i)']$.
Then $t \in \text{dom}\,[(R \circ i)'] = \text{dom}\,[f']$, as desired.
End of proof of Claim.

Since $x \in J$, by the Claim and the MVT,
choose $\alpha \in [0|x]$ s.t. $f_x - f_0 = (f'_\alpha) \cdot (x - 0)$.
Want: $R|_{(0,0)}^{(x,0)} \;=\; ((\partial_1^{VW} R)(\alpha, 0)) \cdot x$.

We have $R|_{(0,0)}^{(x,0)} = [R(x,0)] - [R(0,0)] = f_x - f_0 = (f'_\alpha)\cdot(x-0) = (f'_\alpha)\cdot x$.
Want: $f'_\alpha = (\partial_1^{VW} R)(\alpha, 0)$.
We have $D_\alpha^{\mathbb{R}W} f \;=\; D_\alpha^{\mathbb{R}W}(R \circ i) \;=^*\; (D_{i(\alpha)}^{VW} R) \circ (D_\alpha^{\mathbb{R}V} i)$.
Also, $f'_\alpha = (D_\alpha^{\mathbb{R}W} f)(1)$ and $i'_\alpha = (D_\alpha^{\mathbb{R}V} i)(1)$. Let $u := \varepsilon_1^V$.
Then $(D_\alpha^{\mathbb{R}V} i)(1) = i'_\alpha = i'(\alpha) = C_{\mathbb{R}}^{(1,0)}(\alpha) = (1,0) = \varepsilon_1^V = u$.
Then $f'_\alpha \;=\; (D_\alpha^{\mathbb{R}W} f)(1) \;=^*\; ((D_{i(\alpha)}^{VW} R) \circ (D_\alpha^{\mathbb{R}V} i))(1) \;=\; (D_{i(\alpha)}^{VW} R)(u)$.
Since $0, x \in J$ and since J is an interval, we get $[0|x] \subseteq J$.
Then $\alpha \in [0|x] \subseteq J$. So, as $0 \in J$, we get $(\alpha, 0) \in J^2$.
Then $i(\alpha) = (\alpha, 0) \in J^2 \subseteq \operatorname{dom}[R']$, so $D_{i(\alpha)}^{VW} R \in \mathcal{L}_V^W$.
Then $(D_{i(\alpha)}^{VW} R)(u) \in W$, so $(D_{i(\alpha)}^{VW} R)(u) \neq$ ☹.
Since $f'_\alpha \;=^*\; (D_{i(\alpha)}^{VW} R)(u) \;\neq\;$ ☹, we conclude that $f'_\alpha \;=\; (D_{i(\alpha)}^{VW} R)(u)$.
Since $u = \varepsilon_1^V$, we get $\partial_u^{VW} R = \partial_1^{VW} R$.
We have $(\partial_u^{VW} R)(i(\alpha)) \;=^*\; (D_{i(\alpha)}^{VW} R)_u \;=\; (D_{i(\alpha)}^{VW} R)(u)$.
Then $(\partial_u^{VW} R)(i(\alpha)) \;=^*\; (D_{i(\alpha)}^{VW} R)(u) \;\neq\;$ ☹,
so $(\partial_u^{VW} R)(i(\alpha)) \;=\; (D_{i(\alpha)}^{VW} R)(u)$.
Then $f'_\alpha = (D_{i(\alpha)}^{VW} R)(u) = (\partial_u^{VW} R)(\alpha, 0) = (\partial_1^{VW} R)(\alpha, 0)$. QED

Homework 7: Due on Tuesday 12 March

7-1. Let $V, W \in$ TNSR and let $L, M \in \mathcal{L}_V^W$.
Assume: $[L]_V^W = [M]_V^W$. Show: $L = M$.

Proof: Want: $\forall x \in V$, $L_x = M_x$.
Given $x \in V$. Want: $L_x = M_x$.
Let $A := \mathrm{s}\ell_L^{VW}$. By Theorem 85.6, $L_x \;=\; A \;*_{\mathbb{R}VW}\; x$.
Let $B := \mathrm{s}\ell_M^{VW}$. By Theorem 85.6, $M_x \;=\; B \;*_{\mathbb{R}VW}\; x$.
We have $A = \mathrm{s}\ell_L^{VW} = [L]_V^W = [M]_V^W = \mathrm{s}\ell_M^{VW} = B$, so $A = B$.
Then $L_x \;=\; A \;*_{\mathbb{R}VW}\; x \;=\; B \;*_{\mathbb{R}VW}\; x \;=\; M_x$. QED

7-2. Let $V, W \in$ TNSR and let $A \in W \otimes V$.
Show: $\exists L \in \mathcal{L}_V^W$ s.t. $[L]_V^W = A$.

Proof: Define $L \in \mathcal{L}_V^W$ by $L_x \;=\; A \;*_{\mathbb{R}VW}\; x$. Want: $[L]_V^W = A$.
We have $[L]_V^W = \mathrm{s}\ell_L^{VW} = (\mathrm{s}\ell_\bullet^{VW})(L)$.
Want: $(\mathrm{s}\ell_\bullet^{VW})(L) = A$. Want: $L = (\mathrm{s}\ell_\bullet^{VW})^{-1}(A)$.
Let $M := (\mathrm{s}\ell_\bullet^{VW})^{-1}(A)$. Want: $L = M$.
Want: $\forall x \in V$, $L_x = M_x$. Given $x \in V$. Want: $L_x = M_x$.
Let $B := \mathrm{s}\ell_M^{VW}$. By Theorem 85.6, $M_x \;=\; B \;*_{\mathbb{R}VW}\; x$.

Since $M = (\mathrm{s}\ell_\bullet^{VW})^{-1}(A)$, we get $(\mathrm{s}\ell_\bullet^{VW})(M) = A$.
Then $B = \mathrm{s}\ell_M^{VW} = (\mathrm{s}\ell_\bullet^{VW})(M) = A$.
Then $L_x = A *_{\mathbb{R}VW} x = B *_{\mathbb{R}VW} x = M_x$, as desired. QED

7-3. Let I be a finite set and let $z \in \mathbb{R}^I$.

Show: $$\sum_{j \in I} |z_j| \quad \leqslant \quad \sqrt{\# I} \cdot \sqrt{\sum_{j \in I} z_j^2}.$$

Proof: Define $x \in \mathbb{R}^I$ by $x_j = 1$. Then, $\forall j \in I$, $x_j^2 = 1$.

Then $\displaystyle\sum_{j \in I} x_j^2 = \#I$, so $\displaystyle\sqrt{\sum_{j \in I} x_j^2} = \sqrt{\#I}$.

Define $y \in \mathbb{R}^I$ by $y_j = |z_j|$. Then, $\forall j \in I$, $y_j^2 = z_j^2$.

Then $\displaystyle\sum_{j \in I} y_j^2 = \sum_{j \in I} z_j^2$, so $\displaystyle\sqrt{\sum_{j \in I} y_j^2} = \sqrt{\sum_{j \in I} z_j^2}$.

We have: $\forall j \in I$, $x_j y_j = 1 \cdot |z_j| = |z_j|$.

Then $\displaystyle\sum_{j \in I} x_j y_j = \sum_{j \in I} |z_j|$.

By Theorem 84.11, $\displaystyle\sum_{j \in I} x_j y_j \leqslant \sqrt{\sum_{j \in I} x_j^2} \cdot \sqrt{\sum_{j \in I} y_j^2}$.

Then $$\sum_{j \in I} |z_j| = \sum_{j \in I} x_j y_j \leqslant \sqrt{\sum_{j \in I} x_j^2} \cdot \sqrt{\sum_{j \in I} y_j^2} = \sqrt{\# I} \cdot \sqrt{\sum_{j \in I} z_j^2},$$
as desired. QED

7-4. Let $U, V, W \in \mathrm{TNSR}$ and let $B \in \mathcal{B}_{UV}^W$.

Show: $\exists C \geqslant 0$ s.t., $\forall p \in U$, $\forall q \in V$,
$$|B(p,q)|_W \quad \leqslant \quad C \cdot |p|_U \cdot |q|_V.$$

Proof: Let $K := \max\{\, |\, B(\varepsilon_i^U, \varepsilon_j^V)\,|_W \text{ s.t. } i \in \mathcal{I}_U,\ j \in \mathcal{I}_V \}$.
Then $K \geqslant 0$. Let $C := K \cdot \sqrt{\#\mathcal{I}_U} \cdot \sqrt{\#\mathcal{I}_V}$. Then $C \geqslant 0$.
Want: $\forall p \in U$, $\forall q \in V$, $|B(p,q)|_W \leqslant C \cdot |p|_U \cdot |q|_V$.
Given $p \in U$, $q \in V$. Want: $|B(p,q)|_W \leqslant C \cdot |p|_U \cdot |q|_V$.
Since $p = \displaystyle\sum_{i \in \mathcal{I}_U} p_i \varepsilon_i^U$ and $q = \displaystyle\sum_{j \in \mathcal{I}_V} q_j \varepsilon_j^V$, we get
$$B(p,q) = \sum_{i \in \mathcal{I}_U} \sum_{j \in \mathcal{I}_V} \bigg(p_i \cdot q_j \cdot B(\varepsilon_i^U, \varepsilon_j^V) \bigg).$$
So, by subadditivity and absolute homogeneity of $|\bullet|_W$, we get
$$|\, B(p,q)\,|_W \leqslant \sum_{i \in \mathcal{I}_U} \sum_{j \in \mathcal{I}_V} \bigg(|p_i| \cdot |q_j| \cdot |\, B(\varepsilon_i^U, \varepsilon_j^V)\,|_W \bigg).$$

So, by the choice of K, we get

$$|\,B(p,q)\,|_W \;\leqslant\; \sum_{i\in\mathcal{I}_U}\sum_{j\in\mathcal{I}_V}\left(|p_i|\cdot|q_j|\cdot K\right).$$

So, as $K\cdot\left(\sum_{i\in\mathcal{I}_U}|p_i|\right)\cdot\left(\sum_{j\in\mathcal{I}_V}|q_j|\right) \;=\; \sum_{i\in\mathcal{I}_U}\sum_{j\in\mathcal{I}_V}\left(|p_i|\cdot|q_j|\cdot K\right)$, we get

$$|\,B(p,q)\,|_W \;\leqslant\; K\cdot\left(\sum_{i\in\mathcal{I}_U}|p_i|\right)\cdot\left(\sum_{j\in\mathcal{I}_V}|q_j|\right).$$

By HW#7-3, we get $\sum_{i\in\mathcal{I}_U}|p_i| \leqslant \sqrt{\#\mathcal{I}_U}\cdot\sqrt{\sum_{i\in\mathcal{I}_U}p_i^2}$.

Also, by HW#7-3, we get $\sum_{j\in\mathcal{I}_V}|q_i| \leqslant \sqrt{\#\mathcal{I}_V}\cdot\sqrt{\sum_{j\in\mathcal{I}_V}q_j^2}$.

By definition of $|\bullet|_U$, we have $|p|_U = \sqrt{\sum_{i\in\mathcal{I}_U}p_i^2}$.

By definition of $|\bullet|_V$, we have $|q|_V = \sqrt{\sum_{j\in\mathcal{I}_V}q_j^2}$.

Then
$$\begin{aligned}
|\,B(p,q)\,|_W &\leqslant K\cdot\left(\sum_{i\in\mathcal{I}_U}|p_i|\right)\cdot\left(\sum_{j\in\mathcal{I}_V}|q_j|\right)\\
&\leqslant K\cdot\sqrt{\#\mathcal{I}_U}\cdot\sqrt{\sum_{i\in\mathcal{I}_U}p_i^2}\cdot\sqrt{\#\mathcal{I}_V}\cdot\sqrt{\sum_{j\in\mathcal{I}_V}q_j^2}\\
&\leqslant K\cdot\sqrt{\#\mathcal{I}_U}\cdot|p|_U\cdot\sqrt{\#\mathcal{I}_V}\cdot|q|_V\\
&= K\cdot\sqrt{\#\mathcal{I}_U}\cdot\sqrt{\#\mathcal{I}_V}\cdot|p|_U\cdot|q|_V\\
&= C\cdot|p|_U\cdot|q|_V, \quad \text{as desired. QED}
\end{aligned}$$

7-5. Let $T,U,V,W\in\text{TNSR}$ and let $*\in\mathcal{B}_{UV}^{W}$.

Show: $\text{BNZ}_{TU} \;*\; \text{BNZ}_{TV}\subseteq\text{BNZ}_{TW}$.

Proof: Want: $\forall\alpha\in\text{BNZ}_{TU}\;*\;\text{BNZ}_{TV}$, $\alpha\in\text{BNZ}_{TW}$.
Given $\alpha\in\text{BNZ}_{TU}\;*\;\text{BNZ}_{TV}$. Want: $\alpha\in\text{BNZ}_{TW}$.
Choose $\beta\in\text{BNZ}_{TU}$ and $\gamma\in\text{BNZ}_{TV}$ s.t. $\alpha=\beta*\gamma$.
By HW#7-4, choose $K\geqslant 0$ s.t. $\forall p\in U$, $\forall q\in V$,

$$|p*q|_W \;\leqslant\; K\cdot|p|_U\cdot|q|_V.$$

Since $\beta\in\text{BNZ}_{TU}\subseteq\text{DNZ}_{TU}$, choose $B\in\mathcal{B}_T(0_T)$ s.t. $B\subseteq\text{dom}\,[\beta]$.
Since $\gamma\in\text{BNZ}_{TV}\subseteq\text{DNZ}_{TV}$, choose $C\in\mathcal{B}_T(0_T)$ s.t. $C\subseteq\text{dom}\,[\gamma]$.
Then $B\cap C\subseteq(\text{dom}\,[\beta])\cap(\text{dom}\,[\alpha])=\text{dom}\,[\beta*\gamma]=\text{dom}\,[\alpha]$.
So, since $B\cap C\in\{B,C\}\subseteq\mathcal{B}_T(0_T)$, we conclude that $\alpha\in\text{DNZ}_{TW}$.

Want: $\exists A \in \mathcal{B}_T(0_T)$ s.t. $\alpha_*(A)$ is bounded in W.
Since $\beta \in \mathrm{BNZ}_{TU}$, choose $D \in \mathcal{B}_T(0_T)$ s.t. $\beta_*(D)$ is bounded in U.
Since $\beta_*(D)$ is bounded in U, choose $E \in \mathcal{B}_U$ s.t. $\beta_*(D) \subseteq E$.
Since $E \in \mathcal{B}_U$, by the Superset Recentering Lemma,
choose $F \in \mathcal{B}_U(0_U)$ s.t. $E \subseteq F$.
Since $F \in \mathcal{B}_U(0_U)$, choose $r > 0$ s.t. $F = B_U(0_U, r)$.
Since $\gamma \in \mathrm{BNZ}_{TU}$, choose $X \in \mathcal{B}_T(0_T)$ s.t. $\gamma_*(X)$ is bounded in V.
Since $\gamma_*(X)$ is bounded in V, choose $Y \in \mathcal{B}_V$ s.t. $\gamma_*(X) \subseteq Y$.
Since $Y \in \mathcal{B}_V$, by the Superset Recentering Lemma,
choose $Z \in \mathcal{B}_V(0_V)$ s.t. $Y \subseteq Z$.
Since $Z \in \mathcal{B}_V(0_V)$, choose $s > 0$ s.t. $Z = B_V(0_V, s)$.
Let $A := D \cap X$. Then $A \in \{D, X\} \subseteq \mathcal{B}_T(0_T)$.
Want: $\alpha_*(A)$ is bounded in W. Want: $\exists Q \in \mathcal{B}_W$ s.t. $\alpha_*(A) \subseteq Q$.
Let $Q := B_W(0_W, Krs + 1)$. Then $Q \in \mathcal{B}_W(0_W) \subseteq \mathcal{B}_W$.
Want: $\alpha_*(A) \subseteq Q$. Want: $\forall w \in \alpha_*(A)$, $w \in Q$.
Given $w \in \alpha_*(A)$. Want: $w \in Q$. Want: $|w|_W < Krs + 1$.
Since $w \in \alpha_*(A)$, choose $t \in (\mathrm{dom}\,[\alpha]) \cap A$ s.t. $\alpha(t) = w$.
Since $t \in (\mathrm{dom}\,[\alpha]) \cap A$, we get $t \in \mathrm{dom}\,[\alpha]$ and $t \in A$.
We have $t \in \mathrm{dom}\,[\alpha] = \mathrm{dom}\,[\beta * \gamma] = (\mathrm{dom}\,[\beta]) \cap (\mathrm{dom}\,[\gamma])$,
so $t \in \mathrm{dom}\,[\beta]$ and $t \in \mathrm{dom}\,[\gamma]$.
Also, we have $t \in A = D \cap X$,
so $t \in D$ and $t \in X$.
Since $t \in \mathrm{dom}\,[\beta]$ and $t \in D$, we get $\beta(t) \in \beta_*(D)$.
So, since $\beta_*(D) \subseteq E \subseteq F = B_U(0_U, r)$,
we get $\beta(t) \in B_U(0_U, r)$, so $|\beta(t)|_U < r$.
Since $t \in \mathrm{dom}\,[\gamma]$ and $t \in X$, we get $\gamma(t) \in \gamma_*(X)$.
So, since $\gamma_*(X) \subseteq Y \subseteq Z = B_V(0_V, s)$,
we get $\gamma(t) \in B_V(0_V, s)$, so $|\gamma(t)|_V < s$.
Since $t \in \mathrm{dom}\,[\beta]$, we get $\beta(t) \in \mathrm{im}\,[\beta]$.
Since $\beta \in \mathrm{BNZ}_{TU}$, we get $\mathrm{im}\,[\beta] \subseteq U$.
Since $t \in \mathrm{dom}\,[\gamma]$, we get $\gamma(t) \in \mathrm{im}\,[\gamma]$.
Since $\gamma \in \mathrm{BNZ}_{TV}$, we get $\mathrm{im}\,[\gamma] \subseteq V$.
Since $\beta(t) \in \mathrm{im}\,[\beta] \subseteq U$ and $\gamma(t) \in \mathrm{im}\,[\gamma] \subseteq V$,
it follows, from the choice of K,
that $|(\beta(t)) * (\gamma(t))|_W \leqslant K \cdot |\beta(t)|_U \cdot |\gamma(t)|_V$.
We have $w = \alpha(t) = (\beta * \gamma)(t) = (\beta(t)) * (\gamma(t))$.
Then $|w|_W = |(\beta(t)) * (\gamma(t))|_W \leqslant K \cdot |\beta(t)|_U \cdot |\gamma(t)|_V \leqslant Krs$.
Then $|w|_W \leqslant Krs < Krs + 1$, as desired. QED

Homework 6: Due on Tuesday 5 March

6-1. Let $T \in \text{TNSR}$, $v \in T$. Show: $v = \sum_{j \in \mathcal{I}_T} v_j \varepsilon_j^T$.

Proof: Let $w := \sum_{j \in \mathcal{I}_T} v_j \varepsilon_j^T$. Want $v = w$.

Want: $\forall i \in \mathcal{I}_T$, $v_i = w_i$. Given $i \in \mathcal{I}_T$. Want: $v_i = w_i$.

We have:

$$\begin{aligned} w_i &= \sum_{j \in \mathcal{I}_T} v_j \cdot ((\varepsilon_j^T)_i) \\ &= (v_i \cdot ((\varepsilon_i)_i)) + \left(\sum_{j \in \mathcal{I}_T \setminus \{i\}} v_j \cdot ((\varepsilon_j^T)_i) \right) \\ &= (v_i \cdot 1) + \left(\sum_{j \in \mathcal{I}_T \setminus \{i\}} v_j \cdot 0 \right) = v_i + 0. \end{aligned}$$

Then $v_i = v_i + 0 = w_i$, as desired. QED

6-2. Let $S, T \in \text{TNSR}$, $A \in T \otimes S$, $L := \text{Lin}_A^{ST}$, $i \in \mathcal{I}_S$.

Show: $L(\varepsilon_i^S) = \sum_{j \in \mathcal{I}_T} A_{j\|i} \cdot \varepsilon_j^T$.

Proof: Let $v := L(\varepsilon_i^S)$. By HW#6-1, we have: $v = \sum_{j \in \mathcal{I}_T} v_j \varepsilon_j^T$.

By Theorem 83.5, p. 202, we have: $\forall j \in \mathcal{I}_T$, $A_{j\|i} = (L(\varepsilon_i^S))_j$.

Then

$$\begin{aligned} L(\varepsilon_i^S) = v &= \sum_{j \in \mathcal{I}_T} v_j \varepsilon_j^T = \sum_{j \in \mathcal{I}_T} (L(\varepsilon_i^S))_j \cdot \varepsilon_j^T \\ &= \sum_{j \in \mathcal{I}_T} A_{j\|i} \cdot \varepsilon_j^T, \end{aligned}$$

as desired. QED

6-3. Let $S, T \in \text{TNSR}^+$, $C \in \mathcal{C}_S^T$, $p \in S$. Show: $D_p^{ST} C = \mathbf{0}_S^T$.

Proof: Since $C \in \mathcal{C}_S^T$, choose $w \in T$ s.t. $C = C_S^w$.

We have: $\forall h \in S$,

$$\begin{aligned} C_p^T(h) &= [C(p+h)] - [C(p)] \\ &= [C_S^w(p+h)] - [C_S^w(p)] \\ &= w - w = 0_T = \mathbf{0}_S^T(h). \end{aligned}$$

Then $C_p^T = \mathbf{0}_S^T$. Let $L := \mathbf{0}_S^T$.

Then $C_p^T - L = \mathbf{0}_S^T - \mathbf{0}_S^T = \mathbf{0}_S^T \in \mathcal{O}_1^{ST}$.

Then $L \in \text{LINS}_p C$, and so $D_p^{ST} C = L$.

Then $D_p^{ST} C = L = \mathbf{0}_S^T$, as desired. QED

6-4. Let $S, T \in \mathrm{TNSR}^+$, $L \in \mathcal{L}_S^T$, $p \in S$. Show: $D_p^{ST} L = L$.

Proof: We have: $\forall h \in S$, $L_p^T(h) = [L(p+h)] - [L(p)]$
$= [L(p)] + [L(h)] - [L(p)] = L(h)$.
Then $L_p^T = L$. Then $L_p^T - L = \mathbf{0}_S^T \in \mathcal{o}_1^{ST}$.
Then $L \in \mathrm{LINS}_p L$, and so $D_p^{ST} L = L$, as desired. QED

6-5. Let $V, W \in \mathrm{TNSR}^+$, $Q \in \mathcal{Q}_V^W$, $B \in \mathcal{SB}_V^W$.
Assume: $Q = B(\bullet, \bullet)$. Show: $\forall x \in V$, $D_x^{VW} Q = 2 \cdot (B(x, \bullet))$.

Proof: Given $x \in V$. Want: $D_x^{VW} Q = 2 \cdot (B(x, \bullet))$.
We have: $\forall h \in S$,

$$\begin{aligned} Q_x^T(h) &= (Q(x+h)) - (Q(x)) \\ &= (B(x+h, x+h)) - (B(x,x)) \\ &= (B(x,x)) + (B(x,h)) + (B(h,x)) + (B(h,h)) - (B(x,x)) \\ &= (B(x,x)) - (B(x,x)) + (B(x,h)) + (B(x,h)) + (B(h,h)) \\ &= 2 \cdot (B(x,h)) + (B(h,h)) \\ &= (\,2 \cdot (B(x,\bullet)) + (B(\bullet,\bullet))\,)(h). \end{aligned}$$

Then $Q_x^T = 2 \cdot (B(x, \bullet)) + (B(\bullet, \bullet))$.
Let $L := 2 \cdot (B(x, \bullet))$ and let $R := B(\bullet, \bullet)$.
Then $Q_x^T = L + R$ and $L \in \mathcal{L}_V^W$ and $R \in \mathcal{Q}_V^W$.
Since $L \in \mathcal{L}_V^W$, we get $L - L = \mathbf{0}_S^T$, and so $L - L + R = \mathbf{0}_S^T + R$.
Then $Q_x^T - L = L + R - L = L - L + R = \mathbf{0}_V^W + R = R$.
Then $Q_x^T - L = R \in \mathcal{Q}_V^W \subseteq \widehat{\mathcal{O}}_2^{VW} \subseteq \mathcal{o}_1^{VW}$.
So, since $L \in \mathcal{L}_V^W$, we get $L \in \mathrm{LINS}_x Q$, and so $D_x^{VW} Q = L$.
Then $D_x^{VW} Q = L = 2 \cdot (B(x, \bullet))$, as desired. QED

Homework 5: Due on Tuesday 26 February

5-1. Let $S := \mathbb{R}^2$ and let $T := \mathbb{R}$. Show: $\mathcal{Q}_{ST} \subseteq \widehat{\mathcal{O}}_2^{ST}$.

Proof: Want: $\forall Q \in \mathcal{Q}_{ST}$, $Q \in \widehat{\mathcal{O}}_2^{ST}$.
Given $Q \in \mathcal{Q}_{ST}$. Want: $Q \in \widehat{\mathcal{O}}_2^{ST}$.
Since $Q \in \mathcal{Q}_{ST}$, it follows that $\mathrm{dom}\,[Q] = S$.
Let $\alpha := \mathrm{adj}_{0_S}^{0_T}\left(\dfrac{Q}{|\bullet|_S^2}\right)$. Then $\alpha(0_S) = 0_T$.
Also, $\mathrm{dom}\,[\alpha] = ((\mathrm{dom}\,[Q])_{0_S}^{\times})_{0_S}^{+} = (\mathrm{dom}\,[Q])_{0_S}^{+} = S_{0_S}^{+} = S$.
Also, $\forall x \in S \backslash \{0_S\}$, we have $Q(x) = (\alpha(x)) \cdot (|x|_S^2)$.

Since $Q \in \mathcal{Q}_{ST}$, it follows that $Q(0_S) = 0_T$.
Recall that $T = \mathbb{R}$. Then $0_T = 0$.
Then $\alpha(0_S) = 0_T = 0$ and $Q(0_S) = 0_T = 0$.
Then $Q(0_S) = 0 = 0 \cdot 0^2 = (\alpha(0_S)) \cdot (|0_S|_S^2)$.
Then $\forall x \in S$, we have $Q(x) = (\alpha(x)) \cdot (|x|_S^2)$. Then $Q = \alpha \cdot (|\bullet|_S^2)$.
Want: $Q \in (\mathrm{BNZ}_{ST}) \cdot (|\bullet|_S^2)$. Want: $\alpha \in \mathrm{BNZ}_{ST}$.
Since $\mathrm{dom}\,[\alpha] = S$, we see that $\alpha \in \mathrm{DNZ}_{ST}$.
Want: $\exists B \in \mathcal{B}_S(0_S)$ s.t. $\alpha_*(B)$ is bounded in T.
Want: $\mathrm{im}\,[\alpha]$ is bounded in T.
Recall that $T = \mathbb{R}$. Want: $\mathrm{im}\,[\alpha]$ is bounded in $\mathbb{R}$.
Want: $\exists r > 0$ s.t. $\mathrm{im}\,[\alpha] \subseteq B_{\mathbb{R}}(0, r)$.
Let $C := \{v \in S \text{ s.t. } |v|_S = 1\}$.
Since C is closed and bounded in S, we see that C is compact.
Since $Q \in \mathcal{Q}_{ST}$, it follows that Q is continuous.
Then $Q|C$ is continuous.
So, since C is compact, by the Extreme Value Theorem,
we conclude: $\min(Q|C) \neq ☹ \neq \max(Q|C)$.
Let $K := \min(Q|C)$ and $L := \max(Q|C)$.
Then $(\, K, L \in \mathbb{R}\,) \,\&\, (\, K \leqslant Q \leqslant L \text{ on } C\,)$.
Let $r := 1 + [\max\{|K|, |L|\}]$. Then $r > 0$ and $-r < K$ and $L < r$.
Want: $\mathrm{im}\,[\alpha] \subseteq B_{\mathbb{R}}(0, r)$.
Want: $\forall y \in \mathrm{im}\,[\alpha]$, $y \in B_{\mathbb{R}}(0, r)$.
Given $y \in \mathrm{im}\,[\alpha]$. Want: $y \in B_{\mathbb{R}}(0, r)$.
Since $y \in \mathrm{im}\,[\alpha]$, choose $x \in \mathrm{dom}\,[\alpha]$ s.t. $y = \alpha(x)$.
Exactly one of the following is true:
(1) $x = 0_S$ or (2) $x \neq 0_S$.

Case (1):
We have $y = \alpha(x) = \alpha(0_S) = 0 \in B_{\mathbb{R}}(0, r)$, as desired.
End of Case (1).

Case (2):
Since $x \neq 0_S$, by definition of α, we get: $\alpha(x) = \dfrac{Q(x)}{|x|_S^2}$.
Also, since $x \neq 0_S$, we get $|x|_S > 0$.
Let $b := |x|_S$. Then $b > 0$, so $\dfrac{b}{|b|} = 1$.

Let $u := \dfrac{x}{b}$. Then $|u|_S = \dfrac{|x|_S}{|b|} = \dfrac{b}{|b|} = 1$.
Then $u \in C$, so, since $K \leqslant Q \leqslant L$ on C, we get: $K \leqslant Q(u) \leqslant L$.
So, since $-r < K$ and $L < r$, we get: $-r < Q(u) < r$.
Then $Q(u) \in (-r; r) = B_{\mathbb{R}}(0, r)$.
Since $Q \in \mathcal{Q}_{ST}$, we have $Q\left(\dfrac{x}{b}\right) = \dfrac{Q(x)}{b^2}$.
Then $Q(u) = Q\left(\dfrac{x}{b}\right) = \dfrac{Q(x)}{b^2} = \dfrac{Q(x)}{|x|_S^2} = \alpha(x)$.
Then $y = \alpha(x) = Q(u) \in B_{\mathbb{R}}(0, r)$, as desired.
End of Case (2). QED

5-2. Let $S, T \in \text{TNSR}^+$. Show: $(\text{BNZ}_{ST}) \cdot (\text{CVZ}_{S\mathbb{R}}) \subseteq \text{CVZ}_{ST}$.

Proof: Want: $\forall \gamma \in (\text{BNZ}_{ST}) \cdot (\text{CVZ}_{S\mathbb{R}})$, $\gamma \in \text{CVZ}_{ST}$.
Given $\gamma \in (\text{BNZ}_{ST}) \cdot (\text{CVZ}_{S\mathbb{R}})$. Want: $\gamma \in \text{CVZ}_{ST}$.
Since $\gamma \in (\text{BNZ}_{ST}) \cdot (\text{CVZ}_{S\mathbb{R}})$,
choose $\alpha \in \text{BNZ}_{ST}$, $\beta \in \text{CVZ}_{S\mathbb{R}}$ s.t. $\gamma = \alpha \cdot \beta$.
Since $\alpha \in \text{BNZ}_{ST} \subseteq \text{DNZ}_{ST}$ and since $\beta \in \text{CVZ}_{S\mathbb{R}} \subseteq \text{DNZ}_{S\mathbb{R}}$,
choose $P, Q \in \mathcal{B}_S(0_S)$ s.t. $P \subseteq \text{dom}\,[\alpha]$ and $Q \subseteq \text{dom}\,[\beta]$.
Then $P \cap Q \in \{P, Q\} \subseteq \mathcal{B}_S(0_S)$.
So, since $P \cap Q \subseteq (\text{dom}\,[\alpha]) \cap (\text{dom}\,[\beta]) = \text{dom}\,[\alpha \cdot \beta] = \text{dom}\,[\gamma]$,
we conclude that $\gamma \in \text{DNZ}_{ST}$.
Want: $\gamma(0_S) = 0_T$ and γ is continuous at 0_S.
Since $P \in \mathcal{B}_S(0_S)$, we get $0_S \in P$.
Since $\alpha \in \text{DNZ}_{ST}$, we get $\alpha : S \dashrightarrow T$.
So, since $0_S \in P \subseteq \text{dom}\,[\alpha]$, we get $\alpha(0_S) \in T$, so $[\alpha(0_S)] \cdot 0 = 0_T$.
Since $\beta \in \text{CVZ}_{S\mathbb{R}}$, we get $\beta(0_S) = 0$.
Then $\gamma(0_S) = (\alpha \cdot \beta)(0_S) = [\alpha(0_S)] \cdot [\beta(0_S)] = [\alpha(0_S)] \cdot 0 = 0_T$.
Want: γ is continuous at 0_S.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \text{dom}\,[\gamma]$,
$[\ |x - 0_S|_S < \delta\ ] \Rightarrow [\ |[\gamma(x)] - [\gamma(0_S)]|_T < \varepsilon\ ]$.
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall x \in \text{dom}\,[\gamma]$,
$[\ |x - 0_S|_S < \delta\ ] \Rightarrow [\ |[\gamma(x)] - [\gamma(0_S)]|_T < \varepsilon\ ]$.
Since $\alpha \in \text{BNZ}_{ST}$, choose $A \in \mathcal{B}_S(0_S)$ s.t. $\alpha_*(A)$ is bounded in T.
Since $\alpha_*(A)$ is bounded in T, choose $Y \in \mathcal{B}_T$ s.t. $\alpha_*(A) \subseteq Y$.
By the Superset Recentering Lemma (Theorem 38.17),
choose $Z \in \mathcal{B}_T(0_T)$ s.t. $Y \subseteq Z$.
Choose $r > 0$ s.t. $Z = B_T(0_T, r)$.

Since $\beta \in \text{CVZ}_{S\mathbb{R}}$, it follows that β is continuous at 0_S,

so choose $\eta > 0$ s.t., $\forall x \in \text{dom}\,[\beta]$,

$$[\ |x - 0_S|_S < \eta\ ] \ \Rightarrow\ [\ |[\beta(x)] - [\beta(0)]| < \varepsilon/r\ ].$$

Let $B := B_S(0_S, \eta)$. Then $A, B \in \mathcal{B}_S(0_S)$. Let $Q := A \cap B$.

Then $Q \in \{A, B\} \subseteq \mathcal{B}_S(0_S)$. Choose $\delta > 0$ s.t. $Q = B_S(0_S, \delta)$.

Want: $\forall x \in \text{dom}\,[\gamma]$, ($[\ |x{-}0_S|_S < \delta\ ] \ \Rightarrow\ [\ |[\gamma(x)]{-}[\gamma(0_S)]|_T < \varepsilon\ ]$).

Given $x \in \text{dom}\,[\gamma]$.

Want: $[\ |x - 0_S|_S < \delta\ ] \ \Rightarrow\ [\ |[\gamma(x)] - [\gamma(0_S)]|_T < \varepsilon\ ]$.

Assume: $|x - 0_S|_S < \delta$. Want: $|[\gamma(x)] - [\gamma(0_S)]|_T < \varepsilon$.

We have $x \in \text{dom}\,[\gamma] = \text{dom}\,[\alpha \cdot \beta] = (\text{dom}\,[\alpha]) \cap (\text{dom}\,[\beta])$.

Then $x \in (\text{dom}\,[\alpha]) \cap (\text{dom}\,[\beta]) \subseteq \text{dom}\,[\beta]$.

Also, since $|x - 0_S|_S < \delta$, we have $x \in B_S(0_S, \delta)$.

Then $x \in B_S(0_S, \delta) = Q = A \cap B$.

Then $x \in A \cap B \subseteq B = B_S(0_S, \eta)$, so $|x - 0_S|_S < \eta$.

So, since $x \in \text{dom}\,[\beta]$, by choice of η, we get: $|[\beta(x)] - [\beta(0_S)]| < \varepsilon/r$.

Since $\beta \in \text{CVZ}_{S\mathbb{R}}$, we get $\beta(0_S) = 0$. Then $[\beta(x)]{-}[\beta(0_S)] = \beta(x)$.

Then $|\beta(x)| = |[\beta(x)] - [\beta(0_S)]| < \varepsilon/r$, so $|\beta(x)| < \varepsilon/r$.

Since $x \in (\text{dom}\,[\alpha]) \cap (\text{dom}\,[\beta]) \subseteq \text{dom}\,[\alpha]$ and since $x \in A \cap B \subseteq A$,

we conclude that $\alpha(x) \in \alpha_*(A)$.

Then $\alpha(x) \in \alpha_*(A) \subseteq Y \subseteq Z = B_T(0_T, r)$, so $|\alpha(x)|_T < r$.

Since $0 \leqslant |\alpha(x)|_T < r$ and since $0 \leqslant |\beta(x)| < \varepsilon/r$,

it follows that $|\alpha(x)|_T \cdot |\beta(x)| < r \cdot (\varepsilon/r)$.

Recall that $\gamma(0_S) = 0_T$. We have $\gamma(x) = (\alpha \cdot \beta)(x) = [\alpha(x)] \cdot [\beta(x)]$.

Then $|[\gamma(x)] - [\gamma(0_S)]|_T = |[\gamma(x)] - 0_T|_T = |\gamma(x)|_T = |[\alpha(x)] \cdot [\beta(x)]|_T$

$= |\alpha(x)|_T \cdot |\beta(x)| < r \cdot (\varepsilon/r) = \varepsilon$, as desired. QED

5-3. Let $S \in \text{TNSR}^+$, $f, g : \mathbb{R} \dashrightarrow S$, $p \in \mathbb{R}$.

Show: $(f \bullet g)_p^T \ =\ (f_p^T) \bullet (g_p) \ +\ (f_p) \bullet (g_p^T) \ +\ (f_p^T) \bullet (g_p^T)$.

Proof: Want: $\forall h \in \mathbb{R}$,

$$(f \bullet g)_p^T(h) = ((f_p^T) \bullet (g_p) + (f_p) \bullet (g_p^T) + (f_p^T) \bullet (g_p^T))(h).$$

Given $h \in \mathbb{R}$.

Want: $(f \bullet g)_p^T(h) = ((f_p^T) \bullet (g_p) + (f_p) \bullet (g_p^T) + (f_p^T) \bullet (g_p^T))(h)$.

Let $A := f_{p+h}$, $B := g_{p+h}$.

Then $A - a = f_{p+h} - f_p = f_p^T(h)$.

Also, $B - b = g_{p+h} - g_p = g_p^T(h)$.

We have $A \bullet B - a \bullet b \ =\ a \bullet (B - b) \ +\ b \bullet (A - a)$

$+ (A - a) \bullet (B - b)$.

Then $(f \bullet g)_p^T(h) = [(f \bullet g)_{p+h}] - [(f \bullet g)_p] = f_{p+h} \bullet g_{p+h} - f_p \bullet g_p$
$= A \bullet B - a \bullet b = (A - a) \bullet b + a \bullet (B - b) + (A - a) \bullet (B - b)$
$= (f_p^T(h)) \bullet (g_p) + (f_p) \bullet (g_p^T(h)) + (f_p^T(h)) \bullet (g_p^T(h))$
$= ((f_p^T) \bullet (g_p) + (f_p) \bullet (g_p^T) + (f_p^T) \bullet (g_p^T))(h)$,
as desired. QED

5-4. Let $S, T, U \in \text{TNSR}^+$, $\quad f : S \dashrightarrow T$, $\quad g : T \dashrightarrow U$.
Let $p \in S$, $\quad q := f_p$. $\qquad$ Show: $(f \circ g)_p^T = (g_q^T) \circ (f_p^T)$

Proof: Want: $\forall h \in S$, $((g \circ f)_p^T)(h) = ((g_q^T) \circ (f_p^T))(h)$.
Given $h \in S$. $\qquad$ Want: $((g \circ f)_p^T)(h) = ((g_q^T) \circ (f_p^T))(h)$.
We compute $((g \circ f)_p^T)(h) = [(g \circ f)(p + h)] - [(g \circ f)(p)]$
$= [g(f_{p+h})] - [g(f_p)] = [g(f_{p+h})] - [g(q)]$.
Exactly one of the following is true:
(1) $q = ☹$ $\qquad$ or $\qquad$ (2) $q \neq ☹$.

Case (1):
Since $q = ☹$, we get:
both $\quad [g(f_{p+h})] - [g(q)] = ☹ \quad$ and $\quad ((g_q^T) \circ (f_p^T))(h) = ☹$.
Then $((g \circ f)_p^T)(h) = [g(f_{p+h})] - [g(q)] = ☹ = ((g_q^T) \circ (f_p^T))(h)$.
End of Case (1).

Case (2):
Since $f_p = q \neq ☹$, we get $f_p \in \text{im}[f] \subseteq \mathbb{R}$.
Then $q = f_p \in T$, so $q - q = 0_T$.
Let $k := (f_p^T)(h)$. $\qquad$ Then $k = f_{p+h} - f_p = f_{p+h} - q$.
Then $q + k = q + (f_{p+h} - q) = f_{p+h} + (q - q) = f_{p+h} + 0_T = f_{p+h}$.
Then $((g \circ f)_p^T)(h) = [g(f_{p+h})] - [g(q)] = [g(q + k)] - [g(q)]$
$= g_q^T(k) = (g_q^T)((f_p^T)(h)) = ((g_q^T) \circ (f_p^T))(h)$,
as desired. *End of Case (2).* QED

5-5. Let $S, T \in \text{TNSR}$, $\quad i \in \mathcal{I}_S$, $\quad j \in \mathcal{I}_T$. $\qquad$ Show: $i \| j \in \mathcal{I}_{S \otimes T}$.

Proof: Choose $\sigma, \tau \in \mathbb{N}$ s.t. $S \in \text{TNSR}_\sigma$ and $T \in \text{TNSR}_\tau$.
Choose $m \in \mathbb{R}^\sigma$ and $n \in \mathbb{R}^\tau$ s.t. $S = \mathbb{R}^m$ and $T = \mathbb{R}^n$.
We have $\quad S = \mathbb{R}^m$, $\quad$ so $\quad \mathcal{I}_S = [m]$.
Also, $\quad T = \mathbb{R}^n$, $\quad$ so $\quad \mathcal{I}_S = [n]$.
Also, $\quad S \otimes T = \mathbb{R}^{m\|n}$, $\quad$ so $\quad \mathcal{I}_{S \otimes T} = [m\|n]$.
We have: $\quad m = (m_1, \ldots, m_\sigma)$,

and $\quad n = (n_1, \ldots, n_\tau)$

and $\quad m\|n = (m_1, \ldots, m_\sigma, n_1, \ldots, n_\tau)$.

Then: $\quad [m] = [1..m_1] \times \cdots \times [1..m_\sigma]$,

and $\quad [n] = [1..n_1] \times \cdots \times [1..n_\tau]$

and $\quad [m\|n] = [1..m_1] \times \cdots \times [1..m_\sigma] \times [1..n_1] \times \cdots \times [1..n_\tau]$.

Then: $\quad i \in \mathcal{I}_S = [m] = [1..m_1] \times \cdots \times [1..m_\sigma]$

and $\quad j \in \mathcal{I}_T = [n] = [1..n_1] \times \cdots \times [1..n_\tau]$,

and so $\quad i\|j \in [1..m_1] \times \cdots \times [1..m_\sigma] \times [1..n_1] \times \cdots \times [1..n_\tau]$.

Then $i\|j \in [m\|n] = \mathcal{I}_{S \otimes T}$, as desired. QED

Homework 4: Due on Tuesday 19 February

4-1. Let $k \in \mathbb{N}_0$, $\alpha \in \widehat{\mathcal{O}}_k$ and $\lambda := \mathrm{adj}_0^0\left(\dfrac{\alpha}{|\bullet|^k}\right)$. Show: $\lambda \in$ BNZ.

Proof: Since $\alpha \in \widehat{\mathcal{O}}_k = (\mathrm{BNZ}) \cdot (|\bullet|^k)$,

choose $\mu \in$ BNZ s.t. $\alpha = \mu \cdot (|\bullet|^k)$.

We have $\mathrm{dom}\,[\alpha] = (\mathrm{dom}\,[\mu]) \cap (\mathrm{dom}\,[|\bullet|^k])$.

So, since $\mathrm{dom}\,[\mu] \subseteq \mathbb{R} = \mathrm{dom}\,[|\bullet|^k]$, we see that $\mathrm{dom}\,[\alpha] = \mathrm{dom}\,[\mu]$.

Since $\mu \in \mathrm{BNZ} \subseteq \mathrm{DNZ}$, choose $A \in \mathcal{B}_\mathbb{R}(0)$ s.t. $A \subseteq \mathrm{dom}\,[\mu]$.

Then $A \subseteq \mathrm{dom}\,[\mu] = \mathrm{dom}\,[\alpha]$, so $A \subseteq \mathrm{dom}\,[\alpha]$.

So, since $A \in \mathcal{B}_\mathbb{R}(0)$, we see that $\alpha \in$ DNZ.

We have $\mathrm{dom}\,[\lambda] = \left(\mathrm{dom}\left[\dfrac{\alpha}{|\bullet|^k}\right]\right)_0^+ = ((\mathrm{dom}\,[\alpha])_0^\times)_0^+ = (\mathrm{dom}\,[\alpha])_0^+$.

Then $A \subseteq \mathrm{dom}\,[\alpha] \subseteq (\mathrm{dom}\,[\alpha])_0^+ = \mathrm{dom}\,[\lambda]$, so $A \subseteq \mathrm{dom}\,[\lambda]$.

So, since $A \in \mathcal{B}_\mathbb{R}(0)$, we conclude that $\lambda \in$ DNZ.

Want: $\exists B \in \mathcal{B}_\mathbb{R}(0)$ s.t. $\lambda_*(B)$ is bounded in $\mathbb{R}$.

Since $\mu \in$ BNZ, choose $C \in \mathcal{B}_\mathbb{R}(0)$ s.t. $\mu_*(C)$ is bounded in $\mathbb{R}$.

Since $A, C \in \mathcal{B}_\mathbb{R}(0)$, we get $A \cap C \in \{A, C\}$.

Let $B := A \cap C$. Then $B = A \cap C \in \{A, C\} \subseteq \mathcal{B}_\mathbb{R}(0)$.

Want: $\lambda_*(B)$ is bounded in $\mathbb{R}$.

Since $\mu_*(C)$ is bounded in $\mathbb{R}$, choose $D \in \mathcal{B}_\mathbb{R}$ s.t. $\mu_*(C) \subseteq D$.

By the Superset Recentering Lemma, choose $E \in \mathcal{B}_\mathbb{R}(0)$ s.t. $D \subseteq E$.

Since $E \in \mathcal{B}_\mathbb{R}(0)$, it follows that $0 \in E$.

Since $\mu_*(C) \subseteq D \subseteq E$ and since $0 \in E$,

we conclude that $(\mu_*(C)) \cup \{0\} \subseteq E$.

So, as $E \in \mathcal{B}_\mathbb{R}$, it suffices to show: $\lambda_*(B) \subseteq E$.

Want: $\forall y \in \lambda_*(B)$, $y \in E$.

Given $y \in \lambda_*(B)$. Want: $y \in E$.
Since $y \in \lambda_*(B)$, choose $x \in B \cap (\mathrm{dom}\,[\lambda])$ s.t. $y = \lambda_x$.
Exactly one of the following is true:

(1) $x = 0$ or (2) $x \neq 0$.

Case (1):
We have $y = \lambda_x = \lambda_0 = \left(\mathrm{adj}_0^0\left(\frac{\alpha}{|\bullet|^k}\right)\right)_0 = 0 \in E$, as desired.
End of Case (1).

Case (2):
Since $x \neq 0$, we have $\lambda_x = \left(\mathrm{adj}_0^0\left(\frac{\alpha}{|\bullet|^k}\right)\right)_x = \left(\frac{\alpha}{|\bullet|^k}\right)_x = \frac{\alpha_x}{|x|^k}$.
Since $\alpha = \mu \cdot (|\bullet|^k)$, we get: $\alpha_x = (\mu \cdot (|\bullet|^k))_x$.
Then $\alpha_x = (\mu \cdot (|\bullet|^k))_x = \mu_x \cdot (|x|^k)$.
Since $B \in \mathcal{B}_{\mathbb{R}}(0)$, we get $B \subseteq \mathbb{R}$. Then $x \in B \subseteq \mathbb{R}$.
So, since $x \neq 0$, we get $\frac{|x|^k}{|x|^k} = 1$.
Then $\lambda_x = \frac{\alpha_x}{|x|^k} = \frac{\mu_x \cdot (|x|^k)}{|x|^k} = \mu_x$.
We have $x \in B \cap (\mathrm{dom}\,[\lambda]) \subseteq B = A \cap C$, so $x \in A$ and $x \in C$.
Since $x \in A \subseteq \mathrm{dom}\,[\mu]$ and since $x \in C$, we get $\mu_x \in \mu_*(C)$.
Then $y = \lambda_x = \mu_x \in \mu_*(C) \subseteq D \subseteq E$, as desired.
End of Case (2). QED

4-2. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $U \in \mathbb{R}$.
Assume that U is open in $\mathbb{R}$ and that $U \subseteq \mathrm{dom}\,[f']$.
Assume that f is semi-increasing on U.
Let $T := f'_*(U)$. Show: $T \geqslant 0$.

Proof: Want: $\forall m \in T$, $m \geqslant 0$.
Given $m \in T$. Want: $m \geqslant 0$.
Assume $m < 0$. Want: Contradiction.
Since $m \in T = f'_*(U)$, choose $p \in U \cap (\mathrm{dom}\,[f'])$ s.t. $m = f'_p$.
Then $p \in U \cap (\mathrm{dom}\,[f]) \subseteq U$, so $p \in U$.
Since U is open in $\mathbb{R}$, we get: $U = \mathrm{Int}_{\mathbb{R}} U$.
Since $p \in U = \mathrm{Int}_{\mathbb{R}} U$, choose $B \in \mathcal{B}_{\mathbb{R}}(p)$ s.t. $B \subseteq U$.
Since $B \in \mathcal{B}_{\mathbb{R}}(p)$, choose $r > 0$ s.t. $B = B_{\mathbb{R}}(p, r)$.
Since $f'_p = m < 0$, by Theorem 71.5, choose $\delta > 0$ s.t.

(1) $f < f_p$ on $(p; p+\delta)$ and
(2) $f > f_p$ on $(p-\delta; p)$.

Let $\gamma := [1/2] \cdot [\min\{\delta, r\}]$. Then $0 < \gamma < \delta$ and $-r < \gamma < r$.
Let $q := p + \gamma$. Then $p < q < p + \delta$ and $p - r < q < p + r$.
Then $q \in (p; p+\delta)$ and $q \in (p-r; p+r)$.
We have $q \in (p-r; p+r) = B_{\mathbb{R}}(p, r) = B \subseteq U$.
Then $p, q \in U$ and $p \leqslant q$.
So, since f is semi-increasing on U, we get: $f_p \leqslant f_q$.
Since $q \in (p; p+\delta)$, by (1), we get: $f_q < f_p$.
Then $f_q < f_p \leqslant f_q$, so $f_q < f_q$, so $f_q \neq f_q$. Contradiction. QED

4-3. Let $f : \mathbb{R} \to \mathbb{R}$ and let J be an interval.
Assume that f is c/d on J. Let $I := \mathrm{Int}_{\mathbb{R}} J$ and $T := f'_*(I)$.
Assume $T \geqslant 0$. Show: f is semi-increasing on J.

Proof: Want: $\forall a, b \in J$, $[\,(\,a \leqslant b\,) \Rightarrow (\,f_a \leqslant f_b\,)\,]$.
Given $a, b \in \mathbb{T}$. Want: $(\,a \leqslant b\,) \Rightarrow (\,f_a \leqslant f_b\,)$.
Assume: $a \leqslant b$. Want: $f_a \leqslant f_b$.
Assume $f_a > f_b$. Want: Contradiction.
Since $f_a > f_b$, we get $f_a \neq f_b$, and so $a \neq b$.
Since $a \neq b$ and $a \leqslant b$, we get: $a < b$. Then $b - a > 0$.
Since $f_a > f_b$, we get: $f_b - f_a < 0$.
So, since $b - a > 0$, we get $\dfrac{f_b - f_a}{b - a} < 0$.
Then $\mathrm{DQ}_f(a, b) = \dfrac{f_b - f_a}{b - a} < 0$, so $\mathrm{DQ}_f(a, b) < 0$.
By Theorem 72.22, we have $\mathrm{DQ}_f(a, b) \in f'_*(I)$.
Then $\mathrm{DQ}_f(a, b) \in f'_*(I) = T \geqslant 0$, so $0 \leqslant \mathrm{DQ}_f(a, b)$.
Then $0 \leqslant \mathrm{DQ}_f(a, b) < 0$, so $0 < 0$. Contradiction. QED

4-4. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{R}$.
Show: $(f(p + \bullet))' = f'(p + \bullet)$.

Proof: Let $g := f(p + \bullet)$. Want: $g' = f'(p + \bullet)$.
Want: $\forall q \in \mathbb{R}$, $g'(q) = f'(p + q)$.
Given $q \in \mathbb{R}$. Want: $g'(q) = f'(p + q)$. Let $R := \{\mathrm{s}\ell_L \,|\, L \in \mathrm{LINS}_q g\}$, $S := \{\mathrm{s}\ell_L \,|\, L \in \mathrm{LINS}_{p+q} f\}$.
Then $g'(q) = \mathrm{UE}\, R$ and $f'(p + q) = \mathrm{UE}\, S$. Want: $R = S$.
Want: $\mathrm{LINS}_q g = \mathrm{LINS}_{p+q} f$.
Want: $\{L \in \mathcal{L} \,|\, g_q^T - L \in \mathcal{O}_1\} = \{L \in \mathcal{L} \,|\, f_{p+q}^T - L \in \mathcal{O}_1\}$.

It suffices to show: $g_q^T = f_{p+q}^T$.
Want: $\forall h \in \mathbb{R}$, $(g_q^T)(h) = (f_{p+q}^T)(h)$.
Given $h \in \mathbb{R}$. Want: $(g_q^T)(h) = (f_{p+q}^T)(h)$.
Want: $[g(q+h)] - [g(q)] = [f(p+q+h)] - [f(p+q)]$.
We have $g(q+h) = (f(p+\bullet))(q+h) = f(p+q+h)$
and $g(q) = (f(p+\bullet))(q) = f(p+q)$.
Then $g(q+h) = f(p+q+h)$
and $g(q) = f(p+q)$,
so $[g(q+h)] - [g(q)] = [f(p+q+h)] - [f(p+q)]$. QED

4-5. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathrm{dom}\,[f]$.
Show: [(f has a strict local minimum at p in $\mathbb{R}$)
$\Leftrightarrow$ (f_p^T has a strict local minimum at 0 in $\mathbb{R}$)].

Proof: Let $g := f_p^T$.
Since $p \in \mathrm{dom}\,[f]$, we get $f_p \in \mathrm{im}\,[f]$.
Then $f_p \in \mathrm{im}\,[f] \subseteq \mathbb{R}$, so $f_p \in \mathbb{R}$.

Proof of $\Rightarrow$:
Assume: f has a strict local minimum at p in $\mathbb{R}$.
Want: f_p^T has a strict local minimum at 0 in $\mathbb{R}$.
Want: g has a strict local minimum at 0 in $\mathbb{R}$.
Choose $B \in \mathcal{B}_{\mathbb{R}}(p)$ s.t. $f > f_p$ on $B_p^\times$.
Since $B \in \mathcal{B}_{\mathbb{R}}(p)$, we get $B - p \in \mathcal{B}_{\mathbb{R}}(0)$.
Let $A := B - p$. Then $A \in \mathcal{B}_{\mathbb{R}}(0)$.
It therefore suffices to show: $g > g_0$ on $A_0^\times$.
Want: $\forall h \in A_0^\times$, $g_h > g_0$.
Given $h \in A_0^\times$. Want: $g_h > g_0$.
Since $A = B - p$, it follows that $p + A = B$.
Since $h \in A_0^\times$, we get $h \in A$ and $h \neq 0$.
Then $p + h \in p + A$ and $p + h \neq p$.
Then $p + h \in B$ and $p + h \neq p$, so $p + h \in B_p^\times$.
So, by the choice of B, it follows that $f_{p+h} > f_p$.
So, since $f_p \in \mathbb{R}$, we get: $f_{p+h} - f_p > f_p - f_p$.
Then $g_h = (f_p^T)_h = f_{p+h} - f_p > f_p - f_p = f_{p+0} - f_p = g_0$, as desired.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$:
Assume: f_p^T has a strict local minimum at 0 in $\mathbb{R}$.

Want: f has a strict local minimum at p in $\mathbb{R}$.
Know: g has a strict local minimum at 0 in $\mathbb{R}$.
Choose $A \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $g > g_0$ on $A_0^\times$.
Since $A \in \mathcal{B}_{\mathbb{R}}(0)$, we get $A + p \in \mathcal{B}_{\mathbb{R}}(p)$.
Let $B := A + p$. Then $B \in \mathcal{B}_{\mathbb{R}}(p)$.
It therefore suffices to show: $f > f_p$ on $B_p^\times$.
Want: $\forall x \in B_p^\times$, $f_x > f_p$.
Given $x \in B_p^\times$. Want: $f_x > f_p$.
Since $x \in B_p^\times$, we get $x \in B$ and $x \neq p$.
Then $x - p \in B - p$ and $x - p \neq 0$. Let $h := x - p$.
Then $h \in B - p$ and $h \neq 0$.
Since $B = A + p$, it follows that $B - p = A$.
Then $h \in A$ and $h \neq 0$. Then $h \in A_0^\times$.
So, by the choice of A, it follows that $g_h > g_0$.
So, as $f_p \in \mathbb{R}$, we get: $g_h + f_p > g_0 + f_p$.
We have $g_h = f_{p+h} - f_p$ and $g_0 = f_{p+0} - f_p$,
so, as $f_p \in \mathbb{R}$, we get $g_h + f_p = f_{p+h}$ and $g_0 + f_p = f_{p+0}$.
Since $h = x - p$, we get $p + h = x$.
Then $f_x = f_{p+h} = g_h + f_p > g_0 + f_p = f_{p+0} = f_p$, as desired.
End of proof of $\Leftarrow$. QED

Homework 3: Due on Tuesday 12 February

3-1. Define $\alpha : \mathbb{R} \dashrightarrow \mathbb{R}$ by $\alpha_h = \dfrac{h^2}{9 \cdot (3 + h)}$. Show: $\alpha \in \widehat{\mathcal{O}}_2$.

Proof: Let $\lambda := \mathrm{adj}_0^0\left(\dfrac{\alpha}{|\bullet|^2}\right)$.
Since $\alpha_0 = 0$, it follows, from Theorem 69.5, that $\alpha = \lambda \cdot (|\bullet|^2)$.
Want: $\alpha \in (\mathrm{BNZ}) \cdot (|\bullet|^2)$. Want: $\lambda \in \mathrm{BNZ}$.
We have $\mathrm{dom}\,[\lambda] = \left(\mathrm{dom}\left[\dfrac{\alpha}{|\bullet|^k}\right]\right)_0^+ = ((\mathrm{dom}\,[\alpha])_0^\times)_0^+ \supseteq \mathrm{dom}\,[\alpha]$.
So, since $\alpha \in \mathrm{DNZ}$, it follows that $\lambda \in \mathrm{DNZ}$.
Want: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\lambda_*(B)$ is bounded in $\mathbb{R}$.
Let $B := B_{\mathbb{R}}(0, 1)$. Then $B \in \mathcal{B}_{\mathbb{R}}(0)$.
Want: $\lambda_*(B)$ is bounded in $\mathbb{R}$. Want: $\lambda_*(B) \subseteq B_{\mathbb{R}}(0, 100)$.
Want: $\forall y \in \lambda_*(B)$, $y \in B_{\mathbb{R}}(0, 100)$.
Given $y \in \lambda_*(B)$. Want $y \in B_{\mathbb{R}}(0, 100)$.
Since $y \in \lambda_*(B)$, choose $h \in B$ s.t. $y = \lambda_h$.

Exactly one of the following is true:

(1) $h = 0$ or (2) $h \neq 0$.

Case (1):

We have $y = \lambda_h = \lambda_0 = \left(\mathrm{adj}_0^0\left(\frac{\alpha}{|\bullet|^2}\right)\right)_0 = 0 \in B_{\mathbb{R}}(0, 100)$, as desired.

End of Case (1).

Case (2):

We have $y = \lambda_h = \left(\mathrm{adj}_0^0\left(\frac{\alpha}{|\bullet|^2}\right)\right)_h = \left(\frac{\alpha}{|\bullet|^2}\right)_h = \frac{\alpha_h}{|h|^2}$.

Then $|y| = \frac{|\alpha_h|}{|h|^2}$. Also, $|\alpha_h| = \frac{|h|^2}{9 \cdot |3+h|}$. Then $|y| = \frac{1}{9 \cdot |3+h|}$.

Since $h \in B = B_{\mathbb{R}}(0,1)$, it follows that $d_{\mathbb{R}}(h,0) < 1$.

Then $|h - 0| = d_{\mathbb{R}}(h,0) < 1$, so $-1 < h < 1$.

Then $3 - 1 < 3 + h < 3 + 1$, so $2 < 3 + h < 4$.

Since $3 + h > 2 > 0$, we get $|3 + h| = 3 + h$ and $\frac{1}{3+h} < \frac{1}{2}$.

Since $\frac{1}{3+h} < \frac{1}{2}$, we see that $\frac{1}{9} \cdot \frac{1}{3+h} < \frac{1}{9} \cdot \frac{1}{2}$.

$$\text{Then } d_{\mathbb{R}}(y,0) = |y - 0| = |y| = \frac{1}{9 \cdot |3+h|} = \frac{1}{9 \cdot (3+h)}$$
$$= \frac{1}{9} \cdot \frac{1}{3+h} < \frac{1}{9} \cdot \frac{1}{2} < 100.$$

Then $y \in B_{\mathbb{R}}(0, 100)$. *End of Case (2).* QED

3-2. Define $r : \mathbb{R} \dashrightarrow \mathbb{R}$ by $r_x = 1/x$. Show: $r_3' = -1/9$.

Proof: Define $L \in \mathcal{L}$ by $L_h = -h/9$.

Then $\mathrm{s}\ell_L = L_1 = -1/9$. Want $r_3' = \mathrm{s}\ell_L$.

By Theorem 64.49, it suffices to show: $r_3^T - L \in \mathcal{O}_1$.

Define $\alpha : \mathbb{R} \dashrightarrow \mathbb{R}$ by $\alpha_h = \frac{h^2}{9 \cdot (3+h)}$.

By HW#3-1, $\alpha \in \widehat{\mathcal{O}}_2$, so, since $\widehat{\mathcal{O}}_2 \subseteq \mathcal{O}_1$, we see that $\alpha \in \mathcal{O}_1$.

It therefore suffices to show: $r_3^T - L = \alpha$.

Want: $\forall h \in \mathbb{R}$, $(r_3^T - L)_h = \alpha_h$.

Given $h \in \mathbb{R}$. Want: $(r_3^T - L)_h = \alpha_h$.

$$\text{We compute: } (r_3^T - L)_h = [(r_3^T)_h] - [L_h] = [r_{3+h} - r_3] - [-h/9]$$
$$= r_{3+h} - r_3 + \frac{h}{9} = \frac{1}{3+h} - \frac{1}{3} + \frac{h}{9}$$

$$= \frac{9}{9\cdot(3+h)} - \frac{3\cdot(3+h)}{9\cdot(3+h)} + \frac{h\cdot(3+h)}{9\cdot(3+h)}$$
$$= \frac{9-9-3h+3h+h^2}{9\cdot(3+h)} = \frac{h^2}{9\cdot(3+h)}$$
$$= \alpha_h, \qquad \text{as desired. QED}$$

3-3. Define $L \in \mathcal{L}$ by $L_x = 7x$. Show: $L' = C_{\mathbb{R}}^7$.

Proof: Want: $\forall x \in \mathbb{R}$, $L'_x = (C_{\mathbb{R}}^7)_x$.
Given $x \in \mathbb{R}$. Want: $L'_x = (C_{\mathbb{R}}^7)_x$. Want: $L'_x = 7$.
We have $\mathrm{s}\ell_L = L_1 = 7 \cdot 1 = 7$. Want: $L'_x = \mathrm{s}\ell_L$.
By Theorem 64.49, it suffices to show: $L_x^T - L \in \mathcal{O}_1$.
Since $\mathbf{0} \in \mathcal{O}_1$, it suffices to show: $L_x^T - L = \mathbf{0}$.
Want: $\forall h \in \mathbb{R}$, $(L_x^T - L)_h = \mathbf{0}_h$.
Given $h \in \mathbb{R}$. Want: $(L_x^T - L)_h = \mathbf{0}_h$.
We have $(L_x^T - L)_h = [(L_x^T)_h] - L_h = [L_{x+h} - L_x] - L_h$
$= [7\cdot(x+h) - 7x] - 7h = 0 = \mathbf{0}_h$, as desired. QED

3-4. Let $a, b \in \mathbb{R}$. Assume $a < b$. Let $I := (a; b)$, $J := [a; b]$.
Show: $\mathrm{Int}_{\mathbb{R}} J = I$ and $\mathrm{C}\ell_{\mathbb{R}} I = J$.

Proof: We wish to prove all of the following:
(1) $\mathrm{Int}_{\mathbb{R}} J \subseteq I$ and (2) $\mathrm{Int}_{\mathbb{R}} J \supseteq I$ and
(3) $\mathrm{C}\ell_{\mathbb{R}} I \subseteq J$ and (4) $\mathrm{C}\ell_{\mathbb{R}} I \supseteq J$.

Proof of (1):
Want: $\forall x \in \mathrm{Int}_{\mathbb{R}} J$, $x \in I$. Given $x \in \mathrm{Int}_{\mathbb{R}} J$. Want: $x \in I$.

Claim A: $x \neq a$.
Proof of Claim A:
Assume $x = a$. Want: Contradiction.
Since $a = x \in \mathrm{Int}_{\mathbb{R}} J$, choose $U \in \mathcal{B}_{\mathbb{R}}(a)$ s.t. $U \subseteq J$.
Since $U \in \mathcal{B}_{\mathbb{R}}(a)$, choose $r > 0$ s.t. $U = B_{\mathbb{R}}(a, r)$.
Let $y := a - (r/2)$. Then $y < a$.
Also, $d_{\mathbb{R}}(y, a) = |a - y| = r/2 < r$, so $y \in B_{\mathbb{R}}(a, r)$.
Then $y \in B_{\mathbb{R}}(a, r) = U \subseteq J = [a; b]$, so $y \geqslant a$.
Then $y < a$ and $y \geqslant a$. Contradiction.
End of proof of Claim A.

Claim B: $x \neq b$.

Proof of Claim B:
Assume $x = b$. Want: Contradiction.
Since $b = x \in \mathrm{Int}_{\mathbb{R}} J$, choose $U \in \mathcal{B}_{\mathbb{R}}(b)$ s.t. $U \subseteq J$.
Since $U \in \mathcal{B}_{\mathbb{R}}(b)$, choose $r > 0$ s.t. $U = B_{\mathbb{R}}(b, r)$.
Let $y := b + (r/2)$. Then $y > b$.
Also, $d_{\mathbb{R}}(y, b) = |b - y| = r/2 < r$, so $y \in B_{\mathbb{R}}(b, r)$.
Then $y \in B_{\mathbb{R}}(b, r) = U \subseteq J = [a; b]$, so $y \leqslant b$.
Then $y > b$ and $y \leqslant b$. Contradiction.
End of proof of Claim B.

Since $x \in J = [a; b]$, it follows, from Claim A and Claim B,
that $x \in [a; b] \backslash \{a, b\}$.
Then $x \in [a; b] \backslash \{a, b\} = (a; b) = I$, as desired.
End of proof of (1).

Proof of (2):
Want: $\forall x \in I$, $x \in \mathrm{Int}_{\mathbb{R}} J$. Given $x \in I$. Want: $x \in \mathrm{Int}_{\mathbb{R}} J$.
Want: $\exists U \in \mathcal{B}_{\mathbb{R}}(x)$ s.t. $U \subseteq J$.
Since $x \in I = (a; b)$, we have $a < x < b$.
Then $x - a > 0$ and $b - x > 0$, and so $\min\{x - a, b - x\} > 0$.
Let $r := \min\{x - a, b - x\}$. Then $r > 0$ and $r \leqslant x - a$ and $r \leqslant b - x$.
Let $U := B_{\mathbb{R}}(x, r)$. Then $U \in \mathcal{B}_{\mathbb{R}}(x)$. Want: $U \subseteq J$.
Want: $\forall y \in U$, $y \in J$.
Given $y \in U$. Want: $y \in J$.
Since $y \in U = B_{\mathbb{R}}(x, r)$, we get $d_{\mathbb{R}}(y, x) < r$.
Then $|y - x| = d_{\mathbb{R}}(y, x) < r$, so $x - r < y < x + r$.
Since $r \leqslant x - a$, we get $x - r \geqslant x - (x - a)$, and so $x - r \geqslant a$.
Since $r \leqslant b - x$, we get $x + r \leqslant x + (b - x)$, and so $x + r \leqslant b$.
We have $a \leqslant x - r < y$, so $a < y$. Also, $y < x + r \leqslant b$, so $y < b$.
Then $a < y < b$, so $y \in (a; b)$.
So, since $(a; b) = I$, we get $y \in I$, as desired.
End of proof of (2).

Proof of (3):
Want: $\forall x \in \mathrm{C}\ell_{\mathbb{R}} I$, $x \in J$. Given $x \in \mathrm{C}\ell_{\mathbb{R}} I$. Want: $x \in J$.
Since $x \in \mathrm{C}\ell_{\mathbb{R}} I$, choose $s \in I^{\mathbb{N}}$ s.t. $s_\bullet \to x$ in $\mathbb{R}$.
We have: $\forall j \in \mathbb{N}$, $s_j \in I = (a; b) \subseteq [a; b]$, so $a \leqslant s_j \leqslant b$.
Since $s_\bullet \to x$ in $\mathbb{R}$ and since, $\forall j \in \mathbb{N}$, $s_j \leqslant b$,

it follows, from Theorem 47.1, that $x \leqslant b$.
Since $s_\bullet \to x$ in $\mathbb{R}$ and since, $\forall j \in \mathbb{N}$, $s_j \geqslant a$,
it follows, from Theorem 47.2, that $x \geqslant a$.
Since $a \leqslant x \leqslant b$, we get $x \in [a;b]$.
So, since $[a;b] = J$, we get $x \in J$, as desired.
End of proof of (3).

Proof of (4):
Want: $\forall x \in J$, $x \in \mathrm{C}\ell_{\mathbb{R}} I$. Given $x \in J$. Want: $x \in \mathrm{C}\ell_{\mathbb{R}} I$.
Want: $\exists s \in I^{\mathbb{N}}$ s.t. $s_\bullet \to x$ in $\mathbb{R}$.
Exactly one of the following is true:
(α) $x = a$ or (β) $x \neq a$.

Case (α):
Since $a < b$, we see that $b - a > 0$. Then $(b-a)/2 > 0$.
Let $z := (b-a)/2$. Then $z > 0$.
Also, as $b - a > 0$, we get $(b-a)/2 < b - a$.
Then $z < b - a$, and so $a + z < b$.
Define $s \in \mathbb{R}^{\mathbb{N}}$ by $s_j = a + (z/j)$.

Claim X: $s \in I^{\mathbb{N}}$.
Proof of Claim X:
We have $\mathrm{dom}\,[s] = \mathbb{N}$. Want: $\mathrm{im}\,[s] \subseteq I$.
Want: $\forall q \in \mathrm{im}\,[s]$, $q \in I$.
Given $q \in \mathrm{im}\,[s]$. Want: $q \in I$.
Since $q \in \mathrm{im}\,[s]$, choose $j \in \mathbb{N}$ s.t. $s_j = q$. Want: $s_j \in I$.
We have $j > 0$, so, as $z > 0$,
we get $z/j > 0$, so $a + (z/j) > a + 0$.
Then $s_j = a + (z/j) > a + 0 = a$, so $a < s_j$.
We have $j \geqslant 1$, so, as $z > 0$,
we get $z/j \leqslant z/1$, so $a + (z/j) \leqslant a + (z/1)$.
Then $s_j = a + (z/j) \leqslant a + (z/) = a + z < b$, so $s_j < b$.
Since $a < s_j < b$, we get $s_j \in (a;b)$. Then $s_j \in (a;b) = I$.
End of proof of Claim X.

By Claim X, $s \in I^{\mathbb{N}}$. Want: $s_\bullet \to x$ in $\mathbb{R}$.
By definition of Case (α), we know that $x = a$.
Define $r \in \mathbb{R}^{\mathbb{N}}$ by $r_j = 1/j$. Then, by Theorem 39.7, $r_\bullet \to 0$ in $\mathbb{R}$.

Then, by Theorem 40.2, $(z \cdot r)_\bullet \to z \cdot 0$ in $\mathbb{R}$.
Let $c := C^a_{\mathbb{N}}$. Then, by Theorem 40.3, $c_\bullet \to a$ in $\mathbb{R}$.
Then, by Theorem 40.1, $(c + (z \cdot r))_\bullet \to a + (z \cdot 0)$ in $\mathbb{R}$.
So, since $a + (z \cdot 0) = a + 0 = a = x$, we get $(c + (z \cdot r))_\bullet \to x$ in $\mathbb{R}$.
It therefore suffices to show: $s = c + (z \cdot r)$.
Want: $\forall j \in \mathbb{N}$, $s_j = (c + (z \cdot r))_j$.
Given $j \in \mathbb{N}$. Want: $s_j = (c + (z \cdot r))_j$.
We have $c_j = (C^a_{\mathbb{N}})_j = a$.
Also, $(z \cdot r)_j = z \cdot (r_j) = z \cdot (1/j) = z/j$.
Then $s_j = a + (z/j) = c_j + [(z \cdot r)_j] = (c + (z \cdot r))_j$.
End of Case (α).

Case (β):
Since $x \in J = [a; b]$, we get $a \leqslant x \leqslant b$.
Since $a \leqslant x$ and $x \neq a$, we get $a < x$. Then $x - a > 0$.
Then $(x - a)/2 > 0$. Let $z := (x - a)/2$. Then $z > 0$.
Also, as $x - a > 0$, we get $(x - a)/2 < x - a$.
Then $z < x - a$, and so $x - z > a$.
Define $s \in \mathbb{R}^{\mathbb{N}}$ by $s_j = x - (z/j)$.

Claim Y: $s \in I^{\mathbb{N}}$.
Proof of Claim Y:
We have $\mathrm{dom}\,[s] = \mathbb{N}$. Want: $\mathrm{im}\,[s] \subseteq I$.
Want: $\forall q \in \mathrm{im}\,[s]$, $q \in I$.
Given $q \in \mathrm{im}\,[s]$. Want: $q \in I$.
Since $q \in \mathrm{im}\,[s]$, choose $j \in \mathbb{N}$ s.t. $s_j = q$. Want: $s_j \in I$.
We have $j > 0$, so, as $z > 0$,
we get $z/j > 0$, so $x - (z/j) < x - 0$.
Then $s_j = x - (z/j) < x - 0 = x \leqslant b$, so $s_j < b$.
We have $j \geqslant 1$, so, as $z > 0$,
we get $z/j \leqslant z/1$, so $x - (z/j) \geqslant x - (z/1)$.
Then $s_j = x - (z/j) \geqslant x - (z/1) = x - z \geqslant a$, so $s_j > a$, so $a < s_j$.
Since $a < s_j < b$, we get $s_j \in (a; b)$. Then $s_j \in (a; b) = I$.
End of proof of Claim Y.

By Claim Y, $s \in I^{\mathbb{N}}$. Want: $s_\bullet \to x$ in $\mathbb{R}$.
Define $r \in \mathbb{R}^{\mathbb{N}}$ by $r_j = 1/j$. Then, by Theorem 39.7, $r_\bullet \to 0$ in $\mathbb{R}$.
Then, by Theorem 40.2, $(-z \cdot r)_\bullet \to -z \cdot 0$ in $\mathbb{R}$.

Let $c := C_{\mathbb{N}}^x$. Then, by Theorem 40.3, $c_\bullet \to x$ in $\mathbb{R}$.
Then, by Theorem 40.1, $(c + (-z \cdot r))_\bullet \to x - (z \cdot 0)$ in $\mathbb{R}$.
So, since $x - (z \cdot 0) = x - 0 = x$, we get $(c + (-z \cdot r))_\bullet \to x$ in $\mathbb{R}$.
It therefore suffices to show: $s = c + (-z \cdot r)$.
Want: $\forall j \in \mathbb{N}$, $s_j = (c + (-z \cdot r))_j$.
Given $j \in \mathbb{N}$. Want: $s_j = (c + (-z \cdot r))_j$.
We have $c_j = (C_{\mathbb{N}}^x)_j = x$.
Also, $(-z \cdot r)_j = -z \cdot (r_j) = -z \cdot (1/j) = -z/j$.
Then $s_j = x - (z/j) = x + (-z/j) = c_j + [(-z \cdot r)_j] = (c + (-z \cdot r))_j$.
End of Case (β).

End of proof of (4). QED

3-5. Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that
$$[\ f(-2) = f(2)\ ]\ \&\ [\ \forall x \in (-2;2),\ f'_x \neq 0\ ].$$

Solution: Let $f := |\bullet|$.

Homework 2: Due on Tuesday 5 February

2-1. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a \in \mathbb{R}$, $p \in \operatorname{dom}[f]$, $S \subseteq \mathbb{R}$ and $q := f_p$.
Show: $(\ f > a$ on $S\ ) \Leftrightarrow (\ f_p^T > a - q$ on $S - p\ )$.

Proof: Proof of $\Rightarrow$: Assume: $f > a$ on S. Want: $f_p^T > a - q$ on $S - p$.
Want: $\forall h \in S - p$, $f_p^T(h) > a - q$.
Given $h \in S - p$. Want: $f_p^T(h) > a - q$.
Since $h \in S - p$, choose $x \in S$ s.t. $h = x - p$. Then $p + h = x$.
Since $x \in S$ and since $f > a$ on S, we get: $f_x > a$.
Then $f_x - q > a - q$.
Then $f_p^T(h) = f_{p+h} - f_p = f_x - q > a - q$, as desired. *End of proof of $\Rightarrow$.*

Proof of $\Leftarrow$: Assume: $f_p^T > a - q$ on $S - p$. Want: $f > a$ on S.
Want: $\forall x \in S$, $f_x > a$.
Given $x \in S$. Want: $f_x > a$.
Since $x \in S$, we get: $x - p \in S - p$.
Let $h := x - p$. Then $h \in S - p$ and $p + h = x$.
Since $h \in S - p$ and since $f_p^T > a - q$ on $S - p$, we get: $f_p^T(h) > a - q$.
Then $[f_p^T(h)] + q > a - q + q$. Also, $-f_p + q = -q + q = 0$.

Then $f_x = f_x - f_p + q = f_{p+h} - f_p + q = [f_p^T(h)] + q > a - q + q = a$, as desired. *End of proof of* $\Leftarrow$. QED

2-2. Show: (BNZ) $\circ$ (CVZ) $\subseteq$ BNZ.

Proof: Want: $\forall \alpha \in$ (BNZ) $\circ$ (CVZ), $\alpha \in$ BNZ.
Given $\alpha \in$ (BNZ) $\circ$ (CVZ). Want: $\alpha \in$ BNZ.
Choose $\beta \in$ BNZ, $\gamma \in$ CVZ s.t. $\alpha = \beta \circ \gamma$.
Since $\beta \in$ BNZ, we get $\beta \in$ DNZ, so choose $A \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $A \subseteq \text{dom}[\beta]$.
Since $\gamma \in$ CVZ, we get $\gamma \in$ DNZ, so choose $D \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $D \subseteq \text{dom}[\gamma]$.
Since $\gamma \in$ CVZ, we get: γ is continuous at 0 and $\gamma_0 = 0$.
Since γ is continuous at 0 and $A \in \mathcal{B}_{\mathbb{R}}(0) = \mathcal{B}_{\mathbb{R}}(\gamma_0)$, choose $E \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\gamma_*(E) \subseteq A$.
Let $F := D \cap E$. Then $F \in \mathcal{B}_{\mathbb{R}}(0)$.

Claim 1: $F \subseteq \text{dom}[\alpha]$.
Proof of Claim 1:
Want: $\forall x \in F$, $x \in \text{dom}[\alpha]$.
Given $x \in F$. Want: $x \in \text{dom}[\alpha]$.
We have $x \in F \subseteq D \subseteq \text{dom}[\gamma]$. Also, $x \in F \subseteq E$.
Since $x \in \text{dom}[\gamma]$ and since $x \in E$, we get: $\gamma(x) \in \gamma_*(E)$.
Then $\gamma(x) \in \gamma_*(E) \subseteq A \subseteq \text{dom}[\beta]$, so $\beta(\gamma(x)) \neq ☹$.
Then $\alpha(x) = (\beta \circ \gamma)(x) = \beta(\gamma(x)) \neq ☹$, so $x \in \text{dom}[\alpha]$, as desired.
End of Proof of Claim 1.

Since $F \in \mathcal{B}_{\mathbb{R}}(0)$, by Claim 1, we conclude that $\alpha \in$ DNZ.
Want: $\exists Z \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\alpha_*(Z)$ is bounded in $\mathbb{R}$.
Since $\beta \in$ BNZ, choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\beta_*(B)$ is bounded in $\mathbb{R}$.
Since γ is continuous at 0 and $B \in \mathcal{B}_{\mathbb{R}}(0) = \mathcal{B}_{\mathbb{R}}(\gamma_0)$, choose $Z \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\gamma_*(Z) \subseteq B$.
Want: $\alpha_*(Z)$ is bounded in $\mathbb{R}$.
Since $\beta_*(B)$ is bounded in $\mathbb{R}$, it suffices to prove: $\alpha_*(Z) \subseteq \beta_*(B)$.
Want: $\forall q \in \alpha_*(Z)$, $q \in \beta_*(B)$.
Given $q \in \alpha_*(Z)$. Want: $q \in \beta_*(B)$.
Since $q \in \alpha_*(Z)$, choose $p \in Z \cap (\text{dom}[\alpha])$ s.t. $q = \alpha(p)$.
We have: $p \in Z$ and $p \in \text{dom}[\alpha]$.
Since $p \in \text{dom}[\alpha]$, we get: $\alpha(p) \neq ☹$.
Also, $\beta(\gamma(p)) = (\beta \circ \gamma)(p) = \alpha(p)$.
Then $\beta(\gamma(p)) = \alpha(p) \neq ☹$, so $\gamma(p) \in \text{dom}[\beta]$.

Then $\gamma(p) \neq ☹$, so $p \in \operatorname{dom}[\gamma]$.
Since $p \in \operatorname{dom}[\gamma]$ and since $p \in Z$, we conclude that $\gamma(p) \in \gamma_*(Z)$.
So, since $\gamma_*(Z) \subseteq B$, it follows that $\gamma(p) \in B$.
Since $\gamma(p) \in \operatorname{dom}[\beta]$ and since $\gamma(p) \in B$, we see that $\beta(\gamma(p)) \in \beta_*(B)$.
Then $q = \alpha(p) = \beta(\gamma(p)) \in \beta_*(B)$, as desired. QED

2-3. Let $j, k \in \mathbb{N}$. Show: $\widehat{\mathcal{O}}_j \circ \mathcal{O}_k \subseteq \mathcal{O}_{jk}$.

Proof: Want: $\forall \alpha \in \widehat{\mathcal{O}}_j \circ \mathcal{O}_k$, $\alpha \in \mathcal{O}_{jk}$.
Given $\alpha \in \alpha \in \widehat{\mathcal{O}}_j \circ \mathcal{O}_k$. Want: $\alpha \in \mathcal{O}_{jk}$.
Choose $\beta \in \widehat{\mathcal{O}}_j$, $\gamma \in \mathcal{O}_k$ s.t. $\alpha = \beta \circ \gamma$.
Since $\beta \in \widehat{\mathcal{O}}_j = (\mathrm{BNZ}) \cdot (|\bullet|^j)$, choose $\phi \in \mathrm{BNZ}$ s.t. $\beta = \phi \cdot (|\bullet|^j)$.
Since $\gamma \in \mathcal{O}_k = (\mathrm{CVZ}) \cdot (|\bullet|^k)$, choose $\psi \in \mathrm{CVZ}$ s.t. $\gamma = \psi \cdot (|\bullet|^k)$.
We have: $\forall x \in \mathbb{R}$,

$$\begin{aligned}
\alpha(x) &= (\beta \circ \gamma)(x) = \beta(\gamma(x)) = (\phi \cdot (|\bullet|^j))(\gamma(x)) \\
&= [\phi(\gamma(x))] \cdot [\ |\ \gamma(x)\ |^j\ ] \\
&= [(\phi \circ \gamma)(x)] \cdot [\ |\ (\psi \cdot (|\bullet|^k))(x)\ |^j] \\
&= [(\phi \circ \gamma)(x)] \cdot [\ |\ [\psi(x)] \cdot [|x|^k]\ |^j\ ] \\
&= [(\phi \circ \gamma)(x)] \cdot [\ |\ \psi(x)\ |^j\ ] \cdot [\ |\ |x|^k\ |^j\ ] \\
&= [(\phi \circ \gamma)(x)] \cdot [|\psi(x)|^j] \cdot [|x|^{jk}] \\
&= ([\phi \circ \gamma] \cdot [|\psi^j|] \cdot [|\bullet|^{jk}])(x).
\end{aligned}$$

Then $\alpha = [\phi \circ \gamma] \cdot [|\psi^j|] \cdot [|\bullet|^{jk}]$.
We have $\phi \in \mathrm{BNZ}$ and $\gamma \in \mathcal{O}_k \subseteq \mathcal{O}_0 = \mathrm{CVZ}$, so $\phi \circ \gamma \in (\mathrm{BNZ}) \circ (\mathrm{CVZ})$.
By HW#2-2, $(\mathrm{BNZ}) \circ (\mathrm{CVZ}) \subseteq \mathrm{BNZ}$. Then $\phi \circ \gamma \in \mathrm{BNZ}$.
Since $\psi \in \mathrm{CVZ}$, we get $|\psi| \in |\mathrm{CVZ}| \subseteq \mathrm{CVZ}$.
So, since $j \in \mathbb{N}$, it follows that $|\psi|^j \in \mathrm{CVZ}$.
Then $[\phi \circ \gamma] \cdot [|\psi^j|] \in (\mathrm{BNZ}) \cdot (\mathrm{CVZ})$.
By HW#1-4, $(\mathrm{BNZ}) \cdot (\mathrm{CVZ}) \subseteq \mathrm{CVZ}$. Then $[\phi \circ \gamma] \cdot [|\psi^j|] \in \mathrm{CVZ}$.
Then $\alpha = [\phi \circ \gamma] \cdot [|\psi^j|] \cdot [|\bullet|^{jk}] \in (\mathrm{CVZ}) \cdot (|\bullet|^{jk}) = \mathcal{O}_{jk}$. QED

2-4. Let $\phi \in \mathrm{CVZ}$, $\varepsilon > 0$. Show: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $|\phi| < \varepsilon$ on B.

Proof: Since $\phi \in \mathrm{CVZ}$, we know:

$$\phi \in \mathrm{DNZ} \quad \text{and} \quad \phi_0 = 0 \quad \text{and} \quad \phi \text{ is continuous at } 0.$$

Let $C := B_{\mathbb{R}}(0, \varepsilon)$. Then $C \in \mathcal{B}_{\mathbb{R}}(0) = \mathcal{B}_{\mathbb{R}}(\phi_0)$.
So, since ϕ is continuous at 0, choose $A \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\phi_*(A) \subseteq C$.
Since $\phi \in \mathrm{DNZ}$, choose $D \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $D \subseteq \operatorname{dom}[\phi]$.
Since $A, D \in \mathcal{B}_{\mathbb{R}}(0)$, we get $A \cap D \in \{A, D\}$.
Let $B := A \cap D$. Then $B \in \{A, D\} \subseteq \mathcal{B}_{\mathbb{R}}(0)$.
Want: $|\phi| < \varepsilon$ on B. Want: $\forall x \in B$, $(|\phi|)(x) < \varepsilon$.

Given $x \in B$. Want: $(|\phi|)(x) < \varepsilon$.
We have $x \in B = A \cap D \subseteq D \subseteq \mathrm{dom}\,[\phi]$.
Also, $x \in B = A \cap D \subseteq A$.
Since $x \in A$ and $x \in \mathrm{dom}\,[\phi]$, we get $\phi(x) \in \phi_*(A)$.
Then $\phi(x) \in \phi_*(A) \subseteq C = B_{\mathbb{R}}(0, \varepsilon)$, so $|[\phi(x)] - 0| < \varepsilon$.
Then $(|\phi|)(x) = |\phi(x)| = |[\phi(x)] - 0| < \varepsilon$, as desired. QED

2-5. Let $\alpha \in \mathcal{O}_1$, $L \in \mathcal{L}\backslash\{\mathbf{0}\}$. Show: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $|\alpha| \leqslant |L|$ on B.

Proof: Since $L \in \mathcal{L}$, choose $m \in \mathbb{R}$ s.t. $L = m \cdot \mathrm{id}_{\mathbb{R}}$.
Then, for all $x \in \mathbb{R}$, we have: $L_x = mx$.
Since $L \neq \mathbf{0}$, we see that $m \neq 0$, so $m \in \mathbb{R}_0^\times$, so $|m| > 0$.
Let $\varepsilon := |m|$. Then $\varepsilon > 0$.
Since $\alpha \in \mathcal{O}_1 = \mathrm{CVZ} \cdot (|\bullet|)$, choose $\phi \in \mathrm{CVZ}$ s.t. $\alpha = \phi \cdot (|\bullet|)$.
By HW#2-4, choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $|\phi| < \varepsilon$ on B.
Want: $|\alpha| \leqslant |L|$ on B. Want: $\forall x \in B$, $|\alpha|_x \leqslant |L|_x$.
Given $x \in B$. Want: $|\alpha|_x \leqslant |L|_x$.
Since $x \in B$ and $|\phi| < \varepsilon$ on B, we get: $|\phi|_x < \varepsilon$.
Since $|\phi|_x \leqslant \varepsilon$ and $|x| \geqslant 0$, we conclude that $|\phi|_x \cdot |x| \leqslant \varepsilon \cdot |x|$.
Since $\alpha = \phi \cdot (|\bullet|)$, it follows that $\alpha_x = \phi_x \cdot |x|$, and so $|\alpha_x| = |\phi_x| \cdot |x|$.
Then $|\alpha|_x = |\alpha_x| = |\phi_x| \cdot |x| = |\phi|_x \cdot |x| \leqslant \varepsilon \cdot |x|$
$= |m| \cdot |x| = |mx| = |L_x| = |L|_x$, as desired. QED

Homework 1: Due on Tuesday 29 January

1-1. Let $j \in \mathbb{N}_0$. Show: $\mathcal{O}_j \supseteq (\mathrm{CVZ}) \cdot (|\bullet|^j)$.

Proof: Want: $\forall \alpha \in (\mathrm{CVZ}) \cdot (|\bullet|^j)$, $\alpha \in \mathcal{O}_j$.
Given $\alpha \in (\mathrm{CVZ}) \cdot (|\bullet|^j)$. Want: $\alpha \in \mathcal{O}_j$. Let $\beta := \dfrac{\alpha}{|\bullet|^j}$.
Want: $(\alpha \in \mathrm{DNZ})\,\&\,(\alpha_0 = 0)\,\&\,(\beta \to 0 \text{ near } 0)$.
Since $\alpha \in (\mathrm{CVZ}) \cdot (|\bullet|^j)$, choose $\phi \in \mathrm{CVZ}$ s.t. $\alpha = \phi \cdot (|\bullet|^j)$.
Since $\mathrm{dom}\,[|\bullet|^j] = \mathbb{R}$, we get $|\bullet|^j \in \mathrm{DNZ}$.
Then $\alpha = \phi \cdot (|\bullet|^j) \in (\mathrm{DNZ}) \cdot (\mathrm{DNZ}) \subseteq \mathrm{DNZ}$.
Want: $(\alpha_0 = 0)\,\&\,(\beta \to 0 \text{ near } 0)$.
Since $\phi \in \mathrm{CVZ}$, it follows that $\phi_0 = 0$.
Then $\alpha_0 = (\phi \cdot |\bullet|^j)_0 = (\phi_0) \cdot (|0|^j) = 0 \cdot 0 = 0$.
Want: $\beta \to 0$ near 0.

Claim: $\beta = \phi$ on $\mathbb{R}_0^\times$.

Proof of claim:
Want: $\forall x \in \mathbb{R}_0^\times$, $\beta_x = \phi_x$.
Given $x \in \mathbb{R}_0^\times$. Want: $\beta_x = \phi_x$.
Since $x \in \mathbb{R}_0^\times$, it follows that $\dfrac{|x|^j}{|x|^j} = 1$.
We have: $\alpha_x = (\phi \cdot (|\bullet|^j))_x = (\phi_x) \cdot (|x|^j)$.
Then: $\beta_x = \left(\dfrac{\alpha}{|\bullet|^j}\right)_x = \dfrac{\alpha_x}{|x|^j} = \dfrac{(\phi_x)\cdot(|x|^j)}{|x|^j} = (\phi_x) \cdot 1 = \phi_x$, as desired.
End of proof of claim.

Since $\phi \in$ CVZ, it follows that $\phi \to \phi_0$ near 0.
So, since $\phi_0 = 0$, we get: $\phi \to 0$ near 0.
So, since $\beta = \phi$ on $\mathbb{R}_0^\times$,
we conclude that $\beta \to 0$ near 0, as desired. QED

1-2. Show: CVZ $\subseteq$ BNZ.

Proof: Want: $\forall \alpha \in$ CVZ, $\alpha \in$ BNZ.
Given $\alpha \in$ CVZ. Want: $\alpha \in$ BNZ.
We have $\alpha \in$ CVZ $\subseteq$ DNZ.
Want: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\alpha_*(B)$ is bounded in $\mathbb{R}$.
Since $\alpha \in$ CVZ, it follows that α is continuous at 0.
Choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\alpha_*(B) \subseteq B_{\mathbb{R}}(0,1)$.
Want: $\alpha_*(B)$ is bounded in $\mathbb{R}$.
Since $\alpha_*(B) \subseteq B_{\mathbb{R}}(0,1)$, it follows that $\alpha_*(B)$ is bounded in $\mathbb{R}$. QED

1-3. Show: (BNZ) $\cdot$ (BNZ) $\subseteq$ BNZ.

Proof: Want: $\forall \gamma \in$ (BNZ) $\cdot$ (BNZ), $\gamma \in$ BNZ.
Given $\gamma \in$ (BNZ) $\cdot$ (BNZ). Want: $\gamma \in$ BNZ.
Since $\gamma \in$ (BNZ) $\cdot$ (BNZ), choose $\alpha, \beta \in$ BNZ s.t. $\gamma = \alpha \cdot \beta$.
Since $\alpha, \beta \in$ BNZ $\subseteq$ DNZ,
choose $P, Q \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $P \subseteq \mathrm{dom}\,[\alpha]$ and $Q \in \mathrm{dom}\,[\beta]$.
Then $P \cap Q \in \mathcal{B}_{\mathbb{R}}(0)$.
So, since $P \cap Q \subseteq (\mathrm{dom}\,[\alpha]) \cap (\mathrm{dom}\,[\beta]) = \mathrm{dom}\,[\alpha \cdot \beta] = \mathrm{dom}\,[\gamma]$,
we conclude that $\gamma \in$ DNZ.
Want: $\exists C \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\gamma_*(C)$ is bounded in $\mathbb{R}$.
Since $\alpha, \beta \in$ BNZ, choose $A, B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t.
$\alpha_*(A)$ and $\beta_*(B)$ are bounded in $\mathbb{R}$.
Let $C := A \cap B$. Then $C \in \mathcal{B}_{\mathbb{R}}(0)$.

Want: $\gamma_*(C)$ is bounded in $\mathbb{R}$.
Since $\alpha_*(A)$ and $\beta_*(B)$ are bounded in $\mathbb{R}$,
choose $T, U \in \mathcal{B}_\mathbb{R}$ s.t. $\alpha_*(A) \subseteq T$ and $\beta_*(B) \subseteq U$.
By the Superset Recentering Lemma (Theorem 38.17),
choose $V, W \in \mathcal{B}_\mathbb{R}(0)$ s.t. $T \subseteq V$ and $U \subseteq W$.
Choose $r, s > 0$ s.t. $V = B_\mathbb{R}(0, r)$ and $W = B_\mathbb{R}(0, s)$.
Want: $\gamma_*(C) \subseteq B_\mathbb{R}(0, rs)$.
Want: $\forall y \in \gamma_*(C)$, $y \in B_\mathbb{R}(0, rs)$.
Given $y \in \gamma_*(C)$. Want: $y \in B_\mathbb{R}(0, rs)$. Want: $|y| < rs$.
Since $y \in \gamma_*(C)$, choose $x \in C \cap (\text{dom}\,[\gamma])$ s.t. $\gamma(x) = y$.
We have $x \in \text{dom}\,[\gamma] = \text{dom}\,[\alpha \cdot \beta] = (\text{dom}\,[\alpha]) \cap (\text{dom}\,[\beta])$.
Since $x \in \text{dom}\,[\alpha]$ and $x \in C = A \cap B \subseteq A$, we get: $\alpha(x) \in \alpha_*(A)$.
Since $x \in \text{dom}\,[\beta]$ and $x \in C = A \cap B \subseteq B$, we get: $\beta(x) \in \beta_*(B)$.
Then $\alpha(x) \in \alpha_*(A) \subseteq T \subseteq V = B_\mathbb{R}(0, r)$, so $|\alpha(x)| < r$.
Also, $\beta(x) \in \beta_*(A) \subseteq U \subseteq W = B_\mathbb{R}(0, s)$, so $|\beta(x)| < s$.
Since $0 \leqslant |\alpha(x)| < r$ and $0 \leqslant |\beta(x)| < s$, we get $|\alpha(x)| \cdot |\beta(x)| < rs$.
We have $\gamma(x) = (\alpha \cdot \beta)(x) = [\alpha(x)] \cdot [\beta(x)]$, so $|\gamma(x)| = |\alpha(x)| \cdot |\beta(x)|$.
Then $|y| = |\gamma(x)| = |\alpha(x)| \cdot |\beta(x)| < rs$, as desired. QED

1-4. Show: $(\text{BNZ}) \cdot (\text{CVZ}) \subseteq \text{CVZ}$.

Proof: Want: $\forall \gamma \in (\text{BNZ}) \cdot (\text{CVZ})$, $\gamma \in \text{CVZ}$.
Given $\gamma \in (\text{BNZ}) \cdot (\text{CVZ})$. Want: $\gamma \in \text{CVZ}$.
Since $\gamma \in (\text{BNZ}) \cdot (\text{CVZ})$, choose $\alpha \in \text{BNZ}$, $\beta \in \text{CVZ}$ s.t. $\gamma = \alpha \cdot \beta$.
Since $\alpha \in \text{BNZ} \subseteq \text{DNZ}$ and since $\beta \in \text{CVZ} \subseteq \text{DNZ}$,
choose $P, Q \in \mathcal{B}_\mathbb{R}(0)$ s.t. $P \subseteq \text{dom}\,[\alpha]$ and $Q \subseteq \text{dom}\,[\beta]$.
Then $P \cap Q \in \mathcal{B}_\mathbb{R}(0)$.
So, since $P \cap Q \in (\text{dom}\,[\alpha]) \cap (\text{dom}\,[\beta]) = \text{dom}\,[\alpha \cdot \beta] = \text{dom}\,[\gamma]$,
we conclude that $\gamma \in \text{DNZ}$.
Want: $\gamma(0) = 0$ and γ is continuous at 0.
Since $P \in \mathcal{B}_\mathbb{R}(0)$, we get $0 \in P$.
Since $\alpha \in \text{DNZ}$, we get $\alpha : \mathbb{R} \dashrightarrow \mathbb{R}$.
So, since $0 \in P \subseteq \text{dom}\,[\alpha]$, we get $\alpha(0) \in \mathbb{R}$, so $[\alpha(0)] \cdot 0 = 0$.
Since $\beta \in \text{CVZ}$, we get $\beta(0) = 0$.
Then $\gamma(0) = (\alpha \cdot \beta)(0) = [\alpha(0)] \cdot [\beta(0)] = [\alpha(0)] \cdot 0 = 0$.
Want: γ is continuous at 0.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \text{dom}\,[\gamma]$,

$$[\ |x - 0| < \delta\ ] \Rightarrow [\ |[\gamma(x)] - [\gamma(0)]| < \varepsilon\ ].$$

Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall x \in \text{dom}\,[\gamma]$,

$$[\ |x-0|<\delta\ ]\ \Rightarrow\ [\ |[\gamma(x)]-[\gamma(0)]|<\varepsilon\ ].$$

Since $\alpha \in \text{BNZ}$, choose $A \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\alpha_*(A)$ is bounded in $\mathbb{R}$.
Since $\alpha_*(A)$ is bounded in $\mathbb{R}$,
choose $T \in \mathcal{B}_{\mathbb{R}}$ s.t. $\alpha_*(A) \subseteq T$.
By the Superset Recentering Lemma (Theorem 38.17),
choose $U \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $T \subseteq U$.
Choose $r > 0$ s.t. $U = B_{\mathbb{R}}(0, r)$.
Since $\beta \in \text{CVZ}$, it follows that β is continuous at 0,
so choose $\eta > 0$ s.t., $\forall x \in \text{dom}\,[\beta]$,

$$[\ |x-0|<\eta\ ]\ \Rightarrow\ [\ |[\beta(x)]-[\beta(0)]|<\varepsilon/r\ ].$$

Let $B := B_{\mathbb{R}}(0, \eta)$. Then $A, B \in \mathcal{B}_{\mathbb{R}}(0)$. Let $Q := A \cap B$.
Then $Q \in \mathcal{B}_{\mathbb{R}}(0)$. Choose $\delta > 0$ s.t. $Q = B_{\mathbb{R}}(0, \delta)$.
Want: $\forall x \in \text{dom}\,[\gamma]$, $(\ [\ |x-0|<\delta\ ]\ \Rightarrow\ [\ |[\gamma(x)]-[\gamma(0)]|<\varepsilon\ ]\ )$.
Given $x \in \text{dom}\,[\gamma]$.
Want: $[\ |x-0|<\delta\ ]\ \Rightarrow\ [\ |[\gamma(x)]-[\gamma(0)]|<\varepsilon\ ]$.
Assume: $|x-0|<\delta$. Want: $|[\gamma(x)]-[\gamma(0)]|<\varepsilon$.
We have $x \in \text{dom}\,[\gamma] = \text{dom}\,[\alpha\cdot\beta] = (\text{dom}\,[\alpha]) \cap (\text{dom}\,[\beta])$.
Then $x \in (\text{dom}\,[\alpha]) \cap (\text{dom}\,[\beta]) \subseteq \text{dom}\,[\beta]$.
Also, since $|x-0|<\delta$, we have $x \in B_{\mathbb{R}}(0,\delta)$.
Then $x \in B_{\mathbb{R}}(0,\delta) = Q = A \cap B$.
Then $x \in A \cap B \subseteq B = B_{\mathbb{R}}(0,\eta)$, so $|x-0|<\eta$.
So, since $x \in \text{dom}\,[\beta]$, by choice of η, we get: $|[\beta(x)]-[\beta(0)]|<\varepsilon/r$.
Since $\beta \in \text{CVZ}$, we get $\beta(0)=0$. Then $[\beta(x)]-[\beta(0)] = \beta(x)$.
Then $|\beta(x)| = |[\beta(x)]-[\beta(0)]| < \varepsilon/r$, so $|\beta(x)|<\varepsilon/r$.
Since $x \in (\text{dom}\,[\alpha]) \cap (\text{dom}\,[\beta]) \subseteq \text{dom}\,[\alpha]$ and since $x \in A \cap B \subseteq A$,
we conclude that $\alpha(x) \in \alpha_*(A)$.
Then $\alpha(x) \in \alpha_*(A) \subseteq T \subseteq U = B_{\mathbb{R}}(0,r)$, so $|\alpha(x)|<r$.
Since $0 \leqslant |\alpha(x)| < r$ and since $0 \leqslant |\beta(x)| < \varepsilon/r$,
it follows that $|\alpha(x)|\cdot|\beta(x)| < r\cdot(\varepsilon/r)$.
Recall that $\gamma(0)=0$. We have $\gamma(x) = (\alpha\cdot\beta)(x) = [\alpha(x)]\cdot[\beta(x)]$.
Then $|[\gamma(x)]-[\gamma(0)]| = |[\gamma(x)]-0| = |\gamma(x)| = |[\alpha(x)]\cdot[\beta(x)]|$
$= |\alpha(x)|\cdot|\beta(x)| < r\cdot(\varepsilon/r) = \varepsilon$, as desired. QED

1-5. Let $j \in \mathbb{N}_0$. Show: $\mathcal{o}_j \subseteq \widehat{\mathcal{O}}_j$.

Proof: By HW#1-2, CVZ $\subseteq$ BNZ.
Then $(\text{CVZ})\cdot(|\bullet|^j) \subseteq (\text{BNZ})\cdot(|\bullet|^j)$.
Then $\mathcal{o}_j \subseteq \widehat{\mathcal{O}}_j$, as desired. QED
