CLASS NOTES

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Contents

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1. Foundations

1.1. **Abbreviations.**

We will be using the following abbreviations:

$\forall$	forall (or, sometimes, for any)
$\exists$	there exists (or, sometimes, there exist)
s.t.	such that
$\neg$	not
&	and
$\vee$	or
$\Rightarrow$	implies

For any text strings A and B,

$A \Leftrightarrow B$ means $[\, A \Rightarrow B \,]\ \&\ [\, B \Rightarrow A \,]$.

For any text string A,

$\nexists\, A$ means $\neg\, (\, \exists\, A)$.

1.2. **The logic purist.**

Mathematics is Truth.

We won't be purists about this, but we do describe briefly what how the purist would like mathematics to be organized, beginning with a finite alphabet which would include:

lowercase Roman letters: a,b,c,...,z ;
uppercase Roman letters: A,B,C,...,Z ;
lowercase italic Roman letters: $a,b,c,\ldots,z$;
uppercase italic Roman letters: $A,B,C,\ldots,Z$;
uppercase script letters: $\mathcal{A},\mathcal{B},\mathcal{C},\ldots,\mathcal{Z}$;
uppercase blackboard bold letters: $\mathbb{A},\mathbb{B},\mathbb{C},\ldots,\mathbb{Z}$;
lowercase Greek letters: $\alpha,\beta,\gamma,\ldots,\omega$;
some uppercase Greek letters: $\Gamma,\Delta,\Theta,\Lambda,\Xi,\Pi,\Sigma,\Phi,\Psi,\Omega$;
digits: 0,...,9 ; the symbol ∞ ;
the abbreviations from the preceding section: $\forall,\ldots,\nexists$;
more special characters: $\in,=,<,(,)$;
a blank space to separate words ;
the symbol ☹ .

According to the purist, the entire alphabet should be explained at the start and no characters added later.

We are not purists; later, we'll, in fact, add many characters, *e.g.*

$$>, \leqslant, \geqslant, \varnothing, \notin, \bigcup, \bigcap, \subseteq, \qquad \textit{etc.}$$

Note that $\in$ is not the same as the Greek letter ε.
Note that $\varnothing$ is not the same as the Greek letter ϕ.

The purist would have us give a list of strings called **axioms**. There should be one string on each line. The list may be infinite, but there should be an algorithm (the **axiom algorithm**) that prints it out. For example, the first two axioms might be:

$\forall x,\ x = x$
$\forall x,\ x/0 = ☹$

The purist would ask for an algorithm (the **inference algorithm**) that would take, as input, a FINITE list of strings (one per line), and then produce, as ouput, a FINITE list of strings (one per line).

Typically, for example, if the two axioms above were input into a typical inference algorithm, then, somewhere in the output, we would find a line that reads:

$(\ \forall x,\ x = x\ )\ \&\ (\ \forall x,\ x/0 = ☹\ )$

The typical inference algorithm would take each pair of input lines and, somewhere in the output, produce a line obtained by surrounding each of the two by parentheses and then concatenating them with an ampersand, &, in between.

This carries the idea that, if you know two things separately, then you know both of them are true together.

The purist also asks that every line of input to the inference algorithm is one of the lines of output.

This carries the idea that, if you know something, then you know it.

We won't go into details about all the requirements of the inference algorithm, and it might vary depending on the exact nature of the mathematics you're trying to develop. The basic idea is that the purist wants complete clarity on what kinds of statements can be inferred when other statements are assumed.

Once the axiom algorithm and inference algorithm are written, we write a **theorem algorithm** that does the following:

create the first 100 axioms,
and call them **the first batch of axioms**,

input the first batch of axioms into the inference algorithm,
and call the output the **first batch of theorems**,
print out the first batch of theorems,
create the next 100 axioms,
and call them **the second batch of axioms**,
append the second batch of axioms
to the end of the first batch of theorems,
and call the result the **first appended list**,
input the first appended list into the inference algorithm,
and call the output the **second batch of theorems**,
print out the second batch of theorems,
create the next 100 axioms,
and call them **the third batch of axioms**,
append the third batch of axioms
to the end of the second batch of theorems,
and call the result the **second appended list**,
input the second appended list into the inference algorithm,
and call the output the **third batch of theorems**,
print out the third batch of theorems,
create the next 100 axioms,
and call them **the fourth batch of axioms**,
append the fourth batch of axioms
to the end of the third batch of theorems,
and call the result the **third appended list**,
input the third appended list into the inference algorithm,
and call the output the **fourth batch of theorems**,
print out the fourth batch of theorems,
etc.

The theorem algorithm produces, as output, an infinite list of strings (one per line). These strings are called **theorems**.

The logic purist says, if you think some string is true, you just have to wait to see if it appears in the list of theorems.

If it's NOT true, you have to wait forever to find that out.

We are not logic purists, but the possibility of such a pure system underlies everything that follows.

We start over.

1.3. **Some basic set theory.**

DEFINITION 1.3.1. *By an* **object***, we mean:*

a set *or* *a real number* *or* ∞ *or* $-\infty$ *or* ☹.

DEFINITION 1.3.2. *Let x be an object and let S be a set. By $x \notin S$, we mean:* $\neg\,(\,x \in S\,)$.

DEFINITION 1.3.3. *Let S and T be sets. Then:*

$S \bigcup T := \{x \,|\, (x \in S) \,\vee\, (x \in T)\}$,

$S \bigcap T := \{x \,|\, (x \in S) \,\&\, (x \in T)\}$ *and*

$S \backslash T := \{x \,|\, (x \in S) \,\&\, (x \notin T)\}$.

THEOREM 1.3.4. *Let $S := \{1, 2, 3\}$, $T := \{3, 4, 5\}$. Then:*

$S \bigcup T = \{1, 2, 3, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$ &

$S \bigcap T = \{3\}$ & $S \backslash T = \{1, 2\}$ & $T \backslash S = \{4, 5\}$.

Proof. Omitted. □

DEFINITION 1.3.5. *We define:*

$\varnothing := \{\ \}$,

$\mathbb{N} := \{1, 2, 3, \ldots\}$,

$\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$,

$\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$,

$\mathbb{Q} := \{k/\ell \,|\, k \in \mathbb{Z},\ \ell \in \mathbb{N}\}$ *and*

$\mathbb{R} := \{$ *real numbers* $\}$.

It is a nontrivial theorem that there are elements of $\mathbb{R}$ that are not elements of $\mathbb{Q}$. In particular, $\sqrt{2} \in \mathbb{R} \backslash \mathbb{Q}$. Proving this would require us to delve more deeply into the axioms of $\mathbb{R}$ than we have time for.

1.4. **Some axioms.**

DEFINITION 1.4.1. *Let S and T be sets.*

Then $S \subseteq T$ means: $\forall x \in S,\ x \in T$.

Also, $T \supseteq S$ means: $S \subseteq T$.

Also, $S \nsubseteq T$ means: $\neg(S \subseteq T)$.

Also, $T \nsupseteq S$ means: $S \nsubseteq T$.

Also, $S \subsetneq T$ means: $(\,S \subseteq T\,) \,\&\, (\,S \neq T\,)$.

Also, $T \supsetneq S$ means: $S \subsetneq T$.

The next result is an axiom, called the **Axiom of Extensionality**:

AXIOM 1.4.2. *Let S and T be sets.*
Then: $[\ S = T\ ] \Leftrightarrow [\ (S \subseteq T)\ \&\ (T \subseteq S)\ ]$.

THEOREM 1.4.3. *Let* $A := \{1,2,3\}$, $B := \{1,2,3,4\}$, $C := \{2,3,4\}$.
Then: $(\ A \subsetneq B\ )\ \&\ (\ B \nsubseteq A\ )$
$\&\ (\ B \nsubseteq C\ )\ \&\ (\ C \subsetneq B\ )$
$\&\ (\ A \nsubseteq C\ )\ \&\ (\ C \nsubseteq A\ )$.

In the preceding theorem, the sets A and C are said to be **incomparable** exactly because: $(\ A \nsubseteq C\ )\ \&\ (\ C \nsubseteq A\ )$.

THEOREM 1.4.4. $\mathbb{N} \subsetneq \mathbb{N}_0 \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$.

DEFINITION 1.4.5. *We define:*

$$\begin{aligned} \mathbb{R}^* &:= \mathbb{R} \bigcup \{\,\infty\,,\,-\infty\,\} \qquad \textit{and} \\ \mathbb{Z}^* &:= \mathbb{Z} \bigcup \{\,\infty\,,\,-\infty\,\} \qquad \textit{and} \\ \mathbb{N}^* &:= \mathbb{N} \bigcup \{\,\infty\,\} \qquad \textit{and} \\ \mathbb{N}_0^* &:= \mathbb{N}_0 \bigcup \{\,\infty\,\}. \end{aligned}$$

AXIOM 1.4.6. $\forall x, \quad x = x$.

AXIOM 1.4.7. $\forall x, \quad x/0 = ☹$.

AXIOM 1.4.8. $\forall$*set* S, $\quad S \neq ☹ \notin S$.

In the preceding axiom, " $S \neq ☹ \notin S$ " is an abbreviation of:
" $(\ S \neq ☹\ )\ \&\ (\ ☹ \notin S\ )$ ".
Generally, we will make these kinds of abbreviations without comment.

AXIOM 1.4.9. *Let* $a, b, c \in \mathbb{R}^*$.
Assume $a < b < c$. *Then* $a < c$.

AXIOM 1.4.10. *Let* $a, b \in \mathbb{R}^*$.
Assume $a < b$. *Then* $a \neq b$.

If we wish to develop set theory and the theory of the real number system in parallel, also making room for ☹, it requires a number of axioms, which you can read about in the notes from last year's course.

A more standard approach involves developing set theory first, then defining each real number as a specific set, then proving basic results about the real number system, then moving on to real analysis. In this standard approach, the symbol ☹ is never used.

This year, we will not develop the foundations so carefully, and will instead rely on the reader's sense (based on previous learning) of how sets and real numbers work. So, for example, the last two axioms might simply have been omitted, since most readers know that a double strict inequality contracts to a strict inequality, and, also, most readers know that less than implies not equal to.

1.5. **The Axiom of Choice.**

A set S is said to be **nonempty** if $S \neq \varnothing$.
The next axiom is called **the Axiom of Choice**:

AXIOM 1.5.1. *Let S be a nonempty set. Then* $\mathrm{CH}_S \in S$.

We will use $\mathrm{CH}(S)$ as an alternate notation for CH_S.

AXIOM 1.5.2. $\mathrm{CH}_\varnothing = ☹$.

THEOREM 1.5.3. *Let* $S := \{1,2,3\}$, $x := \mathrm{CH}_S$.
Then: $(\ x = 1\ ) \vee (\ x = 2\ ) \vee (\ x = 3\ )$.

THEOREM 1.5.4. *Let* $S := \{\{1,2,3\}\}$, $x := \mathrm{CH}_S$.
Then: $x = \{1,2,3\}$.

1.6. **Singleton sets and the Unique Element operator.**

A singleton set is a set with exactly one element:

DEFINITION 1.6.1. *Let S be an object.*
By S is a **singleton set**, *we mean:*
$(\ S \text{ is a set}\ ) \quad \& \quad (\ \forall x, y \in S,\ x = y\ )$.

DEFINITION 1.6.2. *Let S be a set. Then:*

$$\mathrm{UE}_S := \begin{cases} \mathrm{CH}_S, & \text{if } S \text{ is a singleton set} \\ ☹, & \text{if } S \text{ is not a singleton set.} \end{cases}$$

We sometimes use $\mathrm{UE}(S)$ to denote UE_S, and sometimes even leave out the parentheses in $\mathrm{UE}(S)$:

THEOREM 1.6.3. *We have:* $\mathrm{UE}\{5\} = 5$ *and* $\mathrm{UE}\{\{1,2\}\} = \{1,2\}$
and $\mathrm{UE}\{1,2\} = ☹$ *and* $\mathrm{UE}\varnothing = ☹$.

1.7. **Functions.**

THEOREM 1.7.1. *We have:* $\{5,8\} = \{8,5\}$.
Also, we have: $\{5,5\} = \{5\}$.
Also, we have: $\{\{5\},\{5\}\} = \{\{5\}\}$.

DEFINITION 1.7.2. *Let a and b be objects.*

Then: $\langle\langle a,b\rangle\rangle \;:=\; \{\,\{a\}\;,\;\{a,b\}\,\}$

THEOREM 1.7.3. *We have:*

$$\langle\langle 5,8\rangle\rangle = \{\{5\},\{5,8\}\},$$
$$\langle\langle 8,5\rangle\rangle = \{\{8\},\{8,5\}\} = \{\{8\},\{5,8\}\},$$
$$\langle\langle 5,8\rangle\rangle = \{\{5\},\{5,8\}\} \neq \{\{8\},\{5,8\}\} = \langle\langle 8,5\rangle\rangle,$$
$$\langle\langle 5,5\rangle\rangle = \{\{5\},\{5,5\}\} = \{\{5\},\{5\}\} = \{\{5\}\} \quad \textit{and}$$
$$\langle\langle\, 1\,,\,\{3\}\,\rangle\rangle = \{\ \{1\}\ ,\ \{\,1\,,\,\{3\}\,\}\ \}.$$

THEOREM 1.7.4. *Let a,b,c,d be objects.*

Then: $[\ \ \langle\langle a,b\rangle\rangle\ \ ] \quad\Leftrightarrow\quad [\ \ (a=c)\,\&\,(b=d)\ \ ].$

Also: $[\ \ \{a,b\}=\{c,d\}\ \ ] \quad\Leftrightarrow\quad [\ \ (\,(a=c)\,\&\,(b=d)\,) \vee (\,(a=d)\,\&\,(b=c)\,)\ \ ].$

DEFINITION 1.7.5. *Let R be an object.*

By R is a **relation**, *we mean:*

$(\,R \text{ is a set}\,) \quad \& \quad (\forall c \in \mathbb{R},\ \exists a,b \text{ s.t. } c = \langle\langle a,b\rangle\rangle\,).$

THEOREM 1.7.6.

The set $\{\langle\langle 1,2\rangle\rangle,\langle\langle 1,3\rangle\rangle,\langle\langle 2,3\rangle\rangle\}$ *is a relation.*

The set $\{\langle\langle 1,2\rangle\rangle,\langle\langle 2,3\rangle\rangle\}$ *is a relation.*

We graphed the two relations in the preceding theorem, noting the vertical line test failing for the frist, but not the second.

DEFINITION 1.7.7. *Let f be an object.*

By f is a **function**, *we mean:*

(1) f is a relation *and*

(2) $\forall a,b,c,\ \ [\ (\ \langle\langle a,b\rangle\rangle \in f\ )\ \&\ (\ \langle\langle a,b\rangle\rangle \in f\ )\ ]\ \Rightarrow\ [\ b=c\ ].$

Condition (2) in the Definition 1.7.7 is called the **vertical line test**.

THEOREM 1.7.8. *Let* $R \;:=\; \{\ \langle\langle 1,2\rangle\rangle,\ \langle\langle 1,3\rangle\rangle,\ \langle\langle 2,3\rangle\rangle\ \}$ *and let* $f \;:=\; \{\ \langle\langle 1,2\rangle\rangle,\ \langle\langle 2,3\rangle\rangle\ \}.$

Then: *R is NOT a function* *and* *f IS a function.*

DEFINITION 1.7.9. *Let f be a function and let x be an object.*

Then $f(x) \;:=\; \mathrm{UE}\{\,y\,|\,\langle\langle x,y\rangle\rangle \in f\,\}.$

Also, $f_x \;:=\; f(x).$

THEOREM 1.7.10. *Let* $f \;:=\; \{\langle\langle 1,2\rangle\rangle, \langle\langle 2,3\rangle\rangle\}$.

$$\textit{Then:}\quad f \textit{ is a function} \quad \textit{and} \quad f(1)=2 \quad \textit{and} \quad f_2 = 3 \quad \textit{and} \quad f_3 = ☹.$$

The function in the preceding theorem is denoted $\left(\begin{array}{c} 1 \mapsto 2 \\ 2 \mapsto 3 \end{array}\right)$. From this point forward, instead of writing

$$f \;:=\; \{\langle\langle 1,2\rangle\rangle, \langle\langle 2,3\rangle\rangle\},$$

we strongly prefer

$$f \;:=\; \left(\begin{array}{c} 1 \mapsto 2 \\ 2 \mapsto 3 \end{array}\right).$$

DEFINITION 1.7.11. *Let* f *be a function. Then:*

$$\mathbb{D}_f \;:=\; \{\, x \,|\, f_x \neq ☹ \,\}$$
$$\textit{and} \quad \mathbb{I}_f \;:=\; \{\, y \,|\, \exists x \in \mathbb{D}_f \textit{ s.t. } f_x = y \,\}.$$

In the preceding definition, $\mathbb{D}_f$ is called the **domain** of f, and $\mathbb{I}_f$ is called the **image** of f.

THEOREM 1.7.12. *Let* $f \;:=\; \left(\begin{array}{c} 1 \mapsto 4 \\ 3 \mapsto 2 \\ 4 \mapsto 2 \end{array}\right)$.

$$\textit{Then:} \quad \mathbb{D}_f \;=\; \{1,3,4\} \quad \textit{and} \quad \mathbb{I}_f \;=\; \{4,2,2\} \;=\; \{2,4\}.$$

DEFINITION 1.7.13. *Let* X *and* Y *be sets, and let* f *be an object.*
Then $f : X \dashrightarrow Y$ *means:* $(f \textit{ is a function}) \,\&\, (\mathbb{D}_f \subseteq X) \,\&\, (\mathbb{I}_f \subseteq Y)$.
Also, $f : X \to Y$ *means:* $(f \textit{ is a function}) \,\&\, (\mathbb{D}_f = X) \,\&\, (\mathbb{I}_f \subseteq Y)$.
Also, $f : X \to> Y$ *means:* $(f \textit{ is a function}) \,\&\, (\mathbb{D}_f = X) \,\&\, (\mathbb{I}_f = Y)$.

DEFINITION 1.7.14. *Let* $a, b \in \mathbb{R}^*$.

$$\begin{array}{lrcl}
\textit{Then} & [a;b] & := & \{\, x \in \mathbb{R}^* \,|\, a \leqslant x \leqslant b \,\} \\
\textit{and} & [a;b) & := & \{\, x \in \mathbb{R}^* \,|\, a \leqslant x < b \,\} \\
\textit{and} & (a;b] & := & \{\, x \in \mathbb{R}^* \,|\, a < x \leqslant b \,\} \\
\textit{and} & (a;b) & := & \{\, x \in \mathbb{R}^* \,|\, a < x < b \,\} \\
\textit{and} & [a..b] & := & \{\, x \in \mathbb{Z}^* \,|\, a \leqslant x \leqslant b \,\} \\
\textit{and} & [a..b) & := & \{\, x \in \mathbb{Z}^* \,|\, a \leqslant x < b \,\} \\
\textit{and} & (a..b] & := & \{\, x \in \mathbb{Z}^* \,|\, a < x \leqslant b \,\} \\
\textit{and} & (a..b) & := & \{\, x \in \mathbb{Z}^* \,|\, a < x < b \,\}.
\end{array}$$

THEOREM 1.7.15. *Let* $f := \{\langle\langle x,y\rangle\rangle \,|\, (x,y \in \mathbb{R})\,\&\,(y = x^2)\}$.

Then: f *is a function* *and* $f_2 = 4$ *and* $f_{-2} = 4$
and $\mathbb{D}_f = \mathbb{R}$ *and* $\mathbb{I}_f = [0;\infty)$
and $f : \mathbb{R} \to \mathbb{R}$ *and* $f : \mathbb{R} \to\!> [0;\infty)$.

From this point forward, instead of writing
"Let $f := \{\langle\langle x,y\rangle\rangle \,|\, (x,y \in \mathbb{R})\,\&\,(y = x^2)\}$",
we strongly prefer
"Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = x^2$".
We graphed this function f; the graph is a parabola.

Define $g : \mathbb{R} \dashrightarrow \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $g_x = (1/x) + 5$.
We graphed g; the graph is a hyperbola.

DEFINITION 1.7.16. *Let* S *be a set and* q *and object.*
Then $S_q^+ := S \bigcup \{q\}$ *and* $S_q^\times := S \backslash \{q\}$.

THEOREM 1.7.17. *Let* $S := \{1,2,3\}$, $y := 3$, $z := 4$.
Then $S_y^+ = \{1,2,3\}$ *and* $S_z^+ = \{1,2,3,4\}$
and $S_y^\times = \{1,2\}$ *and* $S_z^\times = \{1,2,3\}$.

THEOREM 1.7.18. *Define* $g : \mathbb{R} \dashrightarrow \mathbb{R}$ *by:* $\forall x \in \mathbb{R}$, $g_x = (1/x) + 5$.
Then $\mathbb{D}_g = \mathbb{R}_0^\times$ *and* $\mathbb{I}_g = \mathbb{R}_5^\times$
and $g : \mathbb{R}_0^\times \to \mathbb{R}$ *and* $g : \mathbb{R}_0^\times \to\!> \mathbb{R}_5^\times$.

THEOREM 1.7.19. *Let* f *and* g *be functions.*
Then: $(f = g) \Leftrightarrow (\forall x\, f_x = g_x)$.

THEOREM 1.7.20. *Let* f *and* g *be functions and let* S *be a set.*
Assume $\mathbb{D}_f \subseteq S$ *and* $\mathbb{D}_g \subseteq S$. *Then:* $(f = g) \Leftrightarrow (\forall x \in S\, f_x = g_x)$.

DEFINITION 1.7.21. *Let* f *and* g *be functions.*
Then $g \circ f$ *is the function defined by:* $\forall x$, $(g \circ f)_x = g_{f_x}$.

Frownie is infective:

DEFINITION 1.7.22. $\forall f$, $(☹ \circ f = ☹)\,\&\,(f \circ ☹ = ☹)$.

THEOREM 1.7.23. *Define* $f, g : \mathbb{R} \to \mathbb{R}$ *by:*
$\forall x \in \mathbb{R}$, $f_x = x + 1$ *and* $g_x = x^2$.
Then: $\forall x \in \mathbb{R}$, $(g \circ f)_x = (x+1)^2 = x^2 + 2x + 1$ *and* $(f \circ g)_x = x^2 + 1$.
Also, we have: $(g \circ f)_1 = 4 \neq 2 = (f \circ g)_1$.
Also, we have: $g \circ f \neq f \circ g$.

THEOREM 1.7.24. *Let* f, g *and* h *be functions.*
Then $(h \circ g) \circ f = h \circ (g \circ f)$.

Proof. Want: $\forall w,\ ((h \circ g) \circ f)_w = (h \circ (g \circ f))_w$.
Given w. Want: $((h \circ g) \circ f)_w = (h \circ (g \circ f))_w$.
Let $x := f_w$. Let $y := g_x$. Let $z := h_y$.
Then $((h \circ g) \circ f)_w = (h \circ g)_{f_w} = (h \circ g)_x = h_{g_x} = h_y = z$.
Want: $(h \circ (g \circ f))_w = z$. We have $(g \circ f)_w = g_{f_w} = g_x = y$.
Then $(h \circ (g \circ f))_w = h_{(g \circ f)_w} = h_y = z$, as desired. □

1.8. **Injective functions and inverse functions.**

DEFINITION 1.8.1. *Let f be a function.*
By f is **one-to-one**, *we mean:*

$$\forall a, b \in \mathbb{D}_f, \quad (\, f_a = f_b \,) \Rightarrow (\, a = b \,).$$

Also, by f is **injective**, *we mean: f is one-to-one.*

DEFINITION 1.8.2. *Let X and Y be sets, and let f be an object.*
Then $f : X \hookrightarrow Y$ means: $(\, f : X \to Y \,)\ \&\, (\, f$ is one-to-one$\,)$.
Also, $f : X \hookrightarrow\!\!> Y$ means: $(\, f : X \to\!\!> Y \,)\,\&\,(\, f$ is one-to-one$\,)$.

THEOREM 1.8.3. *Let* $f \ := \begin{pmatrix} 1 \mapsto 4 \\ 2 \mapsto 5 \\ 3 \mapsto 6 \end{pmatrix}$.
Then: $f : \{1, 2, 3\} \hookrightarrow\!\!> \{4, 5, 6\}$ *and* $f : \{1, 2, 3\} \hookrightarrow \{3, 4, 5, 6, 7\}$.

THEOREM 1.8.4. $\varnothing : \varnothing \hookrightarrow\!\!> \varnothing$ *and* $\varnothing : \varnothing \hookrightarrow \{7, 8\}$.

DEFINITION 1.8.5. *Let R be a relation.*

$$\text{Then:} \qquad R^\wedge \quad := \quad \{\, \langle\langle y, x \rangle\rangle \mid \langle\langle x, y \rangle\rangle \in R \,\}.$$

THEOREM 1.8.6. *Let* $R \ := \ \{\, \langle\langle 1, 2 \rangle\rangle, \langle\langle 2, 2 \rangle\rangle, \langle\langle 2, 4 \rangle\rangle \,\}$.
Then $R^\wedge \ = \ \{\, \langle\langle 2, 1 \rangle\rangle, \langle\langle 2, 2 \rangle\rangle, \langle\langle 4, 2 \rangle\rangle \,\}$.

Let $R := \{\, \langle\langle 1, 2 \rangle\rangle, \langle\langle 2, 2 \rangle\rangle, \langle\langle 2, 4 \rangle\rangle \,\}$.
We graphed R and $R^\wedge$ and observed:
$R^\wedge$ is obtained from R by reflection through the 45-degree line.
Frownie is infective:

DEFINITION 1.8.7. $\forall x$, ☹$_x$:= ☹. *Also,* $\forall x$, ☹(x) = ☹.

DEFINITION 1.8.8. *Let f be a function.*

$$\text{Then } f^{-1} \ := \begin{cases} f^\wedge, & \text{if } f \text{ is one-to-one} \\ \text{☹}, & \text{if } f \text{ is not one-to-one.} \end{cases}$$

THEOREM 1.8.9. *Let* $f := \begin{pmatrix} 1 \mapsto 5 \\ 2 \mapsto 7 \\ 3 \mapsto 4 \end{pmatrix}$, $g := \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 2 \\ 9 \mapsto 1 \end{pmatrix}$.

Then $f^{-1} = \begin{pmatrix} 5 \mapsto 1 \\ 7 \mapsto 2 \\ 4 \mapsto 3 \end{pmatrix}$ *and* $g^{-1} = ☹$
and f^{-1} *is a function and* g^{-1} *is not a function*
and $f_7^{-1} = 2$ *and* $f_1^{-1} = ☹$ *and* $g_2^{-1} = ☹$
and $\forall x,\ g_x^{-1} = ☹$ *and* $\forall x,\ (g^{-1} \circ g)_x = ☹ = (g \circ g^{-1})_x$
and $(f^{-1} \circ f)_3 = 3$ *and* $(f^{-1} \circ f)_4 = ☹$
and $(f \circ f^{-1})_3 = ☹$ *and* $(f \circ f^{-1})_4 = 4$.

DEFINITION 1.8.10. *Let* S *be a set.*
Then $\mathrm{id}^S : S \to S$ *is defined by:* $\forall x \in S,\ \mathrm{id}^S_x = x$.

THEOREM 1.8.11. *Let* $f := \begin{pmatrix} 1 \mapsto 5 \\ 2 \mapsto 7 \\ 3 \mapsto 4 \end{pmatrix}$.
Then: $f^{-1} \circ f = \mathrm{id}^{\mathbb{D}_f}$ *and* $f \circ f^{-1} = \mathrm{id}^{\mathbb{I}_f}$.

THEOREM 1.8.12. *Let* f *be a one-to-one function.*
Then: f^{-1} *is a function and* $\mathbb{D}_{f^{-1}} = \mathbb{I}_f$ *and* $\mathbb{I}_{f^{-1}} = \mathbb{D}_f$
and $f^{-1} \circ f = \mathrm{id}^{\mathbb{D}_f}$ *and* $f \circ f^{-1} = \mathrm{id}^{\mathbb{I}_f}$.

THEOREM 1.8.13. *Let* $g := \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 2 \\ 9 \mapsto 1 \end{pmatrix}$.
Then: $g^{-1} = ☹$ *and* $g \circ g^{-1} = ☹$ *and* $g^{-1} \circ g = ☹$.

THEOREM 1.8.14. *Let* g *be a function.*
Assume g *is not one-to-one.*
Then: $g^{-1} = ☹$ *and* $g \circ g^{-1} = ☹$ *and* $g^{-1} \circ g = ☹$.

DEFINITION 1.8.15. *Let* f *be a function and* S *a set.*
Then $f_*(S) := \{f_x \mid x \in S \cap \mathbb{D}_f\}$
and $f^*(S) := \{x \in \mathbb{D}_f \mid f_x \in S\}$.

We drew a Venn diagram with a downward function to illustrate the preceding definition.

THEOREM 1.8.16. *Let* $f := \begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 8 \\ 3 \mapsto 9 \end{pmatrix}$.
Then $f^*\{6,7,8\} = \{1,2\}$ *and* $f_*\{2,3,4\} = \{8,9\}$.

DEFINITION 1.8.17. *Let* S *be a set and let* a *be an object.*
Then $C_S^a : S \to \{a\}$ *is defined by:* $\forall x \in S,\ C_S^a(x) = a$.

Let S be a set and let a be an object.
Then C_S^a is called the **constant function** on S with value a.

We graphed $C^2_{(0;\infty)}$; it is a horizontal ray to the right out of the point $(0,2)$. It does not include the point $(0,2)$.

DEFINITION 1.8.18. *Let* T *be a set and let* $S \subseteq T$.
Then $\chi_S^T : T \to \{0,1\}$ *is defined by:*

$$\forall q \in T, \quad \chi_S^T(q) \ = \ \begin{cases} 1, & \textit{if } q \in S \\ 0, & \textit{if } q \notin S. \end{cases}$$

Let T be a set and let $S \subseteq T$.
Then χ_S^T is called the **characteristic function** of S in T.

DEFINITION 1.8.19. *Let* f *be a function,* $A \subseteq \mathbb{D}_f$.
Then $f|A$ *is the function defined by:*

$$\forall x, \quad (f|A)_x \ = \ \begin{cases} f_x, & \textit{if } x \in A \\ ☹, & \textit{if } x \notin A. \end{cases}$$

In the preceding definition, $f|A$ is called the **restriction** of f to A.

THEOREM 1.8.20. *Let* $f := \begin{pmatrix} 1 \mapsto 2 \\ 3 \mapsto 4 \\ 5 \mapsto 6 \end{pmatrix}$ *and let* $A := \{1,5\}$.

$$\textit{Then} \quad f|A = \begin{pmatrix} 1 \mapsto 2 \\ 5 \mapsto 6 \end{pmatrix} \quad \textit{and} \quad f_1 = 2 \quad \textit{and} \quad f_3 = 4 \quad \textit{and} \quad f_5 = 6$$
$$\textit{and} \quad (f|A)_1 = 2 \quad \textit{and} \quad (f|A)_3 = ☹ \quad \textit{and} \quad (f|A)_5 = 6.$$

1.9. **Max, min, inf and sup.**

DEFINITION 1.9.1. *Let* $a \in \mathbb{R}^*$, $S \subseteq \mathbb{R}^*$.
Then $a < S$ *means:* $\forall x \in S,\ a < x$.
Also, $a \leqslant S$ *means:* $\forall x \in S,\ a \leqslant x$.
Also, $a > S$ *means:* $\forall x \in S,\ a > x$.
Also, $a \geqslant S$ *means:* $\forall x \in S,\ a \geqslant x$.

Also, $S > a$ *means:* $a < S$.
Also, $S \geqslant a$ *means:* $a \leqslant S$.
Also, $S < a$ *means:* $a > S$.
Also, $S \leqslant a$ *means:* $a \geqslant S$.

DEFINITION 1.9.2. *Let* $S \subseteq \mathbb{R}^*$.
Then $\mathrm{UB}_S \ := \ \{a \in \mathbb{R}^* \,|\, S \leqslant a\}$.
Also, $\mathrm{LB}_S \ := \ \{a \in \mathbb{R}^* \,|\, a \leqslant S\}$.
Also, $\mathrm{UB}(S) \ := \ UB_S$.
Also, $\mathrm{LB}(S) \ := \ LB_S$.

The set LB_S is the set of **lower bounds** of S.
The set UB_S is the set of **upper bounds** of S.

We are sometimes sloppy and omit () in $\mathrm{UB}(S)$ and $\mathrm{LB}(S)$.
So, for example, $\mathrm{UB}\{1, 3\}$ means $\mathrm{UB}(\{1, 3\})$.

THEOREM 1.9.3. $\mathrm{UB}\{1, 3\} = [3; \infty]$ *and* $\mathrm{LB}\{1, 3\} = [-\infty; 1]$.

DEFINITION 1.9.4. *Let* $S \subseteq \mathbb{R}^*$.
Then $\max_S := \mathrm{UE}(S \cap \mathrm{UB}_S)$.
Also, $\min_S := \mathrm{UE}(S \cap \mathrm{LB}_S)$.
Also, $\max(S) := \max_S$.
Also, $\min(S) := \min_S$.

The object $\min_S$ is **minimum** of S.
The object $\max_S$ is the set of **maximum** of S.

We are sometimes sloppy and omit () in $\max(S)$ and $\min(S)$.
So, for example, $\min(1; 3]$ means $\min(\ (1; 3]\ )$.

THEOREM 1.9.5. *We have:*
$(\max\{1, 2\} = 2)\,\&\,(\min\{1, 2\} = 1)\,\&\,(\max(1; 2] = 2)\,\&\,(\min(1; 2] = ☹)$.

The following two theorems will be used repeatedly without comment:

THEOREM 1.9.6. *Let* $A \subseteq \mathbb{R}^*$, $x := \min A$.
Assume $x \neq ☹$. *Then:* $(\, x \in A \,)\,\&\,(\, x \leqslant A \,)$.

THEOREM 1.9.7. *Let* $A \subseteq \mathbb{R}^*$, $x := \max A$.
Assume $x \neq ☹$. *Then:* $(\, x \in A \,)\,\&\,(\, A \leqslant x \,)$.

DEFINITION 1.9.8. *Let* $S \subseteq \mathbb{R}^*$.

Then $\sup_S := \min(\mathrm{UB}_S)$.
Also, $\inf_S := \max(\mathrm{LB}_S)$.
Also, $\sup(S) := \sup_S$.
Also, $\inf(S) := \inf_S$.

The object $\inf_S$ is **infimum** of S.
The object $\sup_S$ is the set of **supremum** of S.

We are sometimes sloppy and omit () in $\inf(S)$ and $\sup(S)$.
So, for example, $\inf(1;3]$ means $\inf(\ (1;3]\ )$.

THEOREM 1.9.9.

We have: $(\ \mathrm{UB}(1;2] = [2;\infty]\ )\quad \&\quad (\ \sup(1;2] = 2\ )$.
Also: $(\ \mathrm{LB}(1;2] = [-\infty;1]\ )\quad \&\quad (\ \inf(1;2] = 1\ )$.

Some examples:

S	LB	UB	min	max	inf	sup
$\{5\}$	$[-\infty;5]$	$[5;\infty]$	5	5	5	5
$[0;1]$	$[\infty;0]$	$[1;\infty]$	0	1	0	1
$(0;1)$	$[\infty;0]$	$[1;\infty]$	☹	☹	0	1
$[0;1)$	$[\infty;0]$	$[1;\infty]$	0	☹	0	1
$(0;1]$	$[\infty;0]$	$[1;\infty]$	☹	1	0	1
$\{0,1\}$	$[\infty;0]$	$[1;\infty]$	0	1	0	1
$\mathbb{R}^*$	$\{-\infty\}$	$\{\infty\}$	$-\infty$	∞	$-\infty$	∞
$\mathbb{R}$	$\{-\infty\}$	$\{\infty\}$	☹	☹	$-\infty$	∞
$\varnothing$	$\mathbb{R}^*$	$\mathbb{R}^*$	☹	☹	∞	$-\infty$

The next result is called the **Completeness Axiom**:

AXIOM 1.9.10. *Let* $S \subseteq \mathbb{R}^*$. *Then* $\sup_S \neq$ ☹ $\neq \inf_S$.

The following theorem will be used repeatedly without comment:

THEOREM 1.9.11. *All of the following are true:*

(1) $\forall S \subseteq \mathbb{R}^*$, $\qquad \inf S \leqslant S \leqslant \sup S$.
(2) $\forall S \subseteq \mathbb{R}^*$, $\forall x \in \mathbb{R}$, $\qquad (\ S \leqslant x\ ) \Rightarrow (\ \sup S \leqslant x\ )$.
(3) $\forall S \subseteq \mathbb{R}^*$, $\forall x \in \mathbb{R}$, $\qquad (\ x \leqslant S\ ) \Rightarrow (\ x \leqslant \inf S\ )$.

Later, we will need:

THEOREM 1.9.12. *Let* $A \subseteq \mathbb{R}$ *and let* $x := \min A$.
Assume $x \neq ☹$. *Then:* $(\, x \in A\,) \,\&\, (\, x - 1 \notin A\,)$.

Idea of proof: From the definition of min, we see that the minimum of any set is either frownie or is an element both of the set and of its set of lower bounds. In particular, it is an element of the set, and so we see that $x \in A$. To show $x - 1 \notin A$, note that, were $x - 1$ to be an element of A, then we'd have $\min A \leqslant x - 1$, which would yield $x \leqslant x - 1$, which would yield $0 \leqslant -1$, which is not true. QED

1.10. **The Archimedean Principle.**

We have many unstated axioms describing the real numbers. We also have many theorems about the real numbers that will be assumed without proof, and used repeatedly without comment. For example:

THEOREM 1.10.1. $\forall a, b, c \in \mathbb{R}$, $(a < b < c) \Rightarrow (a < c)$.

Note: The compound inequality $a < b < c$ means: $(a < b)\&(b < c)$. We will often use these kinds of compounds without comment.

Another basic fact about $\mathbb{R}$ that is used repeately without comment is the statement: $-\infty < \mathbb{R} < \infty$.
In other words: $\forall x \in \mathbb{R}$, $-\infty < x < \infty$.
In other words: $\forall x \in \mathbb{R}$, $(-\infty < x)\&(x < \infty)$.
The following axiom is also basic:

AXIOM 1.10.2. $\sup \mathbb{N} = \infty$.

From the preceding axiom, we get the **Archimedean Principle**:

THEOREM 1.10.3. $\forall x \in \mathbb{R}$, $\exists k \in \mathbb{N}$ *s.t.* $k > x$.

Proof. Given $x \in \mathbb{R}$. Want: $\exists k \in \mathbb{N}$ s.t. $k > x$.
Assume: $\neg(\, \exists k \in \mathbb{N}$ s.t. $k > x\,)$. Want: Contradiction.
We have: $\forall k \in \mathbb{N}$, $k \leqslant x$. Then $\mathbb{N} \leqslant x$. Then $\sup \mathbb{N} \leqslant x$.
Since $\sup \mathbb{N} \leqslant x \in \mathbb{R} < \infty$, we get $\sup \mathbb{N} < \infty$.
Then $\sup \mathbb{N} \neq \infty$. By Axiom 1.10.2, $\sup \mathbb{N} = \infty$. Contradiction. ☐

Theorem 1.10.3 asserts:
any real number admits a positive integer that is greater.
This may seem obvious; for us, this assertion is important enough that it is named; it is called the **Archimedean Principle**.

1.11. The Principle of Mathematical Induction.

The next axiom is called the **Well-ordering axiom**:

AXIOM 1.11.1. $\forall$*nonempty* $T \subseteq \mathbb{N}$, $\quad \min T \neq ☹$.

From this we prove the **Principle of Mathematical Induction**:

THEOREM 1.11.2. *Let* $S \subseteq \mathbb{N}$.
Assume: $(\, 1 \in S \,) \quad \& \quad (\, \forall j \in S,\, j+1 \in S \,)$.
Then: $S = \mathbb{N}$.

Proof. Assume $S \neq \mathbb{N}$. Want: Contradiction.
Since $S \subseteq \mathbb{N}$ and $S \neq \mathbb{N}$, we have $\mathbb{N}\backslash S \neq \varnothing$.
Let $i := \min(\mathbb{N}\backslash S)$. Since $N\backslash S \neq \varnothing$, by Axiom 1.11.1, $i \neq ☹$.
Then, by Theorem 1.9.12, we get: $(\, i \in \mathbb{N}\backslash S \,) \,\&\, (\, i-1 \notin \mathbb{N}\backslash S \,)$.
By hypothesis, $1 \in S$. Since $i \in \mathbb{N}\backslash S$, we get: $i \notin S$.
Since $1 \in S$ and $i \notin S$, we see that $i \neq 1$.
We have $i \in \mathbb{N}\backslash S \subseteq \mathbb{N}$, so $i \in \mathbb{N}$.
Since $i \in \mathbb{N}$ and $i \neq 1$, we conclude that $i - 1 \in \mathbb{N}$.
Since $i - 1 \notin \mathbb{N}\backslash S$ and $i - 1 \in \mathbb{N}$, it follows that $i - 1 \in S$.
By assumption $\forall j \in S$, $j + 1 \in S$.
So, since $i - 1 \in S$, we see that $(i - 1) + 1 \in S$.
Then $i \in S$. Recall: $i \notin S$. Contradiction. ☐

THEOREM 1.11.3. $\forall j \in \mathbb{N}, \quad 1 + \cdots + j = j(j+1)/2$.

Proof. Let $S := \{j \in \mathbb{N} \,|\, 1 + \cdots + j = j(j+1)/2\}$.
Want: $S = \mathbb{N}$.
Since $1 = 1 \cdot (1+1)/2$, it follows that $1 \in S$.
So, by the PMI, it suffices to show: $\forall j \in S$, $j + 1 \in S$.
Given $j \in S$. Want: $j + 1 \in S$.
Know: $1 + \cdots + j = j(j+1)/2$.
Want: $1 + \cdots + j + (j+1) = (j+1)((j+1)+1)/2$.
We have:

$$\begin{aligned} 1 + \cdots + j + (j+1) &= (j(j+1)/2) + (j+1) \\ &= ((j^2+j)/2) + ((2j+2)/2) \\ &= (j^2+3j+2)/2 \;=\; (j+1)(j+2)/2 \\ &= (j+1)((j+1)+1)/2, \end{aligned}$$

as desired. ☐

THEOREM 1.11.4. $\forall j \in \mathbb{N},\ 1^2 + \cdots + j^2 = j(j+1)(2j+1)/6.$

Proof. Let $S := \{j \in \mathbb{N} \,|\, 1^2 + \cdots + j^2 = j(j+1)(2j+1)/6\}$.
Want: $S = \mathbb{N}$. $\quad 1^2 = 1 \cdot 2 \cdot 3/6 = 1 \cdot (1+1) \cdot (2 \cdot 1 + 1)/6$, so $1 \in S$.
By the PMI, $\quad$ Want: S is successor closed.
Want: $\forall j \in S,\ j+1 \in S$.
Given $j \in S$. $\quad$ Want: $j+1 \in S$.
Since $j \in S$, we know $1^2 + \cdots + j^2 = j(j+1)(2j+1)/6$.
Want: $1^2 + \cdots + j^2 + (j+1)^2 = (j+1)((j+1)+1)(2 \cdot (j+1)+1)/6$.
We have

$$\begin{aligned}
1^2 + \cdots + j^2 + (j+1)^2 &= [1^2 + \cdots + j^2] + [(j+1)^2] \\
&= \left[\frac{j(j+1)(2j+1)}{6}\right] + \left[\frac{6(j+1)^2}{6}\right] \\
&= [j+1]\left[\frac{[j(2j+1)] + [6(j+1)]}{6}\right] \\
&= [j+1]\left[\frac{[2j^2+j] + [6j+6]}{6}\right] \\
&= [j+1]\left[\frac{2j^2+7j+6}{6}\right] \\
&= [j+1]\left[\frac{(j+2)(2j+3)}{6}\right] \\
&= (j+1)((j+1)+1)(2 \cdot (j+1)+1)/6,
\end{aligned}$$

as desired. ☐

1.12. Comparing sets by injections and surjections.

DEFINITION 1.12.1. *Let X and Y be sets.*
Then $\exists X \hookrightarrow Y$ means: $\exists f$ s.t. $f : X \hookrightarrow Y$.
Also, $\exists X \to> Y$ means: $\exists f$ s.t. $f : X \to> Y$.
Also, $\exists X \hookrightarrow> Y$ means: $\exists f$ s.t. $f : X \hookrightarrow> Y$.

THEOREM 1.12.2. $\exists\{1,2,3\} \hookrightarrow> \{4,5,6\}$
and $\exists\{1,2,3\} \hookrightarrow \{3,4,5,6,7\}$
and $\nexists\{3,4,5,6,7\} \hookrightarrow \{1,2,3\}$
and $\exists\{3,4,5,6,7\} \to> \{1,2,3\}$
and $\nexists\{1,2,3\} \to> \{3,4,5,6,7\}$
and $\exists\varnothing \to> \{7,8\}$
and $\nexists\{7,8\} \to> \varnothing$.

Let A and B be sets.
Then $\exists A \hookrightarrow B$ indicates that A is "smaller" than B.
Also, $\exists B \twoheadrightarrow A$ indicates that B is "larger" than A,
EXCEPT when $A = \varnothing$.
So, $\hookrightarrow$ works slightly better than $\twoheadrightarrow$ for comparing sets.

The next four theorems will be used, without comment in the proof of Theorem 1.12.7, as well as in future proofs. The proofs of these four theorems are left as unassigned HW.

THEOREM 1.12.3. *Let X, Y be sets, $f : X \to Y$. Let A be a set. Then:* $(x \in f^*(A)) \Leftrightarrow (f(x) \in A)$.

THEOREM 1.12.4. *Let a, b be objects. Then:* $(a \in \{b\}) \Leftrightarrow (a = b)$.

THEOREM 1.12.5. *Let X, Y be sets, let $f : X \to Y$ and let $w \in \mathbb{I}_f$. Then:* $f^*(\{w\}) \neq \varnothing$.

THEOREM 1.12.6. *Let X, Y be sets, $f : X \to Y$. Let p, q be objects. Assume:* $p \in f^*(\{q\})$. *Then:* $f(p) = q$.

THEOREM 1.12.7. *Let S and T be sets.*
Assume: $\exists T \twoheadrightarrow S$.
Then: $\exists S \hookrightarrow T$.

Proof. We know: $\exists f$ s.t. $f : T \twoheadrightarrow S$. We want: $\exists g$ s.t. $g : S \hookrightarrow T$.
Choose f s.t. $f : T \twoheadrightarrow S$.
Define $g : S \to T$ by: $\forall x \in S$, $g_x = \mathrm{CH}(f^*(\{x\}))$. Want: $g : S \hookrightarrow T$.
Since $g : S \to T$, we need only show: g is one-to-one.
Want: $\forall w, x \in S$, $(g_w = g_x) \Rightarrow (w = x)$.
Given $w, x \in S$. Want: $(g_w = g_x) \Rightarrow (w = x)$.
Assume $g_w = g_x$. Want $w = x$.
Since $f : T \twoheadrightarrow S$, we have $\mathbb{I}_f = S$.
Then $w, x \in S = \mathbb{I}_f$, so $f^*(\{w\}) \neq \varnothing \neq f^*(\{x\})$.
Then $\mathrm{CH}(f^*(\{w\})) \in f^*(\{w\})$ and $\mathrm{CH}(f^*(\{x\})) \in f^*(\{x\})$.
So, since $g_w = \mathrm{CH}(f^*(\{w\}))$ and $g_x = \mathrm{CH}(f^*(\{x\}))$,
we get: $g_w \in f^*(\{w\})$ and $g_x \in f^*(\{x\})$.
It follows that: $f(g_w) = w$ and $f(g_x) = x$.
By assumption, $g_w = g_x$. Then $f(g_w) = f(g_x)$.
Then $w = f(g_w) = f(g_x) = x$, as desired. □

For nonempty sets, $\twoheadrightarrow$ gives the same comparison as $\hookrightarrow$:

THEOREM 1.12.8. *Let S and T be nonemptysets.*

$$\textit{Then:}\quad (\ \exists S \hookrightarrow T\ ) \ \Leftrightarrow\ (\ \exists T \to> S\ ).$$

Proof. Omitted. □

$\exists \hookrightarrow>$ is reflexive:

THEOREM 1.12.9. *Let S be a set. Then $\exists S \hookrightarrow> S$.*

Idea of proof: Show that $\mathrm{id}^S : S \hookrightarrow> S$. QED

The preding theorem is also true for $\exists \hookrightarrow$ and for $\exists \to>$. That is:

THEOREM 1.12.10. *Let S be a set. Then $\exists S \hookrightarrow S$ and $\exists S \to> S$.*

$\exists \hookrightarrow>$ is symmetric:

THEOREM 1.12.11. *Let S and T be sets.*

$$\textit{Then:}\quad (\ \exists S \hookrightarrow> T\ ) \ \Leftrightarrow\ (\ \exists T \hookrightarrow> S\ ).$$

Idea of proof: Show $\forall f : S \hookrightarrow> T$, $f^{-1} : T \hookrightarrow> S$. QED

The preceeding theorem is untrue for $\exists \hookrightarrow$, and is also untrue for $\exists \to>$. It only works for $\exists \hookrightarrow>$.

$\exists \hookrightarrow>$ is transitive:

THEOREM 1.12.12. *Let S, T and U be sets.*

$$\textit{Then:}\quad [\,(\,\exists S \hookrightarrow> T\,)\,\&\,(\,\exists T \hookrightarrow> U\,)\,] \ \Rightarrow\ [\,\exists S \hookrightarrow> U\,].$$

Idea of proof: Show $\forall f : S \hookrightarrow> T$, $\forall g : T \hookrightarrow> U$, $g \circ f : S \hookrightarrow> U$. QED

The preceeding theorem is true for $\exists \hookrightarrow$, and is also true for $\exists \to>$. That is, the following two theorems are both true:

THEOREM 1.12.13. *Let S, T and U be sets.*

$$\textit{Then:}\quad [\,(\,\exists S \hookrightarrow T\,)\,\&\,(\,\exists T \hookrightarrow U\,)\,] \ \Rightarrow\ [\,\exists S \hookrightarrow U\,].$$

THEOREM 1.12.14. *Let S, T and U be sets.*

$$\textit{Then:}\quad [\,(\,\exists S \to> T\,)\,\&\,(\,\exists T \to> U\,)\,] \ \Rightarrow\ [\,\exists S \to> U\,].$$

DEFINITION 1.12.15. *Let S be a set. Then:*

$$\#S \quad := \quad \sup\{\,k \in \mathbb{N}_0 \,|\, \exists[1..k] \hookrightarrow S\,\}.$$

We have $[1..0] = \varnothing$, and so: $\forall$ set S, $\exists[1..0] \hookrightarrow S$.

Thus we have: $\forall$set S, $\quad 0 \quad \in \quad \{\,k \in \mathbb{N}_0 \,|\, \exists[1..k] \hookrightarrow S\,\}$,

and so: $\forall$set S, $\quad \varnothing \quad \neq \quad \{\,k \in \mathbb{N}_0 \,|\, \exists[1..k] \hookrightarrow S\,\}$.

THEOREM 1.12.16. *We have:*

$$(\,\#\{5,6,9\}=3\,)\,\&\,(\,\#\varnothing=0\,)\,\&\,(\,\#\mathbb{N}=\infty\,)\,\&\,(\,\#\mathbb{R}=\infty\,)$$
$$\&\,(\,\#\{\,\{1,2\}\,,\,3\}=2\,)\,\&\,(\,\#\{\,\{1,2\}\,\}=1\,)\,\&\,(\,\#\{1,2\}=2\,).$$

DEFINITION 1.12.17. *Let S be a set.*

By S is **finite**, *we mean $\#S<\infty$.*
By S is **infinite**, *we mean $\#S=\infty$.*

The next three theorems are important, because they clarify the sense in which $\exists \hookrightarrow$ and $\exists \twoheadrightarrow$ compare sets for size. The proofs are, unforunately, omitted, for lack of time. In a course on set theory, we would give proofs of all three, but this is a course in real analysis.

THEOREM 1.12.18. *Let S and T be finite sets.*
Then: $(\ \exists S \hookrightarrow T\ ) \ \Leftrightarrow\ (\ \#S \leqslant \#T\ )$.

THEOREM 1.12.19. *Let S and T be finite sets. Assume $S\neq\varnothing$.*
Then: $(\ \exists T \twoheadrightarrow S\ ) \ \Leftrightarrow\ (\ \#T \geqslant \#S\ )$.

Note that, if $T=\{3,4,5\}$ and $S=\varnothing$, then $\nexists T \twoheadrightarrow S$. So, in the preceding theorem, the assumption that S is nonempty is necessary. As we remarked earlier, $\exists \twoheadrightarrow$ is a flawed way to compare sets, and we prefer to work with $\exists \hookrightarrow$.

1.13. **The World of Sets.**

THEOREM 1.13.1. *Let S and T be sets.*
Then both of the following are true:
(A) $(\exists S \hookrightarrow T\ ) \vee (\ \exists T \hookrightarrow S\ )$.
(B) $[\ (\ \exists S \hookrightarrow T\ )\ \&\ (\ \exists T \hookrightarrow S\ )\ ] \ \Leftrightarrow\ [\ \exists S \hookrightarrow\!\!\!> T\ ]$.

In the preceding, (B) is called the **Schroeder-Bernstein Theorem**.

We drew a picture of the World of Sets, in which
two sets are S and T are on the same level iff $\exists S \hookrightarrow\!\!\!> T$ and
a set S is below a set T if $\exists S \hookrightarrow T$ and $\nexists T \hookrightarrow S$.

We find showed the first few levels of finite sets, where
$\varnothing$ is the only set at the bottom level (called the 0th level),
the first level consists of sets S for which $\#S=1$,
the second level consists of sets S for which $\#S=2$,
the third level consists of sets S for which $\#S=3$,
etc.

There's a line with the finite sets below it, and the infinite sets above it. The next theorem asserts that the level containing $\mathbb{N}$ is at the bottom of the infinite sets.

THEOREM 1.13.2. *Let S be an infinite set.*
Then $\exists \mathbb{N} \hookrightarrow S$.

Proof. Omitted. □

DEFINITION 1.13.3. *Let S be a set.*
By S is **countable**, *we mean: $\exists S \hookrightarrow \mathbb{N}$.*
By S is **uncountable**, *we mean: $\nexists S \hookrightarrow \mathbb{N}$.*
By S is **countably infinite**, *we mean: $\exists S \hookrightarrow> \mathbb{N}$.*

The countably infinite sets therefore form the level with $\mathbb{N}$, which is located at the bottom of the infinite sets. The countable sets are all the sets at or below that level. The uncountable sets are all above it.

The next two theorems tell us that $\mathbb{N}_0$ and $\mathbb{Z}$ are all countably infinite. That is, they are both at that level that contains $\mathbb{N}$.

THEOREM 1.13.4. $\exists \mathbb{N} \hookrightarrow> \mathbb{N}_0$.

Idea of proof: $\begin{pmatrix} 1 \mapsto 0 \\ 2 \mapsto 1 \\ 3 \mapsto 2 \\ \vdots \end{pmatrix}$. QED

THEOREM 1.13.5. $\exists \mathbb{N} \hookrightarrow> \mathbb{Z}$.

Idea of proof: $\begin{pmatrix} 1 \mapsto & 0 \\ 2 \mapsto & 1 \\ 3 \mapsto & -1 \\ 4 \mapsto & 2 \\ 5 \mapsto & -2 \\ 6 \mapsto & 3 \\ 7 \mapsto & -3 \\ \vdots & \end{pmatrix}$. QED

The following table lists each integer in the first column and each positive integer in the first row. Each entry on the inside of the table is obtained by dividing

the integer to the left of it by the positive integer above it.

Every rational number appears in the inside of the table (infinitely many times).

Divide	1	2	3	4	5	6	7	$\cdots$
0	0	0	0	0	0	0	0	$\cdots$
1	1	$1/2$	$1/3$	$1/4$	$1/5$	$1/6$	$1/7$	$\cdots$
-1	-1	$-1/2$	$-1/3$	$-1/4$	$-1/5$	$-1/6$	$-1/7$	$\cdots$
2	$2/1$	$2/2$	$2/3$	$2/4$	$2/5$	$2/6$	$2/7$	$\cdots$
-2	$-2/1$	$-2/2$	$-2/3$	$-2/4$	$-2/5$	$-2/6$	$-2/7$	$\cdots$
3	$3/1$	$3/2$	$3/3$	$3/4$	$3/5$	$3/6$	$3/7$	$\cdots$
-3	$-3/1$	$-3/2$	$-3/3$	$-3/4$	$-3/5$	$-3/6$	$-3/7$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We can list every rational number (infinitely many times) as follows:
start at the upper left 0 (on the zeroth diagonal)
move NE along the first diagonal $1, 0$
move NE along the second diagonal $-1, 1/2, 0$
move NE along the third diagonal $2/1, -1/2, 1/3, 0$
move NE along the fourth diagonal $-2/1, 2/2, -1/3, 1/4, 0$
move NE along the fifth diagonal $3/1, -2/2, 2/3, -1/4, 1/5, 0$
move NE along the sixth diagonal $-3/1, 3/2, -2/3, 2/4, -1/5, 1/6, 0$
etc.

Concatenating these lists of diagonals, we get a list:

$$0 \quad , \quad 1, 0 \quad , \quad -1, 1/2, 0 \quad , \quad 2/1, -1/2, 1/3, 0 \quad , \quad \ldots$$

Every rational number appears (infinitely many times) in this list.

THEOREM 1.13.6. $\exists \mathbb{N} \to\!> \mathbb{Q}$.

Idea of proof: $$\begin{pmatrix} 1 & \mapsto & 0 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 0 \\ 4 & \mapsto & -1 \\ 5 & \mapsto & 1/2 \\ 6 & \mapsto & 0 \\ 7 & \mapsto & 2/1 \\ 8 & \mapsto & -1/2 \\ 9 & \mapsto & 1/3 \\ 10 & \mapsto & 0 \\ \vdots & & \end{pmatrix}.$$ QED

We can now prove that $\mathbb{Q}$ is countably infinite,
i.e., that $\mathbb{Q}$ is at the same level as $\mathbb{N}$ and $\mathbb{N}_0$ and $\mathbb{Z}$.

THEOREM 1.13.7. $\exists \mathbb{N} \hookrightarrow\!\!\!> \mathbb{Q}$.

Proof. By Theorem 1.13.6, $\exists \mathbb{N} \to\!\!> \mathbb{Q}$.
So, by Theorem 1.12.7, $\exists \mathbb{Q} \hookrightarrow \mathbb{N}$.
Since $\mathrm{id}^{\mathbb{N}} : \mathbb{N} \hookrightarrow \mathbb{Q}$, we see that $\exists \mathbb{N} \hookrightarrow \mathbb{Q}$.
Since both $\exists \mathbb{Q} \hookrightarrow \mathbb{N}$ and $\exists \mathbb{N} \hookrightarrow \mathbb{Q}$,
by the Schoeder-Bernstein Theorem,
we get $\exists \mathbb{N} \hookrightarrow\!\!> \mathbb{Q}$, as desired. □

1.14. **Power sets.**

DEFINITION 1.14.1. *Let S be a set.*
Then $2^S \;:=\; \{\, \textit{subsets of } S \,\}$.

Let S be a set. Then the set 2^S is the set of all subsets of S; it is therefore a set of sets. It is called the **Power set** of S.

To list all the subsets of $\{7, 8, 9\}$, build a table of YESs and NOs:

7	NO	NO	NO	NO	YES	YES	YES	YES
8	NO	NO	YES	YES	NO	NO	YES	YES
9	NO	YES	NO	YES	NO	YES	NO	YES

Each column gives us a particular subset, by telling us whehter or not a given element of $\{7, 8, 9\}$ should be in the subset or not.

We therefore arrive at eight subsets:

THEOREM 1.14.2. $2^{\{7,8,9\}} \;=\; \{\; \varnothing \;,\; \{9\} \;,$
$\{8\} \;,\; \{8,9\} \;,\; \{7\}$
$\{7,9\} \;,\; \{7,8\} \;,\; \{7,8,9\} \;\}$.

Note that $\#(2^{\{7,8,9\}}) = 8 = 2^3 = 2^{\#\{7,8,9\}}$. This is not a coincidence:

THEOREM 1.14.3. *Let S be a finite set.* *Then* $\#(2^S) = 2^{\#S}$.

1.15. **Sets of functions.**

DEFINITION 1.15.1. *Let S and T be sets.*
Then $T^S \;:=\; \{\, \textit{functions } S \to T \,\}$.

To list all the functions $\{7, 8, 9\} \to \{0, 1\}$, build a table of 1s and 0s:

7	0	0	0	0	1	1	1	1
8	0	0	1	1	0	0	1	1
9	0	1	0	1	0	1	0	1

Each column gives us a particular function, by telling us whehter or not a given element of $\{7, 8, 9\}$ should map to 1 or 0.

We therefore arrive at eight functions:

THEOREM 1.15.2. *We have:*

$$\{0,1\}^{\{7,8,9\}} = \left\{ \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 0 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 0 \\ 9 \mapsto 1 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 1 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 0 \\ 8 \mapsto 1 \\ 9 \mapsto 1 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 0 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 0 \\ 9 \mapsto 1 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 1 \\ 9 \mapsto 0 \end{pmatrix}, \begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 1 \\ 9 \mapsto 1 \end{pmatrix} \right\}.$$

Note that $\#(\{0,1\}^{\{7,8,9\}}) = 8 = 2^3 = (\#\{0,1\})^{\#\{7,8,9\}}$. This is not a coincidence:

THEOREM 1.15.3. *Let S and T be finite sets.*
Then $\#(T^S) = (\#T)^{\#S}$.

I asked one of you to pick an element of $\{0,1\}^{\{7,8,9\}}$ and you picked the fifth element in the list above. That is, you picked $\begin{pmatrix} 7 \mapsto 1 \\ 8 \mapsto 0 \\ 9 \mapsto 0 \end{pmatrix}$.

We then calculated $f^*(\{1\})$ and obtained $\{7\}$. The fifth element of the list in Theorem 1.14.2 is $\{7\}$. Thus we can build a bijection

$$\{0,1\}^{\{7,8,9\}} \quad \hookrightarrow\!\!> \quad 2^{\{7,8,9\}}$$

by mapping a function $f \in \{0,1\}^{\{7,8,9\}}$ to $f^*(\{1\}) \in 2^{\{7,8,9\}}$.

This works for any set S, even infinite sets:

THEOREM 1.15.4. *Let S be a set.*
Then $\exists \{0,1\}^S \hookrightarrow 2^S$.

Idea of proof: Define $\Phi : \{0,1\}^S \to 2^S$ by:

$$\forall f \in \{0,1\}^S, \quad \Phi_f = f^*(\{1\}).$$

Show that: $\Phi : \{0,1\}^S \hookrightarrow\!\!> 2^S$.

That is, show that Φ is one-to-one, and that $\mathbb{I}_\Phi = 2^S$. QED

1.16. **The World of Sets has no top.**

The next result shows that $\mathbb{N}$ is not at the top of the World of Sets, because $\{0,1\}^{\mathbb{N}}$ is higher:

THEOREM 1.16.1. $\nexists 2^{\mathbb{N}} \hookrightarrow \mathbb{N}$.

Proof. Assume $\exists 2^{\mathbb{N}} \hookrightarrow \mathbb{N}$. Want: Contradiction.
By Theorem 1.15.4, $\exists\{0,1\}^{\mathbb{N}} \hookrightarrow\!> 2^{\mathbb{N}}$, so $\exists\{0,1\}^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}$.
Since $\exists\{0,1\}^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}$ and $\exists 2^{\mathbb{N}} \hookrightarrow \mathbb{N}$,
we see that: $\exists\{0,1\}^{\mathbb{N}} \hookrightarrow \mathbb{N}$.
Then, by Theorem 1.12.8, $\exists\mathbb{N} \to\!> \{0,1\}^{\mathbb{N}}$.
Choose Φ s.t. $\Phi : \mathbb{N} \to\!> \{0,1\}^{\mathbb{N}}$.
Define $f : \mathbb{N} \to \{0,1\}$ by: $\forall j \in \mathbb{N}$, $f(j) = 1 - [\Phi_j(j)]$.
Since $\Phi : \mathbb{N} \to\!> \{0,1\}^{\mathbb{N}}$, we get: $\mathbb{D}_\Phi = \mathbb{N}$ and $\mathbb{I}_\Phi = \{0,1\}^{\mathbb{N}}$.
Since $f \in \{0,1\}^{\mathbb{N}} = \mathbb{I}_\Phi$, choose $j \in \mathbb{D}_\Phi$ s.t. $f = \Phi_j$.
Then $f(j) = \Phi_j(j)$. Also, $j \in \mathbb{D}_\Phi = \mathbb{N}$.
By definition of f, we have: $f(j) = 1 - [\Phi_j(j)]$.
Let $x = \Phi_j(j)$. Then $x = \Phi_j(j) = f(j) = 1 - [\Phi_j(j)] = 1 - x$.
Then $x = 1 - x$, so $2x = 1$, so $x = 1/2$.
Since $j \in \mathbb{N}$ and $f : \mathbb{N} \to \{0,1\}$, it follows that $f(j) \in \{0,1\}$.
Then $1/2 = x = \Phi_j(j) = f(j) \in \{0,1\}$. Then $1/2 \in \{0,1\}$.
Since $1/2 \neq 0$ and $1/2 \neq 1$, we get $1/2 \notin \{0,1\}$. Contradiction. ☐

Idea of the preceding proof:
Each element of $\{0,1\}^{\mathbb{N}}$ is a sequence of bits.
In the proof of Theorem 1.16.1, the function Φ gives rise to an infinite matrix M of bits, by the rule:
$\forall j,k \in \mathbb{N}$, the (j,k)-entry of M is $\Phi_j(k)$.
In order to get $\mathbb{I}_\Phi = \{0,1\}^{\mathbb{N}}$,
we'd need that every element of $\{0,1\}^{\mathbb{N}}$ is a row of M.
However, we can make an element $f \in \{0,1\}^{\mathbb{N}}$ by the rule:
$\forall j \in \mathbb{N}$, the j-entry of f is $1 - [\Phi_j(j)]$.
Then f cannot be the first row of M because
the first entry of f is not equal to
the first entry of the first row of M.
Also, f cannot be the second row of M because
the second entry of f is not equal to
the second entry of the second row of M.
Also, f cannot be the third row of M because

the third entry of f is not equal to
the third entry of the second row of M.
Continuing, we see that, in fact f cannot be any row of M. QED

In the preceding argument, great focus is paid to
$\Phi_1(1)$ and $\Phi_2(2)$ and $\Phi_3(3)$ and $\cdots$.
These are the entries along the diagonal of M. Moreover, the argument in the proof of Theorem 1.16.1 was evidently discovered by G. Cantor. For this reason that argument in the proof of Theorem 1.16.1 is sometimes called a **Cantor diagonalization argument**.

The World of Sets has an element at the bottom, namely $\varnothing$.
The next result says that there is no set S that is at the top:

THEOREM 1.16.2. *$\forall$set S, $\nexists 2^S \hookrightarrow S$.*

Idea of proof: If $S = \varnothing$, then $\#2^S = \#\{\varnothing\} = 1$ and $\#S = \#\varnothing = 0$, so $\nexists 2^S \hookrightarrow S$. We therefore only need consider the case where $S \neq \varnothing$. In that case, since $S \neq \varnothing \neq 2^S$, we can simply change $\mathbb{N}$ to S throughout the proof of Theorem 1.16.1, and prove $\nexists 2^S \hookrightarrow S$, as desired. QED

1.17. Placement of $\mathbb{R}$ in the World of Sets.

THEOREM 1.17.1. $\exists\{0,1\}^{\mathbb{N}} \to\!> [0;1]$.

Idea of proof: Define $F : \{0,1\}^{\mathbb{N}} \to [0;1]$ by:
$\forall a \in \{0,1\}^{\mathbb{N}}$, $F(a) = 0.a_1a_2a_3\cdots$ (base two).
Now, $0.0111111\ldots = 0.1000000\ldots$ (base two),
so this function F is not one-to-one.
However, F can be shown to be surjective onto $[0;1]$.
Note that: $F(1,1,1,1,1,1,\ldots) = 0.111111\ldots = 1$ (base two).
Then $F : \{0,1\}^{\mathbb{N}} \to\!> [0;1]$. QED

THEOREM 1.17.2. $\exists\{0,1\}^{\mathbb{N}} \hookrightarrow \mathbb{R}$.

Idea of proof: Define $G : \{0,1\}^{\mathbb{N}} \to [0;1]$ by:
$\forall a \in \{0,1\}^{\mathbb{N}}$, $G(a) = 0.a_1a_2a_3\cdots$ (base ten).
Note that $0.0999999\ldots = 0.1000000\ldots$ (base ten),
but we are not using the digit nine, only 0 and 1.
This map G can be shown to be injective.
Also, $\mathbb{I}_G \subseteq [0;1] \subseteq \mathbb{R}$. Then $G : \{0,1\}^{\mathbb{N}} \hookrightarrow \mathbb{R}$. QED

THEOREM 1.17.3. $\exists\mathbb{R} \hookrightarrow\!> \{0,1\}^{\mathbb{N}}$.

Proof. Define $A : \mathbb{R} \to (-1; 1)$ by $A_t = t/\sqrt{t^2 + 1}$.
Then $A : \mathbb{R} \hookrightarrow (-1; 1)$.
Define $B : (-1; 1) \to [0; 1]$ by $B_t = (t + 1)/2$.
Then $B : (-1; 1) \hookrightarrow [0; 1]$.
By Theorem 1.17.1, $\exists\{0, 1\}^{\mathbb{N}} \to\!> [0; 1]$.
So, by Theorem 1.12.7, $\exists[0; 1] \hookrightarrow \{0, 1\}^{\mathbb{N}}$.
Choose C s.t. $C : [0; 1] \hookrightarrow \{0, 1\}^{\mathbb{N}}$.
Then $C \circ B \circ A : \mathbb{R} \hookrightarrow \{0, 1\}^{\mathbb{N}}$, so $\exists\mathbb{R} \hookrightarrow \{0, 1\}^{\mathbb{N}}$.
By Theorem 1.17.2, $\exists\{0, 1\}^{\mathbb{N}} \hookrightarrow \mathbb{R}$.
Since both $\exists\mathbb{R} \hookrightarrow \{0, 1\}^{\mathbb{N}}$ and $\exists\{0, 1\}^{\mathbb{N}} \hookrightarrow \mathbb{R}$,
by the Schroeder-Bernstein Theorem,
we see that $\exists\mathbb{R} \hookrightarrow\!> \{0, 1\}^{\mathbb{N}}$, as desired. □

This shows that $\mathbb{R}$ belongs on the same level as $\{0, 1\}^{\mathbb{N}}$. By Theorem 1.15.4, this is the same level as $2^{\mathbb{N}}$, and, by Theorem 1.16.1, this is above $\mathbb{N}$.

DEFINITION 1.17.4. *Let S be a set.*
By S has **continuum cardinality***, we mean $\exists\mathbb{R} \hookrightarrow\!> S$.*

We have now shown that the continuum cardinality level is above the countably infinite level. This level contains $\mathbb{R}$ and $\{0, 1\}^{\mathbb{N}}$ and $2^{\mathbb{N}}$.

Using the tools we have been developing, it is not hard to prove:
For any $a \in \mathbb{R}$, for any $b > a$,
the sets $(a; b)$, $[a; b)$, $(a; b]$ and $[a; b]$
all have continuum cardinality.

An interesting question:
Are there any sets strictly between
the continuum cardinality level and
the countably infinite level?

The standard axioms of set theory, used by most mathematicians, are together sometimes called "ZFC", for Zermelo-Fraenkel with Choice.

The assertion that
there are NO sets strictly between
the continuum cardinality level and
the countably infinite level
is sometimes called "CH", for the Continuum Hypothesis.

It turns out that there are models of ZFC for which CH is true, and models of ZFC for which CH is false. One therefore has a choice:
You can work in ZFC+CH or ZFC+($\neg$CH) or plain old ZFC.

In this class,

we choose to work in plain old ZFC,

and the Continuum Hypothesis will be treated as undecidable, and, to us, uninteresting. It turns out that the basics of Real Analysis don't requre that we consider any sets that might be located strictly between $\mathbb{N}$ and $\mathbb{R}$ in the World of Sets.

1.18. **Scalars, vectors, matrices and tensors.**

For all a, we define $(a) := (1 \mapsto a)$.

For all a, b, we define $(a, b) := \begin{pmatrix} 1 \mapsto a \\ 2 \mapsto b \end{pmatrix}$.

For all a, b, c, we define $(a, b, c) := \begin{pmatrix} 1 \mapsto a \\ 2 \mapsto b \\ 3 \mapsto c \end{pmatrix}$.

For all a, b, c, d, we define $(a, b, c, d) := \begin{pmatrix} 1 \mapsto a \\ 2 \mapsto b \\ 3 \mapsto c \\ 4 \mapsto d \end{pmatrix}$.

$$\vdots$$

For all $a, \ldots, z$, we define $(a, \ldots, z) := \begin{pmatrix} 1 \mapsto a \\ \vdots \\ 26 \mapsto z \end{pmatrix}$.

For all sets A, B, we define

$$A \times B := \{(a, b) \,|\, a \in A, b \in B\}.$$

For all sets A, B, C, we define

$$A \times B \times C := \{(a, b, c) \,|\, a \in A, b \in B, c \in C\}.$$

For all sets A, B, C, D, we define

$$A \times B \times C \times D := \{(a, b, c, d) \,|\, a \in A, b \in B, c \in C, d \in D\}.$$

$$\vdots$$

For all sets $A, \ldots, Z$, we define

$$A \times \cdots \times Z := \{(a, \ldots, z) \,|\, a \in A, \ldots, z \in Z\}.$$

DEFINITION 1.18.1. $\forall$*set* S, $\forall k \in \mathbb{N}$, $\quad S^k := S^{[1..k]}$.

Since $[1..k] = \{1, \ldots, k\}$,

the set S^k is the set of all functions $\{1, \ldots, k\} \to S$.

THEOREM 1.18.2. $[1..4] = \{1,2,3,4\}$ *and*
$\mathbb{R}^4 = \mathbb{R}^{[1..4]} = \{\textit{functions } [1..4] \to \mathbb{R}\}$ *and*

$$(8,9,6,6) = \begin{pmatrix} 1 \mapsto 8 \\ 2 \mapsto 9 \\ 3 \mapsto 6 \\ 4 \mapsto 6 \end{pmatrix} \in \mathbb{R}^{[1..4]} = \mathbb{R}^4.$$

THEOREM 1.18.3. *Let* $v := (8,9,6,6)$.
Then $v \in \mathbb{R}^4$ *and* $v_1 = 8$ *and* $v_2 = 9$ *and* $v_3 = 6$ *and* $v_4 = 6$.

THEOREM 1.18.4. *For any set* S,
$\exists S^1 \hookrightarrow\!> S$ *and*
$S^2 = S \times S$ *and*
$S^3 = S \times S \times S$ *and*
$S^4 = S \times S \times S \times S$.

If you wish, you can continue the preceding theorem out to

$$S^{26} = S\times S,$$

For all a, we define $[a] := ((1,1) \mapsto a)$.
For all a, b, we define $\begin{bmatrix} a \\ b \end{bmatrix} := \begin{pmatrix} (1,1) \mapsto a \\ (2,1) \mapsto b \end{pmatrix}$.
For all a, b, we define $\begin{bmatrix} a & b \end{bmatrix} := \begin{pmatrix} (1,1) \mapsto a \\ (1,2) \mapsto b \end{pmatrix}$.

THEOREM 1.18.5. *All of the following are true:*

$$(5;9) = \{x \in \mathbb{R} \,|\, 5 < x < 9\} \qquad \textit{and}$$

$$(5,9) = \begin{pmatrix} 1 \mapsto a \\ 2 \mapsto b \end{pmatrix} \qquad \textit{and}$$

$$\begin{bmatrix} 5 & 9 \end{bmatrix} = \begin{pmatrix} (1,1) \mapsto 5 \\ (1,2) \mapsto 9 \end{pmatrix} \qquad \textit{and}$$

$$\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{pmatrix} (1,1) \mapsto 5 \\ (2,1) \mapsto 9 \end{pmatrix}.$$

For all a, b, c, d, we define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{pmatrix} (1,1) \mapsto a \\ (1,2) \mapsto b \\ (2,1) \mapsto c \\ (2,2) \mapsto d \end{pmatrix}$.

THEOREM 1.18.6. *We have:*

$$\begin{bmatrix} (5;9) & (5,9) \\ \begin{bmatrix} 5 & 9 \end{bmatrix} & \begin{bmatrix} 5 \\ 9 \end{bmatrix} \end{bmatrix} = \begin{pmatrix} (1,1) \mapsto (5;9) \\ (1,2) \mapsto (5,9) \\ (2,1) \mapsto \begin{bmatrix} 5 & 9 \end{bmatrix} \\ (2,2) \mapsto \begin{bmatrix} 5 \\ 9 \end{bmatrix} \end{pmatrix}.$$

We can continue these definitions to 2×3 and 3×2 matrices:

For all a, b, c, d, e, f, we define:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} := \begin{pmatrix} (1,1) \mapsto a \\ (1,2) \mapsto b \\ (1,3) \mapsto c \\ (2,1) \mapsto d \\ (2,2) \mapsto e \\ (2,3) \mapsto f \end{pmatrix} \qquad \text{and}$$

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} := \begin{pmatrix} (1,1) \mapsto a \\ (2,1) \mapsto b \\ (3,1) \mapsto c \\ (1,2) \mapsto d \\ (2,2) \mapsto e \\ (3,2) \mapsto f \end{pmatrix}.$$

THEOREM 1.18.7. *Both of the following are true:*

$$\begin{bmatrix} 4 & 5 & 6 \\ 9 & 8 & 7 \end{bmatrix} := \begin{pmatrix} (1,1) \mapsto 4 \\ (1,2) \mapsto 5 \\ (1,3) \mapsto 6 \\ (2,1) \mapsto 9 \\ (2,2) \mapsto 8 \\ (2,3) \mapsto 7 \end{pmatrix} \qquad \textit{and}$$

$$\begin{bmatrix} 4 & 9 \\ 5 & 8 \\ 6 & 7 \end{bmatrix} := \begin{pmatrix} (1,1) \mapsto 4 \\ (2,1) \mapsto 5 \\ (3,1) \mapsto 6 \\ (1,2) \mapsto 9 \\ (2,2) \mapsto 8 \\ (3,2) \mapsto 7 \end{pmatrix}.$$

We can continue these definitions to 2×4, 4×2, 3×3 matrices, *etc.*, but we leave it to the reader to do this work.

DEFINITION 1.18.8. $\forall i, j \in \mathbb{N}$, $i \times j := [1..i] \times [1..j]$ *and*
$\forall i, j, k \in \mathbb{N}$, $i \times j \times k := [1..i] \times [1..j] \times [1..k]$

The reader can continue these definitions to include

$$i \times j \times k \times \ell \quad \text{and} \quad i \times j \times k \times \ell \times m \quad \text{and so on.}$$

THEOREM 1.18.9. $2 \times 2 = [1..2] \times [1..2] = \{1,2\} \times \{1,2\}$
$= \{(1,1),(1,2),(2,1),(2,2)\}$.

Also, $\mathbb{R}^{2\times 2} = \{ \text{ functions } \{(1,1),(1,2),(2,1),(2,2)\} \to \mathbb{R} \}$.

Also,
$$\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} = \begin{pmatrix} (1,1) \mapsto 6 \\ (1,2) \mapsto 7 \\ (2,1) \mapsto 8 \\ (2,2) \mapsto 9 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

THEOREM 1.18.10.

We have
$$\begin{aligned} 2 \times 2 \times 2 &= [1..2] \times [1..2] \times [1..2] \\ &= \{1,2\} \times \{1,2\} \times \{1,2\} \\ &= \{(1,1,1),(1,1,2),(1,2,1),(1,2,2), \\ &\qquad (2,1,1),(2,1,2),(2,2,1),(2,2,2)\}. \end{aligned}$$

THEOREM 1.18.11. *Let*

$$T := \begin{pmatrix} (1,1,1) \mapsto & 4 \\ (1,1,2) \mapsto & 3 \\ (1,2,1) \mapsto & 9 \\ (1,2,2) \mapsto & -6 \\ (2,1,1) \mapsto & 6 \\ (2,1,2) \mapsto & -9 \\ (2,2,1) \mapsto & 15 \\ (2,2,2) \mapsto & 8 \end{pmatrix}.$$

Then $T \in \mathbb{R}^{2\times 2\times 2}$.

A real number like $\sqrt{2}$ is sometimes called
a **scalar** or a 0-**tensor**.
A one-dimensional array like $(1,2,3)$ is sometimes called
a **vector** or a 1-**tensor**.
A two-dimensional array like $\begin{bmatrix} 4 & 5 & 6 \\ 9 & 8 & 7 \end{bmatrix}$ is sometimes called
a **matrix** or a 2-**tensor**.
The object T from Theorem 1.18.11 is hard to visualize on a page, but can be thought of as a three-dimesional array of real numbers (with shape $2 \times 2 \times 2$). This kind of object is sometimes called
a 3-**tensor**.
Even harder to visualize would be an element of, say $\mathbb{R}^{5\times 7\times 2\times 3}$, which

can be thought of as a four-dimensional array of real numbers (with shape $5 \times 7 \times 2 \times 3$), and is sometimes called

a 4-**tensor**.

Continuing, we have 5-**tensors** and 6-**tensors** and so on.

1.19. **Functionals.**

Frownie is infective:

DEFINITION 1.19.1. *$\forall a$, $a + ☹ := ☹$ and $☹ + a = ☹$ and $a - ☹ = ☹$ and $☹ - a = ☹$ and $a \cdot ☹ = ☹$ and $☹ \cdot a = ☹$ and $a/☹ = ☹$ and $☹/a = ☹$.*

Also, $\sqrt{☹} := ☹$.

Also, $\mathrm{CH}_{☹} := ☹$ and $\mathrm{CH}(☹) := ☹$.

Also, $\mathrm{UE}_{☹} := ☹$ and $\mathrm{UE}(☹) := ☹$.

DEFINITION 1.19.2. *Let f be an object.*

By f is a **functional**, *we mean:*

f is a function and $\mathbb{I}_f \subseteq \mathbb{R}$.

DEFINITION 1.19.3. *Let f be a functional and let $a \in \mathbb{R}$.*

Then af is the functional defined by:

$$\forall x, \quad (af)_x = a \cdot f_x.$$

DEFINITION 1.19.4. *Let f and g be functionals.*

Then $f + g$ is the functional defined by:

$$\forall x, \quad (f + g)_x = f_x + g_x.$$

Also, $f - g$ is the functional defined by:

$$\forall x, \quad (f - g)_x = f_x - g_x.$$

Also, fg is the functional defined by:

$$\forall x, \quad (fg)_x = f_x \cdot g_x.$$

Also, f/g is the functional defined by:

$$\forall x, \quad (f/g)_x = f_x/g_x.$$

THEOREM 1.19.5. $6 \cdot (7, 8, 9) = (42, 48, 54)$.

Proof. We have:

$$6 \cdot (7, 8, 9) = \begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 8 \\ 3 \mapsto 9 \end{pmatrix} = \begin{pmatrix} 1 \mapsto 42 \\ 2 \mapsto 48 \\ 3 \mapsto 54 \end{pmatrix} = (42, 48, 54).$$ ☐

THEOREM 1.19.6. $6 \cdot \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 42 & 48 & 54 \\ 5 & 12 & 18 \end{bmatrix}$

THEOREM 1.19.7. $(3, 1) + (5, 6) = (8, 7)$.

Proof. We have:
$$(3,1)+(5,6) = \begin{pmatrix} 1 \mapsto 3 \\ 2 \mapsto 1 \end{pmatrix} + \begin{pmatrix} 1 \mapsto 5 \\ 2 \mapsto 6 \end{pmatrix} = \begin{pmatrix} 1 \mapsto 8 \\ 2 \mapsto 7 \end{pmatrix} = (8,7). \qquad \square$$

Proof. We have:
$$(3,1)+(5,6) = \begin{pmatrix} 1 \mapsto 3 \\ 2 \mapsto 1 \end{pmatrix} + \begin{pmatrix} 1 \mapsto 5 \\ 2 \mapsto 6 \end{pmatrix} = \begin{pmatrix} 1 \mapsto 8 \\ 2 \mapsto 7 \end{pmatrix} = (8,7). \qquad \square$$

THEOREM 1.19.8.
$$\begin{bmatrix} 3 & 2 & 1 \\ 5 & -1 & 7 \end{bmatrix} + \begin{bmatrix} 8 & -1 & 4 \\ 9 & 16 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 1 & 3 \\ 14 & 15 & 9 \end{bmatrix}.$$

DEFINITION 1.19.9. *Let* $i \in \mathbb{N}$, $V := \mathbb{R}^i$. *Then* $0_V := C^0_{[1..i]}$.

THEOREM 1.19.10. *Let* $V := \mathbb{R}^4$.
$$\textit{Then } 0_V = C^0_{[1..4]} = C^0_{\{1,2,3,4\}} = \begin{pmatrix} 1 \mapsto 0 \\ 2 \mapsto 0 \\ 3 \mapsto 0 \\ 4 \mapsto 0 \end{pmatrix} = (0,0,0,0).$$

DEFINITION 1.19.11. *Let* $i, j \in \mathbb{N}$, $V := \mathbb{R}^{i \times j}$. *Then* $0_V := C^0_{i \times j}$.

Recall: $2 \times 3 = [1..2] \times [1..3] = \{1, 2\} \times \{1, 2, 3\}$.

THEOREM 1.19.12. *Let* $V := \mathbb{R}^{2 \times 3}$.
$$\textit{Then } 0_V = C^0_{2 \times 3} = C^0_{\{1,2\} \times \{1,2,3\}} = \begin{pmatrix} (1,1) \mapsto 0 \\ (1,2) \mapsto 0 \\ (1,3) \mapsto 0 \\ (2,1) \mapsto 0 \\ (2,2) \mapsto 0 \\ (2,3) \mapsto 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

DEFINITION 1.19.13. *Let* $i, j, k \in \mathbb{N}$, $V := \mathbb{R}^{i \times j \times k}$.
Then $0_V := C^0_{i \times j \times k}$.

The reader is invited to continue these definitions, with $i \times j \times k \times \ell$ and $i \times j \times k \times \ell \times m$ and so on.

1.20. Translating and dilating sets of reals.

DEFINITION 1.20.1. *Let* $a \in \mathbb{R}$*,* $S \subseteq \mathbb{R}$*.*
Then $a \cdot S := \{ax \,|\, x \in S\}$*.*
Also, $a + S := \{a + x \,|\, x \in S\}$*.*
Also, $S \cdot a := \{xa \,|\, x \in S\}$*.*
Also, $S + a := \{x + a \,|\, x \in S\}$*.*

THEOREM 1.20.2. $(-4)\cdot[1;2) = (-8;-4]$ *and* $4+(2;5] = (6;9]$ *and* $3\mathbb{N}_0 + 2 = \{2, 5, 8, 11, 14, 17, 20, \ldots\}$*.*

DEFINITION 1.20.3. *Let* s *be an object.*
By s *is a* **sequence***, we mean:*
s *is a function* *and* $\mathbb{D}_s = \mathbb{N}$*.*

Let $s_1, s_2, s_3, \ldots$ be objects. Then, by $(s_1, s_2, s_3, \ldots)$, we mean:

$$\text{the sequence} \quad \begin{pmatrix} 1 \mapsto s_1 \\ 2 \mapsto s_2 \\ 3 \mapsto s_3 \\ \vdots \end{pmatrix}.$$

THEOREM 1.20.4. *Let* $s := (5, 6, 7, 5, 6, 7, 5, 6, 7, 5, 6, 7, 5, 6, 7, \ldots)$*.*
Then s *is a sequence* *and* $s_{1000} = 5$*.*

A purist is uncomfortable with ellipses $(\cdots)$, and would prefer that we replace
Let $s := (5, 6, 7, 5, 6, 7, 5, 6, 7, 5, 6, 7, 5, 6, 7, \ldots)$
with

$$\text{Let } s : \mathbb{N} \to \mathbb{R} \text{ be defined by: } \forall j \in \mathbb{N},\ s_j = \begin{cases} 5, & \text{if } j \in 3\mathbb{N}_0 + 1 \\ 6, & \text{if } j \in 3\mathbb{N}_0 + 2 \\ 7, & \text{if } j \in 3\mathbb{N}_0 + 3. \end{cases}$$

1.21. Absolute value and norms.

DEFINITION 1.21.1. $\forall x \in \mathbb{R}$*,* $|x| := \sqrt{x^2}$*.*

In the preceding definition, the number $|x|$ is called the **absolute value** of x.

THEOREM 1.21.2. $|-7| = \sqrt{(-7)^2} = \sqrt{49} = 7$*.*

THEOREM 1.21.3. *The following are all true:*

(1) $\forall x \in \mathbb{R}, \quad (\ |x| = 0\ ) \Leftrightarrow (\ x = 0\ )$ *and*

(2) $\forall a \in \mathbb{R}, \quad \forall x \in \mathbb{R},\ |ax| = |a| \cdot |x|$ *and*

(3) $\forall x, y \in \mathbb{R}, \quad |x + y| \leqslant |x| + |y|$.

In the preceding theorem, (1) says absolute value "separates 0" (from other real numbers), while (2) says absolute value is "absolute homogeneous", while (3) says absolute value is "subadditive".

DEFINITION 1.21.4. *Let* $i \in \mathbb{N}$, $V := \mathbb{R}^i$. *Then:*

$$\forall x \in V, \quad |x| := \sqrt{\sum_{s=1}^{i} x_s^2}$$

In the preceding definition, the number $|x|$ is called the **norm** of x, or the V-**norm** of x. It is sometimes written $|x|_V$ for clarity.

THEOREM 1.21.5. $|(-4, 3)| = \sqrt{(-4)^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$.

THEOREM 1.21.6. *Let* $i \in \mathbb{N}$, $V := \mathbb{R}^i$. *Then:*

(1) $\forall x \in V, \quad (\ |x|_V = 0\ ) \Leftrightarrow (\ x = 0_V\ )$ *and*

(2) $\forall a \in \mathbb{R}, \quad \forall x \in V,\ |ax| = |a| \cdot |x|_V$ *and*

(3) $\forall x, y \in V, \quad |x + y|_V \leqslant |x|_V + |y|_V$.

In the preceding theorem, (1) says the V-norm "separates 0_V" (from other vectors), while (2) says the V-norm is "absolute homogeneous", while (3) says the V-norm is "subadditive". Subadditivity is tricky to prove, and its proof will be deferred until spring semester.

DEFINITION 1.21.7. *Let* $i, j \in \mathbb{N}$, $V := \mathbb{R}^{i \times j}$. *Then:*

$$\forall x \in V, \quad |x| := \sqrt{\sum_{s \in i \times j} x_s^2}$$

In the preceding definition, the number $|x|$ is called the **norm** of x, or the V-**norm** of x. It is sometimes written $|x|_V$ for clarity.

THEOREM 1.21.8. $\left| \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix} \right| = \sqrt{7^2 + 8^2 + 9^2 + 4^2 + 5^2 + 6^2}$.

THEOREM 1.21.9. *Let* $i, j \in \mathbb{N}$, $V := \mathbb{R}^{i \times j}$. *Then:*

(1) $\forall x \in V, \quad (\ |x|_V = 0\ ) \Leftrightarrow (\ x = 0_V\ )$ *and*

(2) $\forall a \in \mathbb{R}, \quad \forall x \in V,\ |ax| = |a| \cdot |x|_V$ *and*

(3) $\forall x, y \in V, \quad |x + y|_V \leqslant |x|_V + |y|_V$.

In the preceding theorem, (1) says the V-norm "separates 0_V" (from other matrices), while (2) says the V-norm is "absolute homogeneous", while (3) says the V-norm is "subadditive". Subadditivity is tricky to prove, and its proof will be deferred until spring semester.

DEFINITION 1.21.10. *Let $i,j,k \in \mathbb{N}$, $V := \mathbb{R}^{i\times j\times k}$. Then:*

$$\forall x \in V, \qquad |x| := \sqrt{\sum_{s \in i\times j\times k} x_s^2}$$

In the preceding definition, the number $|x|$ is called the **norm** of x, or the V-**norm** of x. It is sometimes written $|x|_V$ for clarity.

THEOREM 1.21.11. *Let $i,j,k \in \mathbb{N}$, $V := \mathbb{R}^{i\times j\times k}$. Then:*

(1) $\forall x \in V, \quad (\ |x|_V = 0\ ) \Leftrightarrow (\ x = 0_V\ )$ *and*
(2) $\forall a \in \mathbb{R}, \quad \forall x \in V,\ |ax| = |a| \cdot |x|_V$ *and*
(3) $\forall x,y \in V, \quad |x+y|_V \leqslant |x|_V + |y|_V$.

In the preceding theorem, (1) says the V-norm "separates 0_V" (from other matrices), while (2) says the V-norm is "absolute homogeneous", while (3) says the V-norm is "subadditive". Subadditivity is tricky to prove, and its proof will be deferred until spring semester.

These definitions can be continued to develop norms for four-tensors, five-tensors, *etc.* In all cases, we have separation of zero, absolute homogeneity and subadditivity. In all cases subadditivity is most easily proved using the "dot product" and Cauchy-Schwarz, both of which will be exposed in the spring semester.

1.22. **Metrics and metric spaces.**

DEFINITION 1.22.1. *Let S be a set, $d : S \times S \to [0;\infty)$.*
By d is a **metric** *on S, we mean:*

(1) $\forall x,y \in S, \quad [\ x = y\ ] \Leftrightarrow [\ d(x,y) = 0\ ]$ *and*
(2) $\forall x,y \in S, \quad d(x,y) = d(y,x)$ *and*
(3) $\forall x,y,z \in S, \quad d(x,z) \leqslant [d(x,y)] + [d(y,z)]$.

In the preceding definition, (1) says that d "separates points", while (2) says that d is "symmetric", while (3) says that d satisfies the "triangle inequality".

DEFINITION 1.22.2. *For any set S, $\mathcal{M}(S) := \{$metrics on $S\}$.*

DEFINITION 1.22.3. *Define* $d_0 : \mathbb{R} \times \mathbb{R} \to [0;\infty)$ *by:*

$$\forall x, y \in \mathbb{R}, \qquad d_0(x,y) = |y - x|.$$

THEOREM 1.22.4. $d_0(2,7) = d_0(7,2) = 5$.

THEOREM 1.22.5. $d_0 \in \mathcal{M}(\mathbb{R})$.

The metric d_0 is the **standard metric** on $\mathbb{R}$.

DEFINITION 1.22.6. *Let* $i \in \mathbb{N}$*,* $V := \mathbb{R}^i$*.*
Define $d_i : V \times V \to [0;\infty)$ *by:* $\quad \forall x, y \in V, \quad d_i(x,y) = |y - x|_V$.

THEOREM 1.22.7. $d_3(\,(4,1,5)\,,\,(4,-2,9)\,) = \sqrt{0^2 + (-3)^2 + 4^2}$.

THEOREM 1.22.8. *Let* $i \in \mathbb{N}$*,* $V := \mathbb{R}^i$*. Then* $d_i \in \mathcal{M}(V)$*.*

Let $i \in \mathbb{N}$. The metric d_i is the **standard metric** on $\mathbb{R}^i$.

DEFINITION 1.22.9. *Let* $i, j \in \mathbb{N}$*,* $V := \mathbb{R}^{i \times j}$*.*
Define $d_{(i,j)} : V \times V \to [0;\infty)$ *by:*

$$\forall x, y \in V, \qquad d_{(i,j)}(x,y) = |y - x|_V.$$

THEOREM 1.22.10.

$$d_{(2,3)}\left(\begin{bmatrix} 1 & 5 & -6 \\ 4 & 10 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 4 & -6 \\ 8 & 10 & 15 \end{bmatrix} \right) = \sqrt{1^2 + 4^2 + 6^2}.$$

THEOREM 1.22.11. *Let* $i, j \in \mathbb{N}$*,* $V := \mathbb{R}^{i \times j}$*. Then* $d_{(i,j)} \in \mathcal{M}(V)$*.*

Let $i, j \in \mathbb{N}$. The metric $d_{(i,j)}$ is the **standard metric** on $\mathbb{R}^{i \times j}$.

DEFINITION 1.22.12. *Let* $i, j, k \in \mathbb{N}$*,* $V := \mathbb{R}^{i \times j \times k}$*.*
Define $d_{(i,j,k)} : V \times V \to [0;\infty)$ *by:*

$$\forall x, y \in V, \qquad d_{(i,j,k)}(x,y) = |y - x|_V.$$

THEOREM 1.22.13. *Let* $i, j, k \in \mathbb{N}$*,* $V := \mathbb{R}^{i \times j \times k}$*.*
Then $d_{(i,j,k)} \in \mathcal{M}(V)$*.*

Let $i, j, k \in \mathbb{N}$. The metric $d_{(i,j,k)}$ is the **standard metric** on $\mathbb{R}^{i \times j \times k}$.

These definitions can be continued to develop metrics for four-tensors, five-tensors, *etc.*

DEFINITION 1.22.14. *By a* **metric space***, we mean: an ordered pair* (S, d) *s.t.:* $\qquad (\ S$ *is a set* $)\ \&\ (\ d \in \mathcal{M}(S)\ )$*.*

THEOREM 1.22.15. $(\mathbb{R}, d_0)$ *is a metric space.*
Also, $\forall i \in \mathbb{N}$*,* $(\mathbb{R}^i, d_i)$ *is a metric space.*

Also, $\forall i,j\in\mathbb{N}$, $(\mathbb{R}^{i\times j},d_{(i,j)})$ *is a metric space.*
Also, $\forall i,j,k\in\mathbb{N}$, $(\mathbb{R}^{i\times j\times k},d_{(i,j,k)})$ *is a metric space.*

THEOREM 1.22.16. *Let* $X:=(\mathbb{R}^5,d_5)$.
Then $X_1=\mathbb{R}^5$ *and* $X_2=d_5$.

DEFINITION 1.22.17. *Let* X *be a metric space.*
Then $X_{\text{set}}:=X_1$ *and* $d_X:=X_2$.

In the preceding definition, X_{set} is called the **underlying set** of X, while d_X is called the **underlying metric** of X.

THEOREM 1.22.18. *Let* $X:=(\mathbb{R}^5,d_5)$.
Then $X_{\text{set}}=\mathbb{R}^5$ *and* $d_X=d_5$.

THEOREM 1.22.19. *Let* $X:=(\mathbb{R}^{9\times 8\times 7},d_{(9,8,7)})$.
Then $X_{\text{set}}=\mathbb{R}^{9\times 8\times 7}$ *and* $d_X=d_{(9,8,7)}$.

Out of sloppiness, we almost always write X for X_{set}.

Out of sloppiness, we sometimes write d for d_X.

When a set S has a standard metric d, then, out of sloppiness, we frequently write S to mean (S,d). So, for example, out of sloppiness, we frequently write $\mathbb{R}^{9\times 8\times 7}$ to mean $(\mathbb{R}^{9\times 8\times 7},d_{(9,8,7)})$.
In particular, when we write $d_{\mathbb{R}}$, we would mean $d_{(\mathbb{R},d_0)}$.
Then $d_{\mathbb{R}}=d_{(\mathbb{R},d_0)}=d_0$.

THEOREM 1.22.20. $d_{\mathbb{R}}(2,7)=d_{\mathbb{R}}(7,2)=5$.

Let $V:=\mathbb{R}^{2\times 3}$. Then $d_V=d_{(\mathbb{R}^{2\times 3},d_{(2,3)})}=d_{(2,3)}$.

THEOREM 1.22.21. *Let* $V:=\mathbb{R}^{2\times 3}$. *Then*

$$\begin{bmatrix}1&5&-6\\4&10&9\end{bmatrix},\begin{bmatrix}1&4&-6\\8&10&15\end{bmatrix}\in V \qquad \textit{and}$$

$$d_V\left(\begin{bmatrix}1&5&-6\\4&10&9\end{bmatrix},\begin{bmatrix}1&4&-6\\8&10&15\end{bmatrix}\right)=\sqrt{1^2+4^2+6^2}.$$

THEOREM 1.22.22. *Let* $X:=\mathbb{R}^{2\times 2}$. *Then*

$$\begin{bmatrix}5&9\\6&7\end{bmatrix},\begin{bmatrix}8&1\\0&0\end{bmatrix}\in X \qquad \textit{and}$$

$$d_X\left(\begin{bmatrix}5&9\\6&7\end{bmatrix},\begin{bmatrix}8&1\\0&0\end{bmatrix}\right)=\sqrt{3^2+8^2+6^2+7^2}.$$

THEOREM 1.22.23. *Let X be a metric space and let $S \subseteq X$.*
Then $d_X|(S \times S) \in \mathcal{M}(S)$.

Let X be a metric space and let $S \subseteq X$.
Then $d_X|(S \times S)$ is called the **relative metric** on S inherited from X.
It is the standard metric on S, and, by sloppiness,
we sometimes write S for the metric space $(S, d_X|(S \times S))$.
So, for example, in the next theorem, when we write
"Let $S := [0; \infty)$",
the meaning is
"Let $S := ([0; \infty) , d_{\mathbb{R}}|([0; \infty) \times [0; \infty)))$".

THEOREM 1.22.24. *Let $S := [0; \infty)$. Then:*
$(d_S(2, 7) = 5) \ \& \ (d_S(4, -3) = ☹) \ \& \ (d_{\mathbb{R}}(4, -3) = 7)$.

THEOREM 1.22.25. *Let $S := [3; 4]$.*
Then: $\forall x, y \in S$, $d_S(x, y) = |y - x|$.

THEOREM 1.22.26. *Let $T := [-8; -6]$.*
Then: $\forall p, q \in T$, $d_T(p, q) = |q - p|$.

DEFINITION 1.22.27. *Let X be a metric space, $q \in X$ and $t \in \mathbb{R}$.*
Then: $B_X(q, t) := \{p \in X \,|\, d_X(p, q) < t\}$
and $\overline{B}_X(q, t) := \{p \in X \,|\, d_X(p, q) \leqslant t\}$
and $S_X(q, t) := \{p \in X \,|\, d_X(p, q) = t\}$.

DEFINITION 1.22.28. *Let X be a metric space and let $q \in X$.*
Then: $\mathcal{B}_X(q) := \{B_X(q, t) \,|\, t > 0\}$.

DEFINITION 1.22.29. *Let X be a metric space.*
Then: $\mathcal{B}_X := \{B_X(q, t) \,|\, q \in X, \ t > 0\}$.

The next theorem is called the **Recentering Theorem**.

THEOREM 1.22.30. *Let X be a metric space, $B \in \mathcal{B}_X$ and $q \in X$.*
Then: (1) $\exists C \in \mathcal{B}_X(q)$ s.t. $B \subseteq C$
and (2) $(q \in B) \Rightarrow (\exists A \in \mathcal{B}_X(q)$ s.t. $A \subseteq B)$.

Proof. Proof of (1):
Since $B \in \mathcal{B}_X$, choose $p \in X$, $r > 0$ s.t. $B = B_X(p, r)$.
Let $s := d_X(p, q)$. Let $C := B_X(q, r + s)$. Then $C \in \mathcal{B}_X(q)$.
Want: $B \subseteq C$. Want $\forall z \in B$, $z \in C$.

Given $z \in B$. Want $z \in C$. Want: $z \in B_X(q, r+s)$.
Since $z \in B = B_X(p,r)$, we get $d_X(z,p) < r$, so $[d_X(z,p)] + s < r + s$.
Then $d_X(z,q) \leqslant [d_X(z,p)] + [d_X(p,q)] = [d_X(z,p)] + s < r + s$,
and so $z \in B_X(q, r+s)$, as desired.
End of proof of (1).

Proof of (2):
Unassigned HW.
End of proof of (2). □

THEOREM 1.22.31. $\forall a \in \mathbb{R}, \forall \varepsilon > 0, \quad B(a,\varepsilon) = (a-\varepsilon; a+\varepsilon)$ *and* $\overline{B}(a,\varepsilon) = [a-\varepsilon; a+\varepsilon]$ *and* $S(a,\varepsilon) = \{a-\varepsilon, a+\varepsilon\}$.

THEOREM 1.22.32. *Let $a, x \in \mathbb{R}$ and let $\varepsilon > 0$.*
Then:

$$\begin{aligned} (\ |x-a| < \varepsilon\ ) &\Leftrightarrow (\ d_{\mathbb{R}}(a,x) < \varepsilon\ ) \\ &\Leftrightarrow (\ x \in B_{\mathbb{R}}(a,\varepsilon)\ ) \\ &\Leftrightarrow (\ x \in (a-\varepsilon; a+\varepsilon)\ ) \\ &\Leftrightarrow (\ a-\varepsilon < x < a+\varepsilon\ ). \end{aligned}$$

THEOREM 1.22.33. *Let $a, x \in \mathbb{R}$ and let $\varepsilon > 0$.*
Then: $(\ |x-a| < \varepsilon\ ) \Leftrightarrow (\ a-\varepsilon < x < a+\varepsilon\ )$.

THEOREM 1.22.34. *Let $x \in \mathbb{R}$ and let $\varepsilon > 0$.*
Then: $(\ |x| < \varepsilon\ ) \Leftrightarrow (\ -\varepsilon < x < \varepsilon\ )$.

THEOREM 1.22.35. *Let $x \in \mathbb{R}$ and let $\varepsilon > 0$.*
Then: $(\ d_{\mathbb{R}}(x,0) < \varepsilon\ ) \Leftrightarrow (\ -\varepsilon < x < \varepsilon\ )$.

THEOREM 1.22.36. *Let $a, x \in \mathbb{R}$ and let $\varepsilon > 0$.*
Then: $(\ |x-a| \leqslant \varepsilon\ ) \Leftrightarrow (\ a-\varepsilon \leqslant x \leqslant a+\varepsilon\ )$.

THEOREM 1.22.37. *Let $x \in \mathbb{R}$ and let $\varepsilon > 0$.*
Then: $(\ |x| \leqslant \varepsilon\ ) \Leftrightarrow (\ -\varepsilon \leqslant x \leqslant \varepsilon\ )$.

THEOREM 1.22.38. *Let $x \in \mathbb{R}$ and let $\varepsilon > 0$.*
Then: $(\ d_{\mathbb{R}}(x,0) \leqslant \varepsilon\ ) \Leftrightarrow (\ -\varepsilon \leqslant x \leqslant \varepsilon\ )$.

THEOREM 1.22.39. $\forall z \in \mathbb{R}, (\,|z-5| < 0.3\,) \Leftrightarrow (\,4.7 < z < 5.3\,)$.

THEOREM 1.22.40. $\forall z \in \mathbb{R}, (\,|z-5| \leqslant 0.3\,) \Leftrightarrow (\,4.7 \leqslant z \leqslant 5.3\,)$.

THEOREM 1.22.41. $\forall q \in \mathbb{R}, (\,|4-q| < 0.01\,) \Leftrightarrow (\,3.99 < q < 4.01\,)$.

THEOREM 1.22.42. $\forall q \in \mathbb{R}, (\,|4-q| \leqslant 0.01\,) \Leftrightarrow (\,3.99 \leqslant q \leqslant 4.01\,)$.

1.23. **Lipschitz functions.**

DEFINITION 1.23.1. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$ and let $K \geqslant 0$.
By f is **Lipschitz**-K *from X to Y, we mean:*
$$\forall p, q \in \mathbb{D}_f, \quad d_Y(f_p, f_q) \leqslant K \cdot (d_X(p, q)).$$

Lipschitz-0 is the same as constant.
Lipschitz-1 is sometimes called "distance semi-decreasing";
Lipschitz-1 means that the function
may move two points closer together,
but it never moves them farther apart.
Recall that a secant line for a function $f : \mathbb{R} \dashrightarrow \mathbb{R}$ is
a line that crosses through least two points of the graph of f.
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Then f is Lipschitz-K means that
every secant line has slope in $[-K; K]$.

THEOREM 1.23.2. *Let $S := [3; 4]$ and $T := [-7; -5]$.*
Define $f : S \to T$ by: $\forall x \in S$, $f_x = -2x + 1$.
Then f is Lipschitz-2 from S to T.

Proof. Want: $\forall x, y \in S$, $d_T(f_x, f_y) \leqslant 2 \cdot [d_S(x, y)]$.
Given $x, y \in S$. Want: $d_T(f_x, f_y) \leqslant 2 \cdot [d_S(x, y)]$.
It suffices to show: $d_T(f_x, f_y) = 2 \cdot [d_S(x, y)]$.
We have: $d_T(f_x, f_y) = |f_y - f_x| = |(-2y + 1) - (-2x + 1)|$
$= |(-2y) - (-2x)| = |(-2) \cdot (y - x)|$
$= | - 2| \cdot |y - x| = 2 \cdot |y - x| = 2 \cdot (d_S(x, y))$. □

DEFINITION 1.23.3. *Define $|\bullet| : \mathbb{R} \to [0; \infty)$ by $\forall x \in \mathbb{R}$, $|\bullet|_x = |x|$.*

THEOREM 1.23.4. *$|\bullet|$ is Lipschitz-1 from $\mathbb{R}$ to $[0; \infty)$.*

Proof. Want: $\forall x, y \in \mathbb{R}, \quad d(\, |\bullet|_x \, , \, |\bullet|_y \,) \leqslant d(x, y)$.
Given $x, y \in \mathbb{R}$. Want: $d(\, |\bullet|_x \, , \, |\bullet|_y) \leqslant d(x, y)$.
Want: $d(|x|, |y|) \leqslant d(x, y)$.
Let $a := |x|$ and $b := |y|$. Want: $d(a, b) \leqslant d(x, y)$.
Want: $|b - a| \leqslant |y - x|$. Let $t := |y - x|$. Then $t = |x - y|$.
Want: $|b - a| \leqslant t$. Want: $a - t \leqslant b \leqslant a + t$.
Since $a = |x| = |y + (x - y)| \leqslant |y| + |x - y| = b + t$,
we get $a \leqslant b + t$, so $a - t \leqslant b$. Want: $b \leqslant a + t$.
We have $b = |y| = |x + (y - x)| \leqslant |x| + |y - x| = a + t$. □

DEFINITION 1.23.5. *Let $i \in \mathbb{N}$ and let $V := \mathbb{R}^i$.*
Define $|\bullet|_V : V \to [0;\infty)$ by: $\forall x \in V, \quad (|\bullet|_V)_x = |x|_V.$

THEOREM 1.23.6. *Let $i \in \mathbb{N}$ and let $V := \mathbb{R}^i$.*
Then $|\bullet|$ is Lipschitz-1 from V to $[0;\infty)$.

DEFINITION 1.23.7. *Let $i, j \in \mathbb{N}$ and let $V := \mathbb{R}^{i\times j}$.*
Define $|\bullet|_V : V \to [0;\infty)$ by: $\forall x \in V, \quad (|\bullet|_V)_x = |x|_V.$

THEOREM 1.23.8. *Let $i, j \in \mathbb{N}$ and let $V := \mathbb{R}^{i\times j}$.*
Then $|\bullet|$ is Lipschitz-1 from V to $[0;\infty)$.

DEFINITION 1.23.9. *Let $i, j, k \in \mathbb{N}$ and let $V := \mathbb{R}^{i\times j\times k}$.*
Define $|\bullet|_V : V \to [0;\infty)$ by: $\forall x \in V, \quad (|\bullet|_V)_x = |x|_V.$

THEOREM 1.23.10. *Let $i, j, k \in \mathbb{N}$ and let $V := \mathbb{R}^{i\times j\times k}$.*
Then $|\bullet|$ is Lipschitz-1 from V to $[0;\infty)$.

These definitions and theorems can be continued to four-tensors, five-tensors, *etc.* Keep in mind that, in all of these theorems, Lipschitz-1 means distance semi-decreasing. This concept comes up in other ways:

THEOREM 1.23.11. *Define $p : \mathbb{R}^2 \to \mathbb{R}$ by:* $\forall v \in \mathbb{R}^2, \quad p_v = v_1.$
Then p is Lipschitz-1.

Proof. Let $X := \mathbb{R}^2$. Want: $\forall u, v \in X, \;\; d_{\mathbb{R}}(p_u, p_v) \leqslant d_X(u, v)$.
Given $u, v \in X$. Want: $d_{\mathbb{R}}(p_u, p_v) \leqslant d_X(u, v)$.
Want: $|p_v - p_u| \leqslant |v - u|_X$. We have $p_u = u_1$ and $p_v = v_1$.
Want: $|v_1 - u_1| \leqslant |v - u|_X$. We have $u = (u_1, u_2)$ and $v = (v_1, v_2)$.
Want: $|v_1 - u_1| \leqslant |(v_1, v_2) - (u_1, u_2)|_X$.
Want: $|v_1 - u_1| \leqslant |(v_1 - u_1, v_2 - u_2)|_X$.
We have $0 \leqslant (v_1 - u_1)^2$ and $0 \leqslant (v_2 - u_2)^2$.
Since $0 \leqslant (v_2 - u_2)^2$, we get $(v_1 - u_1)^2 \leqslant (v_1 - u_1)^2 + (v_2 - u_2)^2$.
Since $0 \leqslant (v_1 - u_1)^2 \leqslant (v_1 - u_1)^2 + (v_2 - u_2)^2$,
we conclude that $\sqrt{(v_1 - u_1)^2} \leqslant \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$.
That is, $|v_1 - u_1| \leqslant |(v_1 - u_1, v_2 - u_2)|_X$, as desired. □

DEFINITION 1.23.12. *Let X and Y be metric spaces, $f : X \dashrightarrow Y$.*
By f is **Lipschitz** *from X to Y, we mean:*
$\exists K \geqslant 0$ s.t. f is Lipschitz-K.

2. Limits and Continuity

2.1. **A doubly quantified statement.**

THEOREM 2.1.1. *$\forall\varepsilon > 0$, $\exists\delta > 0$ s.t. $\delta^2 + \delta \leqslant \varepsilon$.*

Proof. Given $\varepsilon > 0$. Want: $\exists\delta > 0$ s.t. $\delta^2 + \delta \leqslant \varepsilon$.
Let $\delta := \min\{ \sqrt{\varepsilon/2}\ ,\ \varepsilon/2\ \}$. Then $\delta > 0$.
Want: $\delta^2 + \delta \leqslant \varepsilon$.
By the definition of δ, we have
both $\delta \leqslant \sqrt{\varepsilon/2}$ and $\delta \leqslant \varepsilon/2$.
Since $0 < \delta \leqslant \sqrt{\varepsilon/2}$, we get $\delta^2 \leqslant (\sqrt{\varepsilon/2})^2$.
Then $\delta^2 \leqslant \varepsilon/2$. So, since $\delta \leqslant \varepsilon/2$, $\delta^2 + \delta \leqslant (\varepsilon/2) + (\varepsilon/2)$.
Then $\delta^2 + \delta \leqslant (\varepsilon/2) + (\varepsilon/2) = \varepsilon$, as desired. □

2.2. **Limits of sequences.**

DEFINITION 2.2.1. *Let X be a metric space.*
Let $s \in X^{\mathbb{N}}$ and let $z \in X$.
Then $s \to z$ in X means:
$$\forall\varepsilon > 0,\ \exists K \in \mathbb{N} \text{ s.t. } \forall j \in \mathbb{N},\quad (\ j \geqslant K\ ) \ \Rightarrow\ (\ d_X(s_j, z) < \varepsilon\ ).$$

THEOREM 2.2.2. *Define $s \in \mathbb{R}^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $s_j = 1/j$.*
Then $s \to 0$ in $\mathbb{R}$.

Proof. Want: $\forall\varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ d_{\mathbb{R}}(s_j, 0) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ d_{\mathbb{R}}(s_j, 0) < \varepsilon\ ).$$
By the Archimedean Principle, choose $K \in \mathbb{N}$ s.t. $K > 1/\varepsilon$.
Want: $\forall j \in \mathbb{N}$, $(\ j \geqslant K\ ) \ \Rightarrow\ (\ d_{\mathbb{R}}(s_j, 0) < \varepsilon\ )$.
Since $j \geqslant K > 1/\varepsilon$, we get $j > 1/\varepsilon$.
Since $j > 1/\varepsilon > 0$, we get $1/j < \varepsilon$.
Since $j > 0$, we get $1/j > 0$, and so $|1/j| = 1/j$.
Then $d_{\mathbb{R}}(s_j, 0) = |s_j - 0| = |s_j| = |1/j| = 1/j < \varepsilon$, as desired. □

2.3. **Some precalculus.**

The results in this section will be used in the future without comment.

DEFINITION 2.3.1. $\forall k \in \mathbb{N}$, $\forall a \in \mathbb{R}$, $\sqrt[k]{a} := \max\{x \in \mathbb{R} \,|\, x^k = a\}$.

THEOREM 2.3.2. *We have:*
$(\ \forall a \in \mathbb{R},\ \sqrt[2]{a} = \sqrt{a}\ )$ &

$(\ \forall k \in \mathbb{N}, \forall a \geqslant 0 \ (\sqrt[k]{a})^k = a \ ) \ \&$
$(\ \forall k \in 2\mathbb{N}_0 + 1, \forall a \in \mathbb{R} \ (\sqrt[k]{a})^k = a \ ).$

THEOREM 2.3.3. *$\forall \varepsilon \in \mathbb{R}$, $(\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3) = \varepsilon$.*

THEOREM 2.3.4. *$\forall a, b, c, d, e, f \in \mathbb{R}$,*
$$((a \leqslant b)\&(c \leqslant d)\&(e \leqslant f)) \Rightarrow (a + c + e \leqslant b + d + f).$$

THEOREM 2.3.5. *Let $a, b, c \in \mathbb{R}^*$. Let $m := \min\{a, b, c\}$.*
Then: $(\, m \in \{a, b, c\} \,) \,\&\, (\, m \leqslant a \,) \,\&\, (\, m \leqslant b \,) \,\&\, (\, m \leqslant c \,)$.

THEOREM 2.3.6. *$\forall C \geqslant 0$, $\forall \lambda > 0$,* $\dfrac{C}{C+1} \cdot \lambda < \lambda$.

2.4. **The Precalculus Product Rule.**

THEOREM 2.4.1. *$\forall a, b, x, y \in \mathbb{R}$,*
$$xy - ab \quad = \quad (x - a) \cdot b \ + \ a \cdot (y - b) \ + \ (x - a) \cdot (y - b).$$

To prove this theorem, simply expand the right hand side and cancel.

To remember this theorem:

The theorem is true even if some of the variables are negative, and even if $x \leqslant a$ or $y \leqslant b$.

However, the easiest case to picture is the case when $x > a > 0$ and $y > b > 0$. We imagine an $a \times b$ rectangle that grows to an $x \times y$ rectangle, with the lower left corner staying fixed. The change in area is $xy - ab$, and it can be viewed as the sum of the areas of three subrectangles of the $x \times y$ rectangle. This yields the formula in the preceding theorem.

One can think of $x - a$ as the "change to a" and denote it by $\triangle a$.
One can think of $y - b$ as the "change to b" and denote it by $\triangle b$.
One can think of $xy - ab$ as the "change to ab" and denote it by $\triangle(ab)$.
With this notation the formula reads
$$\triangle(ab) \quad = \quad (\triangle a) \cdot b \ + \ a \cdot (\triangle b) \ + \ (\triangle a) \cdot (\triangle b).$$
Buzz phrase: The change to ab is equal to
(the change to a) times b plus
a times (the change to b) plus
the product of the two changes.

The Precalculus Product Rule and Calculus Product Rule are similar.
However, in the Precalculus Product Rule, we need to remember
that third term: "the product of the two changes".

2.5. Sequential limits of sums and products.

THEOREM 2.5.1. *Let* $s, t \in \mathbb{R}^{\mathbb{N}}$.

Assume: ($s \to 2$ *in* $\mathbb{R}$) & ($t \to 3$ *in* $\mathbb{R}$).

Then: *(1)* $s + t \to 5$ *in* $\mathbb{R}$

and *(2)* $st \to 6$ *in* $\mathbb{R}$.

Proof. Since $s \to 2$ in $\mathbb{R}$, we have: $\forall \varepsilon > 0$, $\exists L \in \mathbb{N}$ s.t.

$$(\ j \geqslant L \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}(s_j, 2) < \varepsilon \ ).$$

Also, since $t \to 3$ in $\mathbb{R}$, we have: $\forall \varepsilon > 0$, $\exists M \in \mathbb{N}$ s.t.

$$(\ j \geqslant M \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}(t_j, 3) < \varepsilon \ ).$$

Proof of (1):

Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\ j \geqslant K \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}((s+t)_j, 5) < \varepsilon \ ).$$

Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\ j \geqslant K \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}((s+t)_j, 5) < \varepsilon \ ).$$

Choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\ j \geqslant L \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}(s_j, 2) < \varepsilon/2 \ )$.

Choose $M \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\ j \geqslant M \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}(t_j, 3) < \varepsilon/2 \ )$.

Let $K := \max\{L, M\}$. Then $K \in \mathbb{N}$.

Want: $(\ j \geqslant K \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}((s+t)_j, 5) < \varepsilon \ )$.

Assume $j \geqslant K$. Want: $d_{\mathbb{R}}((s+t)_j, 5) < \varepsilon$.

Since $j \geqslant K \geqslant L$, by choice of L, we get $d_{\mathbb{R}}(s_j, 2) < \varepsilon/2$.

Since $j \geqslant K \geqslant M$, by choice of M, we get $d_{\mathbb{R}}(t_j, 3) < \varepsilon/2$.

Then $d_{\mathbb{R}}((s+t)_j, 5) = |(s+t)_j - 5| = |(s_j + t_j) - (2+3)|$

$$= |(s_j - 2) + (t_j - 3)| \leqslant |s_j - 2| + |t_j - 3|$$

$$= (d(s_j, 2)) + (d(t_j, 3)) < (\varepsilon/2) + (\varepsilon/2) = \varepsilon.$$

End of proof of (1).

Proof of (2):

Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\ j \geqslant K \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}((st)_j, 6) < \varepsilon \ ).$$

Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\ j \geqslant K \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}((st)_j, 6) < \varepsilon \ ).$$

Let $\delta := \min\{\varepsilon/10, \sqrt{\varepsilon/2}\}$. Then $\delta > 0$.

Choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\ j \geqslant L \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}(s_j, 2) < \delta \ )$.

Choose $M \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\ j \geqslant M \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}(t_j, 3) < \delta \ )$.

Let $K := \max\{L, M\}$. Then $K \in \mathbb{N}$.

Want: $(\ j \geqslant K \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}((st)_j, 6) < \varepsilon \ )$.

Assume $j \geqslant K$. Want: $d_{\mathbb{R}}((st)_j, 6) < \varepsilon$.
By definiton of δ, we have: $\delta \leqslant \varepsilon/10$ and $\delta \leqslant \sqrt{\varepsilon/2}$.
Since $\delta < \varepsilon/10$, we get: $5\delta \leqslant \varepsilon/2$.
Since $0 < \delta \leqslant \sqrt{\varepsilon/2}$, we get $\delta^2 \leqslant \varepsilon/2$.
Since $j \geqslant K \geqslant L$, by choice of L, we get $d_{\mathbb{R}}(s_j, 2) < \delta$.
Since $j \geqslant K \geqslant M$, by choice of M, we get $d_{\mathbb{R}}(t_j, 3) < \delta$.
By the Precalculus Product Rule,

$$s_j t_j - 2 \cdot 3 = (s_j - 2) \cdot 3 + 2 \cdot (t_j - 3) + (s_j - 2) \cdot (t_j - 3).$$

Then
$$\begin{aligned} d_{\mathbb{R}}((st)_j, 6) &= |(st)_j - 6| = |s_j t_j - 2 \cdot 3| \\ &= |(s_j - 2) \cdot 3 + 2 \cdot (t_j - 3) + (s_j - 2) \cdot (t_j - 3)| \\ &\leqslant |s_j - 2| \cdot 3 + 2 \cdot |t_j - 3| + |s_j - 2| \cdot |t_j - 3| \\ &= (d_{\mathbb{R}}(s_j, 2)) \cdot 3 + 2 \cdot (d_{\mathbb{R}}(t_j, 3)) + (d_{\mathbb{R}}(s_j, 2)) \cdot (d_{\mathbb{R}}(t_j, 3)) \\ &< \delta \cdot 3 + 2 \cdot \delta + \delta \cdot \delta \;=\; 5\delta + \delta^2 \;\leqslant\; (\varepsilon/2) + (\varepsilon/2) = \varepsilon. \end{aligned}$$

End of proof of (2). □

2.6. **Continuity.**

DEFINITION 2.6.1. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$ and let p be an object.
Then by f is **continuous** *at p from X to Y, we mean:*

$$\begin{aligned} &[\ p \in \mathbb{D}_f\ ] \quad \& \\ &[\ \forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t.},\ \forall w \in \mathbb{D}_f, \\ &\qquad (\ d(w,p) < \delta\ ) \Rightarrow (\ d(f_w, f_p) < \varepsilon\ )\ ]. \end{aligned}$$

THEOREM 2.6.2. *Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = x^2$.*
Then f is continuous at 2 from $\mathbb{R}$ to $\mathbb{R}$.

Proof. Since $f_2 = 4 \neq$ ☹, we get $2 \in \mathbb{D}_f$.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,

$$(\ d(x,2) < \delta\ ) \Rightarrow (\ d(f_x, f_2) < \varepsilon\ ).$$

Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,

$$(\ d(x,2) < \delta\ ) \Rightarrow (\ d(f_x, f_2) < \varepsilon\ ).$$

Let $\delta := \min\{1, \varepsilon/5\}$. Then $\delta > 0$.
Want: $\forall w \in \mathbb{D}_f$, $(\ d(x,2) < \delta\ ) \Rightarrow (\ d(f_x, f_2) < \varepsilon\ )$.
Given $w \in \mathbb{D}_f$. Want: $(\ d(x,2) < \delta\ ) \Rightarrow (\ d(f_w, f_2) < \varepsilon\ )$.
Assume $d(w,2) < \delta$. Want: $d(f_w, f_2) < \varepsilon$.
By definition of δ, we have $\delta \leqslant 1$ and $\delta \leqslant \varepsilon/5$.
Since $\delta \leqslant \varepsilon/5$, we get $5\delta \leqslant \varepsilon$.
We have $|w - 2| = d(w,2) < \delta$.
Then $|w + 2| = |w - 2 + 4| \leqslant |w - 2| + 4 < \delta + 4 \leqslant 1 + 4 = 5$.

Since $0 \leqslant |w+2| < 5$ and $0 \leqslant |w-2| < \delta$, by multiplying,
we get $|w+2| \cdot |w-2| < 5\delta$.
Then $d(f_w, f_2) = |f_w - f_2| = |w^2 - 2^2| = |(w+2) \cdot (w-2)|$
$= |w+2| \cdot |w-2| < 5\delta \leqslant \varepsilon$, as desired. □

THEOREM 2.6.3. *Define $f : \mathbb{R} \dashrightarrow \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = 1/x$. Then f is continuous at 4 from $\mathbb{R}$ to $\mathbb{R}$.*

Proof. Since $f_4 = 1/4 \neq$ ☹, we get $4 \in \mathbb{D}_f$.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,
$$(\ d(x,4) < \delta\ ) \Rightarrow (\ d(f_x, f_4) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,
$$(\ d(x,4) < \delta\ ) \Rightarrow (\ d(f_x, f_4) < \varepsilon\ ).$$
Let $\delta := \min\{1, 12\varepsilon\}$. Then $\delta > 0$.
Want: $\forall w \in \mathbb{D}_f$, $(\ d(x,4) < \delta\ ) \Rightarrow (\ d(f_x, f_4) < \varepsilon\ )$.
Given $w \in \mathbb{D}_f$. Want: $(\ d(x,4) < \delta\ ) \Rightarrow (\ d(f_w, f_4) < \varepsilon\ )$.
Assume $d(w,4) < \delta$. Want: $d(f_w, f_4) < \varepsilon$.
By definition of δ, we have both $\delta \leqslant 1$ and $\delta \leqslant 12\varepsilon$.
It follows both that $4 - \delta \geqslant 3$ and that $\dfrac{\delta}{12} \leqslant \varepsilon$.
Since $|\bullet|$ is Lipschitz-1, we get $d(|w|, |4|) \leqslant d(w,4)$.
Then $d(|w|, 4) = d(|w|, |4|) \leqslant d(w,4) < \delta$, so $d(|w|, 4) < \delta$.
Since $d(|w|, 4) < \delta$, we get: $4 - \delta < |w| < 4 + \delta$.
Then $|w| > 4 - \delta \geqslant 3$, so $|w| > 3$, so $4 \cdot |w| > 12$.
Also, we have $|4 - w| = d(w,4) < \delta$.
Since $0 \leqslant |4 - w| < \delta$ and since $4 \cdot |w| > 12 > 0$,
we conclude that: $\dfrac{|4-w|}{4 \cdot |w|} < \dfrac{\delta}{12}$.
Then $d(f_w, f_4) = |f_w - f_4| = \left| \dfrac{1}{w} - \dfrac{1}{4} \right| = \left| \dfrac{4-w}{4w} \right|$
$= \dfrac{|4-w|}{|4| \cdot |w|} = \dfrac{|4-w|}{4 \cdot |w|} < \dfrac{\delta}{12} \leqslant \varepsilon$, as desired. □

DEFINITION 2.6.4. *Let X and Y be metric spaces. Let $f : X \dashrightarrow Y$ and let S be a set. Then f is* **continuous** *on S means:*
$$\forall p \in S, \quad (\ f \text{ is continuous at } p \text{ from } X \text{ to } Y\ ).$$

DEFINITION 2.6.5. *Let X and Y be metric spaces, $f : X \dashrightarrow Y$. Then f is* **continuous** *means: f is continuous on $\mathbb{D}_f$.*

THEOREM 2.6.6. *Define* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *by:* $\forall x \in \mathbb{R}$, $f_x = 1/x$.
Then f *is continuous from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. Want: f is continuous on $\mathbb{D}_f$ from $\mathbb{R}$ to $\mathbb{R}$.
We have $\mathbb{D}_f = \mathbb{R}_0^\times$. Want: f is continuous on $\mathbb{R}_0^\times$ from $\mathbb{R}$ to $\mathbb{R}$.
Want: $\forall p \in \mathbb{R}_0^\times$, f is continuous at p from $\mathbb{R}$ to $\mathbb{R}$.
Given $p \in \mathbb{R}_0^\times$. Want: f is continuous at p from $\mathbb{R}$ to $\mathbb{R}$.
We have $p \in \mathbb{R}_0^\times = \mathbb{D}_f$.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,

$$(\, d(w,p) < \delta \,) \;\Rightarrow\; (\, d(f_w, f_p) < \varepsilon \,).$$

Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,

$$(\, d(w,p) < \delta \,) \;\Rightarrow\; (\, d(f_w, f_p) < \varepsilon \,).$$

Since $p \in \mathbb{R}_0^\times$, we get $|p| > 0$, and so $\dfrac{|p|}{2} > 0$.
Since $p \in \mathbb{R}_0^\times$, we get $\dfrac{p^2}{2} > 0$. So, since $\varepsilon > 0$, we get $\dfrac{p^2}{2} \cdot \varepsilon > 0$.
Let $\delta := \min\left\{\dfrac{|p|}{2}, \dfrac{p^2}{2} \cdot \varepsilon\right\}$. Then $\delta > 0$.
Want: $\forall w \in \mathbb{D}_f$, $(\, d(w,p) < \delta \,) \;\Rightarrow\; (\, d(f_w, f_p) < \varepsilon \,)$.
Given $w \in \mathbb{D}_f$. Want: $(\, d(w,p) < \delta \,) \;\Rightarrow\; (\, d(f_w, f_p) < \varepsilon \,)$.
Assume $d(w,p) < \delta$. Want: $d(f_w, f_p) < \varepsilon$.
By definition of δ, we have both $\delta \leqslant \dfrac{|p|}{2}$ and $\delta \leqslant \dfrac{p^2}{2} \cdot \varepsilon$.
It follows both that $|p| - \delta \geqslant |p| - \dfrac{|p|}{2}$ and that $\dfrac{2}{p^2} \cdot \delta \leqslant \varepsilon$.
Since $| \bullet |$ is Lipschitz-1, we get $d(\, |p| \, , \, |w| \,) \leqslant d(\, p \, , \, w \,)$.
Since $d(\, |p| \, , \, |w| \,) \leqslant d(\, p \, , \, w \,) < \delta$, we get $|p| - \delta < |w| < |p| + \delta$.
Then $|w| > |p| - \delta > |p| - \dfrac{|p|}{2} = \dfrac{|p|}{2}$, so $|w| > \dfrac{|p|}{2}$, so $|w| \cdot |p| > \dfrac{|p|^2}{2}$.
So, since $|p|^2 = p^2$, we get $|w| \cdot |p| > \dfrac{p^2}{2}$.
Also, we have $|p - w| = d(w,p) < \delta$.
Since $0 \leqslant |p - w| < \delta$ and since $|w| \cdot |p| > \dfrac{p^2}{2} > 0$,

we conclude that: $$\frac{|p-w|}{|p| \cdot |w|} < \frac{\delta}{p^2/2}.$$

Then $$d(f_w, f_p) = |f_w - f_p| = \left| \frac{1}{w} - \frac{1}{p} \right| = \left| \frac{p-w}{wp} \right|$$
$$= \frac{|p-w|}{|w| \cdot |p|} < \frac{\delta}{p^2/2} = \frac{2}{p^2} \cdot \delta \leqslant \varepsilon,$$ as desired. ☐

2.7. **Uniform continuity.**

THEOREM 2.7.1. *Let X and Y be metric spaces, $f : X \dashrightarrow Y$.*
Then: (*f is continuous from X to Y*)

$$\Leftrightarrow \;(\; \forall p \in \mathbb{D}_f,\ f \text{ is continuous at } p \text{ from } X \text{ to } Y \;)$$

$$\Leftrightarrow \;(\; \forall p \in \mathbb{D}_f,\ \forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t.}, \forall q \in \mathbb{D}_f,\ (\, d(p,q) < \delta \,) \Rightarrow (\, d(f_p, f_q) < \varepsilon \,) \;)$$

$$\Leftrightarrow \;(\; \forall \varepsilon > 0,\ \forall p \in \mathbb{D}_f,\ \exists \delta > 0 \text{ s.t.}, \forall q \in \mathbb{D}_f,\ (\, d(p,q) < \delta \,) \Rightarrow (\, d(f_p, f_q) < \varepsilon \,) \;)$$

DEFINITION 2.7.2. *Let X and Y be metric spaces, $f : X \dashrightarrow Y$.*
By f is **uniformly continuous** *from X to Y, we mean:*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t.}, \forall p \in \mathbb{D}_f,\ \forall q \in \mathbb{D}_f,\ (\, d(p,q) < \delta \,) \Rightarrow (\, d(f_p, f_q) < \varepsilon \,)$$

In homework, you'll show that Lipschitz implies uniformly continuous implies continuous.

2.8. Sequential continuity.

DEFINITION 2.8.1. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$ and let p be an object.
By f is **sequentially continuous** *at p from X to Y, we mean:*

$$[\; p \in \mathbb{D}_f \;] \;\;\&\;\; [\; \forall s \in \mathbb{D}_f^{\mathbb{N}},\ (\, s \to p \text{ in } X \,) \Rightarrow (\, f \circ s \to f_p \text{ in } Y \,) \;].$$

THEOREM 2.8.2. *Let X, Y be metric spaces, $f : X \dashrightarrow Y$, $p \in X$.*
Assume that f is continuous at p from X to Y.
Then f is sequentially continuous at p from X to Y.

Proof. Since f is continuous at p from X to Y, we see that $p \in \mathbb{D}_f$.
It remains to show: $\forall s \in \mathbb{D}_f^{\mathbb{N}}$, $(\, s \to p$ in $X\,) \Rightarrow (\, f \circ s \to f_p$ in $Y\,)$.
Given $s \in \mathbb{D}_f^{\mathbb{N}}$. Want: $(\, s \to p$ in $X\,) \Rightarrow (\, f \circ s \to f_p$ in $Y\,)$.
Assume $s \to p$ in X. Want: $f \circ s \to f_p$ in Y.
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\, j \geqslant K \,) \Rightarrow (\, d((f \circ s)_j, f_p) < \varepsilon \,).$$

Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\, j \geqslant K \,) \Rightarrow (\, d((f \circ s)_j, f_p) < \varepsilon \,).$$

Since f is continuous at p from X to Y,
choose $\delta > 0$ s.t., $\forall w \in \mathbb{D}_f$, $(\, d(w,p) < \delta \,) \Rightarrow (\, d(f_w, f_p) < \varepsilon \,)$.
Since $s \to p$ in X, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(j \geqslant K) \Rightarrow (d(s_j, p) < \delta).$$

Then $K \in \mathbb{N}$. Want: $\forall j \in \mathbb{N}$, $(j \geqslant K) \Rightarrow (d((f \circ s)_j, f_p) < \varepsilon)$.
Given $j \in \mathbb{N}$. Want: $(j \geqslant K) \Rightarrow (d((f \circ s)_j, f_p) < \varepsilon)$.
Assume $j \geqslant K$. Want: $d((f \circ s)_j, f_p) < \varepsilon$.
Since $j \geqslant K$, by choice of K, we have: $d(s_j, p) < \delta$.
Since $s \in \mathbb{D}_f^{\mathbb{N}}$, we conclude that $s_j \in \mathbb{D}_f$.
Let $w := s_j$. Then $w \in \mathbb{D}_f$.
So, since $d(w, p) = d(s_j, p) < \delta$, by choice of δ, we have: $d(f_w, f_p) < \varepsilon$.
We have $(f \circ s)_j = f(s_j) = f(w)$.
Then $d((f \circ s)_j, f_p) = d(f_w, f_p) < \varepsilon$, as desired.

□

THEOREM 2.8.3. *Let X be a metric space, $s \in X^{\mathbb{N}}$, $p \in X$. Assume: $\forall j \in \mathbb{N}$, $d(s_j, p) \leqslant 1/j$. Then $s \to p$ in X.*

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(j \geqslant K) \Rightarrow (d(s_j, p) < \varepsilon).$$

Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(j \geqslant K) \Rightarrow (d(s_j, p) < \varepsilon).$$

By the Archimedean Principle, choose $K \in \mathbb{N}$ s.t. $K > 1/\varepsilon$.
Then $K \in \mathbb{N}$. Want: $\forall j \in \mathbb{N}$, $(j \geqslant K) \Rightarrow (d(s_j, p) < \varepsilon)$.
Given $j \in \mathbb{N}$. Want: $(j \geqslant K) \Rightarrow (d(s_j, p) < \varepsilon)$.
Assume $j \geqslant K$. Want: $d(s_j, p) < \varepsilon$.
By hypothesis, $d(s_j, p) \leqslant 1/j$.
Since $j \geqslant K > 0$, we get: $1/j \leqslant 1/K$.
Since $K > 1/\varepsilon > 0$, we get: $1/K < \varepsilon$.
Then $d(s_j, p) \leqslant 1/j \leqslant 1/K < \varepsilon$, as desired. □

THEOREM 2.8.4. *Let X, Y be metric spaces, $f : X \dashrightarrow Y$, $p \in X$. Then: (f is continuous at p from X to Y) $\Leftrightarrow$ (f is sequentially continuous at p from X to Y).*

Proof. By Theorem 2.8.2, we have $\Rightarrow$. Want: $\Leftarrow$.
Assume: f is sequentially continuous at p from X to y.
Want: f is continuous at p from X to Y.
Assume: f is not continuous at p from X to Y. Want: Contradiction.
Choose $\varepsilon > 0$ s.t., $\forall \delta > 0$, $\exists w \in \mathbb{D}_f$ s.t. $(d(w, p) < \delta) \& (d(f_w, f_p) \geqslant \varepsilon)$.
Then $\forall j \in \mathbb{N}$, $\exists w \in \mathbb{D}_f$ s.t. $(d(w, p) < 1/j) \& (d(f_w, f_p) \geqslant \varepsilon)$.
By the Axiom of Choice,

choose $s \in \mathbb{D}_f^{\mathbb{N}}$ s.t., $\forall j \in \mathbb{N}$, $(d(s_j, p) < 1/j) \& (d(f_{s_j}, f_p) \geqslant \varepsilon)$.

Then $\forall j \in \mathbb{N}$, $d(s_j, p) < 1/j$,
and so, by Theorem 2.8.3, we see that: $s \to p$ in X.
So, since $s \in \mathbb{D}_f^{\mathbb{N}}$ and since f is sequentially continuous at p,
we conclude that: $f \circ s \to f_p$ in Y.
So choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\, j \geqslant K \,) \Rightarrow (\, d((f \circ s)_j, f_p) < \varepsilon \,)$.
So, since $K \geqslant K$, we get $d((f \circ s)_K, f_p) < \varepsilon$, and so $\varepsilon > d((f \circ s)_K, f_p)$.
By the choice of s, we have: $d(f_{s_K}, f_p) \geqslant \varepsilon$.
Then $\varepsilon > d((f \circ s)_K, f_p) = d(f_{s_K}, f_p) \geqslant \varepsilon$,
and so $\varepsilon > \varepsilon$. Contradiction. □

2.9. Arithmetic of functionals.

DEFINITION 2.9.1. *Let f and g be functionals.*
Then $f + g$, $f - g$, fg, f/g are the functionals defined by: $\forall x$,

$$(f+g)_x = f_x + g_x \qquad (f-g)_x = f_x - g,$$
$$(fg)_x = f_x \cdot g_x \qquad (f/g)_x = f_x/g.$$

DEFINITION 2.9.2. *Let f be a functional and let $a \in \mathbb{R}$.*
Then af is the functional defined by: $\forall x$,

$$(af)_x = a \cdot f_x.$$

THEOREM 2.9.3. *Let f and g be functionals.*
Then: $\mathbb{D}_{f+g} = \mathbb{D}_{f-g} = \mathbb{D}_{fg} = \mathbb{D}_f \bigcap \mathbb{D}_g$.
Also, $\mathbb{D}_{f/g} = \mathbb{D}_f \bigcap \mathbb{D}_g \bigcap (g^*(\mathbb{R}_0^\times))$.

THEOREM 2.9.4. *Let f be a functional and let $a \in \mathbb{R}$.*
Then: $\mathbb{D}_{af} = \mathbb{D}_f$.

2.10. Pairing of functions.

Frownie is infective. We make the convention that:

$$\forall t, \quad \begin{pmatrix} 1 \mapsto t \\ 2 \mapsto ☹ \end{pmatrix} = ☹ = \begin{pmatrix} 1 \mapsto ☹ \\ 2 \mapsto t \end{pmatrix}.$$

Recall that: $\forall x, y, \quad (x,y) = \begin{pmatrix} 1 \mapsto x \\ 2 \mapsto y \end{pmatrix}$.

It follows that: $\forall t, \quad (t, ☹) = ☹ = (☹, t)$. Frownie is infective.

DEFINITION 2.10.1. *Let f and g be functions.*
then $(f,g)^{\text{fn}}$ is the function defined by:

$$\forall x, \quad (f,g)^{\text{fn}}_x = (f_x, g_x)$$

We are almost always sloppy and write (f,g) for $(f,g)^{\rm fn}$. As a consequence, when we write (f,g), the reader must discern, from context, whether we mean $(f,g)^{\rm fn}$ or $\begin{pmatrix} 1 \mapsto f \\ 2 \mapsto g \end{pmatrix}$.
Hint: When f and g are functions, by (f,g), we usually mean $(f,g)^{\rm fn}$.

THEOREM 2.10.2. *Define* $f,g:\mathbb{R} \dashrightarrow \mathbb{R}$ *by:*

$$\forall t \in \mathbb{R}, \quad ((\, f_t = \sqrt{t}\,) \,\&\, (\, g_t = \sqrt{3-t}\,)).$$

Let $h := (f,g)$. *Then:*

$$\mathbb{D}_f = [0;\infty) \quad \textit{and} \quad \mathbb{D}_g = (-\infty;3] \quad \textit{and} \quad \mathbb{D}_h = [0;3]$$
$$\textit{and} \quad h:\mathbb{R} \dashrightarrow \mathbb{R}^2 \quad \textit{and} \quad \forall t \in \mathbb{R},\ h_t = (\sqrt{t}, \sqrt{3-t}).$$

THEOREM 2.10.3. $\forall$*functions* f,g, $\quad \mathbb{D}_{(f,g)} = \mathbb{D}_f \bigcap \mathbb{D}_g$.

2.11. Properties of continuity.

THEOREM 2.11.1. *Let* X *be a metric space,* $f,g:X \dashrightarrow \mathbb{R}$, $p \in X$.
Assume f *and* g *are both continuous at* p *from* X *to* $\mathbb{R}$.
Then $f+g$ *and* fg *are both continuous at* p *from* X *to* $\mathbb{R}$.

Proof. By Theorem 2.8.4,
f and g are both sequentially continuous at p from X to $\mathbb{R}$.
Also, by Theorem 2.8.4, it suffices to show:
$f+g$ and fg are both sequentially continuous at p from X to $\mathbb{R}$.
Since f and g are both sequentially continuous at p from X to $\mathbb{R}$,
we get: $p \in \mathbb{D}_f$ and $p \in \mathbb{D}_g$, so $p \in \mathbb{D}_f \bigcap \mathbb{D}_g$.
We have: $\mathbb{D}_{f+g} = \mathbb{D}_f \bigcup \mathbb{D}_g = \mathbb{D}_{fg}$. Then: $p \in \mathbb{D}_{f+g}$ and $p \in \mathbb{D}_{fg}$.
It suffices to show: $\forall s \in (\mathbb{D}_f \bigcap \mathbb{D}_g)^{\mathbb{N}}$, $\quad (\ s \to p$ in $X\ ) \quad \Rightarrow$
$(\ (f+g)\circ s \to (f+g)_p$ in $\mathbb{R}$ and $(fg)\circ s \to (fg)_p$ in $\mathbb{R}\ )$.
Given $s \in (\mathbb{D}_f \bigcap \mathbb{D}_g)^{\mathbb{N}}$. Want: $(\ s \to p$ in $X\ ) \quad \Rightarrow$
$(\ (f+g)\circ s \to (f+g)_p$ in $\mathbb{R}$ and $(fg)\circ s \to (fg)_p$ in $\mathbb{R}\ )$.
Assume: $s \to p$ in X.
Want: $(f+g)\circ s \to (f+g)_p$ in $\mathbb{R}$ and $(fg)\circ s \to (fg)_p$ in $\mathbb{R}$.
We have: $(f+g)_p = f_p + g_p$ and $(fg)_p = f_p \cdot g_p$.
Want: $(f+g)\circ s \to f_p + g_p$ in $\mathbb{R}$ and $(fg)\circ s \to f_p \cdot g_p$ in $\mathbb{R}$.
We have $s \in (\mathbb{D}_f \bigcap \mathbb{D}_g)^{\mathbb{N}} \subseteq \mathbb{D}_f^{\mathbb{N}}$ and $s \in (\mathbb{D}_f \bigcap \mathbb{D}_g)^{\mathbb{N}} \subseteq \mathbb{D}_g^{\mathbb{N}}$.
So, since $s \to p$ in X and
since f and g are both sequentially continuous at p from X to $\mathbb{R}$,
we get: $f \circ s \to f_p$ in $\mathbb{R}$ and $g \circ s \to g_p$ in $\mathbb{R}$.
Then, by HW#4-3 and HW#4-4, we get:
$(f\circ s) + (g \circ s) \to f_p + g_p$ in $\mathbb{R}$ and $(f \circ s)\cdot(g\circ s) \to f_p \cdot g_p$ in $\mathbb{R}$.

Want: $(f \circ s) + (g \circ s) = (f + g) \circ s$ and $(f \circ s) \cdot (g \circ s) = (fg) \circ s$.
Want: $\forall j \in \mathbb{N}$, $((f \circ s) + (g \circ s))_j = ((f + g) \circ s)_j$ and
$((f \circ s) \cdot (g \circ s))_j = ((fg) \circ s)_j$.
Given $j \in \mathbb{N}$. Want:
$((f \circ s) + (g \circ s))_j = ((f + g) \circ s)_j$ and $((f \circ s) \cdot (g \circ s))_j = ((fg) \circ s)_j$.
We have $((f \circ s) + (g \circ s))_j = (f \circ s)_j + (g \circ s)_j = f_{s_j} + g_{s_j}$
$= (f + g)_{s_j} = ((f + g) \circ s)_j$.
Want: $((f \circ s) \cdot (g \circ s))_j = ((fg) \circ s)_j$.
We have $((f \circ s) \cdot (g \circ s))_j = (f \circ s)_j \cdot (g \circ s)_j = f_{s_j} \cdot g_{s_j}$
$= (fg)_{s_j} = ((fg) \circ s)_j$, as desired. □

THEOREM 2.11.2. *Let X and Y be metric spaces, $a \in Y$. Then C_X^a is continuous from X to Y.*

Proof. Since C_X^a is Lipschitz-0 from X to Y,
we see that C_X^a is Lipschitz from X to Y.
Then, by HW#5-1, C_X^a is uniformly continuous from X to Y.
Then, by HW#5-2, C_X^a is continuous from X to Y, as desired. □

THEOREM 2.11.3. *Let X be a metric space.*
Let $f : X \dashrightarrow \mathbb{R}$, $p \in X$, $a \in \mathbb{R}$.
Assume that f is continuous at p from X to $\mathbb{R}$.
Then af is continuous at p from X to $\mathbb{R}$.

Proof. Since C_X^a is continuous from X to $\mathbb{R}$ and $\mathbb{D}_{C_X^a} = X$,
we see that C_X^a is continuous on X from X to $\mathbb{R}$.
So, since $p \in X$, we see that C_X^a is continuous at p from X to $\mathbb{R}$.
So, since f is also continuous at p from X to $\mathbb{R}$,
we see that $C_X^a \cdot f$ is continuous at p from X to $\mathbb{R}$.
It therefore suffices to show: $C_X^a \cdot f = af$.
Want: $\forall q \in X$, $(C_X^a \cdot f)_q = (af)_q$.
Given $q \in X$. Want: $(C_X^a \cdot f)_q = (af)_q$.
We have: $(C_X^a \cdot f)_q = (C_X^a)_q \cdot f_q = a \cdot f_q = (af)_q$, as desired. □

THEOREM 2.11.4. *Let X be a metric space, $f, g : X \dashrightarrow \mathbb{R}$, $p \in X$.*
Assume that f and g are both continuous at p from X to $\mathbb{R}$.
Then $f - g$ is continuous at p from X to $\mathbb{R}$.

Proof. Since g is continuous at p from X to $\mathbb{R}$,
we get: $(-1) \cdot g$ is continuous at p from X to $\mathbb{R}$.
So, since f is also continuous at p from X to $\mathbb{R}$,

we get: $f + (-1) \cdot g$ is continuous at p from X to $\mathbb{R}$.
It therefore suffices to show: $f + (-1) \cdot g = f - g$.
Want: $\forall q \in X$, $(f + (-1) \cdot g)_q = (f - g)_q$.
Given $q \in X$. Want: $(f + (-1) \cdot g)_q = (f - g)_q$.
We have: $(f + (-1) \cdot g)_q = f_q + ((-1) \cdot g)_q = f_q + (-1) \cdot g_q$
$= f_q - g_q = (f - g)_q$, as desired. □

The following theorem is called **Absoluteness of Continuity**:

THEOREM 2.11.5. *Let X and Y be metric spaces.*
Let $A \subseteq X$, $B \subseteq Y$, $f : A \dashrightarrow B$.
Then: (f is continuous from A to B)
$\Leftrightarrow$ (f is continuous from X to Y).

Proof. Proof of $\Rightarrow$:
Assume: f is continuous from A to B.
Want: f is continuous from X to Y.
Want: f is continuous on $\mathbb{D}_f$ from X to Y.
Want: $\forall p \in \mathbb{D}_f$, f is continuous at p from X to Y.
Given $p \in \mathbb{D}_f$. Want: f is continuous at p from X to Y.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w \in X$,
$(d_X(w,p) < \delta) \Rightarrow (d_Y(f_w, f_p) < \varepsilon)$.
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall w \in X$,
$(d_X(w,p) < \delta) \Rightarrow (d_Y(f_w, f_p) < \varepsilon)$.
Since f is continuous from A to B,
we get: f is continuous on $\mathbb{D}_f$ from A to B.
So, since $p \in \mathbb{D}_f$, we get: f is continuous at p from A to B.
So choose $\delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,
$(d_A(w,p) < \delta) \Rightarrow (d_B(f_w, f_p) < \varepsilon)$.
Want: $\forall w \in \mathbb{D}_f$, $[(d_X(w,p) < \delta) \Rightarrow (d_Y(f_w, f_p) < \varepsilon)]$.
Given $w \in \mathbb{D}_f$. Want: $(d_X(w,p) < \delta) \Rightarrow (d_Y(f_w, f_p) < \varepsilon)$.
Assume: $d_X(w,p) < \delta$. Want: $d_Y(f_w, f_p) < \varepsilon$.
Since $f : A \dashrightarrow B$, we get $\mathbb{D}_f \subseteq A$ and $\mathbb{I}_f \subseteq B$.
Then $w, x \in \mathbb{D}_f \subseteq A$ and $f_w, f_x \in \mathbb{I}_f \subseteq B$.
So, since $d_A = d_X|(A \times A)$ and since $d_B = d_Y|(B \times B)$,
we get: $d_A(w,p) = d_X(w,p)$ and $d_B(f_w, f_p) = d_Y(f_w, f_p)$.
Then $d_A(w,p) = d_X(w,p) < \delta$,
so, by choice of δ, we get: $d_B(f_w, f_p) < \varepsilon$.
Then $d_Y(f_w, f_p) = d_B(f_w, f_p) < \varepsilon$, as desired.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$:
Assume: f is continuous from X to Y.
Want: f is continuous from A to B.
Want: f is continuous on $\mathbb{D}_f$ from A to B.
Want: $\forall p \in \mathbb{D}_f$, f is continuous at p from A to B.
Given $p \in \mathbb{D}_f$. Want: f is continuous at p from A to B.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w \in X$,

$$(\ d_A(w,p) < \delta\ ) \ \Rightarrow\ (\ d_B(f_w, f_p) < \varepsilon\ ).$$

Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall w \in X$,

$$(\ d_A(w,p) < \delta\ ) \ \Rightarrow\ (\ d_B(f_w, f_p) < \varepsilon\ ).$$

Since f is continuous from X to Y,
we get: f is continuous on $\mathbb{D}_f$ from X to Y.
So, since $p \in \mathbb{D}_f$, we get: f is continuous at p from X to Y.
So choose $\delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,

$$(\ d_X(w,p) < \delta\ ) \ \Rightarrow\ (\ d_Y(f_w, f_p) < \varepsilon\ ).$$

Want: $\forall w \in \mathbb{D}_f$, $[\ (\ d_A(w,p) < \delta\ ) \ \Rightarrow\ (\ d_B(f_w, f_p) < \varepsilon\ )\ ]$.
Given $w \in \mathbb{D}_f$. Want: $(\ d_A(w,p) < \delta\ ) \ \Rightarrow\ (\ d_B(f_w, f_p) < \varepsilon\ )$.
Assume: $d_A(w,p) < \delta$. Want: $d_B(f_w, f_p) < \varepsilon$.
Since $f : A \dashrightarrow B$, we get $\mathbb{D}_f \subseteq A$ and $\mathbb{I}_f \subseteq B$.
Then $w, x \in \mathbb{D}_f \subseteq A$ and $f_w, f_x \in \mathbb{I}_f \subseteq B$.
So, since $d_A = d_X|(A \times A)$ and since $d_B = d_Y|(B \times B)$,
we get: $d_A(w,p) = d_X(w,p)$ and $d_B(f_w, f_p) = d_Y(f_w, f_p)$.
Then $d_X(w,p) = d_A(w,p) < \delta$,
so, by choice of δ, we get: $d_Y(f_w, f_p) < \varepsilon$.
Then $d_B(f_w, f_p) = d_Y(f_w, f_p) < \varepsilon$, as desired.
End of proof of $\Leftarrow$.

□

THEOREM 2.11.6. *Let* f, g, h *be functions.*
Then $(h \circ g) \circ f = h \circ (g \circ f)$.

Proof. Want: $\forall x,\ ((h \circ g) \circ f)_x = (h \circ (g \circ f))_x$.
Given x. Want: $((h \circ g) \circ f)_x = (h \circ (g \circ f))_x$.
We have: $((h \circ g) \circ f)_x = (h \circ g)(f_x) = h(g(f_x))$
$= h((g \circ f)_x) = (h \circ (g \circ f))_x$, as desired. □

THEOREM 2.11.7. *Let* X, Y, Z *be metric spaces,* $p \in X$.
Let $f : X \dashrightarrow Y$ *and* $g : Y \dashrightarrow Z$.
Assume: f *is continuous at* p *from* X *to* Y.

Assume: g is continuous at f_p from Y to Z.
Then $g \circ f$ is continuous at p from X to Z.

Proof. We have: f is sequentially continuous at p from X to Y.
Also, g is sequentially continuous at f_p from Y to Z.
Want: $g \circ f$ is sequentially continuous at p from X to Z.
Want: $\forall s \in \mathbb{D}_{g\circ f}^{\mathbb{N}}$, $(\ s \to p \text{ in } X\ ) \Rightarrow (\ (g \circ f) \circ s \to (g \circ f)_p \text{ in } Z\ )$.
Given $s \in \mathbb{D}_{g\circ f}^{\mathbb{N}}$. Want: $(\ s \to p \text{ in } X\ ) \Rightarrow (\ (g\circ f)\circ s \to (g\circ f)_p \text{ in } Z\ )$.
Assume: $s \to p$ in X. Want: $(g \circ f) \circ s \to (g \circ f)_p$ in Z.
We have: $\forall j \in \mathbb{N}$,
$s_j \in \mathbb{D}_{g\circ f}$, so $(g \circ f)(s_j) \neq ☹$,
so $g(f(s_j)) = (g \circ f)(s_j) \neq ☹$,
so both $f(s_j) \neq ☹$ and $g((f \circ s)_j) = g(f(s_j)) \neq ☹$,
so both $s_j \in \mathbb{D}_f$ and $(f \circ s)_j \in \mathbb{D}_g$.
Then $s \in \mathbb{D}_f^{\mathbb{N}}$ and $f \circ s \in \mathbb{D}_g^{\mathbb{N}}$.
Since $s \in \mathbb{D}_f^{\mathbb{N}}$ and since $s \to p$ in X
and since f is sequentially continuous at p from X to Y,
we get: $f \circ s \to f_p$ in Y.
Since $f \circ s \in \mathbb{D}_g^{\mathbb{N}}$ and since $f \circ s \to f_p$ in Y
and since g is sequentially continuous at f_p from Y to Z,
we get: $g \circ (f \circ s) \to g_{f_p}$ in Z.
So, since $(g \circ f) \circ s = g \circ (f \circ s)$ and since $(g \circ f)_p = g_{f_p}$,
we get: $(g \circ f) \circ s \to (g \circ f)_p$ in Z, as desired. □

Alternate proof:

Proof. Want: $\forall \varepsilon > 0$, $\exists \gamma > 0$ s.t., $\forall v \in \mathbb{D}_{g\circ f}$,
$$(\ d(v, p) < \gamma\ ) \Rightarrow (\ d((g \circ f)_v, (g \circ f)_p) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists \gamma > 0$ s.t., $\forall v \in \mathbb{D}_{g\circ f}$,
$$(\ d(v, p) < \gamma\ ) \Rightarrow (\ d((g \circ f)_v, (g \circ f)_p) < \varepsilon\ ).$$
Since g is continuous at f_p from Y to Z, choose $\delta > 0$ s.t., $\forall w \in \mathbb{D}_g$,
$$(\ d(w, f_p) < \gamma\ ) \Rightarrow (\ d(g_w, g_{f_p}) < \varepsilon\ ).$$
Since f is continuous at p from X to Y, choose $\gamma > 0$ s.t., $\forall v \in \mathbb{D}_f$,
$$(\ d(v, p) < \gamma\ ) \Rightarrow (\ d(f_v, f_p) < \delta\ ).$$
Want: $\forall v \in \mathbb{D}_{g\circ f}$, $(\ d(v, p) < \gamma\ ) \Rightarrow (\ d((g \circ f)_v, (g \circ f)_p) < \varepsilon\ )$.
Given $v \in \mathbb{D}_{g\circ f}$. Want: $(\ d(v, p) < \gamma\ ) \Rightarrow (\ d((g\circ f)_v, (g\circ f)_p) < \varepsilon\ )$.
Assume: $d(v, p) < \gamma$. Want: $d((g \circ f)_v, (g \circ f)_p) < \varepsilon$.
Since $v \in \mathbb{D}_{g\circ f}$, we get: $(g \circ f)_v \neq ☹$.
So, since $g(f_v) = (g \circ f)_v$, we get: $g(f_v) \neq ☹$.
Then $f_v \in \mathbb{D}_g$. Then $f_v \neq ☹$. Then $v \in \mathbb{D}_f$.

Since $v \in \mathbb{D}_f$ and $d(v,p) < \gamma$, by choice of γ, we get: $d(f_v, f_p) < \delta$.
Since $f_v \in \mathbb{D}_g$ and $d(f_v, f_p) < \delta$, by choice of δ, we get $d(g_{f_v}, g_{f_p}) < \varepsilon$.
Then $d((g \circ f)_v, (g \circ f)_p) = d(g_{f_v}, g_{f_p}) < \varepsilon$, as desired. □

THEOREM 2.11.8. *Let X, Y, Z be metric spaces.*
Let $f : X \dashrightarrow Y$ and let $g : Y \dashrightarrow Z$.
Assume: f is continuous from X to Y.
Assume: g is continuous from Y to Z.
Then $g \circ f$ is continuous from X to Z.

Proof. Want: $g \circ f$ is continuous on $\mathbb{D}_{g \circ f}$ from X to Z.
Want: $\forall p \in \mathbb{D}_{g \circ f}$, $g \circ f$ is continuous at p from X to Z.
Given $p \in \mathbb{D}_{g \circ f}$. Want: $g \circ f$ is continuous at p from X to Z.
Since $p \in \mathbb{D}_{g \circ f}$, we get $(g \circ f)_p \neq$ ☹.
Then $g(f_p) = (g \circ f)_p \neq$ ☹, so $f_p \in \mathbb{D}_g$.
Since $f_p \in \mathbb{D}_g$, we get $f_p \neq$ ☹ and so $p \in \mathbb{D}_f$.
Since f is continuous from X to Y, we get:
f is continuous on $\mathbb{D}_f$ from X to Y.
So, since $p \in \mathbb{D}_f$, we get: f is continuous at p from X to Y.
Since g is continuous from Y to Z, we get:
g is continuous on $\mathbb{D}_g$ from Y to Z.
So, since $f_p \in \mathbb{D}_g$, we get: g is continuous at f_p from Y to Z.
Then, by Theorem 2.11.7, $g \circ f$ is continuous at p from X to Z. □

THEOREM 2.11.9. *Let X be a metric space, $f, g : X \dashrightarrow \mathbb{R}$, $p \in X$.*
Assume that f and g are both continuous at p from X to $\mathbb{R}$.
Assume that $g_p \neq 0$. Then: f/g is continuous at p from X to $\mathbb{R}$.

Proof. Define $h : \mathbb{R} \dashrightarrow \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $h_x = 1/x$.
Then h is continuous from $\mathbb{R}$ to $\mathbb{R}$.
That is, h is continuous on $\mathbb{D}_h$ from $\mathbb{R}$ to $\mathbb{R}$.
So, since $g_p \in \mathbb{R}_0^\times = \mathbb{D}_h$, we conclude:
h is continuous at g_p from $\mathbb{R}$ to $\mathbb{R}$.
So, since g is continuous at p from X to $\mathbb{R}$, we conclude:
$h \circ g$ is continuous at p from X to $\mathbb{R}$.
So, since f is continuous at p from X to $\mathbb{R}$, we conclude:
$f \cdot (h \circ g)$ is continuous at p from X to $\mathbb{R}$.
It therefore suffices to show: $f \cdot (h \circ g) = f/g$.
Want: $\forall w \in X$, $(f \cdot (h \circ g))_w = (f/g)_w$.
Given $w \in X$. Want: $(f \cdot (h \circ g))_w = (f/g)_w$.

We have $(f \cdot (h \circ g)_w = f_w \cdot ((h \circ g)_w) = f_w \cdot h_{g_w} = f_w \cdot (1/g_w)$
$= f_w/g_w = (f/g)_w$, as desired. □

THEOREM 2.11.10. *Let X be a metric space, $f, g : X \dashrightarrow \mathbb{R}$. Assume: f and g are both continuous from X to $\mathbb{R}$. Then:*
(1) $f + g$ is continuous from X to $\mathbb{R}$ and
(2) $f - g$ is continuous from X to $\mathbb{R}$ and
(3) fg is continuous from X to $\mathbb{R}$ and
(4) f/g is continuous from X to $\mathbb{R}$.

Proof. Proof of (1),(2),(3):
Unassigned HW.
End of proof of (1),(2),(3).

Proof of (4):
Want: f/g is continuous on $\mathbb{D}_{f/g}$ from X to $\mathbb{R}$.
Want: $\forall p \in \mathbb{D}_{f/g}$, f/g is continuous at p from X to $\mathbb{R}$.
Given $p \in \mathbb{D}_{f/g}$. Want: f/g is continuous at p from X to $\mathbb{R}$.
Since $p \in \mathbb{D}_{f/g}$, we get $(f/g)_p \neq ☹$.
Then $f_p/g_p = (f/g)_p \neq ☹$, so $f_p \neq ☹$ and $g_p \neq ☹$ and $g_p \neq 0$.
Since $f_p \neq ☹$, we get $p \in \mathbb{D}_f$. Since $g_p \neq ☹$, we get $p \in \mathbb{D}_g$.
Since f is continuous from X to $\mathbb{R}$, we get:
f is continuous on $\mathbb{D}_f$ from X to $\mathbb{R}$.
So, since $p \in \mathbb{D}_f$, we get: f is continuous at p from X to $\mathbb{R}$.
Since g is continuous from X to $\mathbb{R}$, we get:
g is continuous on $\mathbb{D}_g$ from X to $\mathbb{R}$.
So, since $p \in \mathbb{D}_g$, we get: g is continuous at p from X to $\mathbb{R}$.
Then, by Theorem 2.11.9, f/g is continuous at p from X to Z.
End of proof of (4). □

THEOREM 2.11.11. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$, let $S \subseteq \mathbb{D}_f$ and let $p \in S$.
Assume that f is continuous at p from X to Y.
Then $f|S$ is continuous at p from X to Y.

Proof. We have $p \in S = \mathbb{D}_{f|S}$.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_{f|S}$,
$$(\ d(w,p) < \delta\ ) \Rightarrow (\ d(\ (f|S)_w\ ,\ (f|S)_p\ ) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_{f|S}$,
$$(\ d(w,p) < \delta\ ) \Rightarrow (\ d(\ (f|S)_w\ ,\ (f|S)_p\ ) < \varepsilon\ ).$$

Since f is continuous at p from X to Y, choose $\delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,

$$(\ d(w,p) < \delta\ ) \ \Rightarrow\ (\ d(f_w, f_p) < \varepsilon\ ).$$

Then $\delta > 0$. Want: $\forall w \in \mathbb{D}_{f|S}$,

$$(\ d(w,p) < \delta\ ) \ \Rightarrow\ (\ d(\,(f|S)_w\,,\,(f|S)_p\,) \ <\ \varepsilon\ ).$$

Given $w \in \mathbb{D}_{f|S}$. Want: $(\,d(w,p) < \delta\,) \Rightarrow (\,d(\,(f|S)_w\,,\,(f|S)_p\,) \ <\ \varepsilon\,)$.
Assume $d(w,p) < \delta$. Want: $d(\,(f|S)_w\,,\,(f|S)_p\,) \ <\ \varepsilon$.
We have: $w \in \mathbb{D}_{f|S} = S$. By hypothesis, $S \subseteq \mathbb{D}_f$. Then $w \in \mathbb{D}_f$.
Since $(\ w \in \mathbb{D}_f\ ) \,\&\, (\ d(w,p) < \delta\ )$, by choice of δ,
we get: $d(f_w, f_p) < \varepsilon$.
We have $p \in S$. Also, by hypothesis, $w \in S$.
Then: $(f|S)_p = f_p$ and $(f|S)_w = f_w$.
Then: $d(\,(f|S)_w\,,\,(f|S)_p\,) \ =\ d(f_w, f_p) \ <\ \varepsilon$, as desired. □

The converse of the preceding theorem is not true:
Let $S := [4;\infty)$ and $f := \chi_S^{\mathbb{R}}$.
Then $f|S = C_S^1$, so $f|S$ is continuous,
so $f|S$ is continuous at 4 from $\mathbb{R}$ to $\mathbb{R}$.
However, by HW#8-2, f is not continuous at 4 from $\mathbb{R}$ to $\mathbb{R}$.

THEOREM 2.11.12. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$ and let $S \subseteq \mathbb{D}_f$.
Assume that f is continuous from X to Y.
Then $f|S$ is continuous from X to Y.

Proof. Want: $f|S$ is continuous on $\mathbb{D}_{f|S}$ from X to Y.
Want: $\forall p \in \mathbb{D}_{f|S}$, $f|S$ is continuous at p from X to Y.
Given $p \in \mathbb{D}_{f|S}$. Want: $f|S$ is continuous at p from X to Y.
We have: $p \in \mathbb{D}_{f|S} = S$. By hypothesis, $S \subseteq \mathbb{D}_f$. Then $p \in \mathbb{D}_f$.
By hypothesis, f is continuous on $\mathbb{D}_f$ from X to Y.
So, since $p \in \mathbb{D}_f$, we get: f is continuous at p from X to Y.
Then, by Theorem 2.11.11, $f|S$ is continuous at p from X to Y. □

THEOREM 2.11.13. *Let $f, g : \mathbb{R} \to \mathbb{R}$, $p \in \mathbb{R}$.*
Assume f and g are both continuous at p from $\mathbb{R}$ to $\mathbb{R}$.
Then (f, g) is continuous at p from $\mathbb{R}$ to $\mathbb{R}$.

Proof. Let $h := (f, g)$. Want: h is continuous at p from $\mathbb{R}$ to $\mathbb{R}$.
Since f and g are both continuous at p from $\mathbb{R}$ to $\mathbb{R}$,
we get: $p \in \mathbb{D}_f$ and $p \in \mathbb{D}_g$.
We have $\mathbb{D}_h = D_f \bigcap \mathbb{D}_g$. Then $p \in \mathbb{D}_f \bigcap \mathbb{D}_g = \mathbb{D}_h$.
It remains to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_h$,

$$(\ d_{\mathbb{R}}(w,p) < \delta \ ) \ \Rightarrow \ (\ d_{\mathbb{R}^2}(h_w,h_p) < \varepsilon \ ).$$

Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_h$,

$$(\ d_{\mathbb{R}}(w,p) < \delta \ ) \ \Rightarrow \ (\ d_{\mathbb{R}^2}(h_w,h_p) < \varepsilon \ ).$$

Let $\lambda := \varepsilon/\sqrt{2}$.

Since f is continuous at p from $\mathbb{R}$ to $\mathbb{R}$, choose $\alpha > 0$ s.t., $\forall w \in \mathbb{D}_f$,

$$(\ d_{\mathbb{R}}(w,p) < \alpha \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}(f_w,f_p) < \lambda \ ).$$

Since g is continuous at p from $\mathbb{R}$ to $\mathbb{R}$, choose $\beta > 0$ s.t., $\forall w \in \mathbb{D}_g$,

$$(\ d_{\mathbb{R}}(w,p) < \beta \ ) \ \Rightarrow \ (\ d_{\mathbb{R}}(g_w,g_p) < \lambda \ ).$$

Let $\delta := \min\{\alpha,\beta\}$. Then $\delta > 0$.

Want: $\forall w \in \mathbb{D}_h$, $(\ d_{\mathbb{R}}(w,p) < \delta \ ) \ \Rightarrow \ (\ d_{\mathbb{R}^2}(h_w,h_p) < \varepsilon \ )$.

Given $w \in \mathbb{D}_h$. Want: $(\ d_{\mathbb{R}}(w,p) < \delta \ ) \ \Rightarrow \ (\ d_{\mathbb{R}^2}(h_w,h_p) < \varepsilon \ )$.

Assume: $d_{\mathbb{R}}(w,p) < \delta$. Want: $d_{\mathbb{R}^2}(h_w,h_p) < \varepsilon$.

Since $w \in \mathbb{D}_h = \mathbb{D}_{(f,g)} = \mathbb{D}_f \bigcap \mathbb{D}_g$, we get: $w \in \mathbb{D}_f$ and $w \in \mathbb{D}_g$.

Since $\delta := \min\{\alpha,\beta\}$, we get: $\delta \leqslant \alpha$ and $\delta \leqslant \beta$.

Since $w \in \mathbb{D}_f$ and since $d_{\mathbb{R}}(w,p) < \delta \leqslant \alpha$, by choice of α,

we get: $d_{\mathbb{R}}(f_w,f_p) < \lambda$.

Since $|f_w - f_p| = d_{\mathbb{R}}(f_w,f_p) < \lambda$, we get: $|f_w - f_p| < \lambda$.

Since $0 \leqslant |f_w - f_p| < \lambda$, we get: $|f_w - f_p|^2 < \lambda^2$.

So, since $|f_w - f_p|^2 = (f_w - f_p)^2$, we get: $(f_w - f_p)^2 < \lambda^2$.

Since $w \in \mathbb{D}_g$ and since $d_{\mathbb{R}}(w,p) < \delta \leqslant \beta$, by choice of β,

we get: $d_{\mathbb{R}}(g_w,g_p) < \lambda$.

Since $|g_w - g_p| = d_{\mathbb{R}}(g_w,g_p) < \lambda$, we get: $|g_w - g_p| < \lambda$.

Since $0 \leqslant |g_w - g_p| < \lambda$, we get: $|g_w - g_p|^2 < \lambda^2$.

So, since $|g_w - g_p|^2 = (g_w - g_p)^2$, we get: $(g_w - g_p)^2 < \lambda^2$.

We compute:
$$\begin{aligned} d_{\mathbb{R}^2}(h_w,h_p) &= d_{\mathbb{R}^2}(\,(f,g)_w\,,\,(f,g)_p\,) \\ &= d_{\mathbb{R}^2}(\,(f_w,g_w)\,,\,(f_p,g_p)\,) \\ &= |\,(f_w,g_w) \,-\, (f_p,g_p)\,|_{\mathbb{R}^2} \\ &= |\,(\,f_w - f_p\,,\,g_w - g_p\,)\,|_{\mathbb{R}^2} \\ &= \sqrt{(f_w - f_p)^2 + (g_w - g_p)^2}. \end{aligned}$$

Since $\lambda = \varepsilon/2$, we get $\sqrt{2} \cdot \lambda = \varepsilon$.

Since $(f_w - f_p)^2 < \lambda^2$ and $(g_w - g_p)^2 < \lambda^2$,

we get: $(f_w - f_p)^2 + (g_w - g_p)^2 < \lambda^2 + \lambda^2$.

Since $0 \leqslant (f_w - f_p)^2 + (g_w - g_p)^2 < \lambda^2 + \lambda^2$,

we get: $\sqrt{(f_w - f_p)^2 + (g_w - g_p)^2} < \sqrt{\lambda^2 + \lambda^2}$.

So, since $d_{\mathbb{R}^2}(h_w,h_p) = \sqrt{(f_w - f_p)^2 + (g_w - g_p)^2}$

and since $\sqrt{\lambda^2 + \lambda^2} = \sqrt{2\lambda^2} = \sqrt{2} \cdot \lambda = \varepsilon$,

we get: $d_{\mathbb{R}^2}(h_w,h_p) < \varepsilon$, as desired. □

2.12. **Comparison of functions.**

We make the following convention: $\forall x \in \mathbb{R}$,

$\neg(\frownie \leqslant x)$ and $\neg(\frownie \geqslant x)$ and
$\neg(\frownie < x)$ and $\neg(\frownie > x)$ and
$\neg(x \leqslant \frownie)$ and $\neg(x \geqslant \frownie)$ and
$\neg(x < \frownie)$ and $\neg(x > \frownie)$.

We also make the following convention:

$\neg(\frownie < \frownie)$ and $\neg(\frownie > \frownie)$.

Also, since $\frownie = \frownie$, we conclude:

$\frownie \leqslant \frownie$ and $\neg(\frownie \geqslant \frownie)$.

DEFINITION 2.12.1. *Let f and g be functions and let S be a set.*

By $f = g$ on S, we mean: $\forall x \in S,\ f_x = g_x$.

For any functions f and g, for any set S, we have:

$(\ f = g \text{ on } S\ ) \Rightarrow (\ \mathbb{D}_f \cap S = \mathbb{D}_g \cap S\ )$.

DEFINITION 2.12.2. *Let f and g be functionals and let S be a set.*

By $f \leqslant g$ on S, we mean: $\forall x \in S,\ f_x \leqslant g_x$.
By $f \geqslant g$ on S, we mean: $\forall x \in S,\ f_x \geqslant g_x$.

For any functionals f and g, for any set S, we have:

$(\ f \leqslant g \text{ on } S\ ) \Rightarrow (\ \mathbb{D}_f \cap S = \mathbb{D}_g \cap S\ )$.

For any functionals f and g, for any set S, we have:

$(\ f \geqslant g \text{ on } S\ ) \Rightarrow (\ \mathbb{D}_f \cap S = \mathbb{D}_g \cap S\ )$.

DEFINITION 2.12.3. *Let f and g be functionals and let S be a set.*

By $f < g$ on S, we mean: $\forall x \in S,\ f_x < g_x$.
By $f > g$ on S, we mean: $\forall x \in S,\ f_x > g_x$.

For any functionals f and g, for any set S, we have:

$(\ f < g \text{ on } S\ ) \Rightarrow (\ (S \subseteq \mathbb{D}_f) \,\&\, (S \subseteq \mathbb{D}_g)\ )$.

For any functionals f and g, for any set S, we have:

$(\ f > g \text{ on } S\ ) \Rightarrow (\ (S \subseteq \mathbb{D}_f) \,\&\, (S \subseteq \mathbb{D}_g)\ )$.

DEFINITION 2.12.4. *Let f be a function, a an object, S a set.*

By $f \equiv a$ on S, we mean: $\forall x \in S,\ f_x = a$.

For any function f, for any object a, for any set S, we have:

$(\ (f \equiv a \text{ on } S\ ) \,\&\, (\ a \neq \frownie\ )\ ) \Rightarrow (\ S \subseteq \mathbb{D}_f\ )$.

For any function f, for any set S, we have:

$(f \equiv ☹ \text{ on } S)) \Rightarrow (S \bigcap \mathbb{D}_f = \varnothing)$.

DEFINITION 2.12.5. *Let f be a functional, $a \in \mathbb{R}$, S a set.*

By $f \leqslant a$ on S, we mean: $\forall x \in S$, $f_x \leqslant a$.
By $f \geqslant a$ on S, we mean: $\forall x \in S$, $f_x \geqslant a$.
By $f < a$ on S, we mean: $\forall x \in S$, $f_x < a$.
By $f > a$ on S, we mean: $\forall x \in S$, $f_x > a$.
By $a \leqslant f$ on S, we mean: $\forall x \in S$, $a \leqslant f_x$.
By $a \geqslant f$ on S, we mean: $\forall x \in S$, $a \geqslant f_x$.
By $a < f$ on S, we mean: $\forall x \in S$, $a < f_x$.
By $a > f$ on S, we mean: $\forall x \in S$, $a > f_x$.

DEFINITION 2.12.6. *Let X be a metric space.*
Let f and g be functions and let $p \in X$.
By $f = g$ near p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $f = g$ on B.

DEFINITION 2.12.7. *Let X be a metric space.*
Let f and g be functionals and let $p \in X$.
By $f \leqslant g$ near p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $f \leqslant g$ on B.
By $f \geqslant g$ near p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $f \geqslant g$ on B.
By $f < g$ near p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $f < g$ on B.
By $f > g$ near p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $f > g$ on B.

DEFINITION 2.12.8. *Let X be a metric space.*
Let f be a functions, let a be an object and let $p \in X$.
By $f \equiv a$ near p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $f \equiv a$ on B.

DEFINITION 2.12.9. *Let X be a metric space.*
Let f be a functional, let $a \in \mathbb{R}$ and let $p \in X$.
By $f \leqslant a$ near p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $f \leqslant a$ on B.
By $f \geqslant a$ near p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $f \geqslant a$ on B.
By $f < a$ near p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $f < a$ on B.
By $f > a$ near p in X, we mean: $\exists B \in \mathcal{B}_X(p)$ s.t. $f > a$ on B.

THEOREM 2.12.10. *Let X and Y be metric spaces.*
Let $f, g : X \dashrightarrow Y$ and let $p \in X$. Assume $f = g$ near p in X.
Assume g is continuous at p from X to Y.
Then f is continuous at p from X to Y.

Proof. Since g is continuous at p from X to Y,

we conclude that $p \in \mathbb{D}_g$, so $g_p \neq ☹$.

Since $f = g$ near p in X,

choose $B \in \mathcal{B}_X(p)$ s.t. $f = g$ on B.

Since $B \in \mathcal{B}_X(p)$, choose $\mu > 0$ s.t. $B = B_X(p, \mu)$.

Then $p \in B_X(p, \mu) = B$, so $p \in B$.

So, since $f = g$ on B, we get: $f_p = g_p$.

Then $f_p = g_p \neq ☹$, and so $p \in \mathbb{D}_f$.

Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,

$$(\ d(w, p) < \delta \ ) \ \Rightarrow \ (\ d(f_w, f_p) < \varepsilon \ ).$$

Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,

$$(\ d(w, p) < \delta \ ) \ \Rightarrow \ (\ d(f_w, f_p) < \varepsilon \ ).$$

Since g is continuous at p from X to Y,

choose $\lambda > 0$ s.t., $\forall w \in \mathbb{D}_g$,

$$(\ d(w, p) < \lambda \ ) \ \Rightarrow \ (\ d(g_w, g_p) < \varepsilon \ ).$$

Let $\delta := \min\{\lambda, \mu\}$. Then $\delta > 0$.

Want: $\forall w \in \mathbb{D}_f$, $(\ d(w, p) < \delta \ ) \ \Rightarrow \ (\ d(f_w, f_p) < \varepsilon \ )$.

Given $w \in \mathbb{D}_f$. Want: $(\ d(w, p) < \delta \ ) \ \Rightarrow \ (\ d(f_w, f_p) < \varepsilon \ )$.

Assume $d(w, p) < \delta$. Want: $d(f_w, f_p) < \varepsilon$.

Since $d(w, p) < \delta \leqslant \mu$, we get: $w \in B_X(p, \mu)$.

So, since $B_X(p, \mu) = B$, we get: $w \in B$.

So, since $f = g$ on B, we get: $f_w = g_w$.

Since $w \in \mathbb{D}_f$, we get $f_w \neq ☹$.

Then $g_w = f_w \neq ☹$, and so $w \in \mathbb{D}_g$.

So, since $d(w, p) < \delta < \lambda$, by choice of λ, we get: $d(g_w, g_p) < \varepsilon$.

So, since $f_w = g_w$ and $f_p = g_p$, we get: $d(f_w, f_p) < \varepsilon$, as desired. □

2.13. **Limits of functions between metric spaces.**

DEFINITION 2.13.1. *Let f be a function and let p and q be objects. Then $\mathrm{adj}_p^q f$ is the function defined by:*

$$\forall x, \quad (\mathrm{adj}_p^q f)_x = \begin{cases} f_x, & \text{if } x \neq p \\ q, & \text{if } x = p. \end{cases}$$

THEOREM 2.13.2. *Let $f := \mathrm{adj}_1^3(C_{\mathbb{R}}^2)$.*

Then: $$\forall x \in \mathbb{R}, \quad f_x = \begin{cases} 2, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1. \end{cases}$$

THEOREM 2.13.3. *Let* $f := \mathrm{adj}_3^2(C_{\mathbb{R}}^1)$.

Then: $\forall x \in \mathbb{R}, \quad f_x = \begin{cases} 1, & \text{if } x \neq 2 \\ 2, & \text{if } x = 3. \end{cases}$

Also, $\mathrm{adj}_3^1 f = C_{\mathbb{R}}^1$.

DEFINITION 2.13.4. *Let* X *and* Y *be metric spaces.*
Let $f : X \dashrightarrow Y$, *let* $p \in X$ *and let* $q \in Y$.
By $f \to q$ *near* p *from* X *to* Y, *we mean:*
$\forall \varepsilon > 0$, $\exists \delta > 0$ *s.t.*, $\forall w \in \mathbb{D}_f$,
$(\ 0 < d(w,p) < \delta \ ) \ \Rightarrow \ (\ d(f_w, q) < \varepsilon \ )$.

THEOREM 2.13.5. *Let* $f := \mathrm{adj}_1^3(C_{\mathbb{R}}^2)$.
Then: $f_1 = 3$ *and* $f \to 2$ *near* 1 *from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. We have: $f_1 = (\mathrm{adj}_1^3(C_{\mathbb{R}}^2))_1 = 3$.
Want: $f \to 2$ near 1 from $\mathbb{R}$ to $\mathbb{R}$.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,
$(\ 0 < d(w,1) < \delta \ ) \ \Rightarrow \ (\ d(f_w, 2) < \varepsilon \ )$.
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_f$,
$(\ 0 < d(w,1) < \delta \ ) \ \Rightarrow \ (\ d(f_w, 2) < \varepsilon \ )$.
Let $\delta = 6$. Then $\delta > 0$.
Want: $\forall w \in \mathbb{D}_f$, $(\ 0 < d(w,1) < \delta \ ) \ \Rightarrow \ (\ d(f_w, 2) < \varepsilon \ )$.
Given $w \in \mathbb{D}_f$. Want: $(\ 0 < d(w,1) < \delta \ ) \ \Rightarrow \ (\ d(f_w, 2) < \varepsilon \ )$.
Assume: $0 < d(w,1) < \delta$. Want: $d(f_w, 2) < \varepsilon$.
Since $d(w,1) > 0$, we get $d(w,1) \neq 0$,
so $w \neq 1$, so $(\mathrm{adj}_1^3(C_{\mathbb{R}}^2))_w = (C_{\mathbb{R}}^2)_w$.
Since $w \in \mathbb{R}$, we have $(C_{\mathbb{R}}^2)_w = 2$.
Then $f_w = (\mathrm{adj}_1^3(C_{\mathbb{R}}^2))_w = (C_{\mathbb{R}}^2)_w = 2$.
Then $d(f_w, 2) = d(2,2) = 0 < \varepsilon$, as desired. □

THEOREM 2.13.6. *Let* X, Y *and* Z *be metric spaces.*
Let $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$. *Let* $a \in X$, $b \in Y$.
Assume that $f \to b$ *near* a *from* X *to* Y
and that g *is continuous at* b *from* Y *to* Z.
Then: $g \circ f \to g_b$ *near* a *from* X *to* Z.

Proof. Want: $\forall \varepsilon > 0$, $\exists \gamma > 0$ s.t., $\forall v \in \mathbb{D}_{g \circ f}$,
$(\ 0 < d(v, a) < \gamma \ ) \ \Rightarrow \ (\ d((g \circ f)_v, g_b) < \varepsilon \ )$.
Given $\varepsilon > 0$. Want: $\exists \gamma > 0$ s.t., $\forall v \in \mathbb{D}_{g \circ f}$,
$(\ 0 < d(v, a) < \gamma \ ) \ \Rightarrow \ (\ d((g \circ f)_v, g_b) < \varepsilon \ )$.

Since g is continuous at b from Y to Z,
choose $\delta > 0$ s.t., $\forall w \in \mathbb{D}_g$,
$$(\, d(\, w \,,\, b \,) \,<\, \delta \,) \;\Rightarrow\; (\, d(\, g_w \,,\, g_b \,) \,<\, \varepsilon \,).$$
Since $f \to b$ near a from Z to Y,
choose $\gamma > 0$ s.t., $\forall v \in \mathbb{D}_f$,
$$(\, 0 \,<\, d(\, v \,,\, a \,) \,<\, \gamma \,) \;\Rightarrow\; (\, d(\, f_v \,,\, b \,) \,<\, \delta \,).$$
Then $\gamma > 0$. Want: $\forall v \in \mathbb{D}_{g \circ f}$,
$$(\, 0 \,<\, d(\, v \,,\, a \,) \,<\, \gamma \,) \;\Rightarrow\; (\, d(\, (g \circ f)_v \,,\, g_b \,) \,<\, \varepsilon \,).$$
Given $v \in \mathbb{D}_{g \circ f}$.
Want: $(\, 0 \,<\, d(\, v \,,\, a \,) \,<\, \gamma \,) \;\Rightarrow\; (\, d(\, (g \circ f)_v \,,\, g_b \,) \,<\, \varepsilon \,)$.
Assume $0 \,<\, d(\, v \,,\, a \,) \,<\, \gamma$. Want: $d(\, (g \circ f)_v \,,\, g_b \,) \,<\, \varepsilon$.
Since $v \in D_{g \circ f}$, we get $(g \circ f)_v \neq ☹$.
Then $g(f_v) = (g \circ f)_v \neq ☹$, and so $f_v \in \mathbb{D}_g$.
Since $f_v \in \mathbb{D}_g$, we see that $f_v \neq ☹$, and so $v \in \mathbb{D}_f$.
Then $v \in \mathbb{D}_f$ and $0 \,<\, d(\, v \,,\, a \,) \,<\, \gamma$,
so, by choice of γ, we get: $d(\, f_v \,,\, b \,) \,<\, \delta$.
Let $w := f_v$. Then $w = f_v \in \mathbb{D}_g$ and $d(\, w \,,\, b \,) \,=\, d(\, f_v \,,\, b \,) \,<\, \delta$,
so, by choice of δ, we get: $d(\, g_w \,,\, g_b \,) \,<\, \varepsilon$.
Then: $d(\, (g \circ f)_v \,,\, g_b \,) \,=\, d(\, g_{f_v} \,,\, g_b \,) \,=\, d(\, g_w \,,\, g_b \,) \,<\, \varepsilon$. □

2.14. **The Hausdorff property for metric spaces.**

THEOREM 2.14.1. *Let Y be a metric space, $q, r \in Y$. Assume $q \neq r$. Then $\exists \varepsilon > 0$ s.t. $(B_X(q, \varepsilon)) \bigcap (B_X(r, \varepsilon)) = \varnothing$.*

Proof. Since $q \neq r$, we get: $d(q, r) \neq 0$.
So, since $d(q, r) \geqslant 0$, we get: $d(q, r) > 0$.
Let $s := d(q, r)$. Then $s > 0$. Let $\varepsilon := s/2$. Then $\varepsilon > 0$.
Want: $(B_X(q, \varepsilon)) \bigcap (B_X(r, \varepsilon)) = \varnothing$.
Assume $(B_X(q, \varepsilon)) \bigcap (B_X(r, \varepsilon)) \neq \varnothing$. Want: Contradiction.
Choose $w \in (B_X(q, \varepsilon)) \bigcap (B_X(r, \varepsilon))$.
Since $w \in B_X(q, \varepsilon)$, we get: $d(q, w) < \varepsilon$.
Since $w \in B_X(r, \varepsilon)$, we get: $d(w, r) < \varepsilon$.
Then $s = d(q, r) \leqslant (d(q, w)) + (d(w, r)) < \varepsilon + \varepsilon = 2\varepsilon$, so $s < 2\varepsilon$.
Since $\varepsilon = s/2$, we get $2\varepsilon = s$.
Then $s < 2\varepsilon = s$, so $s < s$. Contradiction. □

2.15. **Uniqueness of limits of sequences.**

THEOREM 2.15.1. *Let Y be a metric space, $s \in Y^{\mathbb{N}}$, $q, r \in Y$. Assume $s \to q$ in Y and $s \to r$ in Y. Then $q = r$.*

Proof. Assume $q \neq r$. Want: Contradiction.
By Theorem 2.14.1, choose $\varepsilon > 0$ s.t. $(B_X(q,\varepsilon)) \bigcap (B_X(r,\varepsilon)) = \varnothing$.
Since $s \to q$ in Y, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\ j \geqslant L\ ) \Rightarrow (\ d(s_j, q) < \varepsilon\ ).$$

Since $s \to r$ in Y, choose $M \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\ j \geqslant M\ ) \Rightarrow (\ d(s_j, r) < \varepsilon\ ).$$

Let $j := \max\{L, M\}$. Then: $(\ j \in \mathbb{N}\ ) \,\&\, (\ j \geqslant L\ ) \,\&\, (\ j \geqslant M\ )$.
Since $j \in \mathbb{N}$ and $j \geqslant L$, by choice of L,
we get: $d(s_j, q) < \varepsilon$, so $s_j \in B_X(q,\varepsilon)$.
Since $j \in \mathbb{N}$ and $j \geqslant M$, by choice of M,
we get: $d(s_j, r) < \varepsilon$, so $s_j \in B_X(r,\varepsilon)$.
Then $s_j \in (B_X(q,\varepsilon)) \bigcap (B_X(r,\varepsilon)) = \varnothing$,
so $s_j \in \varnothing$. Contradiction. □

THEOREM 2.15.2. *Let* $g := C^3_{[1;2] \bigcup \{9\}}$.
Then: $\forall q \in \mathbb{R}$, $g \to q$ *near* 9.

Proof. Given $q \in \mathbb{R}$. Want: $g \to q$ near 9.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_g$,

$$(\ 0 < d(x,9) < \delta\ ) \Rightarrow (\ d(g_x, q) < \varepsilon\ ).$$

Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_g$,

$$(\ 0 < d(x,9) < \delta\ ) \Rightarrow (\ d(g_x, q) < \varepsilon\ ).$$

Let $\delta := 7$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_g$, $(\ 0 < d(x,9) < \delta\ ) \Rightarrow (\ d(g_x, q) < \varepsilon\ )$.
Given $x \in \mathbb{D}_g$. Want: $(\ 0 < d(x,9) < \delta\ ) \Rightarrow (\ d(g_x, q) < \varepsilon\ )$.
Since $x \in \mathbb{D}_g = [1;2] \bigcup \{9\}$, we get: $(\ x \in [1;2]\ ) \vee (\ x \in \{9\}\ )$.
Then $(\ 1 \leqslant x \leqslant 2\ ) \vee (\ x = 9\ )$.
Then $(\ 9 - x \geqslant 7\ ) \vee (\ 9 - x = 0\ )$.
Then $(\ |9 - x| \geqslant 7\ ) \vee (\ |9 - x| = 0\ )$.
Then $(\ d(x,9) \geqslant 7\ ) \vee (\ d(x,9) = 0\ )$.
Then $\neg(0 < d(x,9) < 7)$. Then $\neg(0 < d(x,9) < \delta)$.
Then $(\ 0 < d(x,9) < \delta\ ) \Rightarrow (\ d(g_x, q) < \varepsilon\ )$, as desired. □

2.16. **Limits and continuity.**

THEOREM 2.16.1. *Let X and Y be metric spaces.*
Let $g : X \dashrightarrow Y$ and let $p \in \mathbb{D}_g$.
Assume: $g \to g_p$ near p from X to Y.
Then: g is continuous at p from X to Y.

Proof. By hypothesis $p \in \mathbb{D}_g$.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_g$,
$$(\ d_X(w,p) < \delta \ ) \ \Rightarrow \ (\ d_Y(g_w, g_p) < \varepsilon \ ).$$
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall w \in \mathbb{D}_g$,
$$(\ d_X(w,p) < \delta \ ) \ \Rightarrow \ (\ d_Y(g_w, g_p) < \varepsilon \ ).$$
Since $g \to g_p$ near p from X to Y,
choose $\delta > 0$ s.t., $\forall w \in \mathbb{D}_g$,
$$(\ 0 < d_X(w,p) < \delta \ ) \ \Rightarrow \ (\ d_Y(g_w, g_p) < \varepsilon \ ).$$
Then $\delta > 0$. Want: $\forall w \in \mathbb{D}_g$,
$$(\ d_X(w,p) < \delta \ ) \ \Rightarrow \ (\ d_Y(g_w, g_p) < \varepsilon \ ).$$
Given $w \in \mathbb{D}_g$. Want: $(\ d_X(w,p) < \delta \ ) \ \Rightarrow \ (\ d_Y(g_w, g_p) < \varepsilon \ )$.
Assume: $d_X(w,p) < \delta$. Want: $d_Y(g_w, g_p) < \varepsilon$.
Exactly one of the following is true:
(1) $d_X(w,p) = 0$ or (2) $d_X(w,p) \neq 0$.

Case 1:
Since $p \in \mathbb{D}_g$, we get $g_p \in \mathbb{I}_g$. Also, $g : X \dashrightarrow Y$, so $\mathbb{I}_g \subseteq Y$.
Then $g_p \in \mathbb{I}_g \subseteq Y$, so $d_Y(g_p, g_p) = 0$.
Since $d_X(w,p) = 0$, we get $w = p$, and so $g_w = g_p$.
We have $d_Y(g_w, g_p) = d_Y(g_p, g_p) = 0 < \varepsilon$, as desired.
End of Case 1.

Case 2:
Since $d_X(w,p) \geqslant 0$ and $d_X(w,p) \neq 0$, we get $d_X(w,p) > 0$.
Then $w \in \mathbb{D}_g$ and $0 < d_X(w,p) < \delta$.
So, by choice of δ, we have: $d_Y(g_w, g_p) < \varepsilon$, as desired.
End of Case 2. □

2.17. **The metric space $\mathbb{N}^*$.**

DEFINITION 2.17.1. *Define $a : \mathbb{N}^* \to \mathbb{R}$ by:*
$$\forall j \in \mathbb{N}^*, \quad a_j = \begin{cases} 1/j, & \text{if } j \neq \infty \\ 0, & \text{if } j = \infty. \end{cases}$$
Define $d_* : \mathbb{N}^* \times \mathbb{N}^* \to [0;\infty)$ by: $\forall j, k \in \mathbb{N}^*$, $d_*(j,k) = |a_k - a_j|$.

THEOREM 2.17.2. *We have:* $d_*(5,7) = |(1/5) - (1/7)| = 2/35$
and $d_*(3,\infty) = |(1/3) - 0| = 1/3$
and $d_*(\infty,\infty) = |0 - 0| = 0$.

THEOREM 2.17.3. *We have: $d_* \in \mathcal{M}(\mathbb{N}^*)$.*

We refer to d_* as the **compatible metric** on $\mathbb{N}^*$.
We are sometimes sloppy and use $\mathbb{N}^*$ to denote the metric space $(\mathbb{N}^*, d_*)$.
Note that $d_{\mathbb{N}^*} = d_{(\mathbb{N}^*, d_*)} = (\mathbb{N}^*, d_*)_2 = d_*$.

THEOREM 2.17.4. *Let X be a set and let $s \in X^{\mathbb{N}}$.*
Then $s : \mathbb{N} \to X$ and $s : \mathbb{N}^* \dashrightarrow X$.

THEOREM 2.17.5. *Let X be a metric space, $s \in X^{\mathbb{N}}$, $q \in X$.*
Assume: $s \to q$ in X. Then: $s \to q$ near ∞ from $\mathbb{N}^*$ to X.

Proof. Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall j \in \mathbb{D}_s$,
$$(\ 0 < d_{\mathbb{N}^*}(j, \infty) < \delta\ ) \Rightarrow (\ d_X(s_j, q) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall j \in \mathbb{D}_s$,
$$(\ 0 < d_{\mathbb{N}^*}(j, \infty) < \delta\ ) \Rightarrow (\ d_X(s_j, q) < \varepsilon\ ).$$
Since $s \to q$ in X, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \Rightarrow (\ d_X(s_j, q) < \varepsilon\ ).$$
Let $\delta := 1/K$. Then $\delta > 0$.
Want: $\forall j \in \mathbb{D}_s$, $(\ 0 < d_{\mathbb{N}^*}(j, \infty) < \delta\ ) \Rightarrow (\ d_X(s_j, q) < \varepsilon\ )$.
Given $j \in \mathbb{D}_s$. Want: $(\ 0 < d_{\mathbb{N}^*}(j, \infty) < \delta\ ) \Rightarrow (\ d_X(s_j, q) < \varepsilon\ )$.
Assume: $0 < d_{\mathbb{N}^*}(j, \infty) < \delta$. Want: $d_X(s_j, q) < \varepsilon$).
We have $d_{\mathbb{N}^*}(j, \infty) = |(1/j) - 0| = |1/j| = 1/j$,
and so $d_{\mathbb{N}^*}(j, \infty) = 1/j$.
Then $1/j = d_{\mathbb{N}^*}(j, \infty) < \delta = 1/K$, and so $1/j < 1/K$.
Since $0 < 1/j < 1/K$, we get: $j > K$. Then $j \geqslant K$.
So, since $j \in \mathbb{D}_s = \mathbb{N}$, by choice of K, we get: $d(s_j, q) < \varepsilon$. □

2.18. The metric space $\mathbb{R}^*$.

DEFINITION 2.18.1. *Define $\alpha : \mathbb{R}^* \to [-1; 1]$ by:*
$$\forall x \in \mathbb{R}^*, \qquad \alpha_x = \begin{cases} -1, & \text{if } x = -\infty \\ x/\sqrt{1+x^2}, & \text{if } -\infty < x < \infty \\ 1, & \text{if } x = \infty. \end{cases}$$
Define $d^* : \mathbb{R}^* \times \mathbb{R}^* \to [0; \infty)$ by: $\forall x, y \in \mathbb{R}^*$, $d^*(x, y) = |\alpha_y - \alpha_x|$.

THEOREM 2.18.2.
$$\begin{aligned} & d^*(5,7) = (7/\sqrt{50}) - (5/\sqrt{26}) && \textit{and} \\ & d^*(3,\infty) = 1 - (3/\sqrt{10}) && \textit{and} \\ & d^*(-\infty, 6) = (6/\sqrt{37}) - (-1) && \\ & \qquad\qquad = (6/\sqrt{37}) + 1 && \textit{and} \\ & d^*(-\infty, \infty) = 1 - (-1) = 2. && \end{aligned}$$

THEOREM 2.18.3. *We have: $d^* \in \mathcal{M}(\mathbb{R}^*)$.*

We refer to d^* as the **compatible metric** on $\mathbb{R}^*$.
We are sometimes sloppy and use $\mathbb{R}^*$ to denote the metric space $(\mathbb{R}^*, d^*)$.
Note that $d_{\mathbb{R}^*} = d_{(\mathbb{R}^*,d^*)} = (\mathbb{R}^*, d^*)_2 = d^*$.

2.19. **The Intermediate Value Theorem.**

THEOREM 2.19.1. *Let X be a metric space.*
Let $f : X \dashrightarrow \mathbb{R}$, let $p \in X$ and let $a \in \mathbb{R}$.
Assume: (f is continuous at p from X to $\mathbb{R}$) & ($f_p > a$).
Then $\exists C \in \mathcal{B}_X(p)$ s.t. ($f > a$ on $C \cap \mathbb{D}_f$).

Proof. Let $\varepsilon := f_p - a$. Then $\varepsilon > 0$.
Since f is continuous at p from X to $\mathbb{R}$,
choose $\delta > 0$ s.t., $\forall w \in \mathbb{D}_f$, $(d(w, p) < \delta) \Rightarrow (d(f_w, f_p) < \varepsilon)$.
Let $C := B_X(p, \delta)$. Then $C \in \mathcal{B}_X(p)$. Want: $f > a$ on $C \cap \mathbb{D}_f$.
Want: $\forall w \in C \cap \mathbb{D}_f$, $f_w > a$. Given $w \in C \cap \mathbb{D}_f$. Want: $f_w > a$.
We have $w \in C \cap \mathbb{D}_f \subseteq C = B_X(p, \delta)$, so $d(w, p) < \delta$).
We have $w \in C \cap \mathbb{D}_f \subseteq \mathbb{D}_f$, so $w \in \mathbb{D}_f$.
Since $w \in \mathbb{D}_f$ and $d(w, p) < \delta$, by choice of δ, we get: $d(f_w, f_p) < \varepsilon$.
Then $f_p - \varepsilon < f_w < f_p + \varepsilon$, and so $f_w > f_p - \varepsilon$.
Since $\varepsilon = f_p - a$, we get $a = f_p - \varepsilon$. Then $f_w > f_p - \varepsilon = a$. □

THEOREM 2.19.2. *Let X be a metric space.*
Let $f : X \dashrightarrow \mathbb{R}$, let $p \in X$ and let $a \in \mathbb{R}$.
Assume: (f is continuous at p from X to $\mathbb{R}$) & ($f_p < a$).
Then $\exists C \in \mathcal{B}_X(p)$ s.t. ($f < a$ on $C \cap \mathbb{D}_f$).

Proof. Unassigned HW. □

THEOREM 2.19.3. *Let $a \in \mathbb{R}$ and $b > a$ and let $I := [a; b]$.*
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $y \in \mathbb{R}$.
Assume: f is continuous on I from $\mathbb{R}$ to $\mathbb{R}$ and $f_a < y < f_b$.
Let $S := \{ v \in I \mid f_v < y \}$ and let $x := \sup S$.
Then: $x \in I$ and $f_x \geqslant y$.

Proof. Since $a \in [a; b] = I$ and $f_a < y$, by definition of S, we get: $a \in S$.
Then $a \in S \leqslant \sup S$, and so $a \leqslant \sup S$.
Since $S \subseteq I = [a; b] \leqslant b$, we get $S \leqslant b$, and so $\sup S \leqslant b$.
Then: $a \leqslant \sup S \leqslant b$. Then: $\sup S \in [a; b]$.
By hypothesis, $x = \sup S$ and $I = [a; b]$. Then: $x \in I$.
It remains to show: $f_x \geqslant y$. Assume $f_x < y$. Want: Contradiction.
Since f is continuous on I from $\mathbb{R}$ to $\mathbb{R}$ and since $x \in I$,

we conclude: f is continuous at x from $\mathbb{R}$ to $\mathbb{R}$.
So, since $f_x < y$, by Theorem 2.19.2,
choose $C \in \mathcal{B}_{\mathbb{R}}(x)$ s.t. $(f < y$ on $C \cap \mathbb{D}_f)$.
Choose $\lambda > 0$ s.t. $C = B_{\mathbb{R}}(x, \lambda)$.
Recall that $f_x < y$. By hypothesis, we have $y < f_b$.
Then $f_x < y < f_b$, so $f_x < f_b$, so $f_x \neq f_b$, so $x \neq b$.
Since $x \in I$ and $x \neq b$, we get $x \in I_b^\times$.
Then $x \in I_b^\times = [a;b]_b^\times = [a;b)$, and so $a \leqslant x < b$.
Let $\mu := b - x$. Since $x < b$, we get $\mu > 0$.
Let $\delta := \min\{\lambda/2\,,\,\mu\}$. Then: $(\delta > 0)\,\&\,(\delta \leqslant \lambda/2)\,\&\,(\delta \leqslant \mu)$.
Let $w := x + \delta$. Then: $(w > x)\,\&\,(w \leqslant x + (\lambda/2))\,\&\,(w \leqslant x + \mu)$.
Since $w > x$, we get: $x < w$.
Since $\lambda > 0$, we get $x - \lambda < x$ and $x + (\lambda/2) < x + \lambda$.
Then $x - \lambda < x < w \leqslant x + (\lambda/2) < x + \lambda$,
and so $x - \lambda < w < x + \lambda$, and so $w \in (x - \lambda; x + \lambda)$.
Then $w \in (x - \lambda; x + \lambda) = B_{\mathbb{R}}(x, \lambda) = C$, and so $w \in C$.
Since $\mu = b - x$, we get: $x + \mu = b$. Recall: $x < w$.
We have $a \leqslant x < w \leqslant x + \mu = b$, so $a < w \leqslant b$.
Then $w \in (a;b] \subseteq [a;b] = I$, so $w \in I$.
By hypothesis, f is continuous on I from $\mathbb{R}$ to $\mathbb{R}$, so $I \subseteq \mathbb{D}_f$.
Then $w \in I \subseteq \mathbb{D}_f$, so, as $w \in C$, we get $w \in C \cap \mathbb{D}_f$.
So, since $f < y$ on $C \cap \mathbb{D}_f$, we get $f_w < y$.
So, since $w \in I$, by definition of S, we get $w \in S$. Recall: $x < w$.
Then $w \in S \leqslant \sup S = x < w$, so $w < w$. Contradiction. □

THEOREM 2.19.4. *Let $a \in \mathbb{R}$ and $b > a$ and let $I := [a;b]$.*
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $y \in \mathbb{R}$.
Assume: f is continuous on I from $\mathbb{R}$ to $\mathbb{R}$ and $f_a < y < f_b$.
Then: $\exists x \in I$ s.t. $f_x = y$.

Proof. Let $S := \{\, v \in I \,|\, f_v < y \,\}$ and let $x := \sup S$.
By Theorem 2.19.3, we have $x \in I$. Want: $f_x = y$.
By Theorem 2.19.3, we have $f_x \geqslant y$. Want: $f_x \leqslant y$.
Assume $f_x > y$. Want: Contradiction.
Since f is continuous on I from $\mathbb{R}$ to $\mathbb{R}$ and $x \in I$,
we conclude that f is continuous at x from $\mathbb{R}$ to $\mathbb{R}$.
So, since $f_x > y$, by Theorem 2.19.1,
choose $C \in \mathcal{B}_{\mathbb{R}}(x)$ s.t. $f > y$ on $C \cap \mathbb{D}_f$.
Choose $\lambda > 0$ s.t. $C = B_{\mathbb{R}}(x, \lambda)$.

Since $\lambda > 0$, we get $x - \lambda < x < x + \lambda$.
Since $\sup S = x > x - \lambda$, we get $\sup S > x - \lambda$.
Then $\neg(\sup S \leqslant x - \lambda)$, so $\neg(S \leqslant x - \lambda)$,
so choose $w \in S$ s.t. $w > x - \lambda$. Then $x - \lambda < w$.
Then $x - \lambda < w \in S \leqslant \sup S = x < x + \lambda$,
so $x - \lambda < w < x + \lambda$, so $w \in (x - \lambda; x + \lambda)$.
Then $w \in (x - \lambda; x + \lambda) = B_{\mathbb{R}}(x, \lambda) = C$, so $w \in C$.
By hypothesis, f is continuous on I from $\mathbb{R}$ to $\mathbb{R}$, so $I \subseteq \mathbb{D}_f$.
Then $w \in S \subseteq I \subseteq \mathbb{D}_f$, so, as $w \in C$, we get $w \in C \cap \mathbb{D}_f$.
So, since $f > y$ on $C \cap \mathbb{D}_f$, we get $f_w > y$.
Since $w \in S$, by definition of S, we get: $f_w < y$. Contradiction. □

The next theorem is our first version of the Intermediate Value Theorem, which we label **IVT#1**.

THEOREM 2.19.5. *Let* $a \in \mathbb{R}$ *and* $b > a$ *and let* $I := [a; b]$.
Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $y \in \mathbb{R}$.
Assume: f *is continuous on* I *from* $\mathbb{R}$ *to* $\mathbb{R}$ *and* $f_a < y < f_b$.
Then: $\exists x \in (a; b)$ *s.t.* $f_x = y$.

Proof. By Theorem 2.19.5, choose $x \in I$ s.t. $f_x = y$.
As $a < y < b$, we get $f_a < f_x < f_b$, so $f_a \neq f_x \neq f_b$, so $a \neq x \neq b$.
Then $x \in I^{\times}_{\{a,b\}} = [a; b]^{\times}_{\{a,b\}} = (a; b)$. Want: $f_x = y$.
By choice of x, we have $f_x = y$, as desired. □

The next theorem is our second version of the Intermediate Value Theorem, which we label **IVT#2**.

THEOREM 2.19.6. *Let* $a \in \mathbb{R}$, $b \geqslant a$ *and* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: f *is continuous on* $[a; b]$ *from* $\mathbb{R}$ *to* $\mathbb{R}$ *and* $f_a \leqslant f_b$.
Then: $[f_a; f_b] \subseteq f_*(\,[a; b]\,)$.

Proof. Want: $\forall y \in [f_a; f_b]$, $y \in f_*(\,[a; b]\,)$.
Given $y \in [f_a; f_b]$. Want: $y \in f_*(\,[a; b]\,)$.
By definition of $f_*(\,[a; b]\,)$, want: $\exists x \in [a; b] \bigcap \mathbb{D}_f$ s.t. $f_x = y$.
By hypothesis, f is continuous on $[a; b]$ from $\mathbb{R}$ to $\mathbb{R}$,
so $[a; b] \subseteq \mathbb{D}_f$, so $[a; b] \bigcap \mathbb{D}_f = [a; b]$.
Want: $\exists x \in [a; b]$ s.t. $f_x = y$.
Since $y \in [f_a; f_b]$, we get $f_a \leqslant y \leqslant f_b$.
Exactly one of the following is true:
(1) $y = f_a$ or (2) $y = f_b$ or (3) $f_a < y < f_b$.

Case 1:
Let $x := a$. Then $x \in [a;b]$. Want: $f_x = y$. Know: $y = f_a$.
We have $f_x = f_a = y$, as desired.
End of Case 1.

Case 2:
Let $x := b$. Then $x \in [a;b]$. Want: $f_x = y$. Know: $y = f_b$.
We have $f_x = f_b = y$, as desired.
End of Case 2.

Case 3:
By hypothesis, $b \geqslant a$. Let $I := [a;b]$. Want: $\exists x \in I$ s.t. $f_x = y$.
Since $f_a < y < f_b$, we get $f_a < f_b$, so $f_a \neq f_b$, so $a \neq b$.
Sicne $b \geqslant a$ and $a \neq b$, we see that $b > a$.
Then, by Theorem 2.19.4, we conclude: $\exists x \in I$ s.t. $f_x = y$.
End of Case 3. □

DEFINITION 2.19.7. *Let* $a, b \in \mathbb{R}^*$, $\alpha := \min\{a, b\}$, $\beta := \max\{a, b\}$.
Then $[a|b] := [\alpha;\beta]$ *and* $(a|b) := (\alpha;\beta)$.

THEOREM 2.19.8. *We have* $[8|6] = [6|8] = [6;8]$
and $[9|2] = [2|9] = [2;9]$
and $(3|5) = (5|3) = (3;5)$.

The next theorem is our third version of the Intermediate Value Theorem, which we label **IVT#3**.

THEOREM 2.19.9. *Let* $a \in \mathbb{R}$, $b \geqslant a$ *and* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: f *is continuous on* $[a;b]$ *from* $\mathbb{R}$ *to* $\mathbb{R}$. *Then:* $[f_a|f_b] \subseteq f_*(\,[a;b]\,)$.

Proof. At least one of the following is true:
(1) $f_a \leqslant f_b$ or (2) $b_a \geqslant f_b$.

Case 1:
By Theorem 2.19.6, $[f_a;f_b] \subseteq f_*(\,[a;b]\,)$.
Since $f_a \leqslant f_b$, we get: $[f_a|f_b] = [f_a;f_b]$.
Then $[f_a|f_b] = [f_a;f_b] \subseteq f_*(\,[a;b]\,)$, as desired.
End of Case 1.

Case 2: Let $g := -f$.
Then g is continuous on $[a;b]$ from $\mathbb{R}$ to $\mathbb{R}$ and $g_a \leqslant g_b$.
Then, by Theorem 2.19.6, $[g_a;g_b] \subseteq g_*([a;b])$.
Multiplying by -1, we get $-([g_a;g_b]) \subseteq -(g_*([a;b]))$.
Then $[-g_b;-g_a] \subseteq (-g_*)([a;b])$.
So, since $f_b = -g_b$ and $f_a = -g_a$ and $f = -g$,
we get $[f_b;f_a] \subseteq f_*([a;b])$.
Since $f_a \geqslant f_b$, we get: $[f_a|f_b] = [f_b;f_a]$.
Then $[f_a|f_b] = [f_b;f_a] \subseteq f_*([a;b])$, as desired.
End of Case 2. □

The next theorem is our final version of
the **Intermediate Value Theorem**.

THEOREM 2.19.10. *Let* $a, b \in \mathbb{R}$ *and* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: f *is continuous on* $[a|b]$ *from* $\mathbb{R}$ *to* $\mathbb{R}$.
Then: $[f_a|f_b] \subseteq f_*([a|b])$.

Proof. At least one of the following is true:
(1) $a \leqslant b$ or (2) $a \geqslant b$.

Case 1:
By Theorem 2.19.9, $[f_a|f_b] \subseteq f_*([a;b])$.
Since $a \leqslant b$, we get: $[a|b] = [a;b]$.
Then $[f_a|f_b] = f_*([a;b]) = f_*([a|b])$, as desired.
End of Case 1.

Case 2: Let $\alpha := b$ and $\beta := a$. Then $\alpha \leqslant \beta$.
By Theorem 2.19.9, $[f_\alpha|f_\beta] \subseteq f_*([\alpha;\beta])$.
Since $a \geqslant b$, we see that $[a|b] = [b;a]$.
Then $[a|b] = [\alpha;\beta]$. Then $f([a|b]) = f([\alpha;\beta])$.
Also, $[f_\alpha|f_\beta] = [f_\beta|f_\alpha] = [f_a|f_b]$.
Then $[f_a|f_b] = [f_\alpha|f_\beta] \subseteq f_*([\alpha;\beta]) = f_*([a|b])$, as desired.
End of Case 2. □

3. Compactness and the Extreme Value Theorem

3.1. Increasing and decreasing.

DEFINITION 3.1.1. *Let* $f : \mathbb{R}^* \dashrightarrow \mathbb{R}^*$, $S \subseteq \mathbb{D}_f$.
By f *is* **strictly-increasing** *on* S, *we mean:*

$$\forall w, x \in S, \quad (w < x) \Rightarrow (f_w < f_x).$$

By f is **strictly-decreasing** *on S, we mean:*

$$\forall w, x \in S, \quad (w < x) \Rightarrow (f_w > f_x).$$

By f is **semi-increasing** *on S, we mean:*

$$\forall w, x \in S, \quad (w \leqslant x) \Rightarrow (f_w \leqslant f_x).$$

By f is **semi-decreasing** *on S, we mean:*

$$\forall w, x \in S, \quad (w \leqslant x) \Rightarrow (f_w \geqslant f_x).$$

DEFINITION 3.1.2. *Let $f : \mathbb{R}^* \dashrightarrow \mathbb{R}^*$.*
By f is **strictly-increasing***, we mean: f is strictly-increasing on $\mathbb{D}_f$.*
By f is **strictly-decreasing***, we mean: f is strictly-decreasing on $\mathbb{D}_f$.*
By f is **semi-increasing***, we mean: f is semi-increasing on $\mathbb{D}_f$.*
By f is **semi-decreasing***, we mean: f is semi-decreasing on $\mathbb{D}_f$.*

We discussed increasing/decreasing and secant slopes.

THEOREM 3.1.3. *Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = x^2$.*
Then f is strictly-decreasing on $(-\infty; 0]$ and
f is strictly-increasing on $[0; \infty)$.

THEOREM 3.1.4. $(1, 1/2, 1/3, \ldots)$ *is strictly-decreasing and*
$(1, 1, 2, 2, 3, 3, 4, 4, \ldots)$ *is semi-increasing.*

THEOREM 3.1.5. *Let $s \in \mathbb{R}^{\mathbb{N}}$.*
Assume: $\forall j \in \mathbb{N}, \quad s_j < s_{j+1}$.
Then: s is strictly-increasing.

Proof. Since s is a sequence, we get: $\mathbb{D}_s = \mathbb{N}$.
Want: s is strictly-increasing on $\mathbb{D}_s$.
Want: s is strictly-increasing on $\mathbb{N}$.
Want: $\forall j, k \in \mathbb{N}, \quad (j < k) \Rightarrow (s_j < s_k)$.
Given $j, k \in \mathbb{N}$. Want: $(j < k) \Rightarrow (s_j < s_k)$.
Assume $j < k$. Want: $s_j < s_k$.
Since $j, k \in \mathbb{Z}$, we get $k - j \in \mathbb{Z}$. Also, since $j < k$, we get $k - j > 0$.
Let $\ell := k - j$. Then $\ell \in \mathbb{Z}$ and $\ell > 0$. Then $\ell \in \mathbb{N}$.
Also, $k = j + \ell$. Want: $s_j < s_{j+\ell}$.
Want: $\forall m \in \mathbb{N}, \quad s_j < s_{j+m}$.
Let $T := \{m \in \mathbb{N} \,|\, s_j < s_{j+m}\}$. Want $T = \mathbb{N}$.
By assumption, $\forall i \in \mathbb{N}$, $s_i < s_{i+1}$. Then $s_j < s_{j+1}$.
Then $1 \in T$. By the PMI, want: $\forall m \in \mathbb{N}$, $m + 1 \in \mathbb{N}$.
Given $m \in \mathbb{N}$. Want: $m + 1 \in \mathbb{N}$.

Since $m \in \mathbb{N}$, we get $s_j < s_{j+m}$. Want: $s_j < s_{j+m+1}$.
By assumption, $\forall i \in \mathbb{N}$, $s_i < s_{i+1}$. Then $s_{j+m} < s_{j+m+1}$.
Then $s_j < s_{j+m} < s_{j+m+1}$, as desired. □

There are three more theorems that are similar to the last:

THEOREM 3.1.6. *Let* $s \in \mathbb{R}^{\mathbb{N}}$.
Assume: $\forall j \in \mathbb{N}, \quad s_j > s_{j+1}$.
Then: s is strictly-decreasing.

Proof. Unassigned HW □

THEOREM 3.1.7. *Let* $s \in \mathbb{R}^{\mathbb{N}}$.
Assume: $\forall j \in \mathbb{N}, \quad s_j \leqslant s_{j+1}$.
Then: s is semi-increasing.

Proof. Unassigned HW □

THEOREM 3.1.8. *Let* $s \in \mathbb{R}^{\mathbb{N}}$.
Assume: $\forall j \in \mathbb{N}, \quad s_j \geqslant s_{j+1}$.
Then: s is semi-decreasing.

Proof. Unassigned HW □

THEOREM 3.1.9. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: f and g are both strictly-increasing.
Then: $g \circ f$ is strictly-increasing.

Proof. Unassigned HW. □

3.2. **Subsequences.**

DEFINITION 3.2.1. *Let s and t be sequences.*
By t is a **subsequence** *of s, we mean:*
$\exists$*strictly-increasing* $\ell \in \mathbb{N}^{\mathbb{N}}$ *s.t.* $t = s \circ \ell$.

THEOREM 3.2.2. *Let* $s := (2, 4, 6, 8, \ldots)$, $t := (2^2, 4^2, 6^2, 8^2, \ldots)$,
$u := (1, 2, 3, 4, \ldots)$, $v := (4, 2, 8, 6, 12, 10, 16, 14, 20, 18, \ldots)$.
Then t is a subsequence of s and s is a subsequence of u and
u is NOT a subsequence of s and
v is NOT a subsequence of u.

THEOREM 3.2.3. *Let s, t, u be sequences.*
Assume: (u is a subsequence of t) & (t is a subsequence of s).
Then: u is a subsequence of s.

Proof. Choose strictly-increasing $\ell, m \in \mathbb{N}^{\mathbb{N}}$ s.t. $u = t \circ \ell$ and $t = s \circ m$.
Then $m \circ \ell \in \mathbb{N}^{\mathbb{N}}$. Also, by Theorem 3.1.9, $m \circ \ell$ is strictly-increasing.
It therefore suffices to show: $u = s \circ (m \circ \ell)$.
We have $u = t \circ \ell = (s \circ m) \circ \ell = s \circ (m \circ \ell)$, as desired. □

DEFINITION 3.2.4. *Let X be a metric space and let $s \in X^{\mathbb{N}}$.*
By s is **convergent** *in X, we mean: $\exists p \in X$ s.t. $s \to p$ in X.*

DEFINITION 3.2.5. *Let X be a metric space and let $s \in X^{\mathbb{N}}$.*
By s is **subconvergent** *in X, we mean:*
$\exists$subsequence t of s s.t. t is convergent in X.

THEOREM 3.2.6. *Let $s := (-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, \ldots)$.*
Then s is subconvergent in $\mathbb{R}$.

Proof. Want: $\exists$subsequence t of s s.t. t is convergent in $\mathbb{R}$.
Let $\ell := (2, 4, 6, 8, \ldots)$. Then $\ell \in \mathbb{N}^{\mathbb{N}}$ and ℓ is strictly-increasing.
Let $t := s \circ \ell$. Then t is a subsequence of s.
Want: t is convergent in $\mathbb{R}$. Want: $\exists p \in \mathbb{R}$ s.t. $t \to p$ in $\mathbb{R}$.
Let $p := 1$. Then $p \in \mathbb{R}$. Want: $t \to p$ in $\mathbb{R}$.
We have: $\forall j \in \mathbb{N}$, $t_j = (s \circ \ell)_j = s_{\ell_j} = s_{2j} = 1$.
Then $t = C_{\mathbb{N}}^1$. Then $t \to 1$ in $\mathbb{R}$. Then $t \to p$ in $\mathbb{R}$. □

THEOREM 3.2.7. *Let $s := (2, 4, 6, 8, \ldots)$.*
Then s is not subconvergent in $\mathbb{R}$.

Proof. Assume s is subconvergent in $\mathbb{R}$. Want: Contradiction.
Choose a subsequence t of s s.t. t is convergent in $\mathbb{R}$.
Chose $p \in \mathbb{R}$ s.t. $t \to p$ in $\mathbb{R}$.
Choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
Since $t \to p$ in $\mathbb{R}$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(j \geqslant K) \Rightarrow (d(t_j, p) < 1)$.
By the Archimedean Principle,
choose $j \in \mathbb{N}$ s.t. $j \geqslant \max\{ (p+1)/2 , K \}$.
Then $j \geqslant (p+1)/2$ and $j \geqslant K$.
Since $j \in \mathbb{N}$ and $j \geqslant K$, by choice of K, we get: $d(t_j, p) < 1$.
Then $p - 1 < t_j < p + 1$. Then $p + 1 > t_j$.
We have $t_j = (s \circ \ell)_j = s_{\ell_j}$, so $t_j = s_{\ell_j}$.
By definition of s, we know: $\forall k \in \mathbb{N}$, $s_k = 2k$. Then $s_{\ell_j} = 2\ell_j$.
By HW#8-3, $\ell_j \geqslant j$. Then $2\ell_j \geqslant 2j$.
Since $j \geqslant (p+1)/2$, we get $2j \geqslant p + 1$.

Then $p+1 > t_j = s_{\ell_j} = 2\ell_j \geqslant 2j \geqslant p+1$,
so $p+1 > p+1$. Contradiction. □

3.3. Product metrics and product metric spaces.

THEOREM 3.3.1. *Let X and Y be metric spaces, $Z := X \times Y$. Define $d : Z \times Z \to [0;\infty)$ by $\forall p, q \in \mathbb{Z}$,*

$$d(p,q) \quad = \quad \sqrt{\,(d_X(p_1,q_1))^2 \;+\; (d_Y(p_2,q_2))^2\,}\,.$$

Then $d \in \mathcal{M}(Z)$.

DEFINITION 3.3.2. *Let X and Y be metric spaces, $Z := X \times Y$. Define $d : Z \times Z \to [0;\infty)$ by $\forall p, q \in \mathbb{Z}$,*

$$d(p,q) \quad = \quad \sqrt{\,(d_X(p_1,q_1))^2 \;+\; (d_Y(p_2,q_2))^2\,}\,.$$

Then $X \;\times_{MS}\; Y \quad := \quad (X \times Y, d)$.

We are often sloppy and write $X \times Y$ for $X \;\times_{MS}\; Y$.

3.4. Stereographic projection.

THEOREM 3.4.1. *Let $V := \mathbb{R}^2$, $C := S_V(0_V, 1)$. Let $p := (1,0)$, $q \in C_p^\times$, $t \in \mathbb{R}$. Assume: p and q and $(0,t)$ collinear (i.e., on a line). Then:*

$$t \;=\; \frac{q_2}{1-q_1} \qquad \text{and} \qquad q \;=\; \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right).$$

The preceding is a precalculus theorem that we proved in detail in class. It describes the importance of the "stereographic projection" maps given by:

$$q \;\mapsto\; \frac{q_2}{1-q_1} \quad : \quad C_p^\times \to \mathbb{R}$$

$$\text{and} \qquad t \;\mapsto\; \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right) \quad : \quad \mathbb{R} \to C_p^\times.$$

These two functions are called f and g in Theorem 3.7.12, below.

3.5. Two basic facts.

THEOREM 3.5.1. *Let $a, b \geqslant 0$. Then $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$.*

Proof. Let $s := \sqrt{a}$ and $t := \sqrt{b}$. Want: $\sqrt{a+b} \leqslant s+t$.
Since $s^2 = a$ and $t^2 = b$, we get $s^2 + t^2 = a + b$.
We have $s, t \geqslant 0$, so $2st \geqslant 0$. Then $s^2 + t^2 \leqslant s^2 + 2st + t^2$.
So, since $s^2 + t^2 = a + b$ and $s^2 + 2st + t^2 = (s+t)^2$,
we see that $a + b \leqslant (s+t)^2$.
Since $a, b \geqslant 0$, we get $a + b \geqslant 0$, so $0 \leqslant a + b$.
Since $0 \leqslant a + b \leqslant (s+t)^2$, we see that $\sqrt{a+b} \leqslant s + t$, as desired. □

THEOREM 3.5.2. *Define* $\alpha : \mathbb{R}^* \to [-1;1]$ *by:* $\forall x \in \mathbb{R}^*$,

$$\alpha_x \quad = \quad \begin{cases} -1, & \textit{if } x = -\infty \\ x/\sqrt{1+x^2}, & \textit{if } -\infty < x < \infty \\ 1, & \textit{if } x = \infty. \end{cases}$$

Define $\beta : [-1;1] \to \mathbb{R}^*$ *by:* $\forall y \in [-1;1]$,

$$\beta_y \quad = \quad \begin{cases} -\infty, & \textit{if } y = -1 \\ y/\sqrt{1-y^2}, & \textit{if } -1 < y < 1 \\ \infty, & \textit{if } y = 1. \end{cases}$$

Then: α *and* β *are both strictly-increasing* *and*
$\alpha \circ \beta = \mathrm{id}_{[-1;1]}$ *and* $\beta \circ \alpha = \mathrm{id}_{\mathbb{R}^*}$.
Also: $\alpha : \mathbb{R}^* \hookrightarrow\!\!> [-1;1]$ *and* $\beta : [-1;1] \hookrightarrow\!\!> \mathbb{R}^*$ *and*
$\alpha^{-1} = \beta$ *and* $\beta^{-1} = \alpha$.

Proof. Unassigned HW. □

3.6. **Limits involving infinite quantities.**

THEOREM 3.6.1. $\forall N \in \mathbb{R}$, $\exists \delta > 0$ *s.t.*, $\forall x \in \mathbb{R}$,

$$(\ d^*(x, -\infty) < \delta\ ) \Rightarrow (\ x < N\ ).$$

Proof. Given $N \in \mathbb{R}$. Want:$\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,

$$(\ d^*(x, -\infty) < \delta\ ) \Rightarrow (\ x < N\ ).$$

Define $\alpha : \mathbb{R}^* \to [-1;1]$ by: $\forall x \in \mathbb{R}^*$,

$$\alpha_x \quad = \quad \begin{cases} -1, & \text{if } x = -\infty \\ x/\sqrt{1+x^2}, & \text{if } -\infty < x < \infty \\ 1, & \text{if } x = \infty. \end{cases}$$

Define $\beta : [-1;1] \to \mathbb{R}^*$ by: $\forall y \in [-1;1]$,

$$\beta_y \quad = \quad \begin{cases} -\infty, & \text{if } y = -1 \\ y/\sqrt{1-y^2}, & \text{if } -1 < y < 1 \\ \infty, & \text{if } y = 1. \end{cases}$$

Since $N > -\infty$ and α is strictly-increasing, we get: $\alpha_N > \alpha_{-\infty}$.
Let $\delta := 1 + \alpha_N$. Then $\delta > 1 + \alpha_{-\infty} = 1 + (-1) = 0$, so $\delta > 0$.
Want: $\forall x \in \mathbb{R}$, $(\ d^*(x, -\infty) < \delta\ ) \Rightarrow (\ x < N\ )$.
Given $x \in \mathbb{R}$. Want: $(\ d^*(x, -\infty) < \delta\ ) \Rightarrow (\ x < N\ )$.
Assume: $d^*(x, -\infty) < \delta$. Want: $x < N$.
Since $x > -\infty$ and α is strictly-increasing, we get: $\alpha_x > \alpha_{-\infty}$.
Then $\alpha_x - \alpha_{-\infty} > 0$, and so $|\alpha_x - \alpha_{-\infty}| = \alpha_x - \alpha_{-\infty}$.
Then $\alpha_x - \alpha_{-\infty} = |\alpha_x - \alpha_{-\infty}| = d^*(x, -\infty) < \delta$.

Then $\alpha_x < \delta + \alpha_{-\infty} = \delta + (-1) = (1 + \alpha_N) + (-1) = \alpha_N$,
so $\alpha_x < \alpha_N$.
So, since β is strictly-increasing, $\beta_{\alpha_x} < \beta_{\alpha_N}$.
Then $x = (\beta \circ \alpha)_x = \beta_{\alpha_x} < \beta_{\alpha_N} = (\beta \circ \alpha)_N = N$, as desired. □

THEOREM 3.6.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{R}$ *s.t.*, $\forall x \in \mathbb{D}_f$,
$$(\ x < N\ ) \ \Rightarrow\ (\ f_x > M\ ).$$
Then $f \to \infty$ *near* $-\infty$ *from* $(\mathbb{R}^*, d^*)$ *to* $(\mathbb{R}^*, d^*)$.

Proof. Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t, $\forall x \in \mathbb{D}_f$,
$$(\ d^*(x, -\infty) < \delta\ ) \ \Rightarrow\ (\ d^*(f_x, \infty) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t, $\forall x \in \mathbb{D}_f$,
$$(\ d^*(x, -\infty) < \delta\ ) \ \Rightarrow\ (\ d^*(f_x, \infty) < \varepsilon\ ).$$
By HW#9-4, choose $M \in \mathbb{R}$ s.t., $\forall y \in \mathbb{R}$,
$$(\ y > M\ ) \ \Rightarrow\ (\ d^*(y, \infty) < \delta\ ).$$
By hypothesis, choose $N \in \mathbb{R}$ s.t., $\forall x \in \mathbb{D}_f$,
$$(\ x < N\ ) \ \Rightarrow\ (\ f_x > M\ ).$$
By Theorem 3.6.1, choose $\delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\ d^*(x, -\infty) < \delta\ ) \ \Rightarrow\ (\ x < N\ ).$$
Want: $\forall x \in \mathbb{D}_f$, $(\ d^*(x, -\infty) < \delta\ ) \ \Rightarrow\ (\ d^*(f_x, \infty) < \varepsilon\ )$.
Given $x \in \mathbb{D}_f$. Want: $(\ d^*(x, -\infty) < \delta\ ) \ \Rightarrow\ (\ d^*(f_x, \infty) < \varepsilon\ )$.
Assume: $d^*(x, -\infty) < \delta$. Want: $d^*(f_x, \infty) < \varepsilon$.
By hypothesis, $f : \mathbb{R} \dashrightarrow \mathbb{R}$, so $\mathbb{D}_f \subseteq \mathbb{R}$ and $\mathbb{I}_f \subseteq \mathbb{R}$.
Since $x \in \mathbb{D}_f \subseteq \mathbb{R}$ and $d^*(x, -\infty) < \delta$, by choice of δ, we get: $x < N$.
Since $x \in \mathbb{D}_f$ and $x < N$, by choice of N, we get $f_x > M$.
Let $y := f_x$. Then $y > M$.
Since $x \in \mathbb{D}_f$, it follows that $f_x \in \mathbb{I}_f$.
Since $y = f_x \in \mathbb{I}_f \subseteq \mathbb{R}$ and $y > M$, by choice of M, we get $d^*(y, \infty) < \delta$.
Then $d^*(f_x, \infty) = d^*(y, \infty) < \varepsilon$, as desired. □

The converse of Theorem 3.6.2 is also true:

THEOREM 3.6.3. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: $f \to \infty$ *near* $-\infty$ *from* $(\mathbb{R}^*, d^*)$ *to* $(\mathbb{R}^*, d^*)$.
Then: $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{R}$ *s.t.*, $\forall x \in \mathbb{D}_f$,
$$(\ x < N\ ) \ \Rightarrow\ (\ f_x > M\ ).$$

Proof. Unassigned HW. □

The preceding two theorems concern limits of ∞ near $-\infty$.
There are also two theorems concerning limits of $-\infty$ near $-\infty$.

There are also two theorems concerning limits of ∞ near ∞.
There are also two theorems concerning limits of $-\infty$ near ∞.
We leave it to the reader to forumlate and prove all these theorems.

3.7. **Isometries and homeomorphisms.**

DEFINITION 3.7.1. *Let X and Y be metric spaces, f an object. By f is an* **isometry** *from X to Y, we mean:*

$$f : X \hookrightarrow> Y \qquad \textit{and}$$
$$\forall p, q \in X,\ d_Y(f_p, f_q) = d_X(p, q) \qquad \textit{and}$$
$$\forall p, q \in Y,\ d_X(f_p^{-1}, f_q^{-1}) = d_X(p, q).$$

THEOREM 3.7.2. *Let X and Y be metric spaces, $f : X \to> Y$. Assume: $\forall p, q \in X,\ d_Y(f_p, f_q) = d_X(p, q)$. Then f is an isometry from X to Y.*

Proof. *Claim: f is 1-1.*

Proof of Claim: Want: $\forall p, q \in X,\ (\ f_p = f_q\ ) \Rightarrow (\ p = q\ )$.
Given $p, q \in X$. Want: $(\ f_p = f_q\ ) \Rightarrow (\ p = q\ )$.
Assume $f_p = f_q$. Want: $p = q$.
Since $f_p = f_q$, we get $d_Y(f_p, f_q) = 0$.
By hypothesis, $d_Y(f_p, f_q) = d_X(p, q)$.
Then $d_X(p, q) = 0$. Then $p = q$, as desired.
End of proof of Claim.

By the Claim f is 1-1. By hypothesis, $f : X \to> Y$.
Then $f : X \hookrightarrow> Y$.
By hypothesis, $\forall p, q \in X,\ d_Y(f_p, f_q) = d_X(p, q)$.
It remains to show: $\forall p, q \in Y,\ d_X(f_p^{-1}, f_q^{-1}) = d_Y(p, q)$.
Given $p, q \in Y$. Want: $d_X(f_p^{-1}, f_q^{-1}) = d_Y(p, q)$.
Let $s := f_p^{-1}$ and $t := f_q^{-1}$. Then $f_s = p$ and $f_q = t$.
Since $s, t \in X$, by hypothesis, we get $d_Y(f_s, f_t) = d_X(s, t)$.
Then $d_X(f_p^{-1}, f_q^{-1}) = d_X(s, t) = d_Y(f_s, f_t) = d_Y(p, q)$. □

DEFINITION 3.7.3. *Let X and Y be metric spaces. By X and Y are* **isometric***, we mean: $\exists f$ s.t. (f is an isometry from X to Y).*

THEOREM 3.7.4. *$(\mathbb{R}^*, d^*)$ and $[-1; 1]$ are isometric.*

Proof. Want: $\exists \alpha$ s.t. (α is an isometry from $(\mathbb{R}^*, d^*)$ to $[-1;1]$).
Define $\alpha : \mathbb{R}^* \to [-1;1]$ by:

$$\forall x \in \mathbb{R}^*, \qquad \alpha_x = \begin{cases} -1, & \text{if } x = -\infty \\ x/\sqrt{1+x^2}, & \text{if } -\infty < x < \infty \\ 1, & \text{if } x = \infty. \end{cases}$$

Want: α is an isometry from $(\mathbb{R}^*, d^*)$ to $[-1;1]$.
Unassigned HW: Show α is an isometry from $(\mathbb{R}^*, d^*)$ to $[-1;1]$. □

DEFINITION 3.7.5. *Let X and Y be metric spaces, f an object. By f is a* **homeomorphism** *from X to Y, we mean:*
$f : X \hookrightarrow\!\!> Y$ *and*
f *is continuous from* X *to* Y *and*
f^{-1} *is continuous from* Y *to* X.

THEOREM 3.7.6. *Let* $X := [1;2) \bigcup (3;4]$ *and* $Y := [7;9]$.
Define $f : X \to Y$ *by:* $\forall w \in X,\ f_w = \begin{cases} w+6, & \text{if } w < 2 \\ w+5, & \text{if } w > 2. \end{cases}$
Then $f : X \hookrightarrow\!\!> Y$ *and*
f *is continuous from* X *to* Y *and*
f^{-1} *is not continuous at* 8 *from* Y *to* X.

The idea of the preceding theorem is expressed by:
"attaching is continuous, but tearing apart is not".
The function f attaches $[1;2)$ to $[3;4]$ to make $[7;9]$.
The function f^{-1} tears apart $[7;9]$ to make $[1;2)$ and $[3;4]$.

DEFINITION 3.7.7. *Let X and Y be metric spaces.*
By X and Y are **homeomorphic**, *we mean:*
$\exists f$ *s.t.* (f *is an homeomorphism from* X *to* Y).

THEOREM 3.7.8. *Let S and T be sets.*
Let $f : S \to T$ *and* $g : T \to S$.
Assume: ($g \circ f = \mathrm{id}_S$) & ($f \circ g = \mathrm{id}_T$).
Then: ($f : S \hookrightarrow\!\!> T$) & ($g : T \hookrightarrow\!\!> S$)
& ($f^{-1} = g$) & ($g^{-1} = f$).

THEOREM 3.7.9. *Define* $s : \mathbb{R} \dashrightarrow \mathbb{R}$ *by* $s_x = \sqrt{x}$.
Then: $\mathbb{D}_s = [0;\infty) = \mathbb{I}_s$
and s *is uniformly continuous from* $\mathbb{R}$ *to* $\mathbb{R}$
and s *is not Lipschitz from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. Unassigned HW.
Hint for uniformly continuity: Let $\delta := \varepsilon^2$. □

THEOREM 3.7.10. *Define* $s : \mathbb{R} \dashrightarrow \mathbb{R}$ *by* $s_x = \sqrt{x}$.
Then: s *is continuous from* $\mathbb{R}$ *to* $\mathbb{R}$
and s *is continuous from* $[0;\infty)$ *to* $[0;\infty)$.

Proof. By Theorem 3.7.9, s is uniformly continuous from $\mathbb{R}$ to $\mathbb{R}$.
It follows that s is continuous from $\mathbb{R}$ to $\mathbb{R}$.
Want: s is continuous from $[0;\infty)$ to $[0;\infty)$.
By Theorem 3.7.9, we have: $\mathbb{D}_s = [0;\infty) = \mathbb{I}_s$.
Then $s : [0;\infty) \twoheadrightarrow [0;\infty)$, and so $s : [0;\infty) \dashrightarrow [0;\infty)$.
Then, by Absoluteness of Continuity,
we get: s is continuous $[0;\infty)$ to $[0;\infty)$, as desired. □

THEOREM 3.7.11. $(-1;1)$ *and* $\mathbb{R}$ *are homeomorphic.*

Proof. Want: $\exists f$ s.t. f is a homeomorphism from $(-1;1)$ onto $\mathbb{R}$.
Define $f : (-1;1) \to \mathbb{R}$ by: $\forall x \in (-1;1)$, $f_x = \dfrac{x}{\sqrt{1-x^2}}$.
Want: f is a homeomorphism from $(-1;1)$ onto $\mathbb{R}$.
Want: $(\, f : (-1;1) \hookrightarrow\!\!\!> \mathbb{R}\,)$ &
$(\, f$ is continuous from $(-1;1)$ to $\mathbb{R}\,)$ &
$(\, f^{-1}$ is continuous from $\mathbb{R}$ to $(-1;1)\,)$.
Define $g : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $g_x = \dfrac{x}{\sqrt{1+x^2}}$.
Then: $(\, \mathbb{I}_g \subseteq (-1;1)\,)$ & $(\, g \circ f = \mathrm{id}_{(-1;1)}\,)$ & $(\, f \circ g = \mathrm{id}_{\mathbb{R}}\,)$.
It follows that: $f : (-1;1) \hookrightarrow\!\!\!> \mathbb{R}$ and $g = f^{-1}$.
Want: $(\, f$ is continuous from $(-1;1)$ to $\mathbb{R}\,)$ &
$(\, g$ is continuous from $\mathbb{R}$ to $(-1;1)\,)$.
By properties of limits, we conclude that:
$(\, f$ is continuous from $\mathbb{R}$ to $\mathbb{R}\,)$ &
$(\, g$ is continuous from $\mathbb{R}$ to $\mathbb{R}\,)$.
Then, by Absoluteness of Continuity, we have:
$(\, f$ is continuous from $(-1;1)$ to $\mathbb{R}\,)$ &
$(\, g$ is continuous from $\mathbb{R}$ to $(-1;1)\,)$,
as desired. □

THEOREM 3.7.12. *Let* $V := \mathbb{R}^2$, $C := S_V(0_V, 1)$, $p := (1,0)$.
Then $C_p^\times$ *and* $\mathbb{R}$ *are homeomorphic.*

Proof. Want: $\exists f$ s.t. f is a homeomorphism from $C_p^\times$ onto $\mathbb{R}$.
Define $\phi : V \dashrightarrow \mathbb{R}$ by: $\forall q \in V$, $\phi_q = \dfrac{q_2}{1 - q_1}$.
By properties of continuity, ϕ is continuous from V to $\mathbb{R}$.
Also, $C_p^\times \subseteq \{q \in V \mid q_1 < 1\} \subseteq \mathbb{D}_\phi$. Let $f := \phi | C_p^\times$.
Want: f is a homeomorphism from $C_p^\times$ onto $\mathbb{R}$.
Since ϕ is continuous from V to $\mathbb{R}$,
it follows that f is continuous from V to $\mathbb{R}$.
Then, by Absoluteness of Continuity, f is continuous from $C_p^\times$ to $\mathbb{R}$.
Want: $f : C_p^\times \hookrightarrow\!\!> \mathbb{R}$ and f^{-1} is continuous from $\mathbb{R}$ to $C_p^\times$.
Define $\lambda, \mu : \mathbb{R} \to \mathbb{R}$ by: $\forall t \in \mathbb{R}$, $\lambda_t = \dfrac{t^2 - 1}{t^2 + 1}$ and $\mu_t = \dfrac{2t}{t^2 + 1}$.
By properties of continuiy, λ and μ are both continuous from $\mathbb{R}$ to $\mathbb{R}$,
and so (λ, μ) is continuous from $\mathbb{R}$ to $\mathbb{R}^2$.
Let $g := (\lambda, \mu)$. Recall: $V = \mathbb{R}^2$. Then g is continuous from $\mathbb{R}$ to V.
Unassigned HW: $\forall t \in \mathbb{R}$, $g_t \in C_p^\times$ and $f_{g_t} = t$
and $\forall q \in C_p^\times$, $g_{f_q} = q$.
Then $\mathbb{I}_g \subseteq C_p^\times$ and $f \circ g = \mathrm{id}_{\mathbb{R}}$ and $g \circ f = \mathrm{id}_{C_p^\times}$.
Then $f : C_p^\times \hookrightarrow\!\!> \mathbb{R}$ and $f^{-1} = g$.
Want: f^{-1} is continuous from $\mathbb{R}$ to $C_p^\times$.
Since g is continuous from $\mathbb{R}$ to V and $\mathbb{I}_g \subseteq C_p^\times$,
by Absoluteness of Continuity, g is continuous from $\mathbb{R}$ to $C_p^\times$.
So, since $f^{-1} = g$, we get: f^{-1} is continuous from $\mathbb{R}$ to $C_p^\times$. □

3.8. **Boundedness and compactness.**

NOTE FOR NEXT YEAR: Define S is a-wide to mean: $\forall p, q \in S$, $d(p, q) < a$. Then bounded means finitely wide, *i.e.*, $\exists a \in \mathbb{R}$ s.t. S is a-wide. Also, Cauchy means: $\forall \varepsilon > 0$, $\exists \varepsilon$-wide tail. Just use bounded on metric spaces, no "bounded in . . . ".

DEFINITION 3.8.1. *Let X be a metric space and let $S \subseteq X$.*
By S is **bounded** *in X, we mean:*
$(\exists B \in \mathcal{B}_X \text{ s.t. } S \subseteq B) \vee (S = X = \varnothing)$.

THEOREM 3.8.2.
$[1; 2)$ *is bounded in* $\mathbb{R}$ *and* $(-\infty; 0)$ *is NOT bounded in* $\mathbb{R}$.

THEOREM 3.8.3. *Let $V := \mathbb{R}^2$, $C := S_V(0_v, 1)$, $p := (0, 1)$.*
Then C and $C_p^\times$ are both bounded in $\mathbb{R}^2$.

THEOREM 3.8.4. $\mathbb{R} \times \{0\}$ *is NOT bounded in* $\mathbb{R}^2$.

THEOREM 3.8.5. *Let X be a metric space.*
Then $\varnothing$ is bounded in X.

DEFINITION 3.8.6. *Let X be a metric space.*
By X is **bounded**, *we mean:*
X is bounded in X.

THEOREM 3.8.7.
$[1;2)$ *is bounded* *and* $(-\infty;0)$ *is NOT bounded.*

THEOREM 3.8.8. *Let $V := \mathbb{R}^2$, $C := S_V(0_v, 1)$, $p := (0,1)$.*
Then C and $C_p^\times$ are both bounded.

THEOREM 3.8.9. $\mathbb{R} \times \{0\}$ *is NOT bounded.*

THEOREM 3.8.10. $\varnothing$ *is bounded.*

THEOREM 3.8.11. *Both $(\mathbb{N}^*, d_*)$ and $(\mathbb{R}^*, d^*)$ are bounded.*

We have shown that
the bounded metric space $(-1;1)$
is homeomorphic to
the unbounded metric space $\mathbb{R}$.

Let $V := \mathbb{R}^2$, $C := S_V(0_V, 1)$, $p := (0,1)$.
We have shown that
the bounded metric space $C_p^\times$
is homeomorphic to
the unbounded metric space $\mathbb{R}$.
Moreover, $\mathbb{R}$ is isometric to $\mathbb{R} \times \{0\}$,
so $\mathbb{R}$ is homeomorhpic to $\mathbb{R} \times \{0\}$.
We conclude that
the bounded metric space $C_p^\times$
is homeomorphic to
the unbounded metric space $\mathbb{R} \times \{0\}$.

Based on these observations, one might think that
boundedness is not a useful concept to topology.
And yet, it is possible for a metric spaces X to be SO bounded that
any metric space homeomorphic to X is bounded.
Let's say that a metric space X is "super-bounded" if:

any metric space homeomorphic to X is bounded.
We will show: C is super-bounded, although $C_p^\times$ is not.

DEFINITION 3.8.12. *Let X be a metric space.*
By X is **compact***, we mean: $\forall s \in X^{\mathbb{N}}$, s is subconvergent in X.*

Let $V := \mathbb{R}^2$, $C := S_V(0_V, 1)$, $p := (0, 1)$.
We will show: C is compact, although $C_p^\times$ is not. We will also show:
that any metric space
that is homeomorphic to a compact metric space
is compact.
In particular, any metric space homeomorphic to C is compact.
We will also show:
any compact metric space is bounded.
Then any metric space homeomorphic to C is bounded.
That is, C is super-bounded.

In fact, any compact metric space is super-bounded,
and it's possible to show the converse:
any super-bounded metric space is compact,
although that's beyond our scope.

3.9. **Basic properties of compactness.**

THEOREM 3.9.1. *Let X be a compact metric space.*
Then X is bounded.

Proof. Assume X is unboudned. Want: Contradiction.
Since $\varnothing$ is bounded, we get: $X \neq \varnothing$.
Choose $p \in X$. $\forall j \in \mathbb{N}$, let $B_j := B_X(p, j)$.
Then, $\forall j \in \mathbb{N}$, we have: $B_j \in \mathcal{B}_X$.
So, since X is unbounded, we conclude: $\forall j \in \mathbb{N}$,
$X \nsubseteq B_j$ and so $\exists q \in X$ s.t. $q \notin B_j$.
So, by the Axiom of Choice, choose $s \in X^{\mathbb{N}}$ s.t., $\forall i \in \mathbb{N}$, $s_i \notin B_i$.
Since X is compact, we see that s is subconvergent.
Choose a subsequence t of s s.t. t is convergent in X.
Since t is convergent in X,
choose $z \in X$ s.t. $t \to z$ in X.
Since t is a subsequence of s,
choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
Since $t \to z$ in X, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(j \geqslant K) \Rightarrow (d(t_j, z) < 1).$$

By the Archimedean Property, choose $j \in \mathbb{N}$ s.t.

$$j > \max \{ K , 1 + (d(z,p)) \}.$$

Then $j > K$ and $j > 1 + (d(z,p))$,
so $j \geqslant K$ and $j \geqslant 1 + (d(z,p))$.

Since $j \geqslant K$, by choice of K, we get: $d(t_j, z) < 1$.

Recall: $\forall i \in \mathbb{N}$, $s_i \notin B_i$. Then $s_{\ell_j} \notin B_{\ell_j}$. Then $t_j = (s \circ \ell)_j = s_{\ell_j} \notin B_{\ell_j}$.

Since $t_j \notin B_{\ell_j} = B_X(p, \ell_j)$, we get: $d(t_j, p) \geqslant \ell_j$.

By HW#8-3, we get $\ell_j \geqslant j$. Then $d(t_j, p) \geqslant \ell_j \geqslant j$, so $j \leqslant d(t_j, p)$.

Recall: $j \geqslant 1 + (d(z,p))$. Then $1 + (d(z,p)) \leqslant j$.

Recall: $d(t_j, z) < 1$. Then $(d(t_j, z)) + (d(z,p)) < 1 + (d(z,p))$.

By the triangle inequality, we have: $d(t_j, p) \leqslant (d(t_j, z)) + (d(z,p))$.

Then $j \leqslant d(t_j, p) \leqslant (d(t_j, z)) + (d(z,p)) < 1 + (d(z,p)) \leqslant j$,
so $j < j$. Contradiction. □

THEOREM 3.9.2. *Let X and Y be sets.*
Let $f : X \to> Y$ and let $s \in Y^{\mathbb{N}}$.
Then: $\exists \sigma \in X^{\mathbb{N}}$ s.t. $f \circ \sigma = s$.

Proof. We have: $\forall j \in \mathbb{N}$, $s_j \in Y = \mathbb{I}_f$,
and so $\exists q \in X$ s.t. $f_q = s_j$.

So, by the Axiom of Choice, choose $\sigma \in X^{\mathbb{N}}$ s.t., $\forall j \in \mathbb{N}$, $f_{\sigma_j} = s_j$.

Want: $f \circ \sigma = s$. Want: $\forall j \in \mathbb{N}$, $(f \circ \sigma)_j = s_j$.

Given $j \in \mathbb{N}$. Want: $(f \circ \sigma)_j = s_j$.

We have $(f \circ \sigma)_j = f_{\sigma_j} = s_j$, as desired. □

THEOREM 3.9.3. *Let X and Y be sets.*
Let $f : X \to Y$ and let $\sigma, \tau \in X^{\mathbb{N}}$.
Assume: τ is a subsequence of σ.
Then: $f \circ \tau$ is a subsequence of $f \circ \sigma$.

Proof. Choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $\tau = \sigma \circ \ell$.

Then $f \circ \tau = f \circ (\sigma \circ \ell) = (f \circ \sigma) \circ \ell$,
and so $f \circ \tau$ is a subsequence of $f \circ \sigma$, as desired. □

THEOREM 3.9.4. *Let X and Y be metric spaces.*
Let $f : X \to Y$ and let $\tau \in X^{\mathbb{N}}$.
Assume: f is continuous from X to Y and τ is convergent in X.
Then: $f \circ \tau$ is convergent in Y.

Proof. Choose $p \in X$ s.t. $\tau \to p$ in X.

Since $p \in X = \mathbb{D}_f$ and f is continuous on $\mathbb{D}_f$ from X to Y,

we conclude that: f is continuous at p from X to Y.
So, since $\tau \to p$ in X, we get: $f \circ \tau \to f_p$ in Y.
Then $f \circ \tau$ is convergent in Y, as desired. $\square$

THEOREM 3.9.5. *Let X and Y be metric spaces.*
Let $f : X \to Y$ and let $\sigma \in X^{\mathbb{N}}$.
Assume: f is continuous from X to Y and σ is subconvergent in X.
Then: $f \circ \sigma$ is subconvergent in Y.

Proof. Choose a subsequence τ of σ s.t. τ is convergent in X.
By Theorem 3.9.3, $f \circ \tau$ is a subsequence of $f \circ \sigma$.
It therefore suffices to show: $f \circ \tau$ is convergent in Y.
By Theorem 3.9.4, $f \circ \tau$ is convergent in Y, as desired. $\square$

THEOREM 3.9.6. *Let X and Y be metric spaces, $f : X \to> Y$.*
Assume: X is compact and f is continuous from X to Y.
Then : Y is compact.

Proof. Want: $\forall s \in Y^{\mathbb{N}}$, s is subconvergent in Y.
Given $s \in Y^{\mathbb{N}}$. Want: s is subconvergent in Y.
By Theorem 3.9.2, choose $\sigma \in X^{\mathbb{N}}$ s.t. $f \circ \sigma = s$.
Since X is compact and $\sigma \in X^{\mathbb{N}}$, σ is subconvergent in X.
Then, by Theorem 3.9.5, $f \circ \sigma$ is subconvergent in Y.
So, since $f \circ \sigma = s$, we get: s is subconvergent in Y. $\square$

THEOREM 3.9.7. $\forall S \subseteq \mathbb{R}^*$, $\inf S , \sup S \in \mathbb{R}^*$.

THEOREM 3.9.8. $\inf \varnothing = \infty > -\infty = \sup \varnothing$.

THEOREM 3.9.9. *Let $S \subseteq \mathbb{R}^*$. Assume $S \neq \varnothing$.*
Then $\inf S \leqslant \sup S$.

Proof. Choose $a \in S$. Then $a \leqslant \sup S$ and $a \geqslant \inf S$.
Then $\inf S \leqslant a \leqslant \sup S$, as desired. $\square$

THEOREM 3.9.10. *Let $S \subseteq \mathbb{R}$.*
Assume: ($S \neq \varnothing$) & (S is bounded in $\mathbb{R}$).
Then: $\inf S , \sup S \in \mathbb{R}$.

Proof. By Theorem 3.9.7, we have: $\inf S , \sup S \in \mathbb{R}^*$.
Choose $B \in \mathbb{B}_{\mathbb{R}}$ s.t. $S \subseteq B$.
Choose $p \in \mathbb{R}$ and $a > 0$ s.t. $B = B_{\mathbb{R}}(p, a)$.
Since $S \subseteq B = B_{\mathbb{R}}(p, a) = (p - a; p + a)$,

we see that: $p-a<S<p+a$.
Then $p-a\leqslant S\leqslant p+a$.
Then $-\infty<p-a\leqslant\inf S\leqslant\sup S\leqslant p+a<\infty$,
so $-\infty<\inf S<\infty$ and $-\infty<\sup S<\infty$,
so $\inf S\notin\{-\infty,\infty\}$ and $\sup S\notin\{-\infty,\infty\}$,
Then $\inf S$, $\sup S\ \in\ (\mathbb{R}^*)^\times_{\{-\infty,\infty\}}=\mathbb{R}$, as desired. □

3.10. **Basics of topology in metric spaces.**

DEFINITION 3.10.1. *Let X be a metric space and $A\subseteq X$. Then $\partial_X A:=\{p\in X\,|\,\forall B\in\mathcal{B}_X(p),\ B\bigcap A\neq\varnothing\neq B\bigcap(X\backslash A)\}$.*

In the preceding definition, note that $B\bigcap(X\backslash A)=B\backslash A$.

THEOREM 3.10.2. *Let $X:=\mathbb{R}^2$, $U:=B_X(0_X,1)$, $C:=S_X(0_X,1)$. Let $H:=[0;\infty)\times\mathbb{R}$, $A:=U\bigcup(C\bigcap H)$. Then $\partial_X A=C$.*

DEFINITION 3.10.3. *Let X be a metric space and $A\subseteq X$. Then:*
$$\partial_X^{\rm seq}A\ :=\ \{\ p\in X\ \mid\ (\exists s\in A^{\mathbb{N}}\ \text{s.t. } s\to p \text{ in } X)\ \&\ (\exists t\in(X\backslash A)^{\mathbb{N}}\ \text{s.t. } t\to p \text{ in } X)\ \}.$$

THEOREM 3.10.4. *Let $X:=\mathbb{R}^2$, $U:=B_X(0_X,1)$, $C:=S_X(0_X,1)$. Let $H:=[0;\infty)\times\mathbb{R}$, $A:=U\bigcup(C\bigcap H)$. Then $\partial_X^{\rm seq}A=C$.*

THEOREM 3.10.5. *Let X be a metric space and $A\subseteq X$. Then $\partial_X A=\partial_X^{\rm seq}A$.*

Proof. By HW#10-3, we have: $\partial_X A\supseteq\partial_X^{\rm seq}A$. Want: $\partial_X A\subseteq\partial_X^{\rm seq}A$.
Want: $\forall p\in\partial_X A$, $p\in\partial_X^{\rm seq}A$. Given $p\in\partial_X A$. Want: $p\in\partial_X^{\rm seq}A$.
Define $B:\mathbb{N}\to 2^X$ by: $\forall j\in\mathbb{N}$, $B_j=B_X(p,1/j)$.
Since $p\in\partial_X A$, we know:
$\forall B\in\mathcal{B}_X(p)$, $B\bigcap A\neq\varnothing\neq B\bigcap(X\backslash A)$.
We know, $\forall j\in\mathbb{N}$, that $B_j\in\mathcal{B}_X(p)$.
Then: $\forall j\in\mathbb{N}$, $B_j\bigcap A\neq\varnothing\neq B_j\bigcap(X\backslash A)$.
We wish to show:
(1) $\exists s\in A^{\mathbb{N}}$ s.t. $s\to p$ in X and
(2) $\exists t\in(X\backslash A)^{\mathbb{N}}$ s.t. $t\to p$ in X

Proof of (1):
Define $s\in A^{\mathbb{N}}$ by: $\forall j\in\mathbb{N}$, $s_j=\mathrm{CH}(B_j\bigcap A)$.
Then $s\in A^{\mathbb{N}}$. Want: $s\to p$ in X.

We have: $\forall j \in \mathbb{N}$, $s_j \in B_j = B_X(p, 1/j)$.
Then: $\forall j \in \mathbb{N}$, $d(s_j, p) < 1/j$.
Then: $\forall j \in \mathbb{N}$, $d(s_j, p) \leqslant 1/j$.
Then, by Theorem 2.8.3, we get $s \to p$ in X, as desired.
End of proof of (1).

Proof of (2):
Define $t \in (X \backslash A)^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $t_j = \mathrm{CH}(B_j \bigcap (X \backslash A))$.
Then $t \in (X \backslash A)^{\mathbb{N}}$. Want: $t \to p$ in X.
We have: $\forall j \in \mathbb{N}$, $t_j \in B_j = B_X(p, 1/j)$.
Then: $\forall j \in \mathbb{N}$, $d(t_j, p) < 1/j$.
Then: $\forall j \in \mathbb{N}$, $d(t_j, p) \leqslant 1/j$.
Then, by Theorem 2.8.3, we get $t \to p$ in X, as desired.
End of proof of (2). □

DEFINITION 3.10.6. *Let X be a metric space and $A \subseteq X$. Then:*
$$\mathrm{Cl}_X A := A \bigcup \partial_X A$$
and
$$\mathrm{Int}_X A := A \setminus \partial_X A.$$

THEOREM 3.10.7. *Let X be a metric space and $A \subseteq X$.*
Then: $\mathrm{Int}_X A \subseteq A \subseteq \mathrm{Cl}_X A$.

THEOREM 3.10.8. *Let $X := \mathbb{R}^2$, $U := B_X(0_X, 1)$, $C := S_X(0_X, 1)$. Let $H := [0; \infty) \times \mathbb{R}$, $A := U \bigcup (C \bigcap H)$.*
Then: $\mathrm{Cl}_X A = U \bigcup C = \overline{B}_X(0_X, 1)$
and $\mathrm{Int}_X A = U = B_X(0_X, 1)$.

THEOREM 3.10.9. *Let $X := \mathbb{R}$ and $A := [3; 5)$.*
Then: $\mathrm{Cl}_X A = [3; 5]$ *and* $\mathrm{Int}_X A = (3; 5)$.

THEOREM 3.10.10. *Let X be a metric space, $A \subseteq X$, $p \in X$.*
Then: (1) $(\, p \in \mathrm{Cl}_X A\,) \Leftrightarrow (\, \forall B \in \mathcal{B}_X(p),\ B \bigcap A \neq \varnothing\,)$.
Also: (2) $(\, p \in \mathrm{Int}_X A\,) \Leftrightarrow (\, \exists B \in \mathcal{B}_X(p),\ B \subseteq A\,)$.

Proof. Proof of (1): Unassigned HW. *End of proof of (1).*

Proof of ⇒ in (2):
Assume: $p \in \mathrm{Int}_X A$. Want: $\exists B \in \mathcal{B}_X(p)$, $B \subseteq A\,)$.
Since $p \in \mathrm{Int}_X A = A \backslash (\partial_X A)$, we get: $p \in A$ and $p \notin \partial_X A$.
Since $p \notin \partial_X A$, by definition of $\partial_X A$,
we conclude $\neg(\, \forall B \in \mathcal{B}_X(p),\ B \bigcap A \neq \varnothing \neq B \bigcap (X \backslash A)\,)$.

Choose $B \in \mathcal{B}_X(p)$ s.t. $\neg(\,B \bigcap A \neq \varnothing \neq B \bigcap (X \backslash A)\,)$.
Then $B \in \mathcal{B}_X(p)$, and we wish to show: $B \subseteq A$.
We have: $(\,B \bigcap A = \varnothing\,) \vee (\,\varnothing = B \bigcap (X \backslash A)\,)$.
Since $B \in \mathcal{B}_X(p)$, we get: $p \in B$.
So, since $p \in A$, we get: $p \in B \bigcap A$. Then $B \bigcap A \neq \varnothing$.
So, since $(\,B \bigcap A = \varnothing\,) \vee (\,\varnothing = B \bigcap (X \backslash A)\,)$,
we conclude that $\varnothing = B \bigcap (X \backslash A)$.
Then $B = B \bigcap X = B \bigcap (A \bigcup (X \backslash A)) = (B \bigcap A) \bigcup (B \bigcap (X \backslash A))$
$= (B \bigcap A) \bigcup \varnothing = B \bigcap A \subseteq A$, as desired.
End of proof of $\Rightarrow$ in (2).

Proof of $\Leftarrow$ in (2): Unassigned HW. *End of proof of $\Leftarrow$ in (2).* □

THEOREM 3.10.11. *Let X be a metric space and $A \subseteq X$. Then $X \backslash (\mathrm{Int}_X A) = \mathrm{Cl}_X(X \backslash A)$.*

Proof. Proof of $\subseteq$:
Want: $\forall p \in X \backslash (\mathrm{Int}_X A)$, $p \in \mathrm{Cl}_X(X \backslash A)$.
Given $p \in X \backslash (\mathrm{Int}_X A)$. Want: $p \in \mathrm{Cl}_X(X \backslash A)$.
Want: $\forall B \in \mathcal{B}_X(p)$, $B \bigcap (X \backslash A) \neq \varnothing$.
Given $B \in \mathcal{B}_X(p)$. Want: $B \bigcap (X \backslash A) \neq \varnothing$.
Assume: $B \bigcap (X \backslash A) = \varnothing$. Want: Contradiction.
We have $B = B \bigcap X = B \bigcap (A \bigcup (X \backslash A)) = (B \bigcap A) \bigcup (B \bigcap (X \backslash A))$
$= (B \bigcap A) \bigcup \varnothing = B \bigcap A \subseteq A$, so $B \subseteq A$.
Since $B \in \mathcal{B}_X(p)$ and $B \subseteq A$, we conclude,
by (2) of Theorem 3.10.10, that $p \in \mathrm{Int}_X A$.
Since $p \in X \backslash (\mathrm{Int}_X A)$, we conclude that $p \notin \mathrm{Int}_X A$. Contradiction.
End of proof of $\subseteq$.

Proof of $\supseteq$: Unassigned HW. *End of proof of $\supseteq$.* □

THEOREM 3.10.12. *Let X be a metric space and $A \subseteq X$. Then $X \backslash (\mathrm{Cl}_X A) = \mathrm{Int}_X(X \backslash A)$.*

Proof. Let $B := X \backslash A$. Want: $X \backslash (\mathrm{Cl}_X A) = \mathrm{Int}_X B$.
Let $U := \mathrm{Int}_X B$. Want: $X \backslash (\mathrm{Cl}_X A) = U$.
Since $A \subseteq X$, we get: $X \backslash (X \backslash A) = A$. Then $X \backslash B = A$.
By Theorem 3.10.11, $X \backslash (\mathrm{Int}_X B) = \mathrm{Cl}_X(X \backslash B)$.
Then $X \backslash U = X \backslash (\mathrm{Int}_X B) = \mathrm{Cl}_X(X \backslash B) = \mathrm{Cl}_X A$,
so $X \backslash U = \mathrm{Cl}_X A$, so $X \backslash (X \backslash U) = X \backslash (\mathrm{Cl}_X A)$.

Since $U \subseteq X$, we get: $X \backslash (X \backslash U) = U$.
Then $X \backslash (\mathrm{Cl}_X A) = X \backslash (X \backslash U) = U$, as desired. $\square$

DEFINITION 3.10.13. *Let X be a metric space and $A \subseteq X$.*
By A is **closed** *in X, we mean:* $\mathrm{Cl}_X A = A$.
By A is **open** *in X, we mean:* $\mathrm{Int}_X A = A$.

The word "clopen" means "closed and open". As set is **clopen** in a metric space if it is both closed and open in it.

We noted that, in $\mathbb{R}$, there are exactly two clopen sets: $\emptyset$ and $\mathbb{R}$.

We noted that, in $\mathbb{R}^2$, there are exactly two clopen sets: $\emptyset$ and $\mathbb{R}^2$.

Let $S := [1; 2]$ and $T := [3; 4]$ and $X := S \bigcup T$. We noted that, in the metric space X (with the relative metric inherited from $\mathbb{R}$), there are exactly four clopen sets:; $\emptyset$, S, T and X.

THEOREM 3.10.14. *Let $X := \mathbb{R}^2$, $U := B_X(0_X, 1)$, $C := S_X(0_X, 1)$. Let $H := [0; \infty) \times \mathbb{R}$, $A := U \bigcup (C \bigcap H)$.*
Then: U is open in X and not closed in X
and $U \bigcup C$ is closed in X and not open in X
and A is neither open nor closed in X
and X is both open and closed in X.

THEOREM 3.10.15. *Let $X := \mathbb{R}$.*
Then: $(3; 5)$ is open in X and not closed in X
and $[3; 5]$ is closed in X and not open in X
and $[3; 5)$ is neither open nor closed in X
and X is both open and closed in X.

THEOREM 3.10.16. *Let X be a metric space, $T \subseteq X$, $S \subseteq T$.*
Then: (A is open in X) $\Leftrightarrow$ ($X \backslash A$ is closed in X).

Proof. Proof of $\Rightarrow$:
Assume: A is open in X. Want: $X \backslash A$ is closed in X.
Want: $\mathrm{Cl}_X(X \backslash A) = X \backslash A$. Since A is open in X, we get: $\mathrm{Int}_X A = A$.
It follows that $X \backslash (\mathrm{Int}_X A) = X \backslash A$.
By Theorem 3.10.11, we have: $X \backslash (\mathrm{Int}_X A) = \mathrm{Cl}_X(X \backslash A)$.
Then $\mathrm{Cl}_X(X \backslash A) = X \backslash (\mathrm{Int}_X A) = X \backslash A$, as desired.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$:
Assume: $X \backslash A$ is closed in X. Want: A is open in X.
Want: $\mathrm{Int}_X A = A$. As $X \backslash A$ is closed in X, we get: $\mathrm{Cl}_X(X \backslash A) = X \backslash A$.
By Theorem 3.10.11, we have: $X \backslash (\mathrm{Int}_X A) = \mathrm{Cl}_X(X \backslash A)$.
Then $X \backslash (\mathrm{Int}_X A) = \mathrm{Cl}_X(X \backslash A) = X \backslash A$, so $X \backslash (\mathrm{Int}_X A) = X \backslash A$,
and it follows that: $X \backslash (X \backslash (\mathrm{Int}_X A)) = X \backslash (X \backslash A)$.
Since $\mathrm{Int}_X A \subseteq X$ and $A \subseteq X$, we conclude:
$X \backslash (X \backslash (\mathrm{Int}_X A)) = \mathrm{Int}_X A$ and $X \backslash (X \backslash A) = A$.
Then $\mathrm{Int}_X A = X \backslash (X \backslash (\mathrm{Int}_X A)) = X \backslash (X \backslash A) = A$, as desired.
End of proof of $\Leftarrow$. □

THEOREM 3.10.17. *Let X be a metric space, $T \subseteq X$, $S \subseteq T$.*
Then: (A is closed in X) $\Leftrightarrow$ ($X \backslash A$ is open in X).

Proof. Since $A \subseteq X$, we get $X \backslash (X \backslash A) = A$.
Let $B := X \backslash A$. Then $X \backslash B = A$.
By Theorem 3.10.16, we have:
(B is open in X) $\Leftrightarrow$ ($X \backslash B$ is closed in X).
Then ($X \backslash A$ is open in X) $\Leftrightarrow$ (A is closed in X).
Then (A is closed in X) $\Leftrightarrow$ ($X \backslash A$ is open in X), as desired. □

THEOREM 3.10.18. *Let X be a metric space and $A \subseteq X$.*
Then: $\mathrm{Int}_X A$ *is open in X.*

Proof. Let $U := \mathrm{Int}_X A$. Want: U is open in X. Want $\mathrm{Int}_X U = U$.
We have $\mathrm{Int}_X U \subseteq U$. Want: $U \subseteq \mathrm{Int}_X U$.
Want: $\forall p \in U$, $p \in \mathrm{Int}_X U$.
Given $p \in U$. Want: $p \in \mathrm{Int}_X U$.
Want: $\exists B \in \mathcal{B}_X(p)$ s.t. $B \subseteq U$.
Since $p \in U = \mathrm{Int}_X A$, choose $B \in \mathcal{B}_X(p)$ s.t. $B \subseteq A$.
Then $B \in \mathcal{B}_X(p)$. Want: $B \subseteq U$.
Since $B \subseteq A$, we get: $\mathrm{Int}_X B \subseteq \mathrm{Int}_X A$.
By HW#10-5, B is open in X, so $\mathrm{Int}_X B = B$.
Then $B = \mathrm{Int}_X B \subseteq \mathrm{Int}_X A = U$, as desired. □

THEOREM 3.10.19. *Let X be a metric space and $A \subseteq X$.*
Then: $\mathrm{Cl}_X A$ *is closed in X.*

Proof. Let $C := \mathrm{Cl}_X A$. Want: C is closed in X.
By Theorem 3.10.12, we have: $X \backslash (\mathrm{Cl}_X A) = \mathrm{Int}_X(X \backslash A)$.
Let $B := X \backslash A$. Then $X \backslash C = X \backslash (\mathrm{Cl}_X A) = \mathrm{Int}_X(X \backslash A) = \mathrm{Int}_X B$.

By Theorem 3.10.18, $\mathrm{Int}_X B$ is open in X.
So, since $X \backslash C = \mathrm{Int}_X B$, we see that: $X \backslash C$ is open in X.
Then, by Theorem 3.10.17, C is closed in X, as desired. □

3.11. **Convergence of bounded semi-monotonic sequences.**

DEFINITION 3.11.1. *Let X be a metric space, S a set.*
Then: $X_{\mathrm{bi}}^S \quad := \quad \{ f \in X^S \,|\, \mathbb{I}_f$ *is bounded in* $X \}$.

THEOREM 3.11.2. *Let $s \in \mathbb{R}_{\mathrm{bi}}^{\mathbb{N}}$. Assume that s is semi-increasing.*
Then s is convergent in $\mathbb{R}$.

Proof. Want: $\exists q \in \mathbb{R}$ s.t. $s \to q$ in $\mathbb{R}$.
Since $\mathbb{D}_s = \mathbb{N} \neq \varnothing$, we get: $\mathbb{I}_s \neq \varnothing$.
So, since $\mathbb{I}_s$ is bounded in $\mathbb{R}$, by Theorem 3.9.10, we get: $\sup \mathbb{I}_s \in \mathbb{R}$.
Let $q := \sup \mathbb{I}_s$. Then $q \in \mathbb{R}$. Want: $s \to q$ in $\mathbb{R}$.
Want: $\forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\ j \geqslant K\ ) \Rightarrow (\ d(s_j, q) < \varepsilon\ )$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $(\ j \geqslant K\ ) \Rightarrow (\ d(s_j, q) < \varepsilon\ )$.
We have $\neg(q \leqslant q - \varepsilon)$. So, as $q = \sup \mathbb{I}_s$, we get $\neg(\sup \mathbb{I}_s \leqslant q - \varepsilon)$.
Then $\neg(\mathbb{I}_s \leqslant q - \varepsilon)$, so choose $y \in \mathbb{I}_s$ s.t. $y > q - \varepsilon$.
Since $y \in \mathbb{I}_s$, choose $K \in \mathbb{D}_s$ s.t. $y = s_K$.
Then $K \in \mathbb{D}_s = \mathbb{N}$. Want: $\forall j \in \mathbb{N}$, $(\ j \geqslant K\ ) \Rightarrow (\ d(s_j, q) < \varepsilon\ )$.
Given $j \in \mathbb{N}$. Want: $(\ j \geqslant K\ ) \Rightarrow (\ d(s_j, q) < \varepsilon\ )$.
Assume: $j \geqslant K$. Want: $d(s_j, q) < \varepsilon$. Want: $q - \varepsilon < s_j < q + \varepsilon$.
We have $s_j \in \mathbb{I}_s \leqslant \sup \mathbb{I}_s = q < q + \varepsilon$. Want: $q - \varepsilon < s_j$.
Since s is semi-increasing and $K \leqslant j$, we get: $s_K \leqslant s_j$.
By choice of y, we have: $q - \varepsilon < y$. By choice of K, we have $y = s_K$.
Then $q - \varepsilon < y = s_K \leqslant s_j$, as desired. □

DEFINITION 3.11.3. *Let $f : \mathbb{R}^* \dashrightarrow \mathbb{R}^*$.*
By f is strictly-monotone, we mean:
f is strictly-increasing or f is strictly-decreasing.
By f is semi-monotone, we mean:
f is semi-increasing or f is semi-decreasing.

THEOREM 3.11.4. *Let $s \in \mathbb{R}_{\mathrm{bi}}^{\mathbb{N}}$. Assume that s is semi-monotone.*
Then s is convergent in $\mathbb{R}$.

Proof. Either (1) s is semi-increasing or (2) s is semi-decreasing.

Case (1):

By Theorem 3.11.2, s is convergent in $\mathbb{R}$, as desired.
End of Case (1).

Case (2):
Let $t := -s$. Then $t \in \mathbb{R}^{\mathbb{N}}_{\mathrm{bi}}$ and t is semi-increasing.
By Theorem 3.11.2, t is convergent in $\mathbb{R}$.
Then $-t$ is convergent in $\mathbb{R}$.
So, since $-t = -(-s) = s$, we see that s is convergent in $\mathbb{R}$, as desired.
End of Case (2). $\square$

3.12. **Basic dynamical systems.**

DEFINITION 3.12.1. *Let f be a function and let $j \in \mathbb{N}$.*
Then $f_\circ^j$ is the function defined by: $\forall x$,

$$f_\circ^j(x) = \mathrm{UE}\{\, y \in \mathbb{I}_f \mid \exists s \in \mathbb{I}_f^n \text{ s.t. } (\, s_1 = f(x)\,)\,\&\, (\,\forall i \in [2..j],\ s_i = f(s_{j-1})\,)\,\&\, (\, y = s_j\,) \,\}.$$

Let f be a function. Then

$$f_\circ^1 = f, \quad f_\circ^2 = f \circ f, \quad f_\circ^3 = f \circ f \circ f, \quad f_\circ^4 = f \circ f \circ f \circ f, \quad \textit{etc.}$$

THEOREM 3.12.2. *Let f be a function and let $j \in \mathbb{N}$.*
Then: $f_\circ^{j+1} = f \circ f_\circ^j$.

THEOREM 3.12.3. *Let P be a set, $f : P \to P$, $m \in P$.*
Define $\ell \in P^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $\ell_j = f_\circ^j(m)$.
Then: $\forall j \in \mathbb{N}$, $\ell_{j+1} = f(\ell_j)$.

Proof. Given $j \in \mathbb{N}$. Want: $\ell_{j+1} = f(\ell_j)$.
By Theorem 3.12.2, we have: quad $f_\circ^{j+1} = f \circ f_\circ^j$.
We have $\ell_{j+1} = f_\circ^{j+1}(m) = (\, f \circ f_\circ^j\,)(m)$
$= f(\, f_\circ^j(m)\,) = f(\ell_j)$, as desired. $\square$

THEOREM 3.12.4. *Let $P \in \mathbb{N}$. Assume P is infinite.*
Then $\exists \ell \in P^{\mathbb{N}}$ s.t. ℓ is strictly-increasing.

Proof. By hypothesis, the set P is infinite.
For all $k \in P$, since $[1..k] = \{1, \ldots, k\}$, we see that $[1..k]$ is finite,
so, since P is infinite, we see that $P \backslash [1..k]$ is infinite,
so, as $\varnothing$ is finite, we get $P \backslash [1..k] \neq \varnothing$,
so, by the Well-Ordering Axiom, $\min(P \backslash [1..k]) \neq ☹$,

and it follows that $\min(P\backslash[1..k]) \in P\backslash[1..k]$.
Define $f : P \dashrightarrow P$ by: $\forall k \in P$, $f(k) = \min(P\backslash[1..k])$.
Then: $\forall k \in P$, $f(k) \neq ☹$ and $f(k) \in P\backslash[1..k]$.
Since $f : P \dashrightarrow P$ we get $\mathbb{D}_f \subseteq P$.
For all $k \in P$, since $f(k) \neq ☹$, we see that $k \in \mathbb{D}_f$. Then $P \subseteq \mathbb{D}_f$.
So, since $\mathbb{D}_f \subseteq P$, we get: $\mathbb{D}_f = P$. Then $f : P \to P$.
Since P is infinite and as $\varnothing$ is finite, we get $P \neq \varnothing$,
so, by the Well-Ordering Axiom, $\min P \neq ☹$,
and it follows that $\min P \in P$.
Let $m := \min P$. Then $m \in P$.
Define $\ell \in P^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $\ell_j = f_\circ^j(m)$.
Then $\ell \in P^{\mathbb{N}}$, and we wish to show: ℓ is strictly-increasing.
Want: $\forall j \in \mathbb{N}$, $\ell_{j+1} > \ell_j$.
Given $j \in \mathbb{N}$. Want: $\ell_{j+1} > \ell_j$.
By Theorem 3.12.3, we have: $\ell_{j+1} = f(\ell_j)$.
Since $\ell \in P^{\mathbb{N}}$, it follows that $\ell_j \in P$.
Recall: $\forall k \in P$, $f(k) \in P\backslash[1..k]$.
Let $k := \ell_j$. Then $k \in P$, so $f(k) \in P\backslash[1..k]$.
Then $\ell_{j+1} = f(\ell_j) = f(k) \in P\backslash[1..k] \subseteq \mathbb{N}\backslash[1..k] > k = \ell_j$. □

THEOREM 3.12.5. *Let $P \subseteq \mathbb{N}$. Assume P is finite.*
Then $\exists m \in \mathbb{N}_0$ s.t. $(m..\infty) \subseteq \mathbb{N}\backslash P$.

Proof. Since $P \subseteq \mathbb{N}$, we get: $P_0^+ \subseteq \mathbb{N}_0^+ = \mathbb{N}_0$.
Since P_0^+ is a nonempty finite subset of $\mathbb{R}$, we get: $\max P_0^+ \neq ☹$.
Let $m := \max P_0^+$. Then $m \neq ☹$, so $m \in P_0^+ \leqslant m$.
Want: $(m..\infty) \subseteq \mathbb{N}\backslash P$. Want: $\forall j \in (m..\infty)$, $j \in \mathbb{N}\backslash P$.
Given $j \in (m..\infty)$. Want: $j \in \mathbb{N}\backslash P$.
Since $m \in \mathbb{N}_0 \geqslant 0$, we get $(m..\infty) \subseteq (0..\infty)$.
Then $j \in (m..\infty) \subseteq (0..\infty) = \mathbb{N}$, and it remains to show: $j \notin P$.
Since $j \in (m..\infty)$, we see that $j > m$.
Then $P \subseteq P_0^+ \leqslant m < j$, so $P < j$. Then $j \notin P$. □

3.13. Properness of the reals.

THEOREM 3.13.1. *Let $s \in \mathbb{R}^{\mathbb{N}}$.*
Then $\exists$subsequence t of s s.t. t is semi-monotone.

Proof. Let $P := \{j \in \mathbb{N} \,|\, \forall k \in (j..\infty),\ s_j \geqslant s_k\}$.
We know: either (1) P is infinite or (2) P is finite.

Case 1:
By Theorem 3.12.4, choose $\ell \in P^{\mathbb{N}}$ s.t. ℓ is strictly-increasing.
Let $t := s \circ \ell$. Then t is a subsequence of s.
Want: t is semi-monontone. Want: t is semi-decreasing.
Want: $\forall i \in \mathbb{N}, \quad t_i \geqslant t_{i+1}$.
Given $i \in \mathbb{N}$. Want: $t_i \geqslant t_{i+1}$.
Since $\ell \in P^{\mathbb{N}}$, we see that $\ell_i \in P$.
Let $j := \ell_i$. Then $j \in P$.
Then, by definition of P, we have: $\forall k \in (j..\infty)$, $s_j \geqslant s_k$.
Since ℓ is strictly-increasing, we have $\ell_{i+1} > \ell_i$.
Also, since $\ell \in P^{\mathbb{N}}$, we have $\ell_{i+1} \in P$. Let $k := \ell_{i+1}$.
Since $k = \ell_{i+1} > \ell_i = j$ and since $k = \ell_{i+1} \in P \subseteq \mathbb{N}$,
we see that $k \in (j..\infty)$. Then $s_j \geqslant s_k$.
Then $t_i = (s \circ \ell)_i = s_{\ell_i} = s_j \geqslant s_k = s_{\ell_{i+1}} = (s \circ \ell)_{i+1} = t_{i+1}$.
End of Case 1.

Case 2:
By Theorem 3.12.5, choose $m \in \mathbb{N}_0$ s.t. $(m..\infty) \subseteq \mathbb{N} \backslash P$.
Then, $\forall j \in (m..\infty)$, because $j \notin P$, we see, by definition of P, that:

$$\neg(\, \forall k \in (j..\infty),\, s_j \geqslant s_k \,).$$

Thus, $\forall j \in (m..\infty), \quad \exists k \in (j..\infty)$ s.t. $s_j < s_k$.
Then, $\forall j \in (m..\infty), \quad \{\, k \in (j..\infty) \,|\, s_j < s_k \,\} \neq \varnothing$.
Then, by the Well-Ordering Axiom, we have:

$$\forall j \in (m..\infty), \quad \min\{\, k \in (j..\infty) \,|\, s_j < s_k \,\} \neq ☹.$$

Define $f : (m..\infty) \to \mathbb{Z}$ by:

$$\forall j \in (m..\infty), \quad f(j) = \min\{\, k \in (j..\infty) \,|\, s_j < s_k \,\}.$$

Then, $\forall j \in (m..\infty), \quad f(j) \in \{\, k \in (j..\infty) \,|\, s_j < s_k \,\}$,
and so $\quad f(j) \in (j..\infty) \quad$ and $\quad s_j < s_{f(j)}$.
Also, $\forall j \in (m..\infty), \quad j > m, \quad$ and so $(j..\infty) \subseteq (m..\infty)$.
Then, $\forall j \in (m..\infty), \quad f(j) \in (j..\infty) \subseteq (m..\infty)$.
So, since $\mathbb{D}_f = (m..\infty)$, we get: $\forall j \in \mathbb{D}_f$, $f_j \in (m..\infty)$.
Then $\mathbb{I}_f \subseteq (m..\infty)$. Then $f : (m..\infty) \to (m..\infty)$.
Define $\ell \in (m..\infty)^{\mathbb{N}}$ by: $\forall i \in \mathbb{N}$, $\ell_i = f_\circ^i(m+1)$.
Then: $\quad \forall i \in \mathbb{N}, \quad f(\ell_i) = \ell_{i+1}$.
We have: $\forall j \in (m..\infty)$, $f(j) \in (j..\infty) > j$, so $f(j) > j$.
Then, $\forall i \in \mathbb{N}$, $\ell_{i+1} = f(\ell_i) > \ell_i$. Then ℓ is strictly-increasing.
Let $t := s \circ \ell$. Then t is a subsequence of s.
Want: t is semi-monontone. Want: t is strictly-increasing.

Want: $\forall i \in \mathbb{N},\ \ t_i < t_{i+1}$. Given $i \in \mathbb{N}$. Want: $t_i < t_{i+1}$.
Recall: $\forall j \in (m..\infty)$, $s_j < s_{f(j)}$. Let $j := \ell_i$.
Then $j \in \mathbb{I}_\ell = (m..\infty)$, so $s_j < s_{f(j)}$. Also, $f(j) = f(\ell_i) = \ell_{i+1}$.
Then $t_i = (s \circ \ell)_i = s_{\ell_i} = s_j < s_{f(j)} = s_{\ell_{j+1}} = (s \circ \ell)_{j+1} = t_{j+1}$.
End of Case 2. □

THEOREM 3.13.2. *Let f and g be functions. Then $\mathbb{I}_{g \circ f} \subseteq \mathbb{I}_g$.*

Proof. Want: $\forall p \in \mathbb{I}_{g \circ f}$, $p \in \mathbb{I}_f$. Given $p \in \mathbb{I}_{g \circ f}$. Want: $p \in \mathbb{I}_g$.
Since $p \in \mathbb{I}_{g \circ f}$, choose $x \in \mathbb{D}_{g \circ f}$ s.t. $(g \circ f)_x = p$.
Let $y := f_x$. Then $g_y = g_{f_x} = (g \circ f)_x = p$.
Since $p \in \mathbb{I}_{g \circ f}$, we get $p \neq ☹$. Then $g_y = p \neq ☹$, so $g_y \in \mathbb{I}_g$.
Then $p = g_y \in \mathbb{I}_g$, as desired. □

THEOREM 3.13.3. *Let X be a metric space and let $s \in X^{\mathbb{N}}_{\text{bi}}$.*
Let t be a subsequence of s. Then $t \in X^{\mathbb{N}}_{\text{bi}}$.

Proof. Choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
By Theorem 3.13.2, $\mathbb{I}_{s \circ \ell} \subseteq \mathbb{I}_s$. Then $\mathbb{I}_t = \mathbb{I}_{s \circ \ell} \subseteq \mathbb{I}_s$, so $\mathbb{I}_t \subseteq \mathbb{I}_s$.
Since $s \in \mathbb{R}^{\mathbb{N}}_{\text{bi}}$, we get: $\mathbb{I}_s$ is bounded in $\mathbb{R}$.
So, since $\mathbb{I}_t \subseteq \mathbb{I}_s$, we get: $\mathbb{I}_t$ is bounded in X. Then $t \in X^{\mathbb{N}}_{\text{bi}}$. □

DEFINITION 3.13.4. *Let X be a metric space.*
By X is **proper**, *we mean: $\forall s \in X^{\mathbb{N}}_{\text{bi}}$, s is subconvergent in X.*

THEOREM 3.13.5. *$\mathbb{R}$ is proper.*

Proof. Want: $\forall s \in \mathbb{R}^{\mathbb{N}}_{\text{bi}}$, s is subconvergent in $\mathbb{R}$.
Given $s \in \mathbb{R}^{\mathbb{N}}_{\text{bi}}$. Want: s is subconvergent in $\mathbb{R}$.
Want: $\exists$subsequence t of s s.t. t is convergent in $\mathbb{R}$.
By Theorem 3.13.1, choose a subsequence t of s s.t. t is semi-monotone.
Then t is a subsequence of s. Want: t is convergent in $\mathbb{R}$.
By Theorem 3.13.3, we get: $t \in \mathbb{R}^{\mathbb{N}}_{\text{bi}}$.
Then, by Theorem 3.11.4, we conclude: t is convergent in $\mathbb{R}$. □

3.14. Properness of products.

THEOREM 3.14.1. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$ and let $T \subseteq X$.
Assume: (T is bounded in X) &(f is Lipschitz from X to Y).
Then $f_*(T)$ is bounded in Y.

Proof. Exactly one of the following must be true:

$$(1)\ \mathbb{D}_f = \varnothing \qquad \text{or} \qquad \mathbb{D}_f \neq \varnothing.$$

Case (1):
We have $f_*(T) = \{f_p \,|\, p \in T \cap \mathbb{D}_f\} = \{f_p \,|\, p \in \varnothing\} = \varnothing$.
So, since $\varnothing$ is bounded in Y, we get: $f_*(T)$ is bounded in Y.
End of Case (1).

Case (2):
Want: $\exists C \in \mathcal{B}_Y$ s.t. $f_*(T) \subseteq C$.
Since $\mathbb{D}_f \neq \varnothing$, choose $z \in \mathbb{D}_f$. Since $z \in \mathbb{D}_f \subseteq X$, we get $X \neq \varnothing$.
So, since T is bounded in X, choose $A \in \mathcal{B}_X$ s.t. $T \subseteq A$.
By the Superset Recentering Lemma, choose $B \in \mathcal{B}_X(z)$ s.t. $A \subseteq B$.
Since $B \in \mathcal{B}_X(z)$, choose $r > 0$ s.t. $B = B_X(z, r)$.
Since f is Lipschitz, choose $K \geqslant 0$ s.t. f is Lipschitz-K.
Let $C := B_Y(f_z, Kr + 1)$. Then $C \in \mathcal{B}_Y$. Want: $f_*(T) \subseteq C$.
Want: $\forall q \in f_*(T)$, $q \in C$. Given $q \in f_*(T)$. Want: $q \in C$.
Since $q \in f_*(T)$, choose $p \in T \cap \mathbb{D}_f$ s.t. $q = f_p$.
We have $p \in T \subseteq A \subseteq B = B_X(z, r)$, so $d_X(p, z) < r$.
Since f is Lipschitz-K, we have $d_Y(f_p, f_z) \leqslant K \cdot (d_X(p, z))$.
Since $K \geqslant 0$ and $d_X(p, z) < r$, we get $K \cdot (d_X(p, z)) \leqslant Kr$.
Then $d_Y(q, f_z) = d_Y(f_p, f_z) \leqslant K \cdot (d_X(p, z)) \leqslant Kr < Kr + 1$,
so $d_Y(q, f_z) < Kr + 1$, so $q \in B_Y(f_z, Kr + 1)$.
Then $q \in B_Y(f_z, Kr + 1) = C$, as desired.
End of Case (2). □

THEOREM 3.14.2. *Let X and Y be metric spaces.*
Let $f : X \to Y$ and let $s \in X^{\mathbb{N}}_{\text{bi}}$.
Assume: f is Lipschitz from X to Y. Then $f \circ s \in Y^{\mathbb{N}}_{\text{bi}}$.

Proof. Since $s \in X^{\mathbb{N}}_{\text{bi}}$, we get: $s : \mathbb{N} \to X$ and $\mathbb{I}_s$ is bounded in X.
Since $s : \mathbb{N} \to X$ and $f : X \to Y$, we get $f \circ s : \mathbb{N} \to Y$.
Then $f \circ s \in Y^{\mathbb{N}}$. Want: $\mathbb{I}_{f \circ s}$ is bounded in Y.
Since $\mathbb{I}_s$ is bounded in X and since f is Lipschitz from X to Y,
by Theorem 3.14.1, we get: $f_*(\mathbb{I}_s)$ is bounded in Y.
So, since $\mathbb{I}_{f \circ s} = f_*(\mathbb{I}_s)$, we get: $\mathbb{I}_{f \circ s}$ is bounded in Y, as desired. □

THEOREM 3.14.3. *Let X be a metric space, $s \in X^{\mathbb{N}}$ and $q \in X$.*
Let t be a subsequence of s.
Assume: $s \to q$ in X. Then: $t \to q$ in X.

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ d(t_j, q) < \varepsilon\ ).$$

Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ d(t_j, q) < \varepsilon\ ).$$

Since $s \to q$ in X, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,

$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ d(s_j, q) < \varepsilon\ ).$$

Then $K \in \mathbb{N}$. Want: $\forall j \in \mathbb{N}$,

$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ d(t_j, q) < \varepsilon\ ).$$

Given $j \in \mathbb{N}$. Want: $(\ j \geqslant K\ ) \ \Rightarrow\ (\ d(t_j, q) < \varepsilon\ )$.

Assume $j \geqslant K$. Want: $d(t_j, q) < \varepsilon$.

Since t is a subsequence of s,

choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.

Then $\ell_j \geqslant j \geqslant K$, so, by choice of K, we get: $d(s_{\ell_j}, q) < \varepsilon$.

Then $d(t_j, q) = d((s \circ \ell)_j, q) = d(s_{\ell_j}, q) < \varepsilon$, as desired. □

THEOREM 3.14.4. *Let X be a metric space, $s \in X^{\mathbb{N}}$.*

Let t be a subsequence of s.

Assume: s is convergent in X. Then: t is convergent in X.

Proof. Since s is convergent in X, choose $q \in X$ s.t. $s \to q$ in X.

By Theorem 3.14.3, $t \to q$ in X. Then t is convergent in X. □

THEOREM 3.14.5. *Let X and Y be metric spaces.*

Let $Z := X \times Y$ and let $s \in Z_{\mathrm{bi}}^{\mathbb{N}}$.

Assume X is proper. Define $\alpha : Z \to X$ by: $\forall p \in Z$, $\alpha_p = p_1$.

Then: $\exists$subsequence t of s s.t. $\alpha \circ t$ is convergent in X.

Proof. Since α is Lipschitz-1 from Z to X and since $s \in Z_{\mathrm{bi}}^{\mathbb{N}}$,

by Theorem 3.14.2, we get: $\alpha \circ s \in X_{\mathrm{bi}}^{\mathbb{N}}$.

So, since X is proper, we get: $\alpha \circ s$ is subconvergent in X.

Choose a subsequence v of $\alpha \circ s$ s.t. v is convergent in X.

Choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $v = (\alpha \circ s) \circ \ell$.

Let $t := s \circ \ell$. Then t is a subsequence of s.

Want: $\alpha \circ t$ is convergent in X.

We have $\alpha \circ t = \alpha \circ (s \circ \ell) = (\alpha \circ s) \circ \ell = v$.

So, since v is convergent in X, we get:

$\alpha \circ t$ is convergent in X, as desired. □

THEOREM 3.14.6. *Let X and Y be metric spaces.*

Let $Z := X \times Y$ and let $t \in Z_{\mathrm{bi}}^{\mathbb{N}}$.

Assume Y is proper. *Define $\beta : Z \to Y$ by: $\forall p \in Z$, $\beta_p = p_2$.*
Then: $\exists$subsequence u of t s.t. $\beta \circ u$ is convergent in X.

Proof. Unassigned HW. □

THEOREM 3.14.7. *Let f be a function and let $t \in \mathbb{D}_f^{\mathbb{N}}$.*
Let u be a subsequence of t. *Then $f \circ u$ is a subsequence of $f \circ t$.*

Proof. Want: $\exists$strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $f \circ u = (f \circ t) \circ \ell$.
Since u is a subsequence of t,
choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $u = t \circ \ell$.
Then ℓ is strictly-increasing and $\ell \in \mathbb{N}^{\mathbb{N}}$. Want: $f \circ u = (f \circ t) \circ \ell$.
We have $f \circ u = f \circ (t \circ \ell) = (f \circ t) \circ \ell$, as desired. □

THEOREM 3.14.8. *Let X and Y be metric spaces.*
Let $Z := X \times Y$ and let $s \in Z^{\mathbb{N}}$.
Define $\alpha : Z \to X$ by: $\forall p \in Z$, $\alpha_p = p_1$.
Define $\beta : Z \to Y$ by: $\forall p \in Z$, $\beta_p = p_2$.
Assume $\alpha \circ u$ is convergent in X and $\beta \circ u$ is convergent in Y.
Then u is convergent in Z.

Proof. Let $s := \alpha \circ u$ and let $t := \beta \circ u$. Then $(s, t) = u$.
Also, s is convergent in X, so choose $a \in X$ s.t. $s \to a$ in X.
Also, t is convergent in Y, so choose $b \in Y$ s.t. $t \to b$ in Y.
Then, by HW#9-3, we get: $(s, t) \to (a, b)$ in $X \times Y$.
So, since $u = (s, t)$ and since $Z = X \times Y$,
we conclude: $u \to (a, b)$ in Z. Then u is convergent in Z. □

THEOREM 3.14.9. *Let X and Y be proper metric spaces.*
Then $X \times Y$ is proper.

Proof. Let $Z := X \times Y$. Want: Z is proper.
Want: $\forall s \in Z_{\text{bi}}^{\mathbb{N}}$, s is subconvergent in Z.
Given $s \in Z_{\text{bi}}^{\mathbb{N}}$. Want: s is subconvergent in Z.
Want: $\exists$subsequence u of s s.t. u is convergent in Z.
Define $\alpha : Z \to X$ by: $\forall p \in Z$, $\alpha_p = p_1$.
By Theorem 3.14.5, choose a subsequence t of s s.t.
$\alpha \circ t$ is convergent in X.
Define $\beta : Z \to Y$ by: $\forall p \in Z$, $\beta_p = p_2$.
Since t is a subsequence of s and since $s \in Z_{\text{bi}}^{\mathbb{N}}$,
by Theorem 3.13.3, we conclude: $t \in Z_{\text{bi}}^{\mathbb{N}}$.
By Theorem 3.14.6, choose a subsequence u of t s.t.

$\beta \circ u$ is convergent in Y.

Since u is a subsequence of t and since t is a subsequence of s,
by Theorem 3.2.3, we see that u is a subsequence of s.
Want: u is convergent in Z.
Since u is a subsequence of t, by Theorem 3.14.7,
we get: $\alpha \circ u$ is a subsequence of $\alpha \circ t$.
So, since $\alpha \circ t$ is convergent in X, by Theorem 3.14.4, we get:
$\alpha \circ u$ is convergent in X.
So, since $\beta \circ u$ is convergent in Y, we get: u is convergent in Z. □

THEOREM 3.14.10. $\mathbb{R}^2$ *is proper.*

Proof. By Theorem 3.13.5, $\mathbb{R}$ is proper.
So, by Theorem 3.14.9, $\mathbb{R} \times \mathbb{R}$ is proper.
So, since $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, we see that $\mathbb{R}^2$ is proper. □

THEOREM 3.14.11. $\mathbb{R}^3$ *is proper.*

Proof. By Theorem 3.13.5, $\mathbb{R}$ is proper.
By Theorem 3.14.10, $\mathbb{R}^2$ is proper.
Then, by Theorem 3.14.9, $\mathbb{R}^2 \times \mathbb{R}$ is proper.
So, since $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, we see that $\mathbb{R}^3$ is proper. □

Using induction, we may show: $\forall k \in \mathbb{N}$, $\mathbb{R}^k$ is proper.

THEOREM 3.14.12. *Let X be a metric space.*
Then: (X *is compact*) $\Leftrightarrow$ (X *is proper and bounded*).

Proof. Proof of $\Rightarrow$*:*
Assume: X is compact. Want: X is proper and bounded.
By Theorem 3.9.1, X is bounded. Want: X is proper.
Want: $\forall s \in X^{\mathbb{N}}_{\mathrm{bi}}$, s is subconvergent in X.
Given $s \in X^{\mathbb{N}}_{\mathrm{bi}}$. Want: s is subconvergent in X.
Since $s \in X^{\mathbb{N}}$ and X is compact, s is subconvergent in X, as desired.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$*:*
Assume: X is proper and bounded. Want: X is compact.
Want: $\forall s \in X^{\mathbb{N}}$, s is subconvergent in X.
Given $s \in X^{\mathbb{N}}$. Want: s is subconvergent in X.
Since X is bounded and $\mathbb{I}_s \subseteq X$, we get: $\mathbb{I}_s$ is bounded. Then $s \in X^{\mathbb{N}}_{\mathrm{bi}}$.

So, since X is proper, we see that s is subconvergent in X, as desired.
End of proof of $\Leftarrow$. □

DEFINITION 3.14.13. *Let X be a metric space and let $a \in \mathbb{R}$. By X is a-**bounded**, we mean:* $\forall p, q \in X$, $d(p,q) < a$.

Note that a sequence in a metric space is Cauchy iff
$\forall \varepsilon > 0$, the sequence has an ε-bounded tail,
by which we mean
$\forall \varepsilon > 0$, some tail of the sequence has ε-bounded image.

THEOREM 3.14.14. *Let X be a metric space. Then:* (X *is bounded*) $\Leftrightarrow$ ($\exists a \geqslant 0$ *s.t.* X *is* a*-bounded*).

THEOREM 3.14.15. *Let X and Y be bounded metric spaces. Then $X \times Y$ is bounded.*

Proof. Choose $a, b \geqslant 0$ s.t. X is a-bounded and Y is b-bounded.
Let $c := \sqrt{a^2 + b^2}$. Then $X \times Y$ is c-bounded.
Then $X \times Y$ is bounded. □

THEOREM 3.14.16. *Let X and Y be compact metric spaces. Then $X \times Y$ is compact.*

Proof. By $\Rightarrow$ of Theorem 3.14.12, X and Y are proper and bounded.
So, by Theorem 3.14.9 and Theorem 3.14.15,
we see that: $X \times Y$ is proper and bounded.
Then, by $\Leftarrow$ of Theorem 3.14.12, we get: $X \times Y$ is compact. □

3.15. **Density of $\mathbb{Q}$ in $\mathbb{R}$.**

DEFINITION 3.15.1. *Let X be a metric space, $T \subseteq X$, $\varepsilon > 0$. By T is ε-**net** in X, we mean:* $\forall w \in X$, $\exists s \in T$ *s.t.* $d(w,s) < \varepsilon$.

THEOREM 3.15.2. $\mathbb{Z}$ *is* 1*-net in* $\mathbb{R}$.

Proof. Want: $\forall w \in \mathbb{R}$, $\exists s \in \mathbb{Z}$ s.t. $d(w,s) < \varepsilon$.
Given $w \in \mathbb{R}$. Want: $\exists s \in \mathbb{Z}$ s.t. $d(w,s) < \varepsilon$.
By the Archimedean Principle, choose $j \in \mathbb{N}$ s.t. $j > 1 - w$.
Then $w + j > 1$. Let $x := w + j$. Then $x > 1$. Also, $x - j = w$.
Let $A := \{i \in \mathbb{N} \,|\, i > x\}$. By the Archimedan Principle, $A \neq \varnothing$.
So, since $A \subseteq \mathbb{N}$, by the Well-Ordering Axiom, we have: min $A \neq$ ☹.
Then min $A \in A$. Let $k :=$ min A. Then $k \in A$.
So, by definition of A, we have: $k \in \mathbb{N}$ and $k > x$.

Since $k > x > 1$ and $k \in \mathbb{N}$, we get $k \in (1..\infty)$, so $k - 1 \in (0..\infty)$.
Since $k - 1 < k = \min A \leqslant A$, we get: $k - 1 \notin A$.
So, since $k \in (0..\infty) = \mathbb{N}$, we get: $\neg(k - 1 > x)$. Then $k - 1 \leqslant x$.
Let $s := k - j - 1$. As $j, k \in \mathbb{N} \subseteq \mathbb{Z}$, $s \in \mathbb{Z}$. Want: $d(w, s) < 1$.
Since $k - 1 \leqslant x < k$, we get: $k - j - 1 \leqslant x - j < k - j$.
So, since $k - j - 1 = s$ and $x - j = w$ and $k - j = s + 1$,
we get $s \leqslant w < s + 1$,
and so $0 \leqslant w - s < 1$.
Since $w - s \geqslant 0$, we get: $|w - s| = w - s$.
Then $d(w, s) = |w - s| = w - s < 1$, as desired. □

THEOREM 3.15.3. *Let X be a metric space, $U \subseteq X$, $T \subseteq U$, $\varepsilon > 0$. Assume T is ε-net in X. Then U is ε-net in X.*

Proof. Unassigned HW. □

DEFINITION 3.15.4. *Let X be a metric space, $T \subseteq X$. By T is* **dense** *in X, we mean: $\mathrm{Cl}_X T = X$.*

THEOREM 3.15.5. *Let X be a metric space, $U \subseteq X$, $T \subseteq U$. Assume T is dense in X. Then U is dense in X.*

Proof. Want: $\mathrm{Cl}_X U = X$. Since $\mathrm{Cl}_X U \subseteq X$, we want: $X \subseteq \mathrm{Cl}_X U$.
Since $T \subseteq U$, $\mathrm{Cl}_X T \subseteq \mathrm{Cl}_X U$. Since T is dense in X, $\mathrm{Cl}_X T = X$.
Then $X = \mathrm{Cl}_X T \subseteq \mathrm{Cl}_X U$, as desired. □

THEOREM 3.15.6. *Let X be a metric space and let $A \subseteq X$.*
Then $\mathrm{Cl}_X^{\mathrm{seq}} A \ := \ \{\, p \in X \mid \exists s \in A^{\mathbb{N}} \text{ s.t. } s \to p \text{ in } X \,\}$.

THEOREM 3.15.7. *Let X be a metric space and let $A \subseteq X$.*
Then $A \ \subseteq \ \mathrm{Cl}_X^{\mathrm{seq}} A$.

Proof. Want: $\forall p \in A$, $p \in \mathrm{Cl}_X^{\mathrm{seq}} A$. Given $p \in A$. Want: $p \in \mathrm{Cl}_X^{\mathrm{seq}} A$.
We have: $C_{\mathbb{N}}^p \to p$ in X. Want: $\exists s \in A^{\mathbb{N}}$ s.t. $s \to p$ in X.
Let $s := C_{\mathbb{N}}^p$. Since $p \in A$, we get $s \in A^{\mathbb{N}}$. Want $s \to p$ in X.
Since $C_{\mathbb{N}}^p \to p$ in X and since $s = C_{\mathbb{N}}^p$, we get: $s \to p$ in X as desired. □

THEOREM 3.15.8. *Let X be a metric space and let $A \subseteq X$.*
Then: $\partial_X^{\mathrm{seq}} X \ \subseteq \ \mathrm{Cl}_X^{\mathrm{seq}} A$.

Proof. Want: $\forall p \in \partial_X^{\mathrm{seq}} X$, $p \in \mathrm{Cl}_X^{\mathrm{seq}} A$.
Given $p \in \partial_X^{\mathrm{seq}} X$. Want: $p \in \mathrm{Cl}_X^{\mathrm{seq}} A$.
Want: $\exists s \in A^{\mathbb{N}}$ s.t. $s \to p$ in X.

Since $p \in \partial_X^{\rm seq} X$, we know:

$\exists s \in A^{\mathbb{N}}$ s.t. $s \to p$ in X and $\exists t \in (X \backslash A)^{\mathbb{N}}$ s.t. $t \to p$ in X.

Then: $\exists s \in A^{\mathbb{N}}$ s.t. $s \to p$ in X, as desired. □

THEOREM 3.15.9. *Let X be a metric space and let $A \subseteq X$. Then:* $\quad \mathrm{Cl}_X^{\rm seq} X \; = \; \mathrm{Cl}_X A$.

Proof. By HW#11-5, $\mathrm{Cl}_X^{\rm seq} X \subseteq \mathrm{Cl}_X A$. Want: $\mathrm{Cl}_X A \subseteq \mathrm{Cl}_X^{\rm seq} A$.
By Theorem 3.15.7, we have: $A \subseteq \mathrm{Cl}_X^{\rm seq} A$.
By Theorem 3.10.5, we have: $\partial_X A = \partial_X^{\rm seq} A$.
By Theorem 3.15.8, we have: $\partial_X^{\rm seq} A \subseteq \mathrm{Cl}_X^{\rm seq} A$.
Since $A \subseteq \mathrm{Cl}_X^{\rm seq} A$ and $\partial_X A = \partial_X^{\rm seq} A \subseteq \mathrm{Cl}_X A$, we get:

$A \bigcup (\partial_X A) \subseteq \mathrm{Cl}_X^{\rm seq} A$.

Then $\mathrm{Cl}_X A = A \bigcup (\partial_X A) \subseteq \mathrm{Cl}_X^{\rm seq} A$, as desired. □

3.16. Compact vs closed and bounded.

THEOREM 3.16.1. *Let Y be a metric space, $X \subseteq Y$, $p \in X$, $r > 0$. Then $B_X(p,r) = (B_Y(p,r)) \cap X$.*

Proof. Unassigned HW. □

The next result is **Absoluteness of Bounded**:

THEOREM 3.16.2. *Let Y be a metric space, $X \subseteq Y$, $K \subseteq X$. Then: (K is bounded in X) $\Leftrightarrow$ (K is bounded in Y).*

Proof. Proof of $\Rightarrow$: Unassigned HW. *End of proof of $\Rightarrow$.*

Proof of $\Leftarrow$:
Assume K is bounded in Y. Want: K is bounded in X.
Assume K is not bounded in X. Want: Contradiction.
Since K is not bounded in X, while $\varnothing$ is bounded in X,
we conclude that $K \neq \varnothing$.
So, since K is bounded in Y, choose $B \in \mathcal{B}_Y$ s.t. $K \subseteq B$.
Since $X \supseteq K \neq \varnothing$, we get: $X \neq \varnothing$, so choose $p \in X$.
By the Superset Recentering Theorem, choose $C \in \mathcal{B}_Y(p)$ s.t. $B \subseteq C$.
Since $C \in \mathcal{B}_Y(p)$, choose $r > 0$ s.t. $C = B_Y(p,r)$.
Let $A := B_X(p,r)$. By Theorem 3.16.1, we have: $A = C \cap X$.
By hypothesis, we have: $K \subseteq X$.
Since $K \subseteq B \subseteq C$ and $K \subseteq X$, we get: $K \subseteq C \cap X$.
Then $K \subseteq C \cap X = A$.

So, since $A \in \mathcal{B}_X(p) \subseteq \mathcal{B}_X$, we see that K is bounded in X.
Recall that K is not bounded in X. Contradiction.
End of proof of $\Rightarrow$. □

The next result is called **Absoluteness of Limit**:

THEOREM 3.16.3. *Let Y be a metric space and let $X \subseteq Y$. Let $s \in X^{\mathbb{N}}$ and let $q \in X$.*
Then: $(s \to q \text{ in } X) \quad \Leftrightarrow \quad (s \to q \text{ in } Y)$.

Proof. Proof of $\Rightarrow$*:* Unassigned HW. *End of proof of* $\Rightarrow$.

Proof of $\Leftarrow$*:*
Assume $s \to q$ in Y. Want: $s \to q$ in X.
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(j \geqslant K) \Rightarrow (d_X(s_j, q) < \varepsilon).$$
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(j \geqslant K) \Rightarrow (d_X(s_j, q) < \varepsilon).$$
Since s is Cauchy in Y, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(j \geqslant K) \Rightarrow (d_Y(s_j, q) < \varepsilon).$$
Then $K \in \mathbb{N}$. Want: $\forall j \in \mathbb{N}$, $(j \geqslant K) \Rightarrow (d_X(s_j, q) < \varepsilon)$.
Given $j \in \mathbb{N}$. Want: $(j \geqslant K) \Rightarrow (d_X(s_j, q) < \varepsilon)$.
Assume $j \geqslant K$. Want: $d_X(s_j, q) < \varepsilon$.
Since $s \in X^{\mathbb{N}}$, we get $s_j \in X$. By hypothesis, we have $q \in X$.
Since $s_j, q \in X$, we conclude that $d_X(s_j, q) = d_Y(s_j, q)$.
Then $d_X(s_j, q) = d_Y(s_j, q) < \varepsilon$, as desired.
End of proof of $\Leftarrow$. □

THEOREM 3.16.4. *Let Y be a metric space, $X \subseteq Y$, $s \in X^{\mathbb{N}}$. Assume s is convergent in X. Then s is convergent in Y.*

Proof. Sicne s is convergent in X, choose $q \in X$ s.t. $s \to q$ in X.
Then, by Absoluteness of Limit, we have: $s \to q$ in Y.
So, since $q \in X \subseteq Y$, we get: s is convergent in Y, as desired. □

THEOREM 3.16.5. *Let $Y := \mathbb{R}$ and let $X := (0; \infty)$. Define $s \in Y^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $s_j = 1/j$. Then s is convergent in Y and s is not convergent in X.*

DEFINITION 3.16.6. *Let X be a metric space.*
Then $\mathcal{T}_X := \{U \subseteq X \,|\, U \text{ is open in } X\}$ *and*
$\mathcal{T}'_X := \{C \subseteq X \,|\, C \text{ is closed in } X\}$.

THEOREM 3.16.7. *Let Y be a metric space, $X \in \mathcal{T}'_Y$, $s \in X^{\mathbb{N}}$. Then:* $($ *s is convergent in X* $) \Leftrightarrow ($ *s is convergent in Y* $)$.

Proof. By Theorem 3.16.4, we have:
(s is convergent in X) $\Rightarrow$ (s is convergent in Y).
Want: (s is convergent in X) $\Leftarrow$ (s is convergent in Y).
Assume: s is convergent in Y. Want: s is convergent in X.
Since s is convergent in Y, choose $q \in Y$ s.t. $s \to q$ in Y.
Since $s \in X^{\mathbb{N}}$ and $s \to q$ in Y, we get: $q \in \mathrm{Cl}_Y^{\mathrm{seq}} X$.
Since $X \in \mathcal{T}'_Y$, we see that X is closed in Y, so $\mathrm{Cl}_Y X = X$.
By Theorem 3.15.9, $\mathrm{Cl}_Y^{\mathrm{seq}} X = \mathrm{Cl}_Y X$. Then $q \in \mathrm{Cl}_Y^{\mathrm{seq}} X = \mathrm{Cl}_Y X = X$.
Since $s \to q$ in Y and $s \in X^{\mathbb{N}}$ and $q \in X$,
it follows, by Absoluteness of Limit, that: $s \to q$ in X. □

THEOREM 3.16.8. *Let Y be a metric space, $X \in \mathcal{T}'_Y$, $s \in X^{\mathbb{N}}$. Then:* $($ *s is subconvergent in X* $) \Leftrightarrow ($ *s is subconvergent in Y* $)$.

Proof. Proof of $\Rightarrow$: Unassigned HW. *End of proof of $\Rightarrow$.*

Proof of $\Leftarrow$:
Want: (s is subconvergent in X) $\Leftarrow$ (s is subconvergent in Y).
Assume: s is subconvergent in Y. Want: s is subconvergent in X.
Since s is subconvergent in Y,
choose a subsequence t of s s.t. t is convergent in Y.
Since $\mathbb{D}_t = \mathbb{N}$ and $\mathbb{I}_t \subseteq \mathbb{I}_s \subseteq X$, we get: $t \in X^{\mathbb{N}}$.
Then, by Theorem 3.16.7, we conclude: t is convergent in X.
So, since t is a subsequence of s, we see that
s is subconvergent in X, as desired.
End of proof of $\Rightarrow$. □

THEOREM 3.16.9. *Let X be a metric space and let $K \subseteq X$. Assume K is compact. Then K is closed and bounded in X.*

Proof. Since K is compact, by Theorem 3.9.1, K is bounded.
Then K is bounded in K, so, by Absoluteness of Bounded,
K is bounded in X. Want: K is closed in X.
Want: $\mathrm{Cl}_X K = K$. Since $K \subseteq \mathrm{Cl}_X K$, we want: $\mathrm{Cl}_X K \subseteq K$.
Want: $\forall p \in \mathrm{Cl}_X K$, $p \in K$. Given $p \in \mathrm{Cl}_X K$. Want: $p \in K$.
Since $p \in \mathrm{Cl}_X K = \mathrm{Cl}_X^{\mathrm{seq}} K$, choose $s \in K^{\mathbb{N}}$ s.t. $s \to p$ in X.
Since $s \in K^{\mathbb{N}}$ and since K is compact, s is subconvergent in K,
so choose a subsequence t of s s.t. t is convergent in K.

Since t is convergent in K, choose $q \in K$ s.t. $t \to q$ in K.
Since $q \in K$ it suffices to show: $p = q$.
Since $s \to p$ in X and since t is a subsequence of s,
by Theorem 3.14.3, we get: $t \to p$ in X.
Since $t \to p$ in X and $t \to q$ in X,
by Theorem 2.15.1, we get: $p = q$, as desired. □

THEOREM 3.16.10. *Let X be a metric space and let $K \subseteq X$. Assume K is bounded in X. Then $K^{\mathbb{N}} \subseteq X_{\mathrm{bi}}^{\mathbb{N}}$.*

Proof. Want: $\forall s \in K^{\mathbb{N}}$, $s \in X_{\mathrm{bi}}^{\mathbb{N}}$. Given $s \in K^{\mathbb{N}}$. Want: $s \in X_{\mathrm{bi}}^{\mathbb{N}}$.
Since $s \in K^{\mathbb{N}} \subseteq X^{\mathbb{N}}$, we want: $\mathbb{I}_s$ is bounded in X.
Since $\mathbb{I}_s \subseteq K$ and since K is bounded in X,
we conclude that $\mathbb{I}_s$ is bounded in X, as desired. □

THEOREM 3.16.11. *Let X be a proper metric space and let $K \subseteq X$. Then: (K is compact) $\Leftrightarrow$ (K is closed and bounded in X).*

Proof. By Theorem 3.16.9, we have:
(K is compact) $\Rightarrow$ (K is closed and bounded in X).
Want: (K is compact) $\Leftarrow$ (K is closed and bounded in X).
Assume: K is closed and bounded in X. Want: K is compact.
Want: $\forall s \in K^{\mathbb{N}}$, s is subconvergent in K.
Given $s \in K^{\mathbb{N}}$. Want: s is subconvergent in K.
Since K is closed in X, we get $K \in \mathcal{T}'_X$.
So, by Theorem 3.16.8, it suffices to show: s is subconvergent in X.
Since K is bounded in X, by Theorem 3.16.10, we get: $K^{\mathbb{N}} \subseteq X_{\mathrm{bi}}^{\mathbb{N}}$.
Since $s \in K^{\mathbb{N}} \subseteq X_{\mathrm{bi}}^{\mathbb{N}}$ and since X is proper,
it follows that: s is subconvergent in X, as desired. □

3.17. **The unit circle is compact.**

DEFINITION 3.17.1. *Let X be a metric space and let $z \in X$. Then $d_X(z, \bullet) : X \to [0; \infty)$ is defined by:*

$$\forall p \in X, \quad (d_X(z, \bullet))_p = d_X(z, p).$$

By sloppiness, we sometimes write $d(z, \bullet)$ for $d_X(z, \bullet)$.

THEOREM 3.17.2. *Let X be a metric space and let $z \in X$. Then $d_X(z, \bullet)$ is Lipschitz-1 from X to $\mathbb{R}$.*

Proof. Let $f := d_X(z,\bullet)$. Want: f is Lipschitz-1 from X to $\mathbb{R}$.
Want: $\forall p,q \in \mathbb{D}_f,\ \ d_{\mathbb{R}}(f_p,f_q) \leqslant d_X(p,q)$.
Given $p,q \in \mathbb{D}_f$. Want: $d_{\mathbb{R}}(f_p,f_q) \leqslant d_X(p,q)$.
Since $f = d_X(z,\bullet)$, we get $\mathbb{D}_f = X$. Then $p,q \in \mathbb{D}_f = X$.
Let $\varepsilon := d_X(p,q)$. Want: $d_{\mathbb{R}}(f_p,f_q) \leqslant \varepsilon$.
Want: $(\ f_p \leqslant f_q + \varepsilon\ )\ \&\ (\ f_q \leqslant f_p + \varepsilon\ )$.
We have $f_p = (d_X(z,\bullet)_p = d_X(z,p) \leqslant (d_X(z,q)) + (d_X(q,p))$
$= (d_X(z,\bullet))_q + (d_X(p,q)) = f_q + \varepsilon$. Want: $f_q \leqslant f_p + \varepsilon$.
We have $f_q = (d_X(z,\bullet)_q = d_X(z,q) \leqslant (d_X(z,p)) + (d_X(p,q))$
$= (d_X(z,\bullet))_p + (d_X(p,q)) = f_p + \varepsilon$, as desired. □

THEOREM 3.17.3. *Let X and Y be metric spaces.*
Let $f : X \to Y$ and let $U \in \mathcal{T}_Y$.
Assume that f is continuous from X to Y. Then $f^*U \in \mathcal{T}_X$.

Proof. Want: f^*U is open in X. Want: $\mathrm{Int}_X(f^*U) = f^*U$.
We have: $\mathrm{Int}_X(f^*U) \subseteq f^*U$. Want: $f^*U \subseteq \mathrm{Int}_X(f^*U)$.
Want: $\forall p \in f^*U,\ \ p \in \mathrm{Int}_X(f^*U)$.
Given $p \in f^*U$. Want: $p \in \mathrm{Int}_X(f^*U)$.
Want: $\exists A \in \mathcal{B}_X(p)$ s.t. $A \subseteq f^*U$.
Since $U \in \mathcal{T}_Y$, we see that U is open in Y, so $\mathrm{Int}_Y U = U$.
Since $p \in f^*U$, we get: $f_p \in U$. Then $f_p \in U = \mathrm{Int}_Y U$.
Since $f_p \in \mathrm{Int}_Y U$, choose $B \in \mathcal{B}_Y(f_p)$ s.t. $B \subseteq U$.
Since $B \in \mathcal{B}_Y(f_p)$, choose $\varepsilon > 0$ s.t. $B = B(f_p,\varepsilon)$.
By hypothesis, f is continuous from X to Y,
so f is continuous on $\mathbb{D}_f$ from X to Y.
As $p \in f^*U$, we get: $p \in \mathbb{D}_f$. Then f is continuous at p from X to Y,
so choose $\delta > 0$ s.t. $\forall q \in \mathbb{D}_f,\ \ (\ d(p,q) < \delta\ ) \Rightarrow (\ d(f_p,f_q) < \varepsilon\ )$.
Let $A := B_X(p,\delta)$. Then $A \in \mathcal{B}_X(p)$. Want: $A \subseteq f^*U$.
Want: $\forall q \in A,\ \ q \in f^*U$. Given $q \in A$. Want: $q \in f^*U$.
Since $q \in A = B_X(p,\delta)$, we see that $q \in X$ and that $d(p,q) < \delta$.
Since $f : X \to Y$, we get $\mathbb{D}_f = X$. Then $q \in X = \mathbb{D}_f$.
Since $q \in \mathbb{D}_f$ and since $d(p,q) < \delta$, by choice of δ,
we see that $d(f_p,f_q) < \varepsilon$, and so $f_q \in B_Y(f_p,\varepsilon)$.
Then $f_q \in B_Y(f_p,\varepsilon) = B \subseteq U$, and so $q \in f^*U$, as desired. □

THEOREM 3.17.4. *Let f be a function. Let A and B be sets.*
Then $f^*(A \bigcup B) = (f^*A) \bigcup (f^*B)$ *and*
$f^*(A \bigcap B) = (f^*A) \bigcap (f^*B)$ *and*

$$f^*(A\backslash B) = (f^*A)\backslash(f^*B).$$
Also, $$f_*(A \bigcup B) = (f_*A) \bigcup (f_*B).$$

THEOREM 3.17.5. *Let X and Y be metric spaces.*
Let $f : X \to Y$ and let $C \in \mathcal{T}'_Y$.
Assume that f is continuous from X to Y. Then $f^*C \in \mathcal{T}'_X$.

Proof. Let $U := Y\backslash C$. Since $C \in \mathcal{T}'_Y$, we get: $U \in \mathcal{T}_Y$.
Then, by Theorem 3.17.3, we get: $f^*U \in \mathcal{T}_X$.
It follows that $X\backslash(f^*U) \in \mathcal{T}'_X$. Want: $f^*C = X\backslash(f^*U)$.
As $C \subseteq Y$, we get: $Y\backslash(Y\backslash C) = C$. As $f : X \to Y$, we get: $f^*Y = X$.
We have $Y\backslash U = Y\backslash(Y\backslash C) = C$, and so $C = Y\backslash U$.
Then $f^*C = f^*(Y\backslash U) = (f^*Y)\backslash(f^*U) = X\backslash(f^*U)$, as desired. □

THEOREM 3.17.6. *Let X and Y be metric spaces, $f : X \to Y$.*
Then: ($\forall U \in \mathcal{T}_Y$, $f^*U \in \mathcal{T}_X$) $\Leftrightarrow$ (f is continuous from X to Y).

Proof. By Theorem 3.17.3, we have $\Leftarrow$. Want: $\Rightarrow$.
By HW#12-2, we have $\Rightarrow$. □

THEOREM 3.17.7. *Let X and Y be metric spaces, $f : X \to Y$.*
Then: ($\forall C \in \mathcal{T}'_Y$, $f^*C \in \mathcal{T}'_X$) $\Leftrightarrow$ (f is continuous from X to Y).

Proof. By Theorem 3.17.5, we have $\Leftarrow$. Want: $\Rightarrow$.
Assume: $\forall C \in \mathcal{T}'_Y$, $f^*C \in \mathcal{T}'_X$. Want: f is continuous from X to Y.
Then, by HW#12-2, it suffices to show: $\forall U \in \mathcal{T}_Y$, $f^*U \in \mathcal{T}_X$.
Given $U \in \mathcal{T}_Y$. Want: $f^*U \in \mathcal{T}_X$.
Since $U \in \mathcal{T}_Y$, we get: $Y\backslash U \in \mathcal{T}'_Y$. Let $C := Y\backslash U$. Then $C \in \mathcal{T}'_Y$.
Then $f^*C \in \mathcal{T}'_X$, so $X\backslash(f^*C) \in \mathcal{T}_X$. Want: $X\backslash(f^*C) = f^*U$.
Since $C = Y\backslash U$, we get $f^*C = (f^*Y)\backslash(f^*U)$.
So, since $f^*Y = X$, we get $f^*C = X\backslash(f^*U)$.
Since $f^*U \subseteq X$, we conclude that: $X\backslash(X\backslash(f^*U)) = f^*U$.
Then $X\backslash(f^*C) = X\backslash(X\backslash(f^*U)) = f^*U$, as desired. □

THEOREM 3.17.8. *Let X be a metric space, $p \in X$. Then $\{p\} \in \mathcal{T}'_X$.*

Proof. Let $A := \{p\}$. Want: $A \in \mathcal{T}'_X$.
Want: A is closed in X. Want: $\mathrm{Cl}_X A = A$.
We have: $A \subseteq \mathrm{Cl}_X A$. Want: $\mathrm{Cl}_X A \subseteq A$.
Want: $\forall q \in \mathrm{Cl}_X A$, $q \in A$. Given $q \in \mathrm{Cl}_X A$. Want: $q \in A$.
We have $q \in \mathrm{Cl}_X A = \mathrm{Cl}^{\mathrm{seq}}_X A$, so choose $s \in A^{\mathbb{N}}$ s.t. $s \to q$ in X.
We have: $\forall j \in \mathbb{N}$, $s_j \in A = \{p\}$, so $s_j = p$. Then $s = C^p_{\mathbb{N}}$.

So, since $C_{\mathbb{N}}^p \to p$ in X, we see that $s \to p$ in X.
So, since $s \to q$ in X, we get: $p = q$. Then $q = p \in \{p\} = A$. □

THEOREM 3.17.9. *Let* $X := \mathbb{R}^2$ *and let* $C := S_X(0_X, 1)$.
Then C *is compact.*

Proof. Since X is proper, by Theorem 3.16.11,
it suffices to show: C is closed and bounded in X.
Since $C = S_X(0_X, 1) \subseteq B_X(0_x, 2)$, we see that C is bounded in X.
Want: C is closed in X. Want: $C \in \mathcal{T}_X'$. Let $f := d_X(0_X, \bullet)$.
By Theorem 3.17.2, f is Lipschitz-1 from X to $\mathbb{R}$,
and it follows that: f is continuous from X to $\mathbb{R}$.
By Theorem 3.17.8, we get: $\{1\} \in \mathcal{T}_{\mathbb{R}}'$.
Then, by Theorem 3.17.5, we get: $f^*(\{1\}) \in \mathcal{T}_X'$. Want: $f^*(\{1\}) = C$.
We have

$$\begin{aligned} f^*(\{1\}) &= \{ p \in X \mid f_p \in \{1\} \} \\ &= \{ p \in X \mid f_p = 1 \} \\ &= \{ p \in X \mid (d(0_X, \bullet))_p = 1 \} \\ &= \{ p \in X \mid d(0_X, p) = 1 \} \\ &= S_X(0_X, 1) = C, \text{ as desired.} \end{aligned}$$

□

3.18. **The Extreme Value Theorem.**

THEOREM 3.18.1. *Let* $A \subseteq \mathbb{R}$ *and let* $x \in \mathbb{R}$.
Assume: $A \leqslant x \in A$.
Then: $x = \max A$.

Proof. Want: $x = \mathrm{UE}(A \bigcap \mathrm{UB}_A)$. Want: $A \bigcap \mathrm{UB}_A = \{x\}$.
As $A \leqslant x$, we get: $X \in \mathrm{UB}_A$. So, since $x \in A$, we get: $x \in A \bigcap \mathrm{UB}_A$.
Then $\{x\} \subseteq A \bigcap \mathrm{UB}_A$. Want: $A \bigcap \mathrm{UB}_A \subseteq \{x\}$.
Given $y \in A \bigcap \mathrm{UB}_A$. Want: $y \in \{x\}$. Want: $y = x$.
Since $y \in A \leqslant x$, we get: $y \leqslant x$. Want: $x \leqslant y$.
Since $y \in \mathrm{UB}_A$, we get: $A \leqslant y$. Then $x \in A \leqslant y$, so $x \leqslant y$. □

THEOREM 3.18.2. *Let* $K \subseteq \mathbb{R}$. *Say* K *is compact and nonempty.*
Then: $\max K \neq$ ☹.

Proof. Since K is compact, by Theorem 3.16.9, we conclude:
K is closed and bounded in $\mathbb{R}$.
By hypothesis, we have: $K \neq \emptyset$.
So, since K is bounded in $\mathbb{R}$, by Theorem 3.9.10, we get: $\sup K \in \mathbb{R}$.
Let $x := \sup K$. Then $x \in \mathbb{R}$, so $x \neq$ ☹. Want: $x = \max K$.
By Theorem 3.18.1, it suffices to show: $K \leqslant x \in K$.

We have $K \leqslant \sup K = x$, and so $K \leqslant x$. Want: $x \in K$.
Since K is closed in $\mathbb{R}$, we get: $\mathrm{Cl}_{\mathbb{R}} K = K$.
Want: $x \in \mathrm{Cl}_{\mathbb{R}} K$. Want: $\forall B \in \mathcal{B}_{\mathbb{R}}(x)$, $B \cap K \neq \varnothing$.
Given $B \in \mathcal{B}_{\mathbb{R}}(x)$. Want: $B \cap K \neq \varnothing$. Want: $\exists y \in K$ s.t. $y \in B$.
Since $B \in \mathcal{B}_{\mathbb{R}}(x)$, choose $\varepsilon > 0$ s.t. $B = B_{\mathbb{R}}(x, \varepsilon)$.
We have $x > x - \varepsilon$, so $\neg(x \leqslant x - \varepsilon)$.
So, as $x = \sup K$, we get $\neg(\sup K \leqslant x - \varepsilon)$.
Then $\neg(K \leqslant x - \varepsilon)$, so choose $y \in K$ s.t. $y > x - \varepsilon$.
Then $y \in K$. Want: $y \in B$.
We have $y \in K \leqslant \sup K = x < x + \varepsilon$, so $y < x + \varepsilon$.
Then $x - \varepsilon < y < x + \varepsilon$, so $y \in (x - \varepsilon; x + \varepsilon)$.
Then $y \in (x - \varepsilon; x + \varepsilon) = B_{\mathbb{R}}(x, \varepsilon) = B$, as desired. □

THEOREM 3.18.3. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$ and let $K \subseteq \mathbb{D}_f$.
Assume: (K is compact) & (f is continuous from X to Y).
Then: $f_* K$ is compact.

Proof. Let $g := f|K$. By Theorem 2.11.12,
g is continuous from X to Y.
So, since $g : K \to \mathbb{I}_g$, by Absoluteness of Continuity,
g is continuous from K to $\mathbb{I}_g$.
So, since $g : K \to\!> \mathbb{I}_g$, by Theorem 3.9.6, we have: $\mathbb{I}_g$ is compact.
It suffices to show: $\mathbb{I}_g = f_* K$. Since $K \subseteq \mathbb{D}_f$, we get $K \cap \mathbb{D}_f = K$.
Since $g = f|K$, we conclude: ($\mathbb{D}_g = K$) & ($\forall p \in K$, $f_p = g_p$).
Then $\mathbb{I}_g = \{ g_p \mid p \in \mathbb{D}_g \} = \{ g_p \mid p \in K \} = \{ f_p \mid p \in K \}$
$= \{ f_p \mid p \in K \cap \mathbb{D}_f \} = f_* K$, as desired. □

THEOREM 3.18.4. *Let $K \subseteq \mathbb{R}$. Then* $\min K = -(\max(-K))$.

Proof. Unassigned HW. □

THEOREM 3.18.5. *Let $K \subseteq \mathbb{R}$. Say K is compact and nonempty.*
Then: $\min K \neq ☹ \neq \max K$.

Proof. By Theorem 3.18.2, $☹ \neq \max K$. Want: $\min K \neq ☹$.
Define $f : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R}$, $f_x = -x$.
Then f is continuous from $\mathbb{R}$ to $\mathbb{R}$ and $K \subseteq \mathbb{R} = \mathbb{D}_f$.
Then, by Theorem 3.18.3, $f_* K$ is compact.
So, since $f_* K = -K$, we see that $-K$ is compact.
Then, by Theorem 3.18.2, we see that $\max(-K) \neq ☹$,

and it follows that $\max(-K) \in -K$.
By Theorem 3.18.4, we have: $\min K = -(\max(-K))$.
Then $\min K = -(\max(-K)) \in -(-K)$, so $\min K \neq \circledcirc$. □

THEOREM 3.18.6. *Let $s \in \mathbb{R}^{\mathbb{N}}$ and let $a, p \in \mathbb{R}$.*
Assume that $s \to p$ in $\mathbb{R}$ and that $\forall j \in \mathbb{N}$, $s_j \geqslant a$.
Then $p \geqslant a$.

Proof. Assume $p < a$. Want: Contradiction.
Let $\varepsilon := a - p$. Then $\varepsilon > 0$.
Since $s \to p$ in $\mathbb{R}$, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(j \geqslant K) \Rightarrow (d(s_j, p) < \varepsilon).$$
Then $d(s_K, p) < \varepsilon$, so $p - \varepsilon < s_k < p + \varepsilon$.
By hypothesis, we have: $\forall j \in \mathbb{N}$, $s_j \geqslant a$. Then $s_K \geqslant a$.
Then $a \leqslant s_K < p + \varepsilon = p + (a - p) = a$, so $a < a$. Contradiction. □

THEOREM 3.18.7. *Let $s \in \mathbb{R}^{\mathbb{N}}$ and let $b, p \in \mathbb{R}$.*
Assume that $s \to p$ in $\mathbb{R}$ and that $\forall j \in \mathbb{N}$, $s_j \leqslant b$.
Then $p \leqslant b$.

Proof. Unassigned HW. □

THEOREM 3.18.8. *Let $b \in \mathbb{R}$, $a \leqslant b$. Then $[a; b]$ is compact.*

Proof. Let $K := [a; b]$. Want: K is compact.
By Theorem 3.13.5, $\mathbb{R}$ is proper.
So, by Theorem 3.16.11, we want: K is closed and bounded in $\mathbb{R}$.
Let $\rho := b - a + 1$. Then $a + \rho = b + a$.
Since $a \leqslant b$, we get $\rho \geqslant 1$. Then $\rho > 0$, so $a - \rho < a$.
Since $a - \rho < a$ and $b < b + 1$, we get $[a; b] \subseteq (a - \rho; b + 1)$.
Then $K = [a; b] \subseteq (a - \rho; b + 1) = (a - \rho; a + \rho) = B_{\mathbb{R}}(a, \rho)$.
Then K is bounded in $\mathbb{R}$. Want: K is closed in $\mathbb{R}$. Want: $\mathrm{Cl}_{\mathbb{R}} K = K$.
Since $K \subseteq \mathrm{Cl}_{\mathbb{R}} K$, it suffices to show: $\mathrm{Cl}_{\mathbb{R}} K \subseteq K$.
Want: $\forall p \in \mathrm{Cl}_{\mathbb{R}} K$, $p \in K$. Given $p \in \mathrm{Cl}_{\mathbb{R}} K$. Want: $p \in K$.
Since $p \in \mathrm{Cl}_{\mathbb{R}} K = \mathrm{Cl}^{\mathrm{seq}}_{\mathbb{R}} K$, choose $s \in K^{\mathbb{N}}$ s.t. $s \to p$ in $\mathbb{R}$.
We have: $\forall j \in \mathbb{N}$, $s_j \in K = [a; b] \geqslant a$, so $s_j \geqslant a$.
Then, by Theorem 3.18.6, we get: $p \geqslant a$.
We have: $\forall j \in \mathbb{N}$, $s_j \in K = [a; b] \leqslant b$, so $s_j \leqslant b$.
Then, by Theorem 3.18.6, we get: $p \leqslant b$.
Since $a \leqslant p \leqslant b$, we get: $p \in [a; b]$. Then $p \in [a; b] = K$. □

The next theorem is the **Extreme Value Theorem**:

THEOREM 3.18.9. *Let X be a metric space.*
Let $f : X \dashrightarrow \mathbb{R}$, let $K \subseteq \mathbb{D}_f$ and let $L := f_* K$.
Assume: K is compact and nonempty.
Assume: f is continuous from X to $\mathbb{R}$.
Then: $\min L \neq ☹ \neq \max L$.

Proof. By Theorem 3.18.3, we get: L is compact.
Then, by Theorem 3.18.5, we get: $\min L \neq ☹ \neq \max L$. □

THEOREM 3.18.10. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $b \in \mathbb{R}$, $a \leqslant b$, $L := f_*([a; b])$.*
Assume: $[a; b] \subseteq \mathbb{D}_f$ and f is continuous from $\mathbb{R}$ to $\mathbb{R}$.
Then: $\min L \neq ☹ \neq \max L$.

Proof. Let $K := [a; b]$. By Theorem 3.18.8, we get: K is compact.
Then, by Theorem 3.18.9, we get: $\min L \neq ☹ \neq \max L$. □

3.19. **Cauchy sequences and complete metric spaces.**

DEFINITION 3.19.1. *Let X be a metric space and let $s \in X^{\mathbb{N}}$.*
Then s is **Cauchy** *in X means: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,*
$$(\ i, j \geqslant K\ ) \Rightarrow (\ d(s_i, s_j) < \varepsilon\ ).$$

THEOREM 3.19.2. *let X be a metric space, $s \in X^{\mathbb{N}}$.*
Assume: $\forall \varepsilon \in (0; 1]$, $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,
$$(\ i, j \geqslant K\ ) \Rightarrow (\ d(s_i, s_j) < \varepsilon\ ).$$
Then s is Cauchy in X.

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,
$$(\ i, j \geqslant K\ ) \Rightarrow (\ d(s_i, s_j) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,
$$(\ i, j \geqslant K\ ) \Rightarrow (\ d(s_i, s_j) < \varepsilon\ ).$$
Let $\delta := \min\{\varepsilon, 1\}$. Then $\delta \leqslant \varepsilon$ and $\delta \leqslant 1$ and $\delta > 0$.
Since $\delta \in (0; 1]$, by hypothesis, choose $K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,
$$(\ i, j \geqslant K\ ) \Rightarrow (\ d(s_i, s_j) < \varepsilon\ ).$$
Then $K \in \mathbb{N}$. Want: $\forall i, j \in \mathbb{N}$, $(\ i, j \geqslant K\ ) \Rightarrow (\ d(s_i, s_j) < \varepsilon\ )$.
Given $i, j \in \mathbb{N}$. Want: $(\ i, j \geqslant K\ ) \Rightarrow (\ d(s_i, s_j) < \varepsilon\ )$.
Assume $i, j \geqslant K$. Want: $d(s_i, s_j) < \varepsilon$.
Since $i, j \geqslant K$, by choice of K, we have $d(s_i, s_j) < \delta$.
Then $d(s_i, s_j) < \delta \leqslant \varepsilon$, as desired. □

THEOREM 3.19.3. *Let X be a metric space and $s \in X^{\mathbb{N}}$. Assume s is convergent in X. Then s is Cauchy in X.*

Proof. Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,
$(\ i, j \geqslant K \ ) \ \Rightarrow \ (\ d(s_i, s_j) < \varepsilon \ )$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,
$(\ i, j \geqslant K \ ) \ \Rightarrow \ (\ d(s_i, s_j) < \varepsilon \ )$.
Choose $q \in X$ s.t. $s \to q$ in X. Choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$(\ j \geqslant K \ ) \ \Rightarrow \ (\ d(s_j, q) < \varepsilon/2 \ )$.
Then $K \in \mathbb{N}$. Want: $\forall i, j \in \mathbb{N}$, $(\ i, j \geqslant K \ ) \ \Rightarrow \ (\ d(s_i, s_j) < \varepsilon \ )$.
Given $i, j \in \mathbb{N}$. Want: $(\ i, j \geqslant K \ ) \ \Rightarrow \ (\ d(s_i, s_j) < \varepsilon \ )$.
Assume $i, j \geqslant K$. Want: $d(s_i, s_j) < \varepsilon$.
Since $i \geqslant K$, by choice of K, we have $d(s_i, q) < \varepsilon/2$.
Since $j \geqslant K$, by choice of K, we have $d(s_j, q) < \varepsilon/2$.
Then $d(q, s_j) = d(s_j, q) < \varepsilon/2$, so $d(q, s_j) < \varepsilon/2$.
Then $d(s_i, s_j) \leqslant (d(s_i, q)) + (d(q, s_j)) < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$. □

The next result is called **Absoluteness of Cauchy**:

THEOREM 3.19.4. *Let Y be a metric space, $X \subseteq Y$, $s \in X^{\mathbb{N}}$.*
Then: (s is Cauchy in X) $\Leftrightarrow$ (s is Cauchy in Y).

Proof. Proof of $\Rightarrow$: Unassigned HW. *End of proof of $\Rightarrow$.*

Proof of $\Leftarrow$:
Assume s is Cauchy in Y. Want: s is Cauchy in X.
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,
$(\ i, j \geqslant K \ ) \ \Rightarrow \ (\ d_X(s_i, s_j) < \varepsilon \ )$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,
$(\ i, j \geqslant K \ ) \ \Rightarrow \ (\ d_X(s_i, s_j) < \varepsilon \ )$.
Since s is Cauchy in Y, choose $K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,
$(\ i, j \geqslant K \ ) \ \Rightarrow \ (\ d_Y(s_i, s_j) < \varepsilon \ )$.
Then $K \in \mathbb{N}$. Want: $\forall i, j \in \mathbb{N}$, $(\ i, j \geqslant K \ ) \ \Rightarrow \ (\ d_X(s_i, s_j) < \varepsilon \ )$.
Given $i, j \in \mathbb{N}$. Want: $(\ i, j \geqslant K \ ) \ \Rightarrow \ (\ d_X(s_i, s_j) < \varepsilon \ )$.
Assume $i, j \geqslant K$. Want: $d_X(s_i, s_j) < \varepsilon$.
Since $s \in X^{\mathbb{N}}$, we get $s_i, s_j \in X$. Then $d_X(s_i, s_j) = d_Y(s_i, s_j)$.
Then $d_X(s_i, s_j) = d_Y(s_i, s_j) < \varepsilon$, as desired.
End of proof of $\Leftarrow$. □

DEFINITION 3.19.5. *Let X be a metric space.*
By X is **complete**, *we mean:*
$\forall s \in X^{\mathbb{N}}$, (s *is Cauchy in* X) $\Rightarrow$ (s *is convergent in* X).

THEOREM 3.19.6. $\mathbb{R}_0^\times$ *is not complete.*

Proof. Let $X := \mathbb{R}_0^\times$. Want: X is not complete.
Define $s \in X^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $s_j = 1/j$. Then $s \to 0$ in $\mathbb{R}$.
Want: s is Cauchy in X and s is not convergent in X.
Since $s \to 0$ in $\mathbb{R}$, we see that s is convergent in $\mathbb{R}$.
Then, by Theorem 3.19.3, s is Cauchy in $\mathbb{R}$.
Then, by Absoluteness of Cauchy, s is Cauchy in X.
Want: s is not convergent in X.
Assume: s is convergent in X. Want: Contradiction.
Choose $q \in \mathbb{R}$ s.t. $s \to q$ in X.
Then, by Absoluteness of Limits, $s \to q$ in $\mathbb{R}$.
So, since $s \to 0$ in $\mathbb{R}$, by Theorem 2.15.1, we get: $q = 0$.
Then $0 = q \in X \in \mathbb{R}_0^\times$, so $0 \neq 0$. Contradiction. □

THEOREM 3.19.7. *Let X be a metric space, $s \in X^{\mathbb{N}}$.*
Assume s is Cauchy in X. Then $s \in X^{\mathbb{N}}_{\text{bi}}$.

Proof. Want: $\mathbb{I}_s$ is bounded in X. Want: $\exists B \in \mathcal{B}_X$ s.t. $\mathbb{I}_s \subseteq B$.
Since s is Cauchy in X, choose $K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,
($i, j \geqslant K$) $\Rightarrow$ ($d(s_i, s_j) < 1$).
Then: $\forall i \in \mathbb{N}$, ($i \geqslant K$) $\Rightarrow$ ($d(s_i, s_K) < 1$).
Let $q := s_K$. Then: $\forall i \in \mathbb{N}$, ($i \geqslant K$) $\Rightarrow$ ($d(s_i, q) < 1$).
Let $M := \max\{ d(s_1, q), \ldots, d(s_K, q) \}$.
Then $\forall i \in \mathbb{N}$, ($i \leqslant K$) $\Rightarrow$ ($d(s_i, q) \leqslant M$).
Also, since $q = s_K$, we get $d(s_K, q) = 0$, so $M \geqslant 0$, so $0 \leqslant M$.
Let $B := B_X(q, M+1)$. Then $B \in \mathcal{B}_X$. Want: $\mathbb{I}_s \subseteq B$.
Want: $\forall p \in \mathbb{I}_s$, $p \in B$. Given $p \in \mathbb{I}_s$. Want: $p \in B$.
Want: $p \in B_X(q, M+1)$. Want: $d(p, q) < M + 1$.
Since $p \in \mathbb{I}_s$, choose $j \in \mathbb{D}_s$ s.t. $p = s_j$. Then $j \in \mathbb{D}_s = \mathbb{N}$.
At least one of the following must be true:
(1) $j \leqslant K$ or (2) $j \geqslant K$.

Case 1:
Recall: $\forall i \in \mathbb{N}$, ($i \leqslant K$) $\Rightarrow$ ($d(s_i, q) \leqslant M$).
So, since $j \leqslant K$, we get $d(s_j, q) \leqslant M$.

Then $d(p,q) = d(s_j,q) \leqslant M < M+1$, as desired.
End of Case 1.

Case 2:
Recall: $\forall i \in \mathbb{N}$, $(\ i \geqslant K\ ) \Rightarrow (\ d(s_i,q) < 1\ )$.
So, since $j \geqslant K$, we get $d(s_j,q) < 1$.
Also, since $0 \leqslant M$, we get $1 \leqslant M+1$.
Then $d(p,q) = d(s_j,q) < 1 \leqslant M+1$, as desired.
End of Case 2. □

THEOREM 3.19.8. *Let X be a metric space and let $s \in X^{\mathbb{N}}$.*
Assume: s is Cauchy and subconvergent in X.
Then: s is convergent in X.

Proof. Choose a subsequence t of s s.t. t is convergent in X.
Choose $q \in X$ s.t. $t \to q$ in X. Want: $s \to q$ in X.
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \Rightarrow (\ d(s_j,q) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \Rightarrow (\ d(s_j,q) < \varepsilon\ ).$$
Since $t \to q$, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant L\ ) \Rightarrow (\ d(t_j,q) < \varepsilon/2\ ).$$
Since s is Cauchy, choose $M \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$$(\ i,j \geqslant M\ ) \Rightarrow (\ d(s_i,s_j) < \varepsilon/2\ ).$$
Let $K := \max\{L,M\}$. Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $(\ j \geqslant K\ ) \Rightarrow (\ d(s_j,q) < \varepsilon\ )$.
Given $j \in \mathbb{N}$. Want: $(\ j \geqslant K\ ) \Rightarrow (\ d(s_j,q) < \varepsilon\ )$.
Assume $j \geqslant K$. Want: $d(s_j,q) < \varepsilon$.
Since t is a subsequence of s,
choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.
By HW#8-3, we have $\ell_j \geqslant j$. Then $j \geqslant K \geqslant M$ and $\ell_j \geqslant j \geqslant K \geqslant M$.
Since $j, \ell_j \geqslant M$, by choice of M, we have: $d(s_j, s_{\ell_j}) < \varepsilon/2$.
So, since $t_j = (s \circ \ell)_j = s_{\ell_j}$, we get: $d(s_j,t_j) < \varepsilon/2$.
Since $j \geqslant K \geqslant L$, by choice of L, we have: $d(t_j,q) < \varepsilon/2$.
Then: $d(s_j,q) \leqslant (d(s_j,t_j)) + (d(t_j,q)) < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$. □

THEOREM 3.19.9. *Let X be a proper metric space.*
Then X is complete.

Proof. Want: $\forall s \in X^{\mathbb{N}}$, ($s$ is Cauchy in X) $\Rightarrow$ (s is convergent in X).
Given $s \in X^{\mathbb{N}}$. Want: (s is Cauchy in X) $\Rightarrow$ (s is convergent in X).

Assume s is Cauchy in X. Want: s is convergent in X.
By Theorem 3.19.7, we have: $s \in X_{\text{bi}}^{\mathbb{N}}$.
So, since X is proper, we get: s is subconvergent in X.
Then, by Theorem 3.19.8, s is convergent in X, as desired. □

THEOREM 3.19.10. *$\mathbb{R}$ is complete.*

Proof. By Theorem 3.13.5, $\mathbb{R}$ is proper.
Then, by Theorem 3.19.9, $\mathbb{R}$ is complete. □

THEOREM 3.19.11. *$\mathbb{R}^2$ is complete.*

Proof. By Theorem 3.14.10, $\mathbb{R}^2$ is proper.
Then, by Theorem 3.19.9, $\mathbb{R}^2$ is complete. □

Using Theorem 3.14.9 and induction,
we may show: $\forall k \in \mathbb{N}$, $\mathbb{R}^k$ is proper.
Then, by Theorem 3.19.9, it follows that: $\forall k \in \mathbb{N}$, $\mathbb{R}^k$ is complete.
Unassigned HW: Show that
a product of two complete metric spaces is complete.

DEFINITION 3.19.12. *Let X be a metric space and let $A := X_{\text{set}}$. Define $\widehat{d} : A \times A \to [0; \infty)$ by: $\forall p, q \in A$, $\widehat{d}(p, q) = \min\{\, d_X(p, q)\,,\, 1\}$.
Then we define: $\widehat{X} := (\, A\,,\, \widehat{d}\,)$.*

THEOREM 3.19.13. *Let X be a metric space and let $A := X_{\text{set}}$.
Then $\widehat{X}_{\text{set}} = A$. Also, $\forall p \in A$, $B_{\widehat{X}}(p, 2) = A$.*

THEOREM 3.19.14. *$\forall$metric space X, we have: $\widehat{X}$ is bounded.*

THEOREM 3.19.15. *Let X be a metric space. Let $p, q \in X_{\text{set}}$.
Let $d := d_X$ and $\widehat{d} := d_{\widehat{X}}$.*
Then: *(1)* $(\, d(p, q) < 1\,) \Rightarrow (\, \widehat{d}(p, q) = d(p, q)\,)$.
Also: *(2)* $(\, d(p, q) \geqslant 1\,) \Rightarrow (\, \widehat{d}(p, q) = 1\,)$.
Also: *(3)* $(\, \widehat{d}(p, q) \neq 1\,) \Rightarrow (\, d(p, q) < 1\,)$.

THEOREM 3.19.16. *Let X be a metric space. Let $p, q \in X_{\text{set}}$.
Let $d := d_X$ and $\widehat{d} := d_{\widehat{X}}$.
Assume: either (1) $d(p, q) < 1$ or (2) $\widehat{d}(p, q) < 1$.
Then: $\widehat{d}(p, q) = d(p, q)$*

Proof. Case 1:
By (1) of Theorem 3.19.15, we have $\widehat{d}(p, q) = d(p, q)$, as desired.

End of Case 1.

Case 2:
Since $\widehat{d}(p,q) < 1$, we get: $\widehat{d}(p,q) \neq 1$.
Then, by (3) of Theorem 3.19.15, we have $d(p,q) < 1$, as desired.
Then, by (1) of Theorem 3.19.15, we have $\widehat{d}(p,q) = d(p,q)$, as desired.
End of Case 2. ☐

THEOREM 3.19.17. *Let X be a metric space, $s \in X_{\text{set}}^{\mathbb{N}}$.*
Then: (*s is Cauchy in X*) $\Leftrightarrow$ (*s is Cauchy in $\widehat{X}$*).

Proof. Let $d := d_X$ and $\widehat{d} := d_{\widehat{X}}$.
Proof of $\Leftarrow$: Unassigned HW. *End of proof of* $\Leftarrow$.

Proof of $\Rightarrow$:
Assume: s is Cauchy in X. Want: s is Cauchy in $\widehat{X}$.
By Theorem 3.19.2, want: $\forall \varepsilon \in (0;1]$, $\exists K \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$(\; i,j \geqslant K \;) \Rightarrow (\; \widehat{d}(s_i,s_j) < \varepsilon \;)$.
Given $\varepsilon \in (0;1]$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$(\; i,j \geqslant K \;) \Rightarrow (\; \widehat{d}(s_i,s_j) < \varepsilon \;)$.
Since s is Cauchy in X, choose $K \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$(\; i,j \geqslant K \;) \Rightarrow (\; d(s_i,s_j) < \varepsilon \;)$.
Then $K \in \mathbb{N}$. Want: $\forall i,j \in \mathbb{N}$, $(\; i,j \geqslant K \;) \Rightarrow (\; \widehat{d}(s_i,s_j) < \varepsilon \;)$.
Given $i,j \in \mathbb{N}$. Want: $(\; i,j \geqslant K \;) \Rightarrow (\; \widehat{d}(s_i,s_j) < \varepsilon \;)$.
Assume: $i,j \geqslant K$. Want: $\widehat{d}(s_i,s_j) < \varepsilon$.
Since $i,j \geqslant K$, by choice of K, we have: $d(s_i,s_j) < \varepsilon$.
Then $d(s_i,s_j) < \varepsilon \in [0;1) < 1$, so $d(s_i,s_j) < 1$,
so, by Theorem 3.19.16, we have: $\widehat{d}(s_i,s_j) = d(s_i,s_j)$.
Then $\widehat{d}(s_i,s_j) = d(s_i,s_j) < \varepsilon$, as desired. *End of proof of* $\Rightarrow$. ☐

THEOREM 3.19.18. *Let X be a metric space, $A := X_{\text{set}}$, $s \in A^{\mathbb{N}}$.*
Then: (*s is convergent in X*) $\Leftrightarrow$ (*s is convergent in $\widehat{X}$*).

Proof. We have $\widehat{X}_{\text{set}} = X_{\text{set}} = A$.

Proof of $\Leftarrow$: Unassigned HW. *End of proof of* $\Leftarrow$.

Proof of $\Rightarrow$:
Assume: s is convergent in X. Want: s is convergent in $\hat{X}$.
Since s is convergent in X, choose $q \in X$ s.t. $s \to q$ in X.

Then, by $\Rightarrow$ of HW#12-5, we get: $s \to q$ in $\widehat{X}$.
Then s is convergent in $\widehat{X}$, as desired.
End of proof of $\Rightarrow$. $\square$

THEOREM 3.19.19. *Let X be a metric space, $A := X_{\text{set}}$, $s \in A^{\mathbb{N}}$.*
Then: (*s is subconvergent in X*) $\Leftrightarrow$ (*s is subconvergent in $\widehat{X}$*).

Proof. We have $\widehat{X}_{\text{set}} = X_{\text{set}} = A$.

Proof of $\Leftarrow$*:* Unassigned HW. *End of proof of* $\Leftarrow$.

Proof of $\Rightarrow$*:*
Assume: s is subconvergent in X. Want: s is subconvergent in $\hat{X}$.
Since s is subconvergent in X, choose $t \in A^{\mathbb{N}}$ s.t. t is convergent in X.
Then, by $\Rightarrow$ of Theorem 3.19.18, we get: t is convergent in X.
Then s is subconvergent in $\widehat{X}$, as desired.
End of proof of $\Rightarrow$. $\square$

THEOREM 3.19.20. *Let X be a metric space.*
Then: (*X is compact*) $\Leftrightarrow$ (*$\widehat{X}$ is compact*).

Proof. Let $A := X_{\text{set}}$. Then $A = \widehat{X}_{\text{set}}$.

Proof of $\Leftarrow$*:* Unassigned HW. *End of proof of* $\Leftarrow$.

Proof of $\Rightarrow$*:*
Assume: X is compact. Want: $\widehat{X}$ is compact.
Want: $\forall s \in A^{\mathbb{N}}$, s is subconvergent in $\widehat{X}$.
Given $s \in A^{\mathbb{N}}$. Want: s is subconvergent in $\widehat{X}$.
Since X is compact, s is subconvergent in X.
Then, by $\Rightarrow$ of Theorem 3.19.19, s is subconvergent in $\widehat{X}$, as desired.
End of proof of $\Rightarrow$. $\square$

THEOREM 3.19.21. *$\widehat{\mathbb{R}}$ is complete.*

Proof. Want: $\forall s \in \widehat{\mathbb{R}}^{\mathbb{N}}$, ($s$ is Cauchy in $\widehat{\mathbb{R}}$) $\Rightarrow$ (s is convergent in $\widehat{\mathbb{R}}$).
Given $s \in \widehat{\mathbb{R}}^{\mathbb{N}}$. Want: ($s$ is Cauchy in $\widehat{\mathbb{R}}$) $\Rightarrow$ (s is convergent in $\widehat{\mathbb{R}}$).
Assume: s is Cauchy in $\widehat{\mathbb{R}}$. Want: s is convergent in $\widehat{\mathbb{R}}$.
Since s is Cauchy in $\widehat{\mathbb{R}}$, by $\Leftarrow$ of Theorem 3.19.17,
we conclude that s is Cauchy in $\mathbb{R}$.
So, since $\mathbb{R}$ is complete, we get: s is convergent in $\mathbb{R}$.
Then, by $\Rightarrow$ of Theorem 3.19.18, s is convergent in $\widehat{\mathbb{R}}$, as desired. $\square$

THEOREM 3.19.22. *$\widehat{\mathbb{R}}$ is nonproper.*

Proof. Assume $\widehat{\mathbb{R}}$ is proper. Want: Contradiction.
By Theorem 3.19.14, we see that $\widehat{\mathbb{R}}$ is bounded.
Since $\widehat{\mathbb{R}}$ is proper and bounded, by $\Leftarrow$ of Theorem 3.14.12,
we see that: $\widehat{\mathbb{R}}$ is compact.
Then, by $\Leftarrow$ of Theorem 3.19.20, we get: $\mathbb{R}$ is compact.
Then, by $\Rightarrow$ of Theorem 3.14.12, we get: $\mathbb{R}$ proper and bounded.
Then $\mathbb{R}$ is bounded. Also, $\mathbb{R}$ is unbounded. Contradiction. □

3.20. **Continuous injections over compacta are homeomorphisms.**

THEOREM 3.20.1. *Let K be a compact metric space, $C \in \mathcal{T}_K'$.*
Then C is compact.

Proof. Want: $\forall s \in C^{\mathbb{N}}$, s is subconvergent in C.
Given $s \in C^{\mathbb{N}}$. Want: s is subconvergent in C.
Since $s \in C^{\mathbb{N}} \subseteq K^{\mathbb{N}}$ and since K is compact,
we conclude: s is subconvergent in K.
So, since $C \in \mathcal{T}_K'$, by $\Leftarrow$ of Theorem 3.16.8,
we see that: s is subconvergent in C. □

Recall (Theorem 3.7.6) that
a continuous bijection may not be a homeomorphism.
The next result says that that doesn't happen when the domain is compact.

THEOREM 3.20.2. *Let K, Y be metric spaces. Let $f : K \hookrightarrow\!> Y$.*
Assume: (K is compact) & (f is continuous from K to Y).
Then: f is a homeomorphism from K onto Y.

Proof. Since $f : K \hookrightarrow\!> Y$ and since f is continuous from K to Y,
it only remains to show: f^{-1} is continuous from Y to K.
Let $g := f^{-1}$. Then $g : Y \hookrightarrow\!> K$. Want: g is continuous from Y to K.
By $\Leftarrow$ of Theorem 3.17.7, want: $\forall C \in \mathcal{T}_K'$, $g^*C \in \mathcal{T}_Y'$.
Given $C \in \mathcal{T}_K'$. Want: $g^*C \in \mathcal{T}_Y'$.
Since K is compact and $C \in \mathcal{T}_K'$,
by Theorem 3.20.1, we get: C is compact.
So, since f is continuous from K to Y,
by Theorem 3.18.3, we get: f_*C is compact.
Then, by Theorem 3.16.9, we get f_*C is closed in Y, and so $f_*C \in \mathcal{T}_Y'$.
Since $g = f^{-1}$, we get: $g^*C = f_*C$. Then $g^*C = f_*C \in \mathcal{T}_Y'$. □

3.21. **Continuous on compact implies uniformly continuous.**

THEOREM 3.21.1. *Let $a, b \in \mathbb{R}$. Then $a + b \leqslant \sqrt{2} \cdot \sqrt{a^2 + b^2}$.*

Proof. We have: $\forall x \in \mathbb{R}$, $x \leqslant |x|$. Then $a + b \leqslant |a + b|$.
Since $a + b \leqslant |a + b| = \sqrt{(a + b)^2}$,
it suffices to show: $\sqrt{(a + b)^2} \leqslant \sqrt{2} \cdot \sqrt{a^2 + b^2}$.
It therefore suffices to show: $0 \leqslant (a + b)^2 \leqslant 2 \cdot (a^2 + b^2)$.
We have: $\forall x \in \mathbb{R}$, $0 \leqslant x^2$. Then $0 \leqslant (a + b)^2$.
It remains to show: $(a + b)^2 \leqslant 2 \cdot (a^2 + b^2)$.
We have: $\forall x \in \mathbb{R}$, $0 \leqslant x^2$. Then $0 \leqslant (a - b)^2$.
Then $0 \leqslant a^2 - 2ab + b^2$, so $2ab \leqslant a^2 + b^2$.
Adding a^2 and b^2 to both sides gives: $a^2 + 2ab + b^2 \leqslant a^2 + a^2 + b^2 + b^2$.
Then $(a + b)^2 = a^2 + 2ab + b^2 \leqslant a^2 + a^2 + b^2 + b^2 = 2 \cdot (a^2 + b^2)$. □

THEOREM 3.21.2. *Let X be a metric space.*
Then d_X is Lipschits-$\sqrt{2}$ from $X \times X$ to $\mathbb{R}$.

Proof. Let $f := d_X$, let $V := X \times X$ and let $K := \sqrt{2}$.
Want: f is Lipschitz-K from V to $\mathbb{R}$.
Want: $\forall s, t \in V$, $d_{\mathbb{R}}(f_s, f_t) \leqslant K \cdot (d_V(s, t))$.
Given $s, t \in V$. Want: $d_{\mathbb{R}}(f_s, f_t) \leqslant K \cdot (d_V(s, t))$.
Let $\varepsilon := K \cdot (d_V(s, t))$. Want: $d_{\mathbb{R}}(f_s, f_t) \leqslant \varepsilon$.
Want: $(\ f_s \leqslant f_t + \varepsilon\ )\ \&\ (\ f_t \leqslant f_s + \varepsilon\ )$.
Let $a := d_X(s_1, t_1)$ and $b := d_X(s_2, t_2)$.
Then $a = d_X(t_1, s_1)$ and $b = d_X(t_2, s_2)$.
Then
$$\begin{aligned} d_V(s, t) &= d_{X \times X}(\,(s_1, s_2)\,,\,(t_1, t_2)\,) \\ &= \sqrt{(\,d_X(s_1, t_1)\,)^2 + (\,d_X(s_2, t_2)\,)^2} = \sqrt{a^2 + b^2}. \end{aligned}$$
By Theorem 3.21.1, we have: $a + b \leqslant \sqrt{2}\sqrt{a^2 + b^2}$.
So, since $\sqrt{2} = K$ and $\sqrt{a^2 + b^2} = d_V(s, t)$, we get: $a + b \leqslant K \cdot (d_V(s, t))$.
So, since $K \cdot (d_V(s, t)) = \varepsilon$, we get: $a + b \leqslant \varepsilon$.
We have $f_s = f(s) = f(s_1, s_2) = d_X(s_1, s_2)$
and $f_t = f(t) = f(t_1, t_2) = d_X(t_1, t_2)$.
We compute
$$\begin{aligned} f_s = d_X(s_1, s_2) &\leqslant (d_X(s_1, t_1)) + (d_X(t_1, t_2)) + (d_X(t_2, s_2)) \\ &= a + f_t + b = f_t + (a + b) \leqslant f_t + \varepsilon. \end{aligned}$$
It remains to show: $f_t \leqslant f_s + \varepsilon$.
We compute
$$\begin{aligned} f_t = d_X(t_1, t_2) &\leqslant (d_X(t_1, s_1)) + (d_X(s_1, s_2)) + (d_X(s_2, t_2)) \\ &= a + f_s + b = f_t + (a + b) \leqslant f_s + \varepsilon. \end{aligned}$$
□

THEOREM 3.21.3. *Let A, X and Y be metric spaces.*
Let $f : A \dashrightarrow X$, let $g : A \dashrightarrow Y$ and let $p \in A$.

Assume that f is continuous at p from A to X.
Assume that g is continuous at p from A to Y.
Then (f,g) is continuous at p from A to $X \times Y$.

Proof. Let $h := (f,g)$ and let $Z := X \times Y$.
We wish to show: h is continuous at p from A to Z.
Since f is continuous at p from A to X, we get: $p \in \mathbb{D}_f$.
Since g is continuous at p from A to Y, we get: $p \in \mathbb{D}_g$.
Then $p \in \mathbb{D}_f \cap \mathbb{D}_g$. Since $h = (f,g)$, we get: $\mathbb{D}_h = \mathbb{D}_f \cap \mathbb{D}_g$.
Then $p \in \mathbb{D}_f \cap \mathbb{D}_g = \mathbb{D}_h$.
It remains to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall q \in \mathbb{D}_h$,
$$(\, d(p,q) < \delta \,) \;\Rightarrow\; (\, d(h_p,h_q) < \varepsilon \,).$$
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall q \in \mathbb{D}_h$,
$$(\, d(p,q) < \delta \,) \;\Rightarrow\; (\, d(h_p,h_q) < \varepsilon \,).$$
Since f is continuous at p from A to X,
choose $\lambda > 0$ s.t., $\forall q \in \mathbb{D}_f$,
$$(\, d(p,q) < \lambda \,) \;\Rightarrow\; (\, d(f_p,f_q) < \varepsilon/\sqrt{2} \,).$$
Since g is continuous at p from A to Y,
choose $\mu > 0$ s.t., $\forall q \in \mathbb{D}_f$,
$$(\, d(p,q) < \mu \,) \;\Rightarrow\; (\, d(g_p,g_q) < \varepsilon/\sqrt{2} \,).$$
Let $\delta := \min\{\lambda, \mu\}$. Then $\delta > 0$.
Want: $\forall q \in \mathbb{D}_h$, $(\, d(p,q) < \delta \,) \Rightarrow (\, d(h_p,h_q) < \varepsilon \,)$.
Given $q \in \mathbb{D}_h$. Want: $(\, d(p,q) < \delta \,) \Rightarrow (\, d(h_p,h_q) < \varepsilon \,)$.
Assume: $d(p,q) < \delta$. Want: $d(h_p,h_q) < \varepsilon$.
Since $d(p,q) < \delta \leqslant \lambda$, by choice of λ, we get: $d(f_p,f_q) < \varepsilon/\sqrt{2}$.
Since $d(p,q) < \delta \leqslant \mu$, by choice of μ, we get: $d(g_p,g_q) < \varepsilon/\sqrt{2}$.
Since $0 \leqslant d(f_p,f_q) < \varepsilon/\sqrt{2}$, we get: $(d(f_p,f_q))^2 < (\varepsilon/\sqrt{2})^2$.
Since $0 \leqslant d(g_p,g_q) < \varepsilon/\sqrt{2}$, we get: $(d(g_p,g_q))^2 < (\varepsilon/\sqrt{2})^2$.
Then
$$\begin{aligned}(d(f_p,f_q))^2 + (d(g_p,g_q))^2 &< (\varepsilon/\sqrt{2})^2 + (\varepsilon/\sqrt{2})^2 \\ &= (\varepsilon^2/2) + \varepsilon^2/2) = \varepsilon^2.\end{aligned}$$
Since $0 \leqslant (d(f_p,f_q))^2 + (d(g_p,g_q))^2 < \varepsilon^2$,
we conclude that: $\sqrt{(d(f_p,f_q))^2 + (d(g_p,g_q))^2} < \sqrt{\varepsilon^2}$.
Then:
$$\begin{aligned} d(\, h_p \, , \, h_q \,) &= d(\, (f,g)_p \, , \, (f,g)_p \,) \\ &= d(\, (f_p,g_p) \, , \, (f_q,g_q) \,) \\ &= \sqrt{(d(f_p,f_q))^2 + (d(g_p,g_q))^2} \;<\; \sqrt{\varepsilon^2} \;=\; \varepsilon. \end{aligned}$$
□

THEOREM 3.21.4. *Let A, X and Y be metric spaces.*
Let $f \in C_X^A$ and $g \in C_Y^A$. Then $(f,g) \in C_{X \times Y}^A$.

Proof. Let $h := (f,g)$ and let $Z := X \times Y$. Want: $h \in C_Z^A$.
Since $f : A \to X$ and $g : A \to Y$, we get: $h : A \to Z$.
Want: h is continuous from A to Z.
Want: h is continuous on $\mathbb{D}_h$ from A to Z.
Want: h is continuous on A from A to Z.
Want: $\forall p \in A$, h is continuous at p from A to Z.
Given $p \in A$. Want: h is continuous at p from A to Z.
We have: f is continuous from A to X.
Then: f is continuous on $\mathbb{D}_f$ from A to X.
Then: f is continuous on A from A to X.
So, since $p \in A$, we get: f is continuous at p from A to X.
We have: g is continuous from A to Y.
Then: g is continuous on $\mathbb{D}_g$ from A to Y.
Then: g is continuous on A from A to Y.
So, since $p \in A$, we get: g is continuous at p from A to Y.
Since f is continuous at p from A to X
and g is continuous at p from A to Y,
it follows, from Theorem 3.21.3, that
(f,g) is continuous at p from A to $X \times Y$.
So, since $h = (f,g)$ and $Z = X \times Y$, we get:
h is continuous at p from A to Z, as desired. □

DEFINITION 3.21.5. *Let f and g be functions.*
Then $f \times_{\text{fn}} g$ is the function defind by:

$$\forall x, \qquad (f \times_{\text{fn}} g)_x = \begin{cases} (f_{x_1}, g_{x_2}), & \text{if } x \in \mathbb{D}_f \times \mathbb{D}_g \\ ☹, & \text{otherwise.} \end{cases}$$

By sloppiness, we typically denote $f \times_{\text{fn}} g$ by: $f \times g$.

THEOREM 3.21.6. *Let f and g be functions. Then:*

$$(\ \mathbb{D}_{(f,g)} = \mathbb{D}_f \bigcap \mathbb{D}_g\ ) \quad \& \quad (\ \mathbb{D}_{f\times g} = \mathbb{D}_f \times \mathbb{D}_g\ ).$$

THEOREM 3.21.7. *Let S, T, X and Y be sets, $f \in X^S$, $g \in Y^T$.*
Then: $(f,g) : S \bigcap T \to X \times Y$ and $f \times g : S \times T \to X \times Y$.

THEOREM 3.21.8. *Let X, Y and Z be metric spaces.*
Let $f \in C_Y^X$ and let $g \in C_Z^Y$. Then $g \circ f \in C_Z^X$.

The next theorem follows from Theorem 2.11.8.

THEOREM 3.21.9. *Let S, T, X and Y be metric spaces.*
Let $f \in C_X^S$ and let $g \in C_Y^T$. Then $f \times g \in C_{X\times Y}^{S\times T}$.

Proof. Let $A := S \times T$, $Z := X \times Y$ and $h := f \times g$. Want: $h \in C_Z^A$.
Define $\sigma : A \to S$ and $\tau : A \to T$ by: $\forall p \in A$, $\sigma_p = p_1$ and $\tau_p = p_2$.
Then σ is Lipschitz-1 from A to S and τ is Lipschitz-1 from A to T.
Then σ is continuous from A to S and τ is continuous from A to T.
Then $\sigma \in C_S^A$ and $\tau \in C_T^A$.
Since $\sigma \in C_S^A$ and $f \in C_X^S$, we get: $f \circ \sigma \in C_X^A$.
Since $\tau \in C_T^A$ and $g \in C_Y^T$, we get: $g \circ \tau \in C_Y^A$.
Then, by Theorem 3.21.4, we have $(f \circ \sigma, g \circ \tau) \in C_{X\times Y}^A$.
So, since $Z = X \times Y$, we get: $(f \circ \sigma, g \circ \tau) \in C_Z^A$.
It therefore suffices to show that $(f \circ \sigma, g \circ \tau) = h$.
Want: $\forall p \in A$, $(f \circ \sigma, g \circ \tau)_p = h_p$.
Given $p \in A$. Want: $(f \circ \sigma, g \circ \tau)_p = h_p$.
Since $p \in A = S \times T = \mathbb{D}_f \times \mathbb{D}_g$, we get: $(f \times g)_p = (f_{p_1}, g_{p_2})$.
Then $(\, f \circ \sigma\, ,\, g \circ \tau\,)_p \;=\; (\, (f \circ \sigma)_p\, ,\, (g \circ \tau)_p\,) \;=\; (\, f_{\sigma_p}\, ,\, g_{\tau_p}\,)$
$= (\, f_{p_1}\, ,\, g_{p_2}\,) \;=\; (f \times g)_p \;=\; h_p$, as desired. □

The next result follows from Absoluteness of Continuity.

THEOREM 3.21.10. *Let A and Y be metric spaces.*
Let $X \subseteq Y$. Then $C_X^A \subseteq C_Y^A$.

The following is a Squeeze Theorem.

THEOREM 3.21.11. *Let $s \in \mathbb{R}^{\mathbb{N}}$.*
Assume: $\forall j \in \mathbb{N}$, $0 \leqslant s_j < 1/j$. Then $s \to 0$ in $\mathbb{R}$.

THEOREM 3.21.12. *Let $V := [0;\infty)$ and $I := [0;\infty)$.*
Define $\alpha, \beta \in C_I^V$ by: $\forall u \in V$, $\alpha_u = \dfrac{u}{1+u^2}$ and $\beta_u = u$.
Then: $\forall u \in V$, $[\ (\ \alpha_u = 0\ ) \Rightarrow (\ \beta_u = 0\ )\ ]$ and
$\neg[\ \forall \varepsilon > 0, \exists \delta > 0$ s.t.$, \forall u \in V,$
$(\ \alpha_u < \delta\ ) \Rightarrow (\ \beta_u < \varepsilon\ )\ ]$.

THEOREM 3.21.13. *Let V be a compact metric space.*
Let $I := [0;\infty)$ and let $\alpha, \beta \in C_I^V$.
Assume: $\forall u \in V$, $(\ \alpha_u = 0\ ) \Rightarrow (\ \beta_u = 0\ )$.
Then: $\forall \varepsilon > 0, \exists \delta > 0$ s.t.$, \forall u \in V,$
$(\ \alpha_u < \delta\ ) \Rightarrow (\ \beta_u < \varepsilon\ )$.

Proof. Assume $\neg[\ \forall \varepsilon > 0, \exists \delta > 0$ s.t.$, \forall u \in V,$
$(\ \alpha_u < \delta\ ) \Rightarrow (\ \beta_u < \varepsilon\ )\ ]$.
Want: Contradiction. Choose $\varepsilon > 0$ s.t., $\forall \delta > 0$, $\exists u \in V$ s.t.

$(\alpha_u < \delta) \ \& \ (\beta_u \geqslant \varepsilon)].$

Then: $\forall j \in \mathbb{N}, \exists u \in V$ s.t. $(\alpha_u < 1/j) \ \& \ (\beta_u \geqslant \varepsilon)].$

By the Axiom of Choice, choose $s \in V^{\mathbb{N}}$ s.t. $\forall j \in \mathbb{N}$,

$(\alpha_{s_j} < 1/j) \ \& \ (\beta_{s_j} \geqslant \varepsilon)].$

Since V is compact, s is subconvergent in V.

Choose a subsequence t of s s.t. t is convergent in V.

Choose a strictly-increasing $\ell \in \mathbb{N}^{\mathbb{N}}$ s.t. $t = s \circ \ell$.

By HW#8-3, we have: $\forall j \in \mathbb{N}, \ell_j \geqslant j$.

Since $\forall j \in \mathbb{N}$, we have $\ell_j \geqslant j > 0$, we get: $\forall j \in \mathbb{N}, 1/\ell_j \leqslant 1/j$.

Also, we have: $\forall j \in \mathbb{N}, t_j = (s \circ \ell)_j = s_{\ell_j}$.

By choice of s, we know: $\forall j \in \mathbb{N}, (\alpha(s_j) < 1/j) \ \& \ (\beta(s_j) \geqslant \varepsilon)$.

It follows that: $\forall j \in \mathbb{N}, (\alpha(s_{\ell_j}) < 1/\ell_j) \ \& \ (\beta(s_{\ell_j}) \geqslant \varepsilon)$.

Then $\forall j \in \mathbb{N}, ((\alpha \circ t)_j = \alpha(t_j) = \alpha(s_{\ell_j}) < 1/\ell_j \leqslant 1/j) \ \&$

$((\beta \circ t)_j = \beta(t_j) = \beta(s_{\ell_j}) \geqslant \varepsilon)$.

Then $\forall j \in \mathbb{N}, ((\alpha \circ t)_j < 1/j) \ \& \ ((\beta \circ t)_j \geqslant \varepsilon)$.

Since t is convergent in V, choose $q \in V$ s.t. $t \to q$ in V.

Since $q \in V$ and $\alpha, \beta \in C_I^V \subseteq C_{\mathbb{R}}^V$, we conclude:

α and β are both continuous at q from V to $\mathbb{R}$.

So, since $t \to q$ in V, we get:

$\alpha \circ t \to \alpha_q$ in $\mathbb{R}$ and $\beta \circ t \to \beta_q$ in $\mathbb{R}$.

Since $\alpha \in C_I^V$, we get $\mathbb{I}_\alpha \subseteq I$. Then $\mathbb{I}_\alpha \subseteq I = [0; \infty) \geqslant 0$.

Then: $\forall j \in \mathbb{N}, (\alpha \circ t)_j = \alpha(t_j) \in \mathbb{I}_\alpha \geqslant 0$.

Since $\forall j \in \mathbb{N}, 0 \leqslant (\alpha \circ t)_j < 1/j$,

by Theorem 3.21.11, we get: $\alpha \circ t \to 0$ in $\mathbb{R}$.

So, since $\alpha \circ t \to \alpha_q$ in $\mathbb{R}$, by Uniqueness of Limits, we have $\alpha_q = 0$.

By hypothesis, we have: $\forall u \in V, (\alpha_u = 0) \Rightarrow (\beta_u = 0)$.

So, since $q \in V$ and $\alpha_q = 0$, we see that $\beta_q = 0$.

Since $\forall j \in \mathbb{N}, (\beta \circ t)_j \geqslant \varepsilon$ and $\beta \circ t \to \beta_q$ in $\mathbb{R}$,

by Theorem 3.18.6, we get: $\beta_q \geqslant \varepsilon$.

Then $0 < \varepsilon \leqslant \beta_q = 0$, so $0 < 0$. Contradiction. □

THEOREM 3.21.14. *Let X be a compact metric space. Let Y be a metric space. Let $f \in C_X^Y$. Then f is uniformaly continuous from X to Y.*

Proof. Let $V := X \times X$, $W := Y \times Y$, $g := f \times f$ and $I := [0; \infty)$.

Since X is compact, by Theorem 3.14.16, we get: V is compact.

Since $f \in C_X^Y$, by Theorem 3.21.9, we get: $g \in C_W^V$.

By Theorem 3.21.2, d_X is Lipschitz-$\sqrt{2}$ from V to $\mathbb{R}$
and d_Y is Lipschitz-$\sqrt{2}$ from W to $\mathbb{R}$.
It follows that: d_X is continuous from V to $\mathbb{R}$
and d_Y is continuous from W to $\mathbb{R}$.
So, since $\mathbb{I}_{d_X} \subseteq I$ and $\mathbb{I}_{d_Y} \subseteq I$, by Absoluteness of Continuity,
we see that: d_X is continuous from V to I
and d_Y is continuous from W to I.
We conclude that: $d_X \in C_I^V$ and $d_Y \in C_I^W$.
Since $g \in C_W^V$ and $d_Y \in C_I^W$, we see that: $d_Y \circ g \in C_I^V$.
Let $\alpha := d_X$ and let $\beta := d_Y \circ g$. Then $\alpha, \beta \in C_I^V$.

Claim: $\forall u \in V$, $(\alpha_u = 0) \Rightarrow (\beta_u = 0)$.
Proof of Claim:
Given $u \in V$. Want: $(\alpha_u = 0) \Rightarrow (\beta_u = 0)$.
Assume $\alpha_u = 0$. Want: $\beta_u = 0$.
We have $d_X(u_1, u_2) = \alpha(u_1, u_2) = \alpha(u) = \alpha_u = 0$, so $d_X(u_1, u_2) = 0$.
Then $u_1 = u_2$. Then $f_{u_1} = f_{u_2}$. Then $d_Y(f_{u_1}, f_{u_2}) = 0$.
We have $g_u = (f \times f)_u = (f_{u_1}, f_{u_2})$, so $g_u = (f_{u_1}, f_{u_2})$.
Then $\beta_u = (d_Y \circ g)_u = d_Y(g_u) = d_Y(f_{u_1}, f_{u_2}) = 0$, as desired.
End of proof of Claim.

Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall p, q \in \mathbb{D}_f$,
$(d_X(p, q) < \delta) \Rightarrow (d_Y(f_p, f_q) < \varepsilon)$.
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall p, q \in \mathbb{D}_f$,
$(d_X(p, q) < \delta) \Rightarrow (d_Y(f_p, f_q) < \varepsilon)$.
Since V is compact, by the Claim and Theorem 3.21.13,
choose $\delta > 0$ s.t. $\forall u \in V$, $(\alpha_u < \delta) \Rightarrow (\beta_u < \varepsilon)$.
Then $\delta > 0$. Want: $\forall p, q \in \mathbb{D}_f$, $(d_X(p, q) < \delta) \Rightarrow (d_Y(f_p, f_q) < \varepsilon)$.
Given $p, q \in \mathbb{D}_f$. Want: $(d_X(p, q) < \delta) \Rightarrow (d_Y(f_p, f_q) < \varepsilon)$.
Assume: $d_X(p, q) < \delta$. Want: $d_Y(f_p, f_q) < \varepsilon$.
We have $p, q \in \mathbb{D}_f = X$, so $(p, q) \in X \times X$.
Let $u := (p, q)$. Then $u \in X \times X = V$.
We have $\alpha_u = \alpha(u) = \alpha(p, q) = d_X(p, q) < \delta$.
Then, by choice of δ, we see that: $\beta_u < \varepsilon$.
Since $u = (p, q)$, we get: $u_1 = p$ and $u_2 = q$.
It follows that: $g_u = (f \times f)_u = (f_{u_1}, f_{u_2}) = (f_p, f_q)$.
Then $d_Y(f_p, f_q) = d_Y(g_u) = (d_Y \circ g)_u = \beta_u < \varepsilon$, as desired. $\square$

3.22. **Uniform limits and continuity.**

DEFINITION 3.22.1. *Let X and Y be sets, $f \in (Y^X)^{\mathbb{N}}$, $p \in X$.*
Then $f_\bullet(p) \in Y^{\mathbb{N}}$ is defined by: $\forall j \in \mathbb{N}$, $(f_\bullet(p))_j = f_j(p)$.

DEFINITION 3.22.2. *Let X be a set and let Y be a metric space.*
Let $f \in (Y^X)^{\mathbb{N}}$ and let $g \in Y^X$.
By $f \to g$ pointwise from X to Y, we mean:
$\forall p \in X$, $f_\bullet(p) \to g(p)$ *in* Y.
By $f \to g$ uniformly from X to Y, we mean:
$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ *s.t.*, $\forall j \in \mathbb{N}$, $\forall p \in X$,
$(j \geqslant K) \Rightarrow (d(f_j(p) , g(p)) < \varepsilon)$.

Up to reordering quantifiers, the definitions of uniform and pointwise limits are the same:

THEOREM 3.22.3. *Let X be a set and let Y be a metric space.*
Let $f \in (Y^X)^{\mathbb{N}}$ and let $g \in Y^X$.
Then: [$f \to g$ *pointwise from* X *to* Y] $\Leftrightarrow$
[$\forall p \in X$, $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ *s.t.*, $\forall j \in \mathbb{N}$,
$(j \geqslant K) \Rightarrow (d(f_j(p) , g(p)) < \varepsilon)$].

THEOREM 3.22.4. *Let $X := [0;1]$ and let $Y := \mathbb{R}$.*
Define $f \in (Y^X)^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $\forall w \in X$, $f_j(w) = w^j$. Let $g := \chi_{\{1\}}^X$.
Then: [$f \to g$ *pointwise from* X *to* Y]
& [$\neg$($f \to g$ *uniformly from* X *to* Y)]
& [$\forall j \in \mathbb{N}$, f_j *is continuous at* 1 *from* X *to* Y]
& [$\neg$(g *is continuous at* 1 *from* X *to* Y)].

THEOREM 3.22.5. *Let X and Y be metric spaces.*
Let $f \in (Y^X)^{\mathbb{N}}$, let $g \in Y^X$ and let $q \in X$.
Assume: [$f \to g$ *uniformly from* X *to* Y]
& [$\forall j \in \mathbb{N}$, f_j *is continuous at* q *from* X *to* Y]
Then: [g *is continuous at* q *from* X *to* Y].

Proof. Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall p \in X$,
$[d(p,q) < \delta] \Rightarrow [d(g(p) , g(q)) < \varepsilon]$.
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall p \in X$,
$[d(p,q) < \delta] \Rightarrow [d(g(p) , g(q)) < \varepsilon]$.
Since $f \to g$ uniformly from X to Y,
choose $K \in \mathbb{N}$ s.t., $\forall j \in X$, $\forall z \in X$,

$[\ j \geqslant K\ ] \ \Rightarrow\ [\ d(\ f_j(z)\ ,\ g(z)\ )\ <\ \varepsilon/3\ ].$

By hypothesis, $\forall j \in \mathbb{N}$,

f_j is continuous at q from X to Y.

Then f_K is continuous at q from X to Y. Let $h := f_K$.

Then h is continuous at q from X to Y, so

choose $\delta > 0$ s.t., $\forall p \in X$,

$[\ d(p,q) < \delta\ ] \ \Rightarrow\ [\ d(\ h(p)\ ,\ h(q)\ )\ <\ \varepsilon/3\ ].$

Then $\delta > 0$. Want: $\forall p \in X$, $[\ d(p,q) < \delta\ ] \ \Rightarrow\ [\ d(\ g(p)\ ,\ g(q)\ )\ <\ \varepsilon\ ]$.

Given $p \in X$. Want: $[\ d(p,q) < \delta\ ] \ \Rightarrow\ [\ d(\ g(p)\ ,\ g(q)\ )\ <\ \varepsilon\ ]$.

Assume: $d(p,q) < \delta$. Want: $d(\ g(p)\ ,\ g(q)\ )\ <\ \varepsilon$.

Since $K \geqslant K$, it follows, from the choice of K, that

$\forall z \in X$, $d(\ f_K(z)\ ,\ g(z)\ )\ <\ \varepsilon/3$.

Then: $d(\ f_K(p)\ ,\ g(p)\ )\ <\ \varepsilon/3$ and $d(\ f_K(q)\ ,\ g(q)\ )\ <\ \varepsilon/3$.

So, since $f_K = h$, we get:

$d(\ h(p)\ ,\ g(p)\ )\ <\ \varepsilon/3$ and $d(\ h(q)\ ,\ g(q)\ )\ <\ \varepsilon/3$.

As $d(\ g(p)\ ,\ h(p)\ )\ =\ d(\ h(p)\ ,\ g(p)\ )\ <\ \varepsilon/3$, $d(\ g(p)\ ,\ h(p)\ )\ <\ \varepsilon/3$.

Since $d(p,q) < \delta$, by choice of δ, we have: $d(\ h(p)\ ,\ h(q)\ )\ <\ \varepsilon/3$.

Then $d(\ g(p)\ ,\ g(q)\ )$

$\leqslant\ (\ d(\ g(p)\ ,\ h(p)\ )\ )\ +\ (\ d(\ h(p)\ ,\ h(q)\ )\ )\ +\ (\ d(\ h(q)\ ,\ g(p)\ )\ )$

$<\ (\ \varepsilon/3\ )\ +\ (\ \varepsilon/3\ )\ +\ (\ \varepsilon/3\ )\ =\ \varepsilon$, as desired. □

THEOREM 3.22.6. *Let X and Y be metric spaces.*

Let $f \in (C_Y^X)^{\mathbb{N}}$ and let $g \in Y^X$.

Assume: $f \to g$ uniformly from X to Y. Then: $g \in C_Y^X$.

Proof. Want: g is continuous from X to Y.

Want: g is continuous on $\mathbb{D}_g$ from X to Y.

Want: g is continuous on X from X to Y.

Want: $\forall q \in X$, g is continuous at q from X to Y.

Given $q \in X$. Want: g is continuous at q from X to Y.

We have: $\forall j \in \mathbb{N}$, $f_j \in C_Y^X$.

Then: $\forall j \in \mathbb{N}$, f_j is continuous from X to Y.

Then: $\forall j \in \mathbb{N}$, f_j is continuous on $\mathbb{D}_{f_j}$ from X to Y.

Then: $\forall j \in \mathbb{N}$, f_j is continuous on X from X to Y.

Then: $\forall j \in \mathbb{N}$, f_j is continuous at q from X to Y.

Then, by Theorem 3.22.5, g is continuous at q from X to Y. □

THEOREM 3.22.7. *Let X and Y be metric spaces.*

Let $f \in (Y^X)^{\mathbb{N}}$ and let $g \in Y^X$.

Assume $f \to g$ uniformly from X to Y.
Then $f \to g$ pointwise from X to Y.

Proof. Want: $\forall p \in X,\ f_\bullet(p) \to g(p)$ in Y.
Given $p \in X$. Want: $f_\bullet(p) \to g(p)$ in Y.
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ d_Y(\,(f_\bullet(p))_j\,,\,g(p)\,)\ <\ \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ d_Y(\,(f_\bullet(p))_j\,,\,g(p)\,)\ <\ \varepsilon\ ).$$
Since $f \to g$ uniformly from X to Y, choose $K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $\forall q \in X$,
$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ d_Y(\,f_j(q)\,,\,g(q)\,)\ <\ \varepsilon\ ).$$
Then $K \in \mathbb{N}$. Want: $\forall j \in \mathbb{N}$, $(\ j \geqslant K\ ) \Rightarrow (\ d_Y(\,(f_\bullet(p))_j\,,\,g(p)\,)\ <\ \varepsilon\ )$.
Given $j \in \mathbb{N}$. Want: $(\ j \geqslant K\ ) \Rightarrow (\ d_Y(\,(f_\bullet(p))_j\,,\,g(p)\,)\ <\ \varepsilon\ )$.
Assume: $j \geqslant K$. Want: $d_Y(\,(f_\bullet(p))_j\,,\,g(p)\,)\ <\ \varepsilon$.
Since $j \in \mathbb{N}$, since $p \in X$ and since $j \geqslant K$, by choice of K,
we see that: $d_Y(\,f_j(p)\,,\,g(p)\,)\ <\ \varepsilon$.
Then $d_Y(\,(f_\bullet(p))_j\,,\,g(p)\,)\ =\ d_Y(\,f_j(p)\,,\,g(p)\,)\ <\ \varepsilon$, as desired. □

In HW#13-2, we see that Lipschitz maps carry Cauchy sequences to Cauchy sequences. Mere continuity is not enough:

THEOREM 3.22.8. *Let $X := (-1;1)$ and let $Y := \mathbb{R}$.*
Define $f : X \to Y$ by $\forall w \in X,\ f_w = w/\sqrt{1-w^2}$.
Define $s \in X^{\mathbb{N}}$ by $\forall j \in \mathbb{N},\ s_j = j/(j+1)$.
Then: f is continuous from X to Y and
s is Cauchy in X and
$f \circ s$ is not Cauchy in Y.

DEFINITION 3.22.9. *Let K and Y be metric spaces.*
Let $Z := C_Y^K$ and $d := d_Y$. Assume: K is compact and nonempty.
Then $d_Y^K : Z \times Z \to [0;\infty)$ is defined by:
$$\forall f, g \in Z, \qquad d_Y^K(f,g) \quad = \quad \max \mathbb{I}_{d \circ (f,g)}.$$

Let K, Y be metric spaces. Assume K is empty. Then $C_Y^K = \{\varnothing\}$. That is, the only function in C_Y^K is the empty function.

We have little use for the empty metric space, but,
for the sake of completeness, we define d_Y^K when K is empty:

DEFINITION 3.22.10. *Let K and Y be metric spaces.*
Let $Z := C_Y^K$. Assume: K is empty.

Then $d_Y^K : Z \times Z \to [0;\infty)$ *is defined by:*

$$\forall f,g \in Z, \qquad d_Y^K(f,g) \quad = \quad 0.$$

THEOREM 3.22.11. *Let* K *and* Y *be metric spaces.*
Let $Z := C_Y^K$. *Assume:* K *is compact.* *Then* $d_Y^K \in \mathcal{M}(C_Y^K)$.

The metric d_Y^K is sometimes called the **uniform metric** on C_Y^K. It is the standard metric on C_Y^K.

By sloppiness, we use C_Y^K to denote the metric space (C_Y^K, d_Y^K). Note that $d_{C_Y^K} = d_Y^K$. By slopiness, we sometimes use d to denote d_Y^K.

THEOREM 3.22.12. *Let* K *and* Y *be metric spaces.*
Let $Z := C_Y^K$, $f \in Z^{\mathbb{N}}$, $g \in Z$. *Assume* K *is compact and nonempty.*
Then: $(\ f \to g$ *in* $Z\ ) \Leftrightarrow (\ f \to g$ *uniformly from* K *to* $Y\ )$.

Proof. Proof of $\Rightarrow$*:* Unassigned Homework. *End of proof of* $\Rightarrow$.

Proof of $\Leftarrow$*:* Let $d := d_Y^K$.
Assume: $f \to g$ uniformly from K to Y. Want: $f \to g$ in Z.
Want: $\forall \varepsilon > 0$, $\exists L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant L\ ) \ \Rightarrow\ (\ d(\ f_j\ ,\ g\ ) < \varepsilon\ ).$$
Given $\varepsilon > 0$. Want: $\exists L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$$(\ j \geqslant L\ ) \ \Rightarrow\ (\ d(\ f_j\ ,\ g\ ) < \varepsilon\ ).$$
Since $f \to g$ uniformly from K to Y, choose $L \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $\forall p \in K$,
$$(\ j \geqslant L\ ) \ \Rightarrow\ (\ d(\ f_j(p)\ ,\ g(p)\ ) < \varepsilon\ ).$$
Then $L \in \mathbb{N}$. Want: $\forall j \in \mathbb{N}$, $(\ j \geqslant L\ ) \Rightarrow (\ d(\ f_j\ ,\ g\ ) < \varepsilon\ )$.
Given $j \in \mathbb{N}$. Want: $(\ j \geqslant L\ ) \Rightarrow (\ d(\ f_j\ ,\ g\ ) < \varepsilon\ )$.
Assume: $j \geqslant L$. Want: $d(\ f_j\ ,\ g\ ) < \varepsilon$.
As $d : Z \times Z \to [0;\infty)$, we get $d(\ f_j\ ,\ g\ ) \in [0;\infty)$, and so $d(\ f_j\ ,\ g\ ) \neq ☹$.
Let $\alpha := d_Y \circ (f,g)$. Then $d(\ f_j\ ,\ g\ ) = \max \mathbb{I}_\alpha$.
Since $\max \mathbb{I}_\alpha = d(\ f_j\ ,\ g\ ) \neq ☹$, we get: $\max \mathbb{I}_\alpha \in \mathbb{I}_\alpha$.
Since $d(\ f_j\ ,\ g\ ) = \max \mathbb{I}_\alpha \in \mathbb{I}_\alpha$,
choose $p \in \mathbb{D}_\alpha$ s.t. $d(\ f_j\ ,\ g\ ) = \alpha_p$. Want: $\alpha_p < \varepsilon$.
Since $\alpha = d_Y \circ (f_j, g)$, it follows that $\mathbb{D}_\alpha \subseteq \mathbb{D}_{(f_j,g)}$.
Since $f_j, g \in C_Y^K$, we get: $\mathbb{D}_{f_j} = K = \mathbb{D}_g$.
Then $p \in \mathbb{D}_\alpha \subseteq \mathbb{D}_{(f_j,g)} = \mathbb{D}_{f_j} \bigcap \mathbb{D}_g = K \bigcap K = K$.
So, since $j \geqslant L$, by the choice of L, we get: $d_Y(\ f_j(p)\ ,\ g(p)\ ) < \varepsilon$.
Then $\alpha_p = (d_Y \circ (f_j, g))_p = d_Y(\ (f_j, g)_p\ ) = d_Y(\ f_j(p)\ ,\ g(p)\ ) < \varepsilon$.
End of proof of $\Leftarrow$. □

DEFINITION 3.22.13. *Let X and Y be sets, $S \subseteq Y^X$ and $p \in X$. Then $\mathrm{ev}_p^S : S \to Y$ is defined by:* $\forall f \in S,\ (\mathrm{ev}_p^S)_f = f_p$.

THEOREM 3.22.14. *Let K and Y be metric spaces.*
Let $f \in (C_Y^K)^{\mathbb{N}}$ and let $g \in Y^K$.
Assume: K is compact and
f is Cauchy in C_Y^K and
$f \to g$ pointwise from K to Y.
Then: $f \to g$ uniformly from K to Y.

Proof. Want: $\forall \varepsilon > 0$, $\exists L \in \mathbb{N}$ s.t., $\forall i \in \mathbb{N}$, $\forall p \in K$,
$$[\ i \geqslant L\ ] \Rightarrow [\ d_Y(\ f_i(p)\ ,\ g(p)\ ) < \varepsilon\ ].$$
Given $\varepsilon > 0$. Want: $\exists L \in \mathbb{N}$ s.t., $\forall i \in \mathbb{N}$, $\forall p \in K$,
$$[\ i \geqslant L\ ] \Rightarrow [\ d_Y(\ f_i(p)\ ,\ g(p)\ ] < \varepsilon\ ).$$
Let $d := d_Y^K$. Since f is Cauchy in C_Y^K, choose $L \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$$[\ i,j \geqslant L\ ] \Rightarrow [\ d(\ f_i\ ,\ f_j\ ) < \varepsilon/2\ ].$$
Then $L \in \mathbb{N}$. Want: $\forall i \in \mathbb{N}$, $\forall p \in K$,
$$[\ i \geqslant L\ ] \Rightarrow [\ d_Y(\ f_i(p)\ ,\ g(p)\ ] < \varepsilon\ ).$$
Given $i \in \mathbb{N}$, $p \in K$. Want: $[\ i \geqslant L\ ] \Rightarrow [\ d_Y(\ f_i(p)\ ,\ g(p)\ ) < \varepsilon\ ]$.
Assume: $i \geqslant L$. Want: $d_Y(\ f_i(p)\ ,\ g(p)\ ) < \varepsilon$.
By assumption, $f \to g$ pointwise from K to Y, so $f_\bullet(p) \to g(p)$ in Y, so choose $M \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $[\ j \geqslant M\ ] \Rightarrow [\ d_Y(\ (f_\bullet(p))_j\ ,\ g(p)\ ) < \varepsilon/2\ ]$.
Let $j := \max\{L, M\}$. Then $j \in \mathbb{N}$ and $j \geqslant L$ and $j \geqslant M$.
Since $j \geqslant L$, by choice of L, we get: $d(\ f_i\ ,\ f_j\ ) < \varepsilon/2$.
Since $j \geqslant M$, by choice of M, we get: $d(\ (f_\bullet(p))_j\ ,\ g(p)\ ) < \varepsilon/2$.
Then: $d(\ f_j(p)\ ,\ g(p)\ ) = d(\ (f_\bullet(p))_j\ ,\ g(p)\ ) < \varepsilon/2$.
Let $E := \mathrm{ev}_p^Z$. By HW#13-4, E is Lipschitz-1 from Z to Y.
Then $d_Y(\ E_{f_i}\ ,\ E_{f_j}\ ) \leqslant 1 \cdot (d(\ f_i\ ,\ f_j\ ))$.
We have: $E_{f_i} = E(f_i) = (\mathrm{ev}_p^K)(f_i) = f_i(p)$
and $E_{f_j} = E(f_j) = (\mathrm{ev}_p^K)(f_j) = f_j(p)$.
Then
$$d_Y(\ f_i(p)\ ,\ f_j(p)\ ) = d_Y(\ E_{f_i}\ ,\ E_{f_j}\ ) \leqslant 1 \cdot (d(\ f_i\ ,\ f_j\ )) = d(\ f_i\ ,\ f_j\ ) < \varepsilon/2.$$
So, since $d_Y(\ f_j(p)\ ,\ g(p)\ ) < \varepsilon/2$, we conclude that:
$$[\,d_Y(\ f_i(p)\ ,\ f_j(p)\ )\,] + [\,d_Y(\ f_j(p)\ ,\ g(p)\ )\,] < [\,\varepsilon/2\,] + [\,\varepsilon/2\,].$$
By the Triangle Inequality,
$$d_Y(\ f_i(p)\ ,\ g(p)\ ) \leqslant [\,d_Y(\ f_i(p)\ ,\ f_j(p)\ )\,] + [\,d_Y(\ f_j(p)\ ,\ g(p)\ )\,].$$
Then
$$d_Y(\ f_i(p)\ ,\ g(p)\ ) \leqslant [\,d_Y(\ f_i(p)\ ,\ f_j(p)\ )\,] + [\,d_Y(\ f_j(p)\ ,\ g(p)\ )\,] < [\,\varepsilon/2\,] + [\,\varepsilon/2\,] = \varepsilon,$$
as desired. ☐

THEOREM 3.22.15. *Let K be a compact metric space. Then K is complete.*

Proof. Since K is compact, by $\Rightarrow$ of Theorem 3.14.12, we conclude that: K proper and bounded. Then K is proper, so, by Theorem 3.19.9, K is complete. □

DEFINITION 3.22.16. *Let Y be a metric space and $s \in Y^{\mathbb{N}}$. Then:* $Y\text{-}\lim s \;:=\; \mathrm{UE}\{q \in Y \mid s \to q \text{ in } Y\}$.

Writers often omit the "Y-" and simply write "$\lim s$". Also, common is to pick an unbound "dummy variable", like j, and write "$\lim\limits_{j\to\infty} s_j$".

THEOREM 3.22.17. *Let Y be a metric space, $s \in Y^{\mathbb{N}}$, $p \in Y$. Then:* $(\; s \to p \text{ in } Y \;) \Leftrightarrow (\; Y\text{-}\lim s = p \;)$.

In the preceding theorem, $\Rightarrow$ follows from uniqueness of limits.
In the preceding theorem, $\Leftarrow$ follows from the definition of Y-lim.

THEOREM 3.22.18. *Let Y be a metric space and let $s \in Y^{\mathbb{N}}$. Assume s is convergent in Y. Then* ☹ $\neq Y\text{-}\lim s \in Y$.

THEOREM 3.22.19. *Let X be a set and let Y be a metric space. Let $f \in (Y^X)^{\mathbb{N}}$ and let $g \in Y^X$.*

Then: $(\; f \to g \text{ pointwise from } X \text{ to } Y \;)$
$\Leftrightarrow (\; \forall p \in X,\; f_\bullet(p) \to g(p) \text{ in } Y \;)$
$\Leftrightarrow (\; \forall p \in X,\; Y\text{-}\lim(f_\bullet(p)) = g(p) \;)$.

Note that, in the following theorem, by Theorem 3.22.15, K is complete. Thus both the domain K and the target Y are complete metric spaces, and the conclusion is that C_Y^K is a complete metric space.

We do not attempt to generalize by weakening the assumption that K is compact; in fact, in this writeup, we do not specify any standard metric C_Y^K, except in the case where K is compact.

In a later theorem (Theorem 3.22.22), we show that C_Y^K may be nonproper, and therefore noncompact.

THEOREM 3.22.20. *Let K and Y be metric spaces. Assume: K is compact and Y is complete. Then: C_Y^K is complete.*

Proof. Let $Z := C_Y^K$. Want: Z is complete.
Want: $\forall f \in Z^{\mathbb{N}}$, (f is Cauchy in Z) $\Rightarrow$ (f is convergent in Z).
Given $f \in Z^{\mathbb{N}}$. Want: (f is Cauchy in Z) $\Rightarrow$ (f is convergent in Z).
Assume: f is Cauchy in Z. Want: f is convergent in Z.
Want: $\exists g \in Z$ s.t. $f \to g$ in Z.
By HW#13-5, $\forall p \in K$, $f_\bullet(p)$ is Cauchy in Y.
So, since Y is complete, we conclude:
$\forall p \in K$, $f_\bullet(p)$ is convergent in Y.
Define $g \in Y^K$ by: $\forall p \in K$, $g(p) = Y\text{-}\lim(f_\bullet(p))$.
Then $g \in Y^K = Z$. Want: $f \to g$ in Z.
Then: $\forall p \in K$, $f_\bullet(p) \to g(p)$ in Y.
Then: $f \to g$ pointwise from K to Y.
So, since f is Cauchy in Z and since $Z = C_Y^K$, it follows,
by Theorem 3.22.14, that $f \to g$ uniformly from K to Y.
Then, by Theorem 3.22.12, $f \to g$ in Z, as desired. □

THEOREM 3.22.21. *Let K and Y be metric spaces.*
Assume: K is compact.
Let $Z := C_Y^K$, let $f \in Z^{\mathbb{N}}$ and let $h \in Y^K \backslash Z$.
Assume: $f \to h$ pointwise from K to Y.
Then: f is not subconvergent in Z.

Proof. Assume f is subconvergent in Z. Want: Contradiction.
Choose a subsequence ϕ of f s.t. ϕ is convergent in Z.
Choose $g \in Z$ s.t. $\phi \to g$ in Z.
Since $g \in Z$ and $h \notin Z$, we see that: $g \neq h$.
Choose $p \in K$ s.t. $g(p) \neq h(p)$.
Since $\phi \to g$ in Z and since $Z = C_Y^K$, it follows,
by Theorem 3.22.14, that $\phi \to g$ uniformly from K to Y.
Then, by Theorem 3.22.7,we have: $\phi \to g$ pointwise from K to Y.
By hypothesis, $f \to h$ pointwise from K to Y.
Then: $\phi_\bullet(p) \to g(p)$ in Y and $f_\bullet(p) \to h(p)$ in Y.
Let $E := \mathrm{ev}_p^Z$. Recall: ϕ is a subsequence of f.
Then: $E \circ \phi$ is a subsequence of $E \circ f$.
So, since $E \circ \phi = \phi_\bullet(p)$ and since $E \circ f = f_\bullet(p)$,
we conclude that $\phi_\bullet(p)$ is a subsequence of $f_\bullet(p)$.
So, since $f_\bullet(p) \to h(p)$ in Y, we see that $\phi_\bullet(p) \to h(p)$ in Y.
Since $\phi_\bullet(p) \to g(p)$ in Y and since $\phi_\bullet(p) \to h(p)$ in Y, we conclude,

by uniqueness of limits, that $g(p) = h(p)$.
By choice of p, we have: $g(p) \neq h(p)$. Contradiction. □

THEOREM 3.22.22. *Let $K := [0;1]$ and let $Y := \mathbb{R}$.*
Then: C_Y^K is nonproper.

Proof. Let $Z := C_Y^K$. Want: Z is nonproper.
Want: $\exists f \in Z_{\text{bi}}^{\mathbb{N}}$ s.t. f is not subconvergent in Z.
Define $f \in Z^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}, \forall x \in K, \quad f_j(x) = x^j$.
Let $g := C_K^0$. Then $g \in C_Y^K = Z$.
Also: $\forall j \in \mathbb{N}$, we have $d_Z(f_j, g) = d_Y^K(f_j, g) = 1$, so $f_j \in B_Z(g, 1)$.
Then $\mathbb{I}_f \subseteq B_Z(g, 1)$. Then $\mathbb{I}_f$ is bounded in Z.
So, as $f \in Z^{\mathbb{N}}$, we get $f \in Z_{\text{bi}}^{\mathbb{N}}$. Want: f is not subconvergent in Z.
Let $h := \chi_{\{1\}}^K$. Then $h \in Y^K$.
As h is not continuous at 1 from K to Y, we get: $h \notin Z$.
Then $f \to h$ pointwise from K to Y and $h \in Y^X \backslash Z$.
Then, by Theorem 3.22.21, we get: f is not subconvergent in Z. □

From Theorem 3.22.20 and Theorem 3.22.22, we see that the metric space $C_{\mathbb{R}}^{[0;1]}$ is both complete and nonproper. So, while proper implies complete, the converse is NOT true; complete does not imply proper.

We already observed this; see Theorem 3.19.21 and Theorem 3.19.22.

4. Derivatives

4.1. DNZ and BNZ and CVZ.

DEFINITION 4.1.1. *Let X be a metric space, f a function, $p \in X$.*
Then f is **defined near** *p in X means:*
$\exists B \in \mathcal{B}_X(p)$ *s.t.* $B \subseteq \mathbb{D}_f$.

DEFINITION 4.1.2. *Let X and Y be metric spaces.*
Let $f : X \dashrightarrow Y$ and let $p \in X$.
Then f is **bounded near** *p from X to Y means:*
$\exists B \in \mathcal{B}_X(p)$ *s.t.* ($B \subseteq \mathbb{D}_f$) & (f_*B *is bounded in* Y).

DEFINITION 4.1.3. *We define:*
$$\begin{aligned} \text{DNZ} &:= \{ f : \mathbb{R} \dashrightarrow \mathbb{R} \mid f \text{ is defined near } 0 \text{ in } \mathbb{R} \} \\ \text{and BNZ} &:= \{ f : \mathbb{R} \dashrightarrow \mathbb{R} \mid f \text{ is bounded near } 0 \text{ from } \mathbb{R} \text{ to } \mathbb{R} \} \\ \text{and CVZ} &:= \{ f \in \text{DNZ} \mid (f \text{ is continuous } 0 \text{ from } \mathbb{R} \text{ to } \mathbb{R}) \,\&\, (h_0 = 0) \}. \end{aligned}$$

THEOREM 4.1.4. *Let* $f : \mathbb{R} \to \mathbb{R}$. *Then:*
$$[\, f \in \text{BNZ} \,] \Leftrightarrow [\, \exists \delta, K > 0 \textit{ s.t.}, \forall x \in \mathbb{R}, (\, |x| < \delta \,) \Rightarrow (\, |f_x| \leqslant K \,)\,].$$

Proof. Proof of $\Leftarrow$*:* Unassigned HW. *End of proof of* $\Leftarrow$.

Proof of $\Rightarrow$*:* Assume: $f \in \text{BNZ}$.
Want: $\exists \delta, K > 0$ s.t., $\forall x \in \mathbb{R}, (\, |x| < \delta \,) \Rightarrow (\, |f_x| < K \,)$.
Since $f \in \text{BNZ}$, we see that f is bounded near 0 from $\mathbb{R}$ to $\mathbb{R}$,
so choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $(\, \mathbb{B} \subseteq \mathbb{D}_f \,)$ & $(\, f_* B$ is bounded in $\mathbb{R}$ $)$.
Since $f_* B$ is bounded in $\mathbb{R}$, choose $C \in \mathcal{B}_{\mathbb{R}}$ s.t. $f_* B \subseteq C$.
By the Recentering Theorem, choose $D \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $C \subseteq D$.
Since $B, D \in \mathcal{B}_{\mathbb{R}}(0)$, choose $\delta, K > 0$ s.t.
$$(\, B = B_{\mathbb{R}}(0, \delta) \,) \;\&\; (\, D = B_{\mathbb{R}}(0, K) \,).$$
Then $\delta, K > 0$. Want: $\forall x \in \mathbb{R}, (\, |x| < \delta \,) \Rightarrow (\, |f_x| < K \,)$.
Given $x \in \mathbb{R}$. Want: $(\, |x| < \delta \,) \Rightarrow (\, |f_x| < K \,)$.
Assume $|x| < \delta$. Want: $|f_x| < K$.
Since $B \subseteq \mathbb{D}_f$, we get: $B \bigcap \mathbb{D}_f = B$.
We have $d_{\mathbb{R}}(x, 0) = |x - 0| = |x| < \delta$, and so $x \in B_{\mathbb{R}}(0, \delta)$.
Then $x \in B_{\mathbb{R}}(0, \delta) = B = B \bigcap \mathbb{D}_f$, so $f_x \in f_* B$.
Then $f_x \in f_* B \subseteq C \subseteq D = B_{\mathbb{R}}(0, K)$, so $d_{\mathbb{R}}(f_x, 0) < K$.
So, since $d_{\mathbb{R}}(f_x, 0) = |f_x - 0| = |f_x|$,
we conclude that $|f_x| < K$, as desired.
End of proof of $\Leftarrow$. □

THEOREM 4.1.5. *Let* $f : \mathbb{R} \to \mathbb{R}$. *Then:*
$$[\, f \in \text{CVZ} \,] \Leftrightarrow [\, \forall \varepsilon > 0, \exists \delta > 0 \textit{ s.t.}, \forall x \in \mathbb{R}, (\, |x| < \delta \,) \Rightarrow (\, |f_x| \leqslant \varepsilon \,)\,].$$

Proof. This is HW#14-1. □

THEOREM 4.1.6. *Let* $f \in \text{CVZ}$ *and let* $B \in \mathcal{B}_{\mathbb{R}}(0)$.
Then: $\exists A \in \mathcal{B}_{\mathbb{R}}(0)$ *s.t.* $(\, A \subseteq \mathbb{D}_f \,)$ & $(\, f_* A \subseteq B \,)$.

Proof. Since $B \in \mathcal{B}_{\mathbb{R}}(0)$, choose $\varepsilon > 0$ s.t. $B = B_{\mathbb{R}}(0, \varepsilon)$.
By Theorem 4.1.5, choose $\delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\, |x| < \delta \,) \;\Rightarrow\; (\, |f_x| < \varepsilon \,).$$
Let $A := B_{\mathbb{R}}(0, \delta)$. Then $A \in \mathcal{B}_{\mathbb{R}}(0)$. Want: $(\, A \subseteq \mathbb{D}_f \,)$ & $(\, f_* A \subseteq B \,)$.

Claim: $A \subseteq \mathbb{D}_f$.
Proof of Claim:
Want: $\forall x \in A$, $x \in \mathbb{D}_f$. Given $x \in A$. Want: $x \in \mathbb{D}_f$.
Since $x \in A = B_{\mathbb{R}}(0, \delta)$, we get $d_{\mathbb{R}}(x, 0) < \delta$.

Then $|x| = |x-0| = d_{\mathbb{R}}(x,0) < \delta$, so, by choice of δ, we get: $|f_x| < \varepsilon$.
Then $f_x \neq ☹$, and so $x \in \mathbb{D}_f$, as desired.
End of proof of Claim.

By the claim, it suffices to show: $f_* A \subseteq B$.
Want: $\forall y \in f_* A$, $y \in B$. Given $y \in f_* A$. Want: $y \in B$.
Since $y \in f_* A$, choose $x \in A \bigcap \mathbb{D}_f$ s.t. $y = f_x$.
Since $x \in A \bigcap \mathbb{D}_f \subseteq A = B_{\mathbb{R}}(0,\delta)$, we get: $d_{\mathbb{R}}(x,0) < \delta$.
Then $|x| = |x-0| = d_{\mathbb{R}}(x,0) < \delta$, so, by choice of δ, we get: $|f_x| < \varepsilon$.
Then $d_{\mathbb{R}}(f_x,0) = |f_x - 0| = |f_x| < \varepsilon$, and so $f_x \in B_{\mathbb{R}}(0,\varepsilon)$.
Then $y = f_x \in B_{\mathbb{R}}(0,\varepsilon) = B$, as desired. □

THEOREM 4.1.7. DNZ $\supseteq$ BNZ $\supseteq$ CVZ.

Proof. By HW#14-2, we have CVZ $\subseteq$ BNZ. Want: BNZ $\subseteq$ DNZ.
Want: $\forall f \in$ BNZ, $f \in$ DNZ. Given $f \in$ BNZ. Want: $f \in$ DNZ.
Since $f \in$ BNZ, we see that f is bounded near 0 from $\mathbb{R}$ to $\mathbb{R}$,
so choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. ($\mathbb{B} \subseteq \mathbb{D}_f$) & ($f_* B$ is bounded in $\mathbb{R}$).
Since $B \in \mathcal{B}_{\mathbb{R}}(0)$ and $B \subseteq \mathbb{D}_f$, we get: f is defined near 0 in $\mathbb{R}$.
Then $f \in$ DNZ, as desired. □

We showed graphs to show that
not every element of BNZ is vanishing at 0 and
not every element of BNZ that vanishes at 0
is continuous at 0 and
not every element of DNZ is bounded near 0.
Consequently, DNZ $\supsetneq$ BNZ $\supsetneq$ CVZ.

DEFINITION 4.1.8. *Let X be a metric space.*
Let $f : X \dashrightarrow \mathbb{R}$, let $g : X \to [0;\infty)$ and let $p \in X$.
By g **envelopes** *f near p in X, we mean:*
$\exists B \in \mathcal{B}_X(p)$ *s.t.* $-g \leqslant f \leqslant g$ *on B.*

THEOREM 4.1.9. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$. Then:*
$$[\,(\,f \in \text{BNZ}\,) \Leftrightarrow (\,\exists K \geqslant 0 \text{ s.t. } C_{\mathbb{R}}^K \text{ envelopes } f \text{ near } 0\,)\,]$$
$$\&\,[\,(\,f \in \text{CVZ}\,) \Leftrightarrow (\,\forall \varepsilon \geqslant 0,\, C_{\mathbb{R}}^{\varepsilon} \text{ envelopes } f \text{ near } 0\,)\,].$$

Proof. Unassigned HW. □

DEFINITION 4.1.10. *Let $S \subseteq \mathbb{R}$ and let T be a set of functionals.*
Then $S \cdot_{\text{setset}} T := \{\, a \cdot f \,|\, (\,a \in S\,) \,\&\, (\,f \in T\,)\,\}$.

We typically omit the "setset", and write "$S \cdot T$".
We often omit the "$\cdot$", and write "ST".

DEFINITION 4.1.11. *Let S and T be a sets of functionals.*

$$\text{Then } S +_{\text{setset}} T \ := \ \{ f{+}g \,|\, (f \in S) \,\&\, (g \in T) \}.$$
$$\text{Also, } S \cdot_{\text{setset}} T \ := \ \{ f \cdot g \,|\, (f \in S) \,\&\, (g \in T) \}.$$

We typically omit the "setset", and write "$S + T$" and "$S \cdot T$".
We often omit the "$\cdot$", and write "ST".
For any two sets S and T of functionals, we have:

$$S + T = T + S \qquad \text{and} \qquad ST = TS.$$

THEOREM 4.1.12. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $g \in \text{DNZ}$.
Assume: $f = g$ *near* 0 *in* $\mathbb{R}$. *Then:* $f \in \text{DNZ}$.

THEOREM 4.1.13. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $g \in \text{BNZ}$.
Assume: $f = g$ *near* 0 *in* $\mathbb{R}$. *Then:* $f \in \text{BNZ}$.

THEOREM 4.1.14. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $g \in \text{CVZ}$.
Assume: $f = g$ *near* 0 *in* $\mathbb{R}$. *Then:* $f \in \text{CVZ}$.

The preceding three theorems express that DNZ, BNZ and CVZ are all "zero-local" conditions. That is, given the restriction f to a ball around zero in $\mathbb{R}$, we can determine whether or not f satisfies the condition. We do not need to know how f behaves far away from zero.

DEFINITION 4.1.15. *Let S be a set of functionals. By S is v.op.-closed, we mean:* $(\, \mathbb{R} \cdot S \subseteq S \,) \,\&\, (\, S + S \subseteq S \,)$.

In the preceding definition, "v.op.-closed" is an abbreviation for "vector-operation-closed". Unassigned HW: Show DNZ is v.op.-closed. From HW#13-3, BNZ and CVZ are both v.op.-closed, as well.

THEOREM 4.1.16. $\text{BNZ} \cdot \text{CVZ} \subseteq \text{CVZ}$.

Proof. Want: $\forall h \in \text{BNZ} \cdot \text{CVZ}$, $h \in \text{CVZ}$.
Given $h \in \text{BNZ} \cdot \text{CVZ}$. Want: $h \in \text{CVZ}$.
By HW#14-1, it suffices to show: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,

$$(\, |x| < \delta \,) \ \Rightarrow \ (\, |h_x| < \varepsilon \,).$$

Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,

$$(\, |x| < \delta \,) \ \Rightarrow \ (\, |h_x| < \varepsilon \,).$$

Since $h \in \text{BNZ} \cdot \text{CVZ}$, choose $f \in \text{BNZ}$ and $g \in \text{CVZ}$ s.t. $h = fg$.
Since $f \in \text{BNZ}$, by Theorem 4.1.4, choose $\lambda, K > 0$ s.t., $\forall x \in \mathbb{R}$,

$$(\ |x| < \lambda \ ) \ \Rightarrow \ (\ |f_x| < K \ ).$$

Since $g \in \mathrm{CVZ}$, by Theorem 4.1.4, choose $\mu > 0$ s.t., $\forall x \in \mathbb{R}$,

$$(\ |x| < \mu \ ) \ \Rightarrow \ (\ |g_x| < \varepsilon/K \ ).$$

Let $\delta := \min\{\lambda, \mu\}$. Then: $\delta > 0$ and $\delta \leqslant \lambda$ and $\delta \leqslant \mu$.

Want: $\forall x \in \mathbb{R}$, $(\ |x| < \delta \ ) \ \Rightarrow \ (\ |h_x| < \varepsilon \ )$.

Given $x \in \mathbb{R}$. Want: $(\ |x| < \delta \ ) \ \Rightarrow \ (\ |h_x| < \varepsilon \ )$.

Since $|x| < \delta \leqslant \lambda$, by choice of λ, we have: $|f_x| < K$.

Since $|x| < \delta \leqslant \mu$, by choice of μ, we have: $|g_x| < \varepsilon/K$.

Since $0 \leqslant |f_x| < K$ and $0 \leqslant |g_x| < \varepsilon/K$, we get: $|f_x| \cdot |g_x| < K \cdot (\varepsilon/K)$.

Then $|h_x| = |(fg)_x| = |f_x \cdot g_x| = |f_x| \cdot |g_x| < K \cdot (\varepsilon/K) = \varepsilon$. $\square$

Unassigned HW: Show $\mathrm{DNZ} \cdot \mathrm{DNZ} \subseteq \mathrm{DNZ}$.

From HW#13-4, $\mathrm{BNZ} \cdot \mathrm{BNZ} \subseteq \mathrm{BNZ}$.

From Theorem 4.1.16, $\mathrm{BNZ} \cdot \mathrm{CVZ} \subseteq \mathrm{CVZ}$. Also $\mathrm{CVZ} \subseteq \mathrm{BNZ}$.

Then $\mathrm{CVZ} \cdot \mathrm{CVZ} \subseteq \mathrm{BNZ} \cdot \mathrm{CVZ} \subseteq \mathrm{CVZ}$,

so $\mathrm{CVZ} \cdot \mathrm{CVZ} \subseteq \mathrm{CVZ}$.

4.2. **Little-o and big-O functions, and homogeneous polynomials.**

Recall that $| \bullet | : \mathbb{R} \to \mathbb{R}$. Also: $\forall x \in \mathbb{R}$, $| \bullet |_x = |x|$.

DEFINITION 4.2.1. $(\bullet) := \mathrm{id}_{\mathbb{R}}$.

THEOREM 4.2.2. *$\forall x \in \mathbb{R}$, $(\bullet)_x = x$.*

It is our convention, in this course, that $0^0 = 1$.

Then, for any functional f, we have $f^0 = C^1_{\mathbb{D}_f}$.

Note, also, that, for any functional f, we have $f^1 = f$.

THEOREM 4.2.3. $(\ | \bullet |^0 = C^1_{\mathbb{R}} = (\bullet)^0 \ )$
$\& \ (\ | \bullet |^1 = | \bullet | \ ) \ \& \ (\ (\bullet)^1 = (\bullet) \ )$
$\& \ (\ | \bullet |^2 = (\bullet)^2 \ ) \ \& \ (\ | \bullet |^4 = (\bullet)^4 \ )$
$\& \ (\ | \bullet |^6 = (\bullet)^6 \ ) \ \& \ (\ | \bullet |^8 = (\bullet)^8 \ )$.

In the preceding theorem, the pattern established, for even powers continues.

We graphed $(\bullet)^3$ and $-(\bullet)^3$ and noted that

the union of the graphs of $| \bullet |^3$ and $-| \bullet |^3$

is the same as

the union of the graphs of $(\bullet)^3$ and $-(\bullet)^3$.

DEFINITION 4.2.4. *Let S be a set of functionals, f a functional. Then $S \cdot_{\text{setfn}} f := S \cdot \{f\}$ and $f \cdot_{\text{fnset}} S := \{f\} \cdot S$.*

We typically omit "setfn" and "fnset", and write "$S \cdot f$" and "$f \cdot S$".
We often omit the "$\cdot$", and write "Sf" and "fS".
For any set S of functionals, for any functional g, we have:

$$Sg = \{fg \,|\, f \in S\} \qquad \text{and} \qquad gS = \{gf \,|\, f \in S\}.$$

For any set S of functionals, for any functional f, we have: $Sf = fS$.

DEFINITION 4.2.5. *Let $j \in \mathbb{N}_0$. Then:*

$$\mathcal{o}_j \; := \; \text{CVZ} \cdot |\bullet|^j \qquad \textit{and} \qquad \widehat{\mathcal{O}}_j \; := \; \text{BNZ} \cdot |\bullet|^j.$$

Note that $\mathcal{o}_0 = \text{CVZ}$ and that $\widehat{\mathcal{O}}_0 = \text{BNZ}$.
Recall that $\text{CVZ} \subsetneq \text{BNZ} \subsetneq \text{DNZ}$.
Then $\mathcal{o}_0 = \text{CVZ} \subseteq \text{BNZ} \subseteq \text{DNZ}$.
Also, $\widehat{\mathcal{O}}_0 = \text{BNZ} \subseteq \text{DNZ}$, and $\widehat{\mathcal{O}}_0 = \text{BNZ} \nsubseteq \text{CVZ}$.
From Theorem 4.1.16, $\text{BNZ} \cdot \text{CVZ} \subseteq \text{CVZ}$.
Also, for all $j \in \mathbb{N}_0$, we have: $|\bullet|^j \in \text{BNZ}$.
Then, for all $j \in \mathbb{N}_0$, we have: $\mathcal{o}_j \subseteq \text{CVZ} \cdot \text{BNZ} = \text{BNZ} \cdot \text{CVZ} \subseteq \text{CVZ}$.
Then, for all $j \in \mathbb{N}_0$, we have: $\mathcal{o}_j \subseteq \text{CVZ} \subseteq \text{BNZ} \subseteq \text{DNZ}$.
Also, for all $j \in \mathbb{N}$, we have $|\bullet|^j \in \text{CVZ}$.
Then, for all $j \in \mathbb{N}$, we have: $\widehat{\mathcal{O}}_j \subseteq \text{BNZ} \cdot \text{CVZ} \subseteq \text{CVZ}$.
Then, for all $j \in \mathbb{N}$, we have $\widehat{\mathcal{O}}_j \subseteq \text{CVZ} \subseteq \text{BNZ} \subseteq \text{DNZ}$.
However, recall that $\widehat{\mathcal{O}}_0 = \text{BNZ} \nsubseteq \text{CVZ}$.
Since $\text{CVZ} \subseteq \text{BNZ}$, we conclude: $\forall j \in \mathbb{N}_0, \;\; \mathcal{o}_j \subseteq \widehat{\mathcal{O}}_j$.

THEOREM 4.2.6. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $j \in \mathbb{N}_0$. Then:*

$$[\, (\, f \in \widehat{\mathcal{O}}_j \,) \Leftrightarrow (\, \exists K \geqslant 0 \textit{ s.t. } K \cdot |\bullet|^j \textit{ envelopes } f \textit{ near } 0 \,)\,]$$
$$\&\, [\, (\, f \in \mathcal{o}_j \,) \Leftrightarrow (\, \forall \varepsilon \geqslant 0,\, \varepsilon \cdot |\bullet|^j \textit{ envelopes } f \textit{ near } 0 \,)\,].$$

For all $f : \mathbb{R} \dashrightarrow \mathbb{R}$, for all $j \in \mathbb{N}$, we have $\dfrac{f}{|\bullet|^j} \notin \text{DNZ} \supseteq \text{BNZ} \supseteq \text{CVZ}$.
We make an adjustment to correct for this difficulty:

DEFINITION 4.2.7. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $j \in \mathbb{Z}$. Then:*

$$f_j^+ := \text{adj}_0^0\,(f \cdot |\bullet|^j) \qquad \textit{and} \qquad f_j^- := f_{-j}^+.$$

For any $f : \mathbb{R} \dashrightarrow \mathbb{R}$, for any $j \in \mathbb{Z}$, we have: $f_j^- := \text{adj}_0^0 \left(\dfrac{f}{|\bullet|^j} \right)$
For any $f : \mathbb{R} \dashrightarrow \mathbb{R}$, we have: $f_0^+ = f = f_0^-$.

Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $j, k \in \mathbb{Z}$. Then:

$$[\ (f_j^+)_k^+ = f_{j+k}^+\ ] \quad \& \quad [\ (f_j^-)_k^- = f_{j+k}^-\ ] \quad \&$$
$$[\ (f_j^+)_k^- = f_{j-k}^+ = f_{k-j}^-\ ] \quad \& \quad [\ (f_j^-)_k^+ = f_{k-j}^+ = f_{j-k}^-\ ].$$

For any $f : \mathbb{R} \dashrightarrow \mathbb{R}$, we have: $(f_j^+)_j^- = \mathrm{adj}_0^0 f = (f_j^-)_j^+$.

THEOREM 4.2.8. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, let $j \in \mathbb{Z}$ and let $g := f_j^-$.*
Then $\mathbb{D}_g = (\mathbb{D}_f)_0^+$. Also, $(\ 0 \in \mathbb{D}_f\ ) \Rightarrow (\ \mathbb{D}_g = \mathbb{D}_f\ )$.

For any $f \in \mathrm{DNZ}$, for any $j \in \mathbb{Z}$, we have: $\mathbb{D}_{f_j^+} = \mathbb{D}_f = \mathbb{D}_{f_j^-}$.
For any $f \in \mathrm{DNZ}$, for any $j \in \mathbb{Z}$, we have: $f_j^+, f_j^- \in \mathrm{DNZ}$.
For any $f \in \mathrm{DNZ}$, we have: $\mathrm{adj}_0^0 f = f_0^+ \in \mathrm{DNZ}$.
For any $f : \mathbb{R} \dashrightarrow \mathbb{R}$, we have: $(\ f_0 = 0\ ) \Rightarrow (\ \mathrm{adj}_0^0 f = f\ )$.

THEOREM 4.2.9. *We have:*
$(\ \forall f \in \mathrm{CVZ},\ \mathrm{adj}_0^0 f = f \in \mathrm{CVZ}\ )\ \&\ (\ \forall f \in \mathrm{BNZ},\ \mathrm{adj}_0^0 f \in \mathrm{BNZ}\ )$.

We graphed $(\bullet)/|\bullet|$ and $(\bullet)^2/|\bullet|$.
We noted that $(\bullet)^2/|\bullet| = |\bullet|^2/|\bullet| \neq |\bullet|$.
We graphed $(\bullet)_1^-$ and $((\bullet)^2)_1^-$.
We noted that $(\bullet)_1^- \notin \mathrm{CVZ}$ and $(\bullet)_1^- \in \mathrm{BNZ} \subseteq \mathrm{DNZ}$.
We noted that $((\bullet)^2)_1^- = (|\bullet|^2)_1^- = |\bullet| \in \mathrm{CVZ} \subseteq \mathrm{BNZ} \subseteq \mathrm{DNZ}$.

THEOREM 4.2.10. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $j \in \mathbb{N}_0$. Assume $f_0 = 0$.*
Then: $(\ f \in \mathcal{O}_j\ ) \Leftrightarrow (\ f_j^- \in \mathrm{CVZ}\ )$.

Proof. Since $f_0 = 0$, it follows that: $\mathrm{adj}_0^0 f = f$.

Proof of $\Rightarrow$:
Assume $f \in \mathcal{O}_j$. Want: $f_j^- \in \mathrm{CVZ}$.
Since $f \in \mathcal{O}_j = \mathrm{CVZ} \cdot |\bullet|^j$, choose $\phi \in \mathrm{CVZ}$ s.t. $f = \phi \cdot |\bullet|^j$.
Since $\phi \in \mathrm{CVZ}$, by Theorem 4.2.9, we get: $\mathrm{adj}_0^0 \phi \in \mathrm{CVZ}$.
Then $f = \mathrm{adj}_0^0 f = \mathrm{adj}_0^0(\phi \cdot |\bullet|^j) = \phi_j^+$.
Then $f_j^- = (\phi_j^+)_j^- = \mathrm{adj}_0^0 \phi \in \mathrm{CVZ}$, as desired.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$:
Assume $f_j^- \in \mathrm{CVZ}$. Want: $f \in \mathcal{O}_j$.
Let $g := f_j^- \cdot |\bullet|^j$. Then $g \in \mathrm{CVZ} \cdot |\bullet|^j = \mathcal{O}_j$. Want: $g = f$.
We have $g_0 = (f_j^- \cdot |\bullet|^j)_0 = (f_j^-)_0 \cdot |0|^j = 0$. Then $\mathrm{adj}_0^0 g = g$.
Then $g = \mathrm{adj}_0^0 g = \mathrm{adj}_0^0(f_j^- \cdot |\bullet|^j) = (f_j^-)_j^+ = \mathrm{adj}_0^0 f = f$, as desired.
End of proof of $\Leftarrow$. ∎

THEOREM 4.2.11. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $j \in \mathbb{N}_0$. *Assume* $f_0 = 0$.
Then: $(f \in \widehat{\mathcal{O}}_j) \Leftrightarrow (f_j^- \in \mathrm{BNZ})$.

Proof. This is HW#14-5. □

THEOREM 4.2.12. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: $f = g$ *near* 0 *in* $\mathbb{R}$.
Then: $(f_0 = 0) \Leftrightarrow (g_0 = 0)$.
Also: $\forall j \in \mathbb{N}_0$, $f_j^- = g_j^-$ *near* 0 *in* $\mathbb{R}$.

THEOREM 4.2.13. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $j \in \mathbb{N}_0$.
Assume: $f = g$ *near* 0 *in* $\mathbb{R}$.
Assume $g \in \mathcal{o}_j$. *Then* $f \in \mathcal{o}_j$.

THEOREM 4.2.14. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $j \in \mathbb{N}_0$.
Assume: $f = g$ *near* 0 *in* $\mathbb{R}$.
Assume $g \in \widehat{\mathcal{O}}_j$. *Then* $f \in \widehat{\mathcal{O}}_j$.

The preceding two theorems express that:
$\forall j \in \mathbb{N}_0$, $\mathcal{o}_j$ and $\widehat{\mathcal{O}}_j$ are both "zero-local" conditions.

DEFINITION 4.2.15. *Define* $\sqrt{\bullet} : \mathbb{R} \dashrightarrow \mathbb{R}$ *by:* $\forall x \in \mathbb{R}$, $(\sqrt{\bullet})_x = \sqrt{x}$.

DEFINITION 4.2.16. *Let* f *be a functional.*
Then we define: $(|f| := |\bullet| \circ f) \;\&\; (\sqrt{f} := \sqrt{\bullet} \circ f)$.

Let f be a functional. Recall: $f^0 = C_{\mathbb{R}}^1$.
Also, $\forall j \in \mathbb{N}$, $f^j = f \cdots f$ is the j-fold product of f with itself.

DEFINITION 4.2.17. *Let* f *be a functional,* $j \in \mathbb{N}$.
Then we define: $f^{-j} := \dfrac{1}{f^j}$.

DEFINITION 4.2.18. *Let* f *be a functional,* $j \in \mathbb{Z}$.
Then we define: $f^{j+(1/2)} := f^j \cdot \sqrt{f}$.

THEOREM 4.2.19. *Let* f *be a functional and let* x *be an object.*
Then: $[\, |f|_x = |f_x| \,] \;\&\; [\, (\sqrt{f})_x = \sqrt{f_x} \,]$
$\&\; [\, \forall j \in \mathbb{Z},\; (f^j)_x = (f_x)^j \,]$
$\&\; [\, \forall j \in \mathbb{Z},\; (f^{j+(1/2)})_x = (f_x)^{j+(1/2)} \,]$.

THEOREM 4.2.20. *Let* $j \in \mathbb{N}_0$. *Then* $(\bullet)^j \in \widehat{\mathcal{O}}_j \backslash \mathcal{o}_j$.

Proof. Let $f := (\bullet)^j$. Want: $f \in \widehat{\mathcal{O}}_j \backslash \mathcal{o}_j$.
We have: either (1) $j = 0$ or (2) $j \neq 0$.

Case (1):
We have $(\bullet)^j = (\bullet)^0 = C^1_{\mathbb{R}} \in \text{BNZ}\backslash\text{CVZ} = \widehat{\mathcal{O}}_0\backslash\mathcal{O}_0 = \widehat{\mathcal{O}}_j\backslash\mathcal{O}_j$.
End of Case (1).

Case (2):
We have $j \in \mathbb{N}_0\backslash\{0\} = \mathbb{N}$. Then $f(0) = 0^j = 0$.
So, by Theorem 4.2.11 and Theorem 4.2.10,
it suffices to show $f_j^- \in \text{BNZ}\backslash\text{CVZ}$.
Since $f_j^- = ((\bullet)^j)_j^- = \text{adj}_0^0\left(\dfrac{f}{|\bullet|^j}\right) = \text{adj}_0^0\left(\dfrac{(\bullet)^j}{|\bullet|^j}\right)$, we see that:

$$\left(\ \forall x > 0, \quad f_j^-(x) = \frac{x^j}{x^j} = 1\ \right) \quad \& \quad \left(\ f_0 = 0\ \right)$$
$$\& \quad \left(\ \forall x < 0, \quad f_j^-(x) = \frac{x^j}{-x^j} = -1\ \right).$$

Then $f_j^- \in \text{BNZ}\backslash\text{CVZ}$, as desired.
End of Case (2). □

THEOREM 4.2.21. *Let $j \in \mathbb{N}$. Then $\mathcal{O}_j \subsetneq \widehat{\mathcal{O}}_j$.*

Proof. We have $\mathcal{O}_j = \text{CVZ} \cdot |\bullet|^j \subseteq \text{BNZ} \cdot |\bullet|^j = \widehat{\mathcal{O}}_j$.
Want: $\exists f$ s.t. ($f \notin \mathcal{O}_j$) & ($f \in \widehat{\mathcal{O}}_j$).
Let $f := |\bullet|^j$. Want: ($f \notin \mathcal{O}_j$) & ($f \in \widehat{\mathcal{O}}_j$).
Since $f_0 = 0$, by Theorem 4.2.11 and Theorem 4.2.10,
it suffices to show $f_j^- \in \text{BNZ}\backslash\text{CVZ}$.
Since $(f_j^-)_0 = 0 = (\text{adj}_0^0(C^1_{\mathbb{R}}))_0$
and since $\forall x \in \mathbb{R}_0^\times$, $(f_j^-)_x = 1 = (\text{adj}_0^0(C^1_R))_x$,
we conclude: $f_j^- = \text{adj}_0^0(C^1_{\mathbb{R}})$.
Then: $f_j^- = \text{adj}_0^0(C^1_{\mathbb{R}}) \in \text{BNZ}\backslash\text{CVZ}$, as desired. □

THEOREM 4.2.22. *Let $j \in \mathbb{N}_0$. Then $\widehat{\mathcal{O}}_{j+1} \subsetneq \mathcal{O}_j$.*

Proof. We have $|\bullet| \in \text{CVZ}$. Recall: $\text{BNZ} \cdot \text{CVZ} \subseteq \text{CVZ}$.
Then: $\widehat{\mathcal{O}}_{j+1} = \text{BNZ} \cdot |\bullet|^{j+1} = \text{BNZ} \cdot |\bullet| \cdot |\bullet|^j$
$\in \text{BNZ} \cdot \text{CVZ} \cdot |\bullet|^j \subseteq \text{CVZ} \cdot |\bullet|^j = \mathcal{O}_j$.
Want: $\exists f$ s.t. ($f \notin \widehat{\mathcal{O}}_{j+1}$) & ($f \in \mathcal{O}_j$).
Let $f := |\bullet|^{j+(1/2)}$. Want: ($f \notin \widehat{\mathcal{O}}_{j+1}$) & ($f \in \mathcal{O}_j$).
Since $f_0 = 0$, by Theorem 4.2.11 and Theorem 4.2.10,
it suffices to show: ($f_{j+1}^- \notin \text{BNZ}$) & ($f_j^- \in \text{CVZ}$).
Since $(f_{j+1}^-)_0 = 0 = (\text{adj}_0^0(|\bullet|^{-1/2}))_0$

and since $\forall x \in \mathbb{R}_0^\times$, $(f_{j+1}^-)_x = |x|^{-1/2} = (\mathrm{adj}_0^0(|\bullet|^{-1/2}))_x$,
we conclude: $f_{j+1}^- = \mathrm{adj}_0^0(|\bullet|^{-1/2})$.
Then: $f_{j+1}^- = \mathrm{adj}_0^0(|\bullet|^{-1/2}) \notin \mathrm{BNZ}$. Want: $f_j^- \in \mathrm{CVZ}$.
Since $(f_j^-)_0 = 0 = (\mathrm{adj}_0^0(|\bullet|^{1/2}))_0$
and since $\forall x \in \mathbb{R}_0^\times$, $(f_j^-)_x = |x|^{1/2} = (\mathrm{adj}_0^0(|\bullet|^{1/2}))_x$,
we conclude: $f_j^- = \mathrm{adj}_0^0(|\bullet|^{1/2})$.
Since $(|\bullet|^{1/2})_0 = |0|^{1/2} = 0$, we get: $\mathrm{adj}_0^0(|\bullet|^{1/2}) = |\bullet|^{1/2}$.
Then: $f_j^- = \mathrm{adj}_0^0(|\bullet|^{1/2}) = |\bullet|^{1/2} \in \mathrm{CVZ}$, as desired. □

THEOREM 4.2.23. *Let $j \in \mathbb{N}_0$. Then $\mathcal{O}_j + \mathcal{O}_j \subseteq \mathcal{O}_j$.*

Proof. Recall: $\mathrm{CVZ} + \mathrm{CVZ} \subseteq \mathrm{CVZ}$.
Then: $\mathcal{O}_j + \mathcal{O}_j = (\mathrm{CVZ} \cdot |\bullet|^j) + (\mathrm{CVZ} \cdot |\bullet|^j)$
$= (\mathrm{CVZ} + \mathrm{CVZ}) \cdot |\bullet|^j \subseteq \mathrm{CVZ} \cdot |\bullet|^j = \mathcal{O}_j$, as desired. □

THEOREM 4.2.24. *Let $j \in \mathbb{N}_0$. Then $\mathcal{O}_j$ and $\widehat{\mathcal{O}}_j$ are v.op.-closed.*

Proof. Unassigned HW. □

THEOREM 4.2.25. *Let $j, k \in \mathbb{N}_0$. Then $\mathcal{O}_j \cdot \widehat{\mathcal{O}}_k \subseteq \mathcal{O}_{j+k}$.*

Proof. Recall: $\mathrm{CVZ} \cdot \mathrm{BNZ} \subseteq \mathrm{CVZ}$.
Then: $\mathcal{O}_j \cdot \widehat{\mathcal{O}}_k = (\mathrm{CVZ} \cdot |\bullet|^j) \cdot (\mathrm{BNZ} \cdot |\bullet|^k)$
$= (\mathrm{CVZ} \cdot \mathrm{BNZ}) \cdot |\bullet|^{j+k} \subseteq \mathrm{CVZ} \cdot |\bullet|^{j+k} = \mathcal{O}_{j+k}$. □

THEOREM 4.2.26. *Let $j, k \in \mathbb{N}_0$. Then:*

$$(\widehat{\mathcal{O}}_j \cdot \widehat{\mathcal{O}}_k \subseteq \widehat{\mathcal{O}}_{j+k}) \,\&\, (\mathcal{O}_j \cdot \widehat{\mathcal{O}}_k \subseteq \mathcal{O}_{j+k})$$
$$\&\, (\widehat{\mathcal{O}}_j \cdot \mathcal{O}_k \subseteq \mathcal{O}_{j+k}) \,\&\, (\mathcal{O}_j \cdot \mathcal{O}_k \subseteq \mathcal{O}_{j+k}).$$

Proof. Unassigned HW. □

DEFINITION 4.2.27. *For all $j \in \mathbb{N}_0$, let $\mathcal{H}_j := \mathbb{R} \cdot (\bullet)^j$.*

We have: $\forall j \in \mathbb{N}_0$,

$$\mathcal{H}_j = \{ f : \mathbb{R} \to \mathbb{R} \mid \exists a \in \mathbb{R} \text{ s.t., } \forall x \in \mathbb{R},\ f_x = ax^j \}.$$

DEFINITION 4.2.28. *We define:*

$$\mathcal{C} := \mathcal{H}_0 \quad \textit{and} \quad \mathcal{L} := \mathcal{H}_1 \quad \textit{and} \quad \mathcal{Q} := \mathcal{H}_2 \quad \textit{and} \quad \mathcal{K} := \mathcal{H}_3.$$

THEOREM 4.2.29. *$\forall j \in \mathbb{N}_0$, $\mathcal{H}_j$ is v.op.-closed.*

THEOREM 4.2.30. *$\forall j, k \in \mathbb{N}_0$, $\mathcal{H}_j \cdot \mathcal{H}_k \subseteq \mathcal{H}_{j+k}$.*

DEFINITION 4.2.31. *We define:* $\mathbf{0} := C_{\mathbb{R}}^0$.

THEOREM 4.2.32. *Let $j \in \mathbb{N}_0$. Then $\mathcal{H}_j \subseteq \widehat{\mathcal{O}}_j$ and $\mathcal{H}_j \bigcap \mathcal{O}_j = \{\mathbf{0}\}$.*

Proof. By Theorem 4.2.20, we have: $(\bullet)^j \in \widehat{\mathcal{O}}_j$.
By Theorem 4.2.24, $\widehat{\mathcal{O}}_j$ and $\mathcal{o}_j$ are v.op.-closed,
and so $\mathbb{R} \cdot \widehat{\mathcal{O}}_j \subseteq \widehat{\mathcal{O}}_j$ and $\mathbb{R} \cdot \mathcal{o}_j \subseteq \mathcal{o}_j$.
Then: $\mathcal{H}_j = \mathbb{R} \cdot (\bullet)^j \subseteq \mathbb{R} \cdot \widehat{\mathcal{O}}_j \subseteq \widehat{\mathcal{O}}_j$. Want: $\mathcal{H}_j \bigcap \mathcal{o}_j = \{\mathbf{0}\}$.
We have: $\mathbf{0} = 0 \cdot (\bullet)^j \in \mathcal{H}_j$. Also: $\mathbf{0} = C_{\mathbb{R}}^0 = \mathbb{C}_{\mathbb{R}}^0 \cdot |\bullet|^j \in \mathrm{CVZ} \cdot |\bullet|^j = \mathcal{o}_j$.
Since $\mathbf{0} \in \mathcal{H}_j$ and $\mathbf{0} \in \mathcal{o}_j$, we get $\mathbf{0} \in \mathcal{H}_j \bigcap \mathcal{o}_j$, so $\{\mathbf{0}\} \subseteq \mathcal{H}_j \bigcap \mathcal{o}_j$.
Want: $\mathcal{H}_j \bigcap \mathcal{o}_j \subseteq \{\mathbf{0}\}$. Want: $\forall f \in \mathcal{H}_j \bigcap \mathcal{o}_j,\ f \in \{\mathbf{0}\}$.
Given $f \in \mathcal{H}_j \bigcap \mathcal{o}_j$. Want: $f \in \{\mathbf{0}\}$. Want: $f = \mathbf{0}$.
Since $f = \mathcal{H}_j = \mathbb{R} \cdot (\bullet)^j$, choose $a \in \mathbb{R}$ s.t. $f = a \cdot (\bullet)^j$.
Want: $a = 0$. Assume $a \neq 0$. Want: Contradiction.
Since $1/a \in \mathbb{R}$ and $f \in \mathcal{o}_j$, we get: $(1/a) \cdot f \in \mathbb{R} \cdot \mathcal{o}_j$.
Since $f = a \cdot (\bullet)^j$, we get: $(1/a) \cdot f = (\bullet)^j$.
Then $(\bullet)^j = (1/a) \cdot f \in \mathbb{R} \cdot \mathcal{o}_j \subseteq \mathcal{o}_j$, and so $(\bullet)^j \in \mathcal{o}_j$.
By Theorem 4.2.20, we have: $(\bullet)^j \notin \mathcal{o}_j$. Contradiction. □

THEOREM 4.2.33. *We have:*
$\mathcal{C} \subseteq \widehat{\mathcal{O}}_0$ *and* $\mathcal{C} \cap \mathcal{o}_0 = \{\mathbf{0}\}$ *and*
$\mathcal{L} \subseteq \widehat{\mathcal{O}}_1$ *and* $\mathcal{L} \cap \mathcal{o}_1 = \{\mathbf{0}\}$ *and*
$\mathcal{Q} \subseteq \widehat{\mathcal{O}}_2$ *and* $\mathcal{Q} \cap \mathcal{o}_2 = \{\mathbf{0}\}$ *and*
$\mathcal{K} \subseteq \widehat{\mathcal{O}}_3$ *and* $\mathcal{K} \cap \mathcal{o}_3 = \{\mathbf{0}\}$.

DEFINITION 4.2.34. *Let S and T be sets of functions.*
Then: $T \circ_{\mathrm{setset}} S := \{\ g \circ f \mid (\ g \in T\ )\ \&\ (\ f \in S\ )\}$.

We typically omit "setset" and simply write "$T \circ S$".

DEFINITION 4.2.35. *Let f be a function, S a set of functions.*
Then: $f \circ_{\mathrm{fnset}} S := \{f\} \circ S$
and $S \circ_{\mathrm{setfn}} f := S \circ \{f\}$.

We typically omit "fnset" and "setfn", and write "$f \circ S$" and "$S \circ f$".

THEOREM 4.2.36. *We have:*
(1) $\mathrm{DNZ} \circ \mathrm{CVZ} \subseteq \mathrm{DNZ}$ *and*
(2) $\mathrm{BNZ} \circ \mathrm{CVZ} \subseteq \mathrm{BNZ}$ *and*
(3) $\mathrm{CVZ} \circ \mathrm{CVZ} \subseteq \mathrm{CVZ}$.

Proof. Proof of (1):
Want: $\forall h \in \mathrm{DNZ} \circ \mathrm{CVZ},\ h \in \mathrm{DNZ}$.
Given $h \in \mathrm{DNZ} \circ \mathrm{CVZ}$. Want: $h \in \mathrm{DNZ}$.

Want: $\exists A \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $A \subseteq \mathbb{D}_h$.
Since $h \in \text{DNZ} \circ \text{CVZ}$, choose $g \in \text{DNZ}$ and $f \in \text{CVZ}$ s.t. $h = g \circ f$.
Since $g \in \text{DNZ}$, choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $B \subseteq \mathbb{D}_g$.
By Theorem 4.1.6, choose $A \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. ($A \subseteq \mathbb{D}_f$) & ($f_*A \subseteq B$).
Then $A \in \mathcal{B}_{\mathbb{R}}(0)$. Want: $A \subseteq \mathbb{D}_h$.
Want: $\forall x \in A$, $x \in \mathbb{D}_h$. Given $x \in A$. Want: $x \in \mathbb{D}_h$.
Since $x \in A \subseteq \mathbb{D}_f$, we get $x \in A \cap \mathbb{D}_f$. Then $f_x \in f_*A$.
Since $f_x \in f_*A \subseteq B \subseteq \mathbb{D}_g$, it follows that $g_{f_x} \neq ☹$.
Since $h_x = (g \circ f)_x = g_{f_x} \neq ☹$, it follows that $x \in \mathbb{D}_h$, as desired.
End of proof of (1).

Proof of (2): Unassigned HW. *End of proof of (2).*

Proof of (3). Unassigned HW. *End of proof of (3).* □

THEOREM 4.2.37. *Let* $g := C^0_{(-1;1)}$.
Define $f : \mathbb{R} \to \mathbb{R}$ *by:* $\forall x \in \mathbb{R}$, $f_x = x + 2$.
Then $f \in \text{BNZ} \subseteq \text{DNZ}$ *and* $g \in \text{CVZ} \subseteq \text{BNZ} \subseteq \text{DNZ}$.
Also, $g \circ f \notin \text{DNZ} \supseteq \text{BNZ} \supseteq \text{CVZ}$.

Proof. Unassigned HW: Show that $f \in \text{BNZ}$ and that $g \in \text{CVZ}$.
Then, since $\text{CVZ} \subseteq \text{BNZ} \subseteq \text{DNZ}$, it remains to show: $g \circ f \notin \text{DNZ}$.
We have $(g \circ f)_0 = g_{f_0} = g_2 = ☹$, so $0 \neq \mathbb{D}_{g \circ f}$.
Then $g \circ f$ is not defined near 0 in $\mathbb{R}$, so $g \circ f \notin \text{DNZ}$. □

THEOREM 4.2.38. *Let* X *be a metric space and let* $A \subseteq X$.
Assume that A *is bounded in* X.
Then $\text{Cl}_X A$ *is bounded in* X.

Proof. Let $C := \text{Cl}_X A$. Want: C is bounded in X.
Want: $\exists t > 0$ s.t. C is t-bounded in X.
Since A is bounded in X, choose $s > 0$ s.t. A is s-bounded in X.
Let $t := s + 2$. Then $t > 0$. Want: C is t-bounded in X.
Want: $\forall p, q \in C$, $d_X(p, q) < t$. Given $p, q \in C$. Want: $d_X(p, q) < t$.
Since $p, q \in C = \text{Cl}_X A$, we get $(B_X(p, 1)) \cap A \neq \varnothing \neq (B_X(q, 1)) \cap A$.
Choose $y \in (B_X(p, 1)) \cap A$ and $z \in (B_X(q, 1)) \cap A$.
Then $y, z \in A$. So, since A is s-bounded, we get: $d_X(y, z) < s$.
Since $y \in B_X(p, 1)$, it follows that $d_X(p, y) < 1$.
Since $z \in B_X(q, 1)$, it follows that $d_X(z, q) < 1$.
By the triangle inequality, $d_X(p, q) \leqslant (d_X(p, y)) + (d_X(y, z)) + (d_X(z, q))$.
Then $d_X(p, q) < 1 + s + 1 = s + 2 = t$, as desired. □

THEOREM 4.2.39. *Let X be a proper metric space and let $A \subseteq X$. Assume: A is bounded in X. Then $\mathrm{Cl}_X A$ is compact.*

Proof. By Theorem 4.2.38, $\mathrm{Cl}_X A$ is bounded in X.
So, since $\mathrm{Cl}_X A$ is closed in X and since X is proper,
it follows, from Theorem 3.16.11, that $\mathrm{Cl}_X A$ is compact. □

Recall: $\forall x, y \in \mathbb{R}$, $d_{\widehat{\mathbb{R}}}(x, y) = \min\{\,|x - y|\,,\,1\,\}$
and $d_{\mathbb{R}}(x, y) = |x - y|$.

THEOREM 4.2.40. *Let $f := \mathrm{id}_{\mathbb{R}}$ and let $A := \mathbb{R}$. Then: f is a homeomorphism from $\widehat{\mathbb{R}}$ to $\mathbb{R}$ and $f_* A = A$. Also: A is bounded in $\widehat{\mathbb{R}}$ and $f_* A$ is unbounded in $\mathbb{R}$.*

Proof. Unassigned HW. *Hint:* Use Theorem 3.19.18. □

THEOREM 4.2.41. *Let X and Y be metric spaces, $A \subseteq X$, $f \in C_Y^X$. Assume: X is proper and A is bounded in X. Then: $f_* A$ is bounded in Y.*

Proof. By Theorem 4.2.39, $\mathrm{Cl}_X A$ is compact.
Let $B := \mathrm{Cl}_X A$. Then B is compact.
Since $f \in C_Y^X$, we have: f is continuous from X to Y.
Then, by Theorem 3.18.3, $f_* B$ is compact.
Then, by Theorem 3.9.1, $f_* B$ is bounded in Y.
Since $A \subseteq \mathrm{Cl}_X A = B$, we get: $f_* A \subseteq f_* B$.
So, since $f_* B$ is bounded in Y, it follows that $f_* A$ is bounded in Y. □

THEOREM 4.2.42. $C_{\mathbb{R}}^{\mathbb{R}} \circ \mathrm{BNZ} \subseteq \mathrm{BNZ}$.

Proof. Want: $\forall h \in C_{\mathbb{R}}^{\mathbb{R}} \circ \mathrm{BNZ}$, $h \in \mathrm{BNZ}$.
Given $h \in C_{\mathbb{R}}^{\mathbb{R}} \circ \mathrm{BNZ}$. Want: $h \in \mathrm{BNZ}$.
Want: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. ($B \subseteq \mathbb{D}_h$) & ($h_* B$ is bounded in $\mathbb{R}$).
Since $h \in C_{\mathbb{R}}^{\mathbb{R}} \circ \mathrm{BNZ}$, choose $g \in \mathbb{C}_{\mathbb{R}}^{\mathbb{R}}$ and $f \in \mathrm{BNZ}$ s.t. $h = g \circ f$.
Since $f \in \mathrm{BNZ}$, we know: f is bounded near 0 in $\mathbb{R}$,
so choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. ($B \subseteq \mathbb{D}_f$) & ($f_* B$ is bounded in $\mathbb{R}$).
Then $B \in \mathcal{B}_{\mathbb{R}}(0)$. Want: ($B \subseteq \mathbb{D}_h$) & ($h_* B$ is bounded in $\mathbb{R}$).
Since $f \in \mathrm{BNZ}$, we get $f : \mathbb{R} \dashrightarrow \mathbb{R}$, and so $\mathbb{I}_f \subseteq \mathbb{R}$.
Since $g \in C_{\mathbb{R}}^{\mathbb{R}}$, we get $g : \mathbb{R} \to \mathbb{R}$, and so $\mathbb{D}_g = \mathbb{R}$.
Since $B \subseteq \mathbb{D}_f$ and since $f_* B \subseteq \mathbb{I}_f \subseteq \mathbb{R} = \mathbb{D}_g$, we conclude: $B \subseteq \mathbb{D}_{g \circ f}$.
Recall: $h = g \circ f$. Then: $B \subseteq \mathbb{D}_{g \circ f} = \mathbb{D}_h$.
It remains to show: $h_* B$ is bounded in $\mathbb{R}$.

Since $g \in C^{\mathbb{R}}_{\mathbb{R}}$ and $\mathbb{R}$ is proper and $f_* B$ is bounded in $\mathbb{R}$,
it follows, from Theorem 4.2.41, that $g_*(f_* B)$ is bounded in $\mathbb{R}$.
So, since $h_* B = (g \circ f)_* B = g_*(f_* B)$, we get: $h_* B$ is bounded in $\mathbb{R}$. □

Warning: Define $f : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $f_x = x + 2$. Let $g := C^0_{(-1;1)}$.
By Theorem 4.2.37, we have: $f \in \mathrm{BNZ}$ and $g \circ f \notin \mathrm{BNZ}$.

Unassigned HW: Show, $\forall j \in \mathbb{N}_0$, that $|\bullet|^j \in C^{\mathbb{R}}_{\mathbb{R}}$.

THEOREM 4.2.43. *Let* $\phi \in \mathrm{BNZ}$ *and let* $j \in \mathbb{N}_0$.
Then $|\phi|^j \in \mathrm{BNZ}$.

Proof. We have: $|\phi|^j = |\bullet|^j \circ \phi \in C^{\mathbb{R}}_{\mathbb{R}} \circ \mathrm{BNZ} \subseteq \mathrm{BNZ}$. □

By Theorem 4.2.37, we have: $\mathrm{CVZ} \circ \mathrm{BNZ} \not\subseteq \mathrm{DNZ}$.
Then $\widehat{\mathcal{O}}_0 \circ \widehat{\mathcal{O}}_0 = \mathrm{BNZ} \circ \mathrm{BNZ} \subseteq \mathrm{CVZ} \circ \mathrm{BNZ} \not\subseteq \mathrm{DNZ} \supseteq \mathrm{BNZ} = \widehat{\mathcal{O}}_0$,
and it follows that: $\widehat{\mathcal{O}}_0 \circ \widehat{\mathcal{O}}_0 \not\subseteq \widehat{\mathcal{O}}_0$.
The next theorem asserts that, for *positive* integers j, k, $\widehat{\mathcal{O}}_k \circ \widehat{\mathcal{O}}_j \subseteq \widehat{\mathcal{O}}_{j+k}$.

THEOREM 4.2.44. *Let* $j, k \in \mathbb{N}_0$. *Then* $\widehat{\mathcal{O}}_k \circ \widehat{\mathcal{O}}_j \subseteq \widehat{\mathcal{O}}_{j+k}$.

Proof. Want: $\forall h \in \widehat{\mathcal{O}}_k \circ \widehat{\mathcal{O}}_j$, $h \in \widehat{\mathcal{O}}_{j+k}$.
Given $h \in \widehat{\mathcal{O}}_k \circ \widehat{\mathcal{O}}_j$. Want: $h \in \widehat{\mathcal{O}}_{j+k}$.
Since $h \in \widehat{\mathcal{O}}_k \circ \widehat{\mathcal{O}}_j$, choose $f \in \widehat{\mathcal{O}}_j$ and $g \in \widehat{\mathcal{O}}_k$ s.t. $h = g \circ f$.
Since $j \in \mathbb{N}$, we get: $\widehat{\mathcal{O}}_j \subseteq \widehat{\mathcal{O}}_1$.
Then $f \in \widehat{\mathcal{O}}_j \subseteq \widehat{\mathcal{O}}_1 \subseteq \mathcal{o}_0 = \mathrm{CVZ}$, so $f \in \mathrm{CVZ}$.
Since $f \in \widehat{\mathcal{O}}_j = \mathrm{BNZ} \circ |\bullet|^j$, choose $\phi \in \mathrm{BNZ}$ s.t. $f = \phi \circ |\bullet|^j$.
Since $g \in \widehat{\mathcal{O}}_k = \mathrm{BNZ} \circ |\bullet|^k$, choose $\psi \in \mathrm{BNZ}$ s.t. $g = \psi \circ |\bullet|^k$.
Then: $\forall x \in \mathbb{R}$,

$$\begin{aligned}
h_x &= (g \circ f)_x = g(f_x) \\
&= (\psi \cdot |\bullet|^k)(f_x) \\
&= [\psi(f_x)] \cdot [\, | \quad f_x \quad |^k \,] \\
&= [\psi(f_x)] \cdot [\, | \ (\phi \cdot |\bullet|^j)_x \ |^k \,] \\
&= [\psi(f_x)] \cdot [\, | \ \phi_x \cdot |x|^j \ |^k \,] \\
&= [\psi(f_x)] \cdot [\, |\phi_x|^k \,] \cdot [\, | \ |x|^j \ |^k \,] \\
&= [\psi(f_x)] \cdot [\, |\phi_x|^k \,] \cdot [\ |x|^{jk} \ ] \\
&= (\psi \circ f)_x \cdot (|\phi|^k)_x \cdot (|\bullet|^{jk})_x \\
&= (\ (\psi \circ f) \cdot |\phi|^k \cdot |\bullet|^{jk}\ )_x
\end{aligned}$$

Then: $h = (\psi \circ f) \cdot |\phi|^k \cdot |\bullet|^{jk}$.
Since $\phi \in \mathrm{BNZ}$, by Theorem 4.2.43, we have: $|\phi|^k \in \mathrm{BNZ}$.
By (2) of Theorem 4.2.36, we have: $\mathrm{BNZ} \circ \mathrm{CVZ} \subseteq \mathrm{BNZ}$.
By HW#13-4, we conclude that: $\mathrm{BNZ} \cdot \mathrm{BNZ} \subseteq \mathrm{BNZ}$.

$$\begin{array}{llll} \text{Then:} & h & = & (\ \psi \ \circ \ f \ ) \cdot \ |\phi|^k \cdot |\bullet|^{jk} \\ & & \in & (\text{BNZ} \circ \text{CVZ}) \cdot \text{BNZ} \cdot |\bullet|^{jk} \\ & & \subseteq & \text{BNZ} \cdot \text{BNZ} \cdot |\bullet|^{jk} \\ & & \subseteq & \text{BNZ} \cdot |\bullet|^{jk}. \end{array}$$

Then: $h \in \text{BNZ} \cdot |\bullet|^{jk} = \widehat{\mathcal{O}}_{jk}$, as desired. □

4.3. **Double-translates of functions $\mathbb{R} \dashrightarrow \mathbb{R}$.**

DEFINITION 4.3.1. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x \in \mathbb{R}$.*
Then $f_x^{\mathbb{T}} : \mathbb{R} \dashrightarrow \mathbb{R}$ is defined by:
$\forall h \in \mathbb{R}$, $f_x^{\mathbb{T}}(h) = [f(x+h)] - [f(x)]$.

Then: $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall x, h \in \mathbb{R}$, we have: $(f_x^{\mathbb{T}})_h = f_{x+h} - f_x$.
Then: $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall x \in \mathbb{R}\backslash\mathbb{D}_f$, $\forall h \in \mathbb{R}$, we have: $(f_x^{\mathbb{T}})_h = ☹$.
Then: $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall x \in \mathbb{R}\backslash\mathbb{D}_f$, we have: $f_x^{\mathbb{T}} = \varnothing$.
Consequently, it is only useful to study $f_x^{\mathbb{T}}$ when $x \in \mathbb{D}_f$.
We will call $f_x^{\mathbb{T}}$ the "double-translate of f based at x.
It is not a scalar; it is a function $\mathbb{R} \dashrightarrow \mathbb{R}$.
The next theorem, in part, asserts that
the graph of the double-translate
passes through the origin in $\mathbb{R}^2$.
It also describes how
the domain of the double-translate
is related to
the domain of the original function.
It also gives two examples of how
properties of the double translate $f_x^{\mathbb{T}}$ near 0
are related to
properties of the original function f near x.

THEOREM 4.3.2. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, let $x \in \mathbb{D}_f$ and let $\phi := f_x^{\mathbb{T}}$.*
Then: $[\ \phi_0 = 0\ ]$ & $[\ \mathbb{D}_\phi = \mathbb{D}_f - x\ ]$
& $[\ (\ \phi$ *is defined near* 0 *in* $\mathbb{R}$)
$\Leftrightarrow$ (f *is defined near* x *in* $\mathbb{R}$) $]$
& $[\ (\ \phi$ *is continuous at* 0 *from* $\mathbb{R}$ *to* $\mathbb{R}$)
$\Leftrightarrow$ (f *is continuous at* x *from* $\mathbb{R}$ *to* $\mathbb{R}$) $]$.

THEOREM 4.3.3. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x \in \mathbb{R}$.*
Then: $(f+g)_x^{\mathbb{T}} = f_x^{\mathbb{T}} + g_x^{\mathbb{T}}$.

Proof. Unassigned HW. □

THEOREM 4.3.4. *Let $a \in \mathbb{R}$, $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $x \in \mathbb{R}$. Then* $(af)_x^{\mathbb{T}} = a \cdot f_x^{\mathbb{T}}$

Proof. Unassigned HW. □

THEOREM 4.3.5. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x \in \mathbb{D}_f \bigcap \mathbb{D}_g$. Then* $(fg)_x^{\mathbb{T}} = f_x \cdot g_x^{\mathbb{T}} + g_x \cdot f_x^{\mathbb{T}} + f_x^{\mathbb{T}} \cdot g_x^{\mathbb{T}}$.

Proof. Let $U := (fg)_x^{\mathbb{T}}$ and let $V := f_x \cdot g_x^{\mathbb{T}} + g_x \cdot f_x^{\mathbb{T}} + f_x^{\mathbb{T}} \cdot g_x^{\mathbb{T}}$.
Want: $U = V$. Want: $\forall h \in \mathbb{R}$, $U_h = V_h$.
Given $h \in \mathbb{R}$. Want: $U_h = V_h$.
Let $a := f_x$, $A := f_{x+h}$, $b := g_x$, $B := g_{x+h}$.
Then: $(f_x^{\mathbb{T}})_h = f_{x+h} - f_x = A - a$ and $(g_x^{\mathbb{T}})_h = g_{x+h} - g_x = B - b$.
Also, $(f_x \cdot g_x^{\mathbb{T}})_h = f_x \cdot [(g_x^{\mathbb{T}})_h] = a(B - b)$.
Also, $(g_x \cdot f_x^{\mathbb{T}})_h = g_x \cdot [(f_x^{\mathbb{T}})_h] = b(A - a)$.
Also, $(f_x^{\mathbb{T}} \cdot g_x^{\mathbb{T}})_h = [(f_x^{\mathbb{T}})_h] \cdot [(g_x^{\mathbb{T}})_h] = (A - a)(B - b)$.
Also, $(fg)_x = f_x \cdot g_x = ab$ and $(fg)_{x+h} = f_{x+h} \cdot g_{x+h} = AB$.
We have

$$\begin{aligned} V_h &= (f_x \cdot g_x^{\mathbb{T}})_h + (g_x \cdot f_x^{\mathbb{T}})_h + (f_x^{\mathbb{T}} \cdot g_x^{\mathbb{T}})_h \\ &= a(B - b) + b(A - a) + (A - a)(B - b) \\ &= aB - ab + bA - ba + AB - Ab - aB + ab \\ &= \qquad\qquad - ba + AB. \end{aligned}$$

Then

$$\begin{aligned} U_h = ((fg)_x^{\mathbb{T}})_h = (fg)_{x+h} - (fg)_x &= AB - ab \\ &= -ba + AB = V_h. \end{aligned}$$

□

THEOREM 4.3.6. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, let $x \in \mathbb{D}_f$ and let $y := f_x$. Then* $(g \circ f)_x^{\mathbb{T}} = g_y^{\mathbb{T}} \circ f_x^{\mathbb{T}}$.

Proof. Let $U := (g \circ f)_x^{\mathbb{T}}$ and let $V := g_y^{\mathbb{T}} \circ f_x^{\mathbb{T}}$.
Want: $U = V$. Want: $\forall h \in \mathbb{R}$, $U_h = V_h$.
Given $h \in \mathbb{R}$. Want: $U_h = V_h$.
Let $k := (f_x^{\mathbb{T}})_h$. Then: $k = f_{x+h} - f_x$.
So, since $y = f_x$, we get: $y + k = f_x + f_{x+h} - f_x = f_{x+h}$.
Since $(f_x^{\mathbb{T}})_h = k$, we get: $(g_y^{\mathbb{T}} \circ f_x^{\mathbb{T}})_h = (g_y^{\mathbb{T}})_k$.
Then $V_h = (g_y^{\mathbb{T}} \circ f_x^{\mathbb{T}})_h = (g_y^{\mathbb{T}})_k = g_{y+k} - g_y$, so $V_h = g_{y+k} - g_y$.
Then $U_h = ((g \circ f)_x^{\mathbb{T}})_h = (g \circ f)_{x+h} - (g \circ f)_x = g_{f_{x+h}} - g_{f_x}$.
So, since $f_{x+h} = y + k$ and $f_x = y$, we get: $U_h = g_{y+k} - g_y$.
Then $U_h = g_{y+k} - g_y = V_h$, as desired. □

4.4. **Linearizations and the D-derivative.**

DEFINITION 4.4.1. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $x \in \mathbb{R}$.
Then: $\mathrm{LINS}_x f \ := \ \{ \ L \in \mathcal{L} \mid f_x^{\mathbb{T}} - L \ \in \ \mathcal{O}_1 \ \}$.
Also, $D_x f \ := \ \mathrm{UE}\,(\,\mathrm{LINS}_x f\,)$.

The set $\mathrm{LINS}_x f$ is the set of "linearizations of f at x".
Eventually, we will show that $\mathrm{LINS}_x f$ has at most one element.
The linear function $D_x f$ is the "D-**derivative** of f at x".
It is not a scalar; it is a linear function from $\mathbb{R}$ to $\mathbb{R}$.
Intuition: Its graph is the tangent line to the graph of $f_x^{\mathbb{T}}$ at the origin.
The D-derivative of f at x is the unique linearization of f at x, provided such a linearization exists.

THEOREM 4.4.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $x \in \mathbb{R}$.
Assume: $\mathrm{LINS}_x f \neq \varnothing$. *Then:* $f_x^{\mathbb{T}} \ \in \ \widehat{\mathcal{O}}_1$.

Proof. Choose $L \in \mathrm{LINS}_x f$. Then $(\ L \in \mathcal{L}\ )$ & $(\ f_x^{\mathbb{T}} - L \in \mathcal{O}_1\ )$.
We have: $f_x^{\mathbb{T}} - L \in \mathcal{O}_1 \subseteq \widehat{\mathcal{O}}_1$ and $L \in \mathcal{L} = \mathcal{H}_1 \subseteq \widehat{\mathcal{O}}_1$.
Then $f_x^{\mathbb{T}} = (f_x^{\mathbb{T}} - L) + L \in \widehat{\mathcal{O}}_1 + \widehat{\mathcal{O}}_1 = \widehat{\mathcal{O}}_1$, as desired. □

THEOREM 4.4.3. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, *let* $x \in \mathbb{R}$ *and let* $L \in \mathcal{L}$.
Assume: $L \in \mathrm{LINS}_x f$. *Then:* $D_x f = L$.

Proof. Since $D_x f = \mathrm{UE}(\mathrm{LINS}_x f)$, we wish to show: $\mathrm{LINS}_x f = \{L\}$.
Since $L \in \mathrm{LINS}_x f$, we get $\{L\} \subseteq \mathrm{LINS}_x f$. Want: $\mathrm{LINS}_x f \subseteq \{L\}$.
Want: $\forall M \in \mathrm{LINS}_x f, \quad M \in \{L\}$.
Given $M \in \mathrm{LINS}_x f$. Want: $M \in \{L\}$.
Since $L \in \mathrm{LINS}_x f$, we get $\mathrm{LINS}_x f \neq \varnothing$, so,
by Theorem 4.4.2, we see that $f_x^{\mathbb{T}} \in \widehat{\mathcal{O}}_1$.
Then $f_x^{\mathbb{T}} \in \widehat{\mathcal{O}}_1 \subseteq \mathcal{O}_0 = \mathrm{CVZ} \subseteq \mathrm{BNZ} \subseteq \mathrm{DNZ}$, so $f_x^{\mathbb{T}} \in \mathrm{DNZ}$.
Since $f_x^{\mathbb{T}} \in \mathrm{DNZ}$, it follows that: $f_x^{\mathbb{T}} - f_x^{\mathbb{T}} = 0$ near 0 in $\mathbb{R}$.
Then $(\ f_x^{\mathbb{T}} - L\ ) - (\ f_x^{\mathbb{T}} - M\ ) = M - L$ near 0 in $\mathbb{R}$.
Since $L, M \in \mathrm{LINS}_x f$, we get $f_x^{\mathbb{T}} - L \in \mathcal{O}_1$ and $f_x^{\mathbb{T}} - M \in \mathcal{O}_1$.
Then $(\ f_x^{\mathbb{T}} - L\ ) - (\ f_x^{\mathbb{T}} - M\ ) \in \mathcal{O}_1 - \mathcal{O}_1 \subseteq \mathcal{O}_1$.
So, since $(\ f_x^{\mathbb{T}} - L\ ) - (\ f_x^{\mathbb{T}} - M\ ) = M - L$ near 0 in $\mathbb{R}$,
and since $\mathcal{O}_1$ is a zero-local condition, we get: $M - L \in \mathcal{O}_1$.
Also, $M - L \in \mathcal{L} - \mathcal{L} \subseteq \mathcal{L} = \mathcal{H}_1$. Recall: $\mathcal{H}_1 \bigcap \mathcal{O}_1 = \{\mathbf{0}\}$.
Then $M - L \in \mathcal{H}_1 \bigcap \mathcal{O}_1 = \{\mathbf{0}\}$, so $M - L = \mathbf{0}$, so $M = L \in \{L\}$. □

4.5. **The prime-derivative.**

THEOREM 4.5.1. *Let* $m \in \mathbb{R}$.

Define $L : \mathbb{R} \to \mathbb{R}$ *by:* $\forall x \in \mathbb{R},\ L_x = mx$.

Then: $L = m \cdot (\bullet) \in \mathbb{R} \cdot (\bullet) = \mathcal{L}$ *and* $L_1 = m \cdot 1 = m$.

The point of the preceding theorem is that,

to get the slope of a linear function L,

you can simply compute L_1.

DEFINITION 4.5.2. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.

Then $f' : \mathbb{R} \to \mathbb{R}$ *is defined by:* $\forall x \in \mathbb{R},\ \ f'_x = (D_x f)_1$.

The function f' is called the **derivative** or **prime-derivative** of f.
The prime-derivative of f at x, denoted f'_x, is a scalar.
Recall that the D-derivative of f at x is not a scalar;
it is a linear function from $\mathbb{R}$ to $\mathbb{R}$.
To compute the prime-derivative of at x,
you can simply take the slope of the D-derivative of f at x.
Recall the intuition:
The graph of the D-derivative of f at x is
the tangent line to the graph of $f_x^{\mathbb{T}}$ at the origin.
So, since that tangent line is parallel to
the tangent line to the graph of f at (x, f_x),
we conclude:
the prime-derivative of f at x is the slope of
the tangent line to the graph of f at (x, f_x).

THEOREM 4.5.3. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by:* $\forall x \in \mathbb{R},\ f_x = x^2$.

Then: $f'_3 = 6$.

Proof. Define $L \in \mathcal{L}$ and $Q \in \mathcal{Q}$ by:

$$\forall h \in \mathbb{R}, \qquad L_h = 6h \quad \text{and} \quad Q_h = h^2.$$

We have:
$$\forall h \in \mathbb{R}, \qquad (f_3^{\mathbb{T}})_h = f_{3+h} - f_3 = (3+h)^2 - 3^2 = (9 + 6h + h^2) - 9 = 6h + h^2 = L_h + Q_h = (L+Q)_h.$$

Then $f_3^{\mathbb{T}} = L + Q$, so $f_3^{\mathbb{T}} - L = Q$.
So, since $Q \in \mathcal{Q} = \mathcal{H}_2 \subseteq \widehat{\mathcal{O}}_2 \subseteq o_1$,
we get: $f_3^{\mathbb{T}} - L \in o_1$, and so $L \in \mathrm{LINS}_3 f$.
Then, by Theorem 4.4.3, $D_3 f = L$.
Then $f'_3 = (D_3 f)_1 = L_1 = 6 \cdot 1 = 6$. □

THEOREM 4.5.4. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by:* $\forall x \in \mathbb{R}$, $f_x = x^2$.
Let $x \in \mathbb{R}$. *Then:* $f'_x = x^2$.

Proof. Unassigned HW. *Hint:* Take the preceding proof and replace:
3 by x, 6 by $2x$ and 9 by x^2. □

THEOREM 4.5.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $x \in \mathbb{R}$.
Assume: $\mathrm{LINS}_x f \neq \varnothing$. *Then* $x \in \mathbb{D}_{f'}$.

Proof. Choose $L \in \mathrm{LINS}_x f$. By Theorem 4.4.3, we have: $D_x f = L$.
We have $L \in \mathrm{LINS}_x f \subseteq \mathcal{L}$, so $L : \mathbb{R} \to \mathbb{R}$, so $L_1 \neq \varnothing$.
Then $f'_x = (D_x f)_1 = L_1 \neq ☹$, so $x \in \mathbb{D}_{f'}$, as desired. □

THEOREM 4.5.6. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $x \in \mathbb{D}_{f'}$.
Then $D_x f \in \mathrm{LINS}_x f \subseteq \mathcal{L}$.

Proof. By definition of $\mathrm{LINS}_x f$, we have $\mathrm{LINS}_x f \subseteq \mathcal{L}$.
It remains to show: $D_x f \in \mathrm{LINS}_x f$. Since $x \in \mathbb{D}_{f'}$, we get $f'_x \neq ☹$.
So, since $f'_x = (D_x f)_1$, we get $(D_x f)_1 \neq ☹$, so $D_x f \neq ☹$.
So, since $D_x f = \mathrm{UE}(\mathrm{LINS}_x f)$, we get $\mathrm{UE}(\mathrm{LINS}_x f) \neq ☹$,
and it follows that $\mathrm{UE}(\mathrm{LINS}_x f) \in \mathrm{LINS}_x f$.
Then $D_x f = \mathrm{UE}(\mathrm{LINS}_x f) \in \mathrm{LINS}_x f$, as desired. □

THEOREM 4.5.7. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $x \in \mathbb{D}_{f'}$.
Then: f *is defined near* x *in* $\mathbb{R}$
and f *is continuous at* x *from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. By Theorem 4.4.2, we have $f_x^{\mathbb{T}} \in \widehat{\mathcal{O}}_1$.
Then $f_x^{\mathbb{T}} \in \widehat{\mathcal{O}}_1 \subseteq \mathcal{O}_0 = \mathrm{CVZ}$, so $f_x^{\mathbb{T}} \in \mathrm{CVZ}$,
so $f_x^{\mathbb{T}}$ is continuous at 0 from $\mathbb{R}$ to $\mathbb{R}$.
Then, by Theorem 4.3.2, f is continuous at x from $\mathbb{R}$ to $\mathbb{R}$.
Want: f is defined near x in $\mathbb{R}$.
We have $f_x^{\mathbb{T}} \in \mathrm{CVZ} \subseteq \mathrm{BNZ} \subseteq \mathrm{DNZ}$, so $f_x^{\mathbb{T}} \in \mathrm{DNZ}$,
so $f_x^{\mathbb{T}}$ is defined near 0 in $\mathbb{R}$.
Then, by Theorem 4.3.2, f is defined near x in $\mathbb{R}$, as desired. □

THEOREM 4.5.8. *Let* $f := C^1_{[2;3]}$. *Then* $f'_2 = ☹ = f'_3$.

Proof. Since $\neg$ (f is defined near 2),
by Theorem 4.5.7, we conclude: $2 \notin \mathbb{D}_{f'}$.
Also, since $\neg$ (f is defined near 3),
by Theorem 4.5.7, we conclude: $3 \notin \mathbb{D}_{f'}$.
Since $2 \notin \mathbb{D}_{f'}$ and $3 \notin \mathbb{D}_{f'}$, we conclude: $f'_2 = ☹ = f'_3$. □

THEOREM 4.5.9. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$. *Then* $\mathbb{D}_{f'} \subseteq \mathbb{D}_f$.

Proof. Want: $\forall x \in \mathbb{D}_{f'}$, $x \in \mathbb{D}_f$. Given $x \in \mathbb{D}_{f'}$. Want: $x \in \mathbb{D}_f$.
By Theorem 4.5.7, we see that f is defined near x,
so choose $B \in \mathcal{B}_{\mathbb{R}}(x)$ s.t. $B \subseteq \mathbb{D}_f$.
Since $B \in \mathcal{B}_{\mathbb{R}}(x)$, we get: $x \in B$. Then $x \in B \subseteq \mathbb{D}_f$. □

4.6. Basic properties of D-derivatives.

THEOREM 4.6.1. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $x \in \mathbb{D}_{f'} \bigcap \mathbb{D}_{g'}$.
Then: *(1)* $D_x(f+g) = D_x f + D_x g$
and *(2)* $D_x(fg) = f_x \cdot D_x g + g_x \cdot D_x f$.

Proof. Proof of (1): Unassigned HW. *End of proof of (1).*

Proof of (2):
By Theorem 4.5.9, $x \in \mathbb{D}_f \bigcap \mathbb{D}_g$.
Then $f_x \in \mathbb{I}_f \subseteq \mathbb{R}$ and $g_x \in \mathbb{I}_g \subseteq \mathbb{R}$.
By Theorem 4.5.6, we have $D_x f \in \mathrm{LINS}_x f \subseteq \mathcal{L}$ and $D_x g \in \mathrm{LINS}_x g \subseteq \mathcal{L}$.
Then $\mathrm{LINS}_x f \neq \varnothing \neq \mathrm{LINS}_x g$, so, by Theorem 4.4.2,
we get: $f_x^{\mathbb{T}} \in \widehat{\mathcal{O}}_1$ and $g_x^{\mathbb{T}} \in \widehat{\mathcal{O}}_1$.
Let $s := f_x$, $t := g_x$, $L := D_x f$, $M := D_x g$.
Then $s, t \in \mathbb{R}$ and $L, M \in \mathcal{L}$.
So, since $\mathcal{L}$ is v.op.-closed, we see that: $sM + tL \in \mathcal{L}$.
Since $f_x \cdot D_x g + g_x \cdot D_x f = sM + tL$,
we wish to prove: $D_x(fg) = sM + tL$.
Then, by Theorem 4.4.3, we wish to show: $sM + tL \in \mathrm{LINS}_x(fg)$.
Since $sM + tL \in \mathcal{L}$, we need only show: $(fg)_x^{\mathbb{T}} - (sM + tL) \in \mathcal{o}_1$.
By Theorem 4.3.5, $(fg)_x^{\mathbb{T}} = f_x \cdot g_x^{\mathbb{T}} + g_x \cdot f_x^{\mathbb{T}} + f_x^{\mathbb{T}} \cdot g_x^{\mathbb{T}}$.
Let $\phi := f_x^{\mathbb{T}}$ and $\psi := g_x^{\mathbb{T}}$. Then $(fg)_x^{\mathbb{T}} = s \cdot \psi + t \cdot \phi + \phi \cdot \psi$.
Then $(fg)_x^{\mathbb{T}} - (sM + tL) = s \cdot (\psi - M) + t \cdot (\phi - M) + \phi \cdot \psi$.
Since $L = D_x f \in \mathrm{LINS}_x f$, we get $f_x^{\mathbb{T}} - L \in \mathcal{o}_1$.
Then $\phi - L = f_x^{\mathbb{T}} - L \in \mathcal{o}_1$.
Since $M = D_x g \in \mathrm{LINS}_x g$, we get $g_x^{\mathbb{T}} - M \in \mathcal{o}_1$.
Then $\psi - M = g_x^{\mathbb{T}} - M \in \mathcal{o}_1$.
Also, we have: $\phi = f_x^{\mathbb{T}} \in \widehat{\mathcal{O}}_1$ and $\psi = g_x^{\mathbb{T}} \in \widehat{\mathcal{O}}_1$.
Then: $\phi \cdot \psi \in \widehat{\mathcal{O}}_1 \cdot \widehat{\mathcal{O}}_1 \subseteq \widehat{\mathcal{O}}_2$.
Then $(fg)_x^{\mathbb{T}} - (sM + tL) = s \cdot (\phi - L) + t \cdot (\psi - M) + \phi \cdot \psi$
$\in \mathbb{R} \cdot \mathcal{o}_1 + \mathbb{R} \cdot \mathcal{o}_1 + \widehat{\mathcal{O}}_2$

$$\subseteq \quad \mathcal{O}_1 \quad + \quad \mathcal{O}_1 \quad + \quad \mathcal{O}_1$$
$$\subseteq \quad \mathcal{O}_1, \quad \text{as desired.}$$

End of proof of (2). □

THEOREM 4.6.2. *Let $a \in \mathbb{R}$, let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x \in \mathbb{D}_{f'}$.*
Then: $D_x(a \cdot f) = a \cdot D_x f$.

Proof. Unassigned HW. □

THEOREM 4.6.3. *Let f and g be functionals. Let h be a function.*
Then $(f + g) \circ h = (f \circ h) + (g \circ h)$.

Proof. Want: $\forall x$, $((f + g) \circ h)_x = ((f \circ h) + (g \circ h))_x$.
Given x. Want: $((f + g) \circ h)_x = ((f \circ h) + (g \circ h))_x$.
We have $((f \circ h) + (g \circ h))_x = (f \circ h)_x + (g \circ h)_x = f_{h_x} + g_{h_x}$.
Then $((f + g) \circ h)_x = (f + g)_{h_x} = f_{h_x} + g_{h_x} = ((f \circ h) + (g \circ h))_x$. □

THEOREM 4.6.4. *Let $f := C_{\mathbb{R}}^3$, $g := C_{\mathbb{R}}^4$, $h := (\bullet)^2$.*
Then $h \circ (f + g) \neq (h \circ f) + (h \circ g)$.

Proof. We have $(f + g)_1 = f_1 + g_1 = 3 + 4 = 7$.
We have $(h \circ f)_1 = h_{f_1} = h_3 = 9$ and $(h \circ g_1 = h_4 = 16$.
Then $((h \circ f) + (h \circ g))_1 = (h \circ f)_1 + (h \circ g)_1 = 9 + 16 = 25$.
Then $(h \circ (f + g))_1 = h_{(f+g)_1} = h_7 = 49 \neq 25 = ((h \circ f) + (h \circ g))_1$, so $(h \circ (f + g))_1 \neq ((h \circ f) + (h \circ g))_1$.
Then $h \circ (f + g) \neq (h \circ f) + (h \circ g)$, as desired. □

THEOREM 4.6.5. *Let $L \in \mathcal{L}$ and $s, t \in \mathbb{R}$. Then $L_{s+t} = L_s + L_t$.*

Proof. Unassigned HW. □

THEOREM 4.6.6. *Let f and g be functionals. Let $L \in \mathcal{L}$.*
Then $L \circ (f + g) = (L \circ f) + (L \circ g)$.

Proof. Want: $\forall x$, $(L \circ (f + g))_x = ((L \circ f) + (L \circ g))_x$.
Given x. Want: $(L \circ (f + g))_x = ((L \circ f) + (L \circ g))_x$.
Let $s := f_x$ and $t := g_x$.
Then $((L \circ f) + (L \circ g))_x = (L \circ f)_x + (L \circ g)_x = L_{f_x} + L_{g_x} = L_s + L_t$.
Also, $(f + g)_x = f_x + g_x = s + t$. By Theorem 4.6.5, $L_{s+t} = L_s + L_t$.
Then $(L \circ (f + g))_x = L_{(f+g)_x} = L_{s+t} = L_s + L_t = ((L \circ f) + (L \circ g))_x$. □

THEOREM 4.6.7. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, let $x \in \mathbb{D}_{f'}$ and let $y := f_x$.*
Assume: $y \in \mathbb{D}_{g'}$. Then: $D_x(g \circ f) = (D_y g) \circ (D_x f)$.

Proof. By Theorem 4.5.9, $x \in \mathbb{D}_f$ and $y \in \mathbb{D}_g$.
Then $f_x \in \mathbb{I}_f \subseteq \mathbb{R}$ and $g_y \in \mathbb{I}_g \subseteq \mathbb{R}$.
By Theorem 4.5.6, we have $D_x f \in \mathrm{LINS}_x f \subseteq \mathcal{L}$ and $D_y g \in \mathrm{LINS}_y g \subseteq \mathcal{L}$.
Then $\mathrm{LINS}_x f \neq \varnothing$, so, by Theorem 4.4.2, we get: $f_x^{\mathbb{T}} \in \widehat{\mathcal{O}}_1$.
Let $L := D_x f$, $M := D_y g$. Then $L, M \in \mathcal{L}$.
Then $M \circ L \in \mathcal{L} \circ \mathcal{L} = \mathcal{H}_1 \circ \mathcal{H}_1 \subseteq \mathcal{H}_1 = \mathcal{L}$.
Since $(D_y g) \circ (D_x f) = M \circ L$, we wish to prove: $D_x(g \circ f) = M \circ L$.
Then, by Theorem 4.4.3, we wish to show: $M \circ L \in \mathrm{LINS}_x(g \circ f)$.
Since $M \circ L \in \mathcal{L}$, we need only show: $(g \circ f)_x^{\mathbb{T}} - (M \circ L) \in \mathcal{O}_1$.
By Theorem 4.3.6, $(g \circ f)_x^{\mathbb{T}} = g_y^{\mathbb{T}} \circ f_x^{\mathbb{T}}$.
Let $\phi := f_x^{\mathbb{T}}$ and $\psi := g_y^{\mathbb{T}}$. Then $(g \circ f)_x^{\mathbb{T}} = \psi \circ \phi$.
Since $L = D_x f \in \mathrm{LINS}_x f$, we get $f_x^{\mathbb{T}} - L \in \mathcal{O}_1$.
Then $\phi - L = f_x^{\mathbb{T}} - L \in \mathcal{O}_1$.
Since $M = D_y g \in \mathrm{LINS}_y g$, we get $g_y^{\mathbb{T}} - M \in \mathcal{O}_1$.
Then $\psi - M = g_y^{\mathbb{T}} - M \in \mathcal{O}_1$.
Let $R := \phi - L$ and $S := \psi - M$. Then $R, S \in \mathcal{O}_1$.
Also, $L + R = \phi$ and $M + S = \psi$.
Since $M \in \mathcal{L}$, by Theorem 4.6.6,
we get: $M \circ (L + R) = (M \circ L) + (M \circ R)$.
Then
$$\begin{aligned}\psi \circ \phi &= (M + S) \circ \phi \\ &= (M \circ \phi) + (S \circ \phi) \\ &= (M \circ (L + R)) + (S \circ \phi) \\ &= (M \circ L) + (M \circ R) + (S \circ \phi).\end{aligned}$$
We have $M \circ R \in \mathcal{L} \circ \mathcal{O}_1 = \mathcal{H}_1 \circ \mathcal{O}_1 \subseteq \widehat{\mathcal{O}}_1 \circ \mathcal{O}_1 \subseteq \mathcal{O}_1$.
Recall: $f_x^{\mathbb{T}} \in \widehat{\mathcal{O}}_1$. Then $S \circ \phi = S \circ f_x^{\mathbb{T}} \in \mathcal{O}_1 \circ \widehat{\mathcal{O}}_1 \subseteq \mathcal{O}_1$.
Then: $(\psi \circ \phi) - (M \circ L) = (M \circ R) + (S \circ \phi) \in \mathcal{O}_1 + \mathcal{O}_1 \subseteq \mathcal{O}_1$. □

4.7. **Basic properties of prime-derivatives.**

THEOREM 4.7.1. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x \in \mathbb{D}_{f'} \bigcap \mathbb{D}_{g'}$.*
Then: *(1)* $(f + g)'_x = f'_x + g'_x$
and *(2)* $(fg)'_x = f_x \cdot g'_x + g_x \cdot f'_x$.

Proof. Proof of (1): Unassigned HW. *End of proof of (1).*

Proof of (2):
By Theorem 4.6.1, we have $D_x(fg) = f_x \cdot D_x g + g_x \cdot D_x f$.
Let $s := f_x$, $t := g_x$, $L := D_x f$, $M := D_x g$.
Then $D_x(fg) = f_x \cdot D_x g + g_x \cdot D_x f = s \cdot M + t \cdot L$.

Also, we have: $f'_x = (D_x f)_1 = L_1$ and $g'_x = (D_x g)_1 = M_1$.
Then $(fg)'_x = (D_x(fg))_1 = (s \cdot M + t \cdot L)_1$

$$= s \cdot M_1 + t \cdot L_1$$
$$= f_x \cdot g'_x + g_x \cdot f'_x.$$

End of proof of (2). ☐

THEOREM 4.7.2. *Let $a \in \mathbb{R}$, let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x \in \mathbb{D}_{f'}$.*
Then: $(a \cdot f)'_x = a \cdot f'_x$.

Proof. Unassigned HW. ☐

THEOREM 4.7.3. *Let $L, M \in \mathcal{L}$. Then $(M \circ L)_1 = M_1 \cdot L_1$.*

Proof. Since $M \in \mathcal{L} = \mathbb{R} \cdot (\bullet)$, choose $t \in \mathbb{R}$ s.t. $M = t \cdot (\bullet)$.
Then: $M_1 = t \cdot 1 = t$ and $M_{L_1} = t \cdot L_1$.
Then $(M \circ L)_1 = M_{L_1} = t \cdot L_1 = M_1 \cdot L_1$, as desired. ☐

THEOREM 4.7.4. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, let $x \in \mathbb{D}_{f'}$ and let $y := f_x$.*
Assume: $y \in \mathbb{D}_{g'}$. Then: $(g \circ f)'_x = g'_y \cdot f'_x$.

Proof. By Theorem 4.6.7, we have $D_x(g \circ f) = (D_y g) \circ (D_x f)$.
Let $L := D_x f$ and $M := D_y g$.
Then $D_x(g \circ f) = (D_y g) \circ (D_x f) = M \circ L$.
By Theorem 4.7.3, we have: $(M \circ L)_1 = M_1 \cdot L_1$.
Also, we have: $f'_x = (D_x f)_1 = L_1$ and $g'_y = (D_y g)_1 = M_1$.
Then: $(g \circ f)'_x = (D_x(g \circ f))_1 = (M \circ L)_1 = M_1 \cdot L_1 = g'_y \cdot f'_x$. ☐

4.8. Fermat's Theorem.

DEFINITION 4.8.1. *Let f be a functional and let $p \in \mathbb{D}_f$.*
Then f has a **global strict-maximum** *means: $f < f_p$ on $(\mathbb{D}_f)^\times_p$.*
Also, f has a **global strict-minimum** *means: $f > f_p$ on $(\mathbb{D}_f)^\times_p$.*
Also, f has a **global semi-maximum** *means: $f \leqslant f_p$ on $(\mathbb{D}_f)^\times_p$.*
Also, f has a **global semi-minimum** *means: $f \geqslant f_p$ on $(\mathbb{D}_f)^\times_p$.*
Also, f has a **global strict-extremum** *means:*
f has a global strict-maximum or global strict-minimum at p.
Also, f has a **global semi-extremum** *means:*
f has a global semi-maximum or global semi-minimum at p.

DEFINITION 4.8.2. *Let X be a metric space, $fX \dashrightarrow \mathbb{R}$, $p \in \mathbb{D}_f$.*
Then f has a **local strict-maximum** *means:*

$$\exists B \in \mathcal{B}_X(p) \text{ s.t. } f < f_p \text{ on } B^\times_p.$$

Also, f has a **local strict-minimum** *means:*

$$\exists B \in \mathcal{B}_X(p) \text{ s.t. } f > f_p \text{ on } B_p^\times.$$

Also, f has a **local semi-maximum** *means:*

$$\exists B \in \mathcal{B}_X(p) \text{ s.t. } f \leqslant f_p \text{ on } B_p.$$

Also, f has a **local semi-minimum** *means:*

$$\exists B \in \mathcal{B}_X(p) \text{ s.t. } f \geqslant f_p \text{ on } B_p.$$

Also, f has a **local strict-extremum** *means:*

f has a local strict-maximum or local strict-minimum at p.

Also, f has a **local semi-extremum** *means:*

f has a local semi-maximum or local semi-minimum at p.

THEOREM 4.8.3. *Let $j \in \mathbb{N}_0$, $\phi \in \mathcal{O}_j$, $\varepsilon > 0$.*
Then $|\phi| \leqslant \varepsilon \cdot |\bullet|^j$ near 0 in $\mathbb{R}$.

Proof. Want: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $|\phi| \leqslant \varepsilon \cdot |\bullet|^j$ on B.
Since $\phi \in \mathcal{O}_j = \mathrm{CVZ} \cdot |\bullet|^j$, choose $\psi \in \mathrm{CVZ}$ s.t. $\phi = \psi \cdot |\bullet|^j$.
Since $\psi \in \mathrm{CVZ}$, by Theorem 4.1.5, choose $\delta > 0$ s.t., $\forall x \in \mathbb{R}$,

$$(\ |x| < \delta\ ) \Rightarrow (\ |\psi_x| < \varepsilon\ ).$$

Let $B := B_{\mathbb{R}}(0, \delta)$. Then $B \in \mathcal{B}_{\mathbb{R}}(0)$. Want: $|\phi| \leqslant \varepsilon \cdot |\bullet|^j$ on B.
Want: $\forall x \in B$, $|\phi|_x \leqslant (\varepsilon \cdot |\bullet|^j)_x$.
Given $x \in B$. Want: $|\phi|_x \leqslant (\varepsilon \cdot |\bullet|^j)_x$.
Since $x \in B = B_{\mathbb{R}}(0, \delta)$, we get: $d_{\mathbb{R}}(x, 0) < \delta$.
Since $|x| = |x - 0| = d_{\mathbb{R}}(x, 0) < \delta$, by choice of δ, we get: $|\psi_x| < \varepsilon$.
Then $|\phi|_x = |\phi_x| = |(\psi \cdot |\bullet|^j)_x| = |\ \psi_x \cdot |x|^j\ | = |\psi_x| \cdot |x|^j$.
Since $|x| \geqslant 0$ and since $|\psi_x| < \varepsilon$, we get $|\psi_x| \cdot |x|^j \leqslant \varepsilon \cdot |x|^j$.
Then $|\phi|_x = |\psi_x| \cdot |x|^j \leqslant \varepsilon \cdot |x|^j = (\varepsilon \cdot |\bullet|^j)_x$, as desired. □

THEOREM 4.8.4. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $c \in \mathbb{D}_{f'}$.*
Assume: f has a local semi-maximum at c in $\mathbb{R}$.
Then: $f'_c = 0$.

Proof. Since $c \in \mathbb{D}_{f'}$, we get: $f'_c \neq ☹$.
Let $L := D_c f$. Then, since $L_1 = f'_c$, we get $L_1 \neq ☹$. Then $L \neq ☹$.
Since $\mathrm{UE}(\mathrm{LINS}_c f) = D_c f = L \neq ☹$, we get $\mathrm{UE}(\mathrm{LINS}_c f) \in \mathrm{LINS}_c f$.
Then $L = D_c f = \mathrm{UE}(\mathrm{LINS}_c f) \in \mathrm{LINS}_c f$, so $L \in \mathrm{LINS}_c f$.
Since $L \in \mathrm{LINS}_c f$ and since f has a local semi-maximum at c in $\mathbb{R}$,
it follows, from, HW#1-3, that: $L = \mathbf{0}$.
Then $f'_c = L_1 = \mathbf{0}_1 = 0$, as desired. □

The next result is called **Fermat's Theorem**.

THEOREM 4.8.5. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $c \in \mathbb{D}_{f'}$.*

Assume: f *has a local semi-extremum at* c *in* $\mathbb{R}$.

Then: $f'_c = 0$.

Proof. Since f has a local semi-extremum at c in $\mathbb{R}$,
at least one of the following must be true:
(1) f has a local semi-maximum at c in $\mathbb{R}$
or (2) f has a local semi-minimum at c in $\mathbb{R}$.

Case 1: By Theorem 4.8.4, $f'_c = 0$, as desired. *End of Case 1.*

Case 2: Let $g := -f$.
Since f has a local semi-minimum at c in $\mathbb{R}$,
it follows that g has a local semi-maximum at c in $\mathbb{R}$.
Since $c \in \mathbb{D}_{f'}$ and $g = -f$, we get $g'_c = -f'_c$. Want: $g'_c = 0$.
Since $c \in \mathbb{D}_{f'}$, we get $f'_c \in \mathbb{I}_{f'}$. So, since $\mathbb{I}_{f'} \subseteq \mathbb{R}$, we get $f'_c \in \mathbb{R}$.
Since $f'_c \in \mathbb{R}$, we see that $-f'_c \in \mathbb{R}$, and so $-f'_c \neq ☹$.
Since $g'_c = -f'_c \neq ☹$, we get $c \in \mathbb{D}_{g'}$. Then, by Theorem 4.8.4, $g'_c = 0$.
End of Case 2. □

4.9. **The Second Derivative Tests.**

The next result is the **Quadratic Taylor Theorem**:

THEOREM 4.9.1. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $x \in \mathbb{D}_{f''}$, $m := f'_x$ and $a := f''_x/2$. Let $L := m \cdot (\bullet)$ and let $Q := a \cdot (\bullet)^2$. Then: $f_p^{\mathbb{T}} - L - Q \in \mathcal{O}_2$.*

Proof. THIS WILL BE PROVED IN A LATER CLASS. □

The next result is the **Positive Second Derivative Test**:

THEOREM 4.9.2. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x \in \mathbb{D}_{f''}$.*

Assume: $(\ f'_x = 0\ )\ \&\ (\ f''_x > 0\ )$.

Then: f *has a local strict-minimum at* x *in* $\mathbb{R}$.

Proof. Let $\phi := f_p^{\mathbb{T}}$. Want: ϕ has a local strict-minimum at 0 in $\mathbb{R}$.
Let $m := f'_x$ and let $a := f''_x/2$. Then: $(\,m = 0\,)\,\&\,(\,a > 0\,)$.
Let $L := m\cdot(\bullet)$ and $Q := a\cdot(\bullet)^2$. By Theorem 4.9.1, $f_p^{\mathbb{T}} - L - Q \in \mathcal{O}_2$.
Since $m = 0$, we get: $L = m \cdot (\bullet) = 0 \cdot (\bullet) = \mathbf{0}$.
Then $\phi - Q = \phi - \mathbf{0} - Q = f_p^{\mathbb{T}} - L - Q \in \mathcal{O}_2$, so $\phi - Q \in \mathcal{O}_2$.
Then, by HW#1-4, ϕ has a local strict-minimum at 0 in $\mathbb{R}$. □

The next result is the **Negative Second Derivative Test**:

THEOREM 4.9.3. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x \in \mathbb{D}_{f''}$.*
Assume: $(f'_x = 0)$ & $(f''_x < 0)$.
Then: *f has a local strict-maximum at x in $\mathbb{R}$.*

Proof. Unassigned HW. □

4.10. **Basic properties of derivatives.**

THEOREM 4.10.1. *Let $s \in \mathbb{R}$. Then $(C_{\mathbb{R}}^s)' = \mathbf{0}$.*

Proof. Let $f := C_{\mathbb{R}}^s$. Want: $f' = \mathbf{0}$.
Want: $\forall x \in \mathbb{R}$, $f'_x = \mathbf{0}_x$. Given $x \in \mathbb{R}$. Want: $f'_x = \mathbf{0}_x$.
We have: $\forall h \in \mathbb{R}$, $(f_x^{\mathbb{T}})_h = f_{x+h} - f_x = s - s = 0 = \mathbf{0}_h$.
Then $f_x^{\mathbb{T}} = \mathbf{0}$. Then $f_x^{\mathbb{T}} - \mathbf{0} = \mathbf{0} - \mathbf{0} = \mathbf{0} \in \mathcal{O}_1$, so $f_x^{\mathbb{T}} - \mathbf{0} \in \mathcal{O}_1$.
So, since $\mathbf{0} = 0 \cdot (\bullet) \in \mathcal{L}$, we conclude that $\mathbf{0} \in \mathrm{LINS}_x f$.
Then $D_x f = \mathbf{0}$. Then $f'_x = (D_x f)_1 = \mathbf{0}_1 = 0 = \mathbf{0}_x$, as desired. □

THEOREM 4.10.2. *Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $c \in \mathbb{R}$.*
Assume: $f = g$ near c in $\mathbb{R}$. Then: $(D_c f = D_c g)$ & $(f'_c = g'_c)$.

Proof. By HW#1-5, we have: $\mathrm{LINS}_c f \subseteq \mathrm{LINS}_c g$.
Also, by HW#1-5 (interchanging f and g), we have: $\mathrm{LINS}_c g \subseteq \mathrm{LINS}_c f$.
Then $\mathrm{LINS}_c f = \mathrm{LINS}_c g$. Then $\mathrm{UE}(\mathrm{LINS}_c f) = \mathrm{UE}(\mathrm{LINS}_c g)$.
Then $D_c f = \mathrm{UE}(\mathrm{LINS}_c f) = \mathrm{UE}(\mathrm{LINS}_c g) = D_c g$.
It remains only to show: $f'_c = g'_c$.
We have $f'_c = (D_c f)_1 = (D_c g)_1 = g'_c$, as desired. □

THEOREM 4.10.3. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $c, s \in \mathbb{R}$.*
Assume: $f = s$ near c in $\mathbb{R}$. Then: $f'_c = 0$.

Proof. Since $f = s$ near c in $\mathbb{R}$, we conclude: $f = C_{\mathbb{R}}^s$ near c in $\mathbb{R}$.
Then, by Theorem 4.10.2, it follows that: $f'_c = (C_{\mathbb{R}}^s)'_c$.
By Theorem 4.10.1, we see that: $(C_{\mathbb{R}}^s)' = \mathbf{0}$.
Then: $f'_c = (C_{\mathbb{R}}^s)'_c = \mathbf{0}'_c = 0$, as desired.
□

4.11. **Rolle's Theorem.**

THEOREM 4.11.1. *Let K be a nonempty compact metric space.*
Let $g : K \to \mathbb{R}$. Assume g is continuous from K to $\mathbb{R}$.
Then: $\exists p, q \in K$ s.t. $g_p \leqslant g \leqslant g_q$ on K.

Proof. Let $L := \mathbb{I}_g$. Then $g : K \to\!> L$.
So, since g is cointuous from K to L,
by Theorem 3.9.6, we see: L is compact,
and it follows that: L is closed and bounded in $\mathbb{R}$.
Also, since $K \neq \varnothing$, it follows that: $L \neq \varnothing$.
Then, by Theorem 3.18.5, we get: $\min L \neq ☹ \neq \max L$.
Let $s := \min L$ and $t := \max L$. Then $s \neq ☹ \neq t$.
Then $s, t \in L$ and $s \leqslant L \leqslant t$. Then $L \leqslant t$ and $L \geqslant s$.
Since $s, t \in L = \mathbb{I}_g$, choose $p, q \in \mathbb{D}_g$ s.t. $g_p = s$ and $g_q = t$.
Then $p, q \in \mathbb{D}_g = K$. Want: $g_p \leqslant g \leqslant g_q$ on K.
Want: $s \leqslant g \leqslant t$ on K.
Want: $\forall x \in K,\ \ s \leqslant g_x \leqslant t$.
Given $x \in K$. Want: $s \leqslant g_x \leqslant t$.
Since $x \in K = \mathbb{D}_g$, it follows that $g_x \in \mathbb{I}_g$.
Then $g_x \in \mathbb{I}_g = L \leqslant t$. Want $g_x \geqslant s$.
We have $g_x \in \mathbb{I}_g = L \geqslant s$, as desired. □

THEOREM 4.11.2. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $b \in \mathbb{R}$, $a < b$, $g : [a; b] \to \mathbb{R}$.*
Assume: g is continuous from $\mathbb{R}$ to $\mathbb{R}$ and $g_a = g_b$.
Then: $\exists c \in (a; b)$ s.t. g has a global semi-extremum at c.

Proof. Let $K := [a; b]$. Since $a < b$, we get: $K \neq \varnothing$.
Since K is closed and bounded in $\mathbb{R}$ and since $\mathbb{R}$ is proper,
we conclude that K is compact.
By hypothesis, we have: $g : K \to \mathbb{R}$.
By absoluteness of continuity, g is continuous from K to $\mathbb{R}$.
By Theorem 4.11.1, choose $p, q \in K$ s.t. $g_p \leqslant g \leqslant g_q$ on K.
Exactly one of the following is true:
(1) $\{p, q\} \subseteq \{a, b\}$ or (2) $\{p, q\} \not\subseteq \{a, b\}$.

Case 1:
Let $c := (a + b)/2$. Then $a < c < b$, so $c \in (a; b)$.
Want: g has a global semi-extremum at c.
Want: g has a global semi-maximum at c.
Want: $g \leqslant g_c$ on $\mathbb{D}_g$. Want: $g = g_c$ on $\mathbb{D}_g$.
Want: $\forall x \in \mathbb{D}_g$, $g_x = g_c$. Given $x \in \mathbb{D}_g$. Want: $g_x = g_c$.
We have: $p \in \{p, q\} \subseteq \{a, b\}$, so $g_p \in \{g_a, g_b\}$.
Also: $q \in \{p, q\} \subseteq \{a, b\}$, so $g_q \in \{g_a, g_b\}$.

Let $y := g_a$. Then $y = g_a = g_b$. Then $\{g_a, g_b\} = \{y\}$.
Then $g_p, g_q \in \{g_a, g_b\} = \{y\}$, so $g_p = y = g_q$.
So, since $g_p \leqslant g \leqslant g_q$ on K, we get: $y \leqslant g \leqslant y$ on K.
Then $g = y$ on K. That is, $\forall s \in K$, $g_s = y$.
So, since $x \in \mathbb{D}_g = K$ and since $c \in (a;b) \subseteq [a;b] = K$,
we conclude: $g_x = y$ and $g_c = y$.
Then $g_x = y = g_c$, as desired.
End of Case 1.

Case 2:
Since $\{p,q\} \nsubseteq \{a,b\}$, choose $c \in \{p,q\}$ s.t. $c \neq \{a,b\}$.
We have $c \in \{p,q\} \subseteq K = [a;b]$ and $c \notin \{a,b\}$.
Then: $c \in [a;b] \backslash \{a,b\} = (a;b)$.
Want: g has a global semi-extremum at c.
Since $g_p \leqslant g \leqslant g_q$ on K, we get:
($g \leqslant g_q$ on K) & ($g \geqslant g_p$ on K).
So, since $c \in \{p,q\}$, we get:
($g \leqslant g_c$ on K) $\vee$ ($g \geqslant g_c$ on K).
So, since $K = \mathbb{D}_g$, we get:
($g \leqslant g_c$ on $\mathbb{D}_g$) $\vee$ ($g \geqslant g_c$ on $\mathbb{D}_g$).
That is, either g has a global semi-maximum at c
or g has a global semi-minimum at c.
Then g has a global semi-extremum at c, as desired.
End of Case 2. □

THEOREM 4.11.3. *Let X be a metric space and $f : X \dashrightarrow \mathbb{R}$.*
Let $K \subseteq \mathbb{D}_f$ and $g := f|K$ and $c \in \mathrm{Int}_X K$.
Assume: g has a global strict-maximum at c.
Then: f has a local strict-maximum at c in X.

Proof. Want: $\exists B \in \mathcal{B}_X(c)$ s.t. $f < f_c$ on $B_c^\times$.
Since $c \in \mathrm{Int}_X K$, choose $B \in \mathcal{B}_X(c)$ s.t. $B \subseteq K$.
Then $B \in \mathcal{B}_X(c)$. Want: $f < f_c$ on $B_c^\times$.
Since $K \subseteq \mathbb{D}_f$ and $g = f|K$, we see that: $\mathbb{D}_g = K$.
Since $g = f|K$, it follows that $g = f$ on K.
So, since $K_c^\times \subseteq K$, we get: $g = f$ on $K_c^\times$.
We have $c \in \mathrm{Int}_X K \subseteq K$, so $(f|K)_c = f_c$. Then $g_c = (f|K)_c = f_c$.
Since $B \subseteq K$, it follows that $B_c^\times \subseteq K_c^\times$.
Since g has a global strict-maximum at c and since $\mathbb{D}_g = K$,

we conclude that: $g < g_c$ on $K_c^\times$.

So, since $g = f$ on $K_c^\times$ and since $g_c = f_c$, we get: $f < f_c$ on $K_c^\times$.

So, since $B_c^\times \subseteq K_c^\times$, we get: $f < f_c$ on $B_c^\times$. □

THEOREM 4.11.4. *Let X be a metric space and $f : X \dashrightarrow \mathbb{R}$. Let $K \subseteq \mathbb{D}_f$ and $g := f|K$ and $c \in \mathrm{Int}_X K$.*
Assume: g has a global strict-minimum at c.
Then: f has a local strict-minimum at c in X.

Proof. Unassigned HW. □

THEOREM 4.11.5. *Let X be a metric space and $f : X \dashrightarrow \mathbb{R}$. Let $K \subseteq \mathbb{D}_f$ and $g := f|K$ and $c \in \mathrm{Int}_X K$.*
Assume: g has a global semi-maximum at c.
Then: f has a local semi-maximum at c in X.

Proof. Unassigned HW. □

THEOREM 4.11.6. *Let X be a metric space and $f : X \dashrightarrow \mathbb{R}$. Let $K \subseteq \mathbb{D}_f$ and $g := f|K$ and $c \in \mathrm{Int}_X K$.*
Assume: g has a global semi-minimum at c.
Then: f has a local semi-minimum at c in X.

Proof. Unassigned HW. □

THEOREM 4.11.7. *Let X be a metric space and $f : X \dashrightarrow \mathbb{R}$. Let $K \subseteq \mathbb{D}_f$ and $g := f|K$ and $c \in \mathrm{Int}_X K$.*
Assume: g has a global semi-extremum at c.
Then: f has a local semi-extremum at c in X.

Proof. Unassigned HW. □

THEOREM 4.11.8. *Let X be a metric space and $f : X \dashrightarrow \mathbb{R}$. Let $K \subseteq \mathbb{D}_f$ and $c \in \mathrm{Int}_X K$.*
Assume that f has a global strict-maximum at c.
Then f has a local strict-maximum at c in X.

Proof. Let $K := \mathbb{D}_f$ and let $g := f|K$. Then $g = f$.

So, since f has a global strict-maximum at c,

we conclude that: g has a global strict-maximum at c.

Then, by Theorem 4.11.3, f has a local strict-maximum at c in X. □

In class, we explained why, in Theorem 4.11.8, it is insufficient to assume only that $c \in K$; we need $c \in \mathrm{Int}_X K$.

Recall: $\forall a, b \in \mathbb{R}$, $[a;b] \in \mathcal{T}'_{\mathbb{R}}$ and $\mathrm{Int}_{\mathbb{R}}[a;b] = (a;b)$.

DEFINITION 4.11.9. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathbb{R}$.*
By f is **c/d** *on S, we mean:*
(*f is continuous on S from $\mathbb{R}$ to $\mathbb{R}$*) & ($\mathrm{Int}_{\mathbb{R}} S \subseteq \mathbb{D}_{f'}$).

That is, $\forall S \subseteq \mathbb{R}$, c/d on S means:
continuous on S and differentiable on $\mathrm{Int}_{\mathbb{R}} S$.
In particular, $\forall a, b \in \mathbb{R}$, c/d on $[a;b]$ means:
continuous on $[a;b]$ and differentiable on $(a;b)$.

The next theorem is **Rolle's Theorem**:

THEOREM 4.11.10. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, let $b \in \mathbb{R}$, and let $a < b$.*
Assume: f is c/d on $[a;b]$ and $f_a = f_b$.
Then: $\exists c \in (a;b)$ s.t. $f'_c = 0$.

Proof. Since f is c/d on $[a;b]$ we get:
both f is continuous on $[a;b]$ from $\mathbb{R}$ to $\mathbb{R}$ and $(a;b) \subseteq \mathbb{D}_{f'}$.
Let $K := [a;b]$. Then f is continuous on K from $\mathbb{R}$ to $\mathbb{R}$,
so, by Theorem 2.11.12, $f|K$ is continuous on K from $\mathbb{R}$ to $\mathbb{R}$.
Let $g := f|K$. Then g is continuous on K from $\mathbb{R}$ to $\mathbb{R}$.
Since $a, b \in [a;b] = K$ and $g = f|K$, we get: $g_a = f_a$ and $g_b = f_b$.
By hypothesis, $f_a = f_b$. Then $g_a = f_a = f_b = g_b$, so $g_a = g_b$.
Then, by Theorem 4.11.2, choose $c \in (a;b)$ s.t.
g has a global semi-extremum at c.
Then $c \in (a;b)$. Want: $f'_c = 0$.
We have $c \in (a;b) = \mathrm{Int}_{\mathbb{R}}[a;b] = \mathrm{Int}_{\mathbb{R}} K$.
Then, by Theorem 4.11.7, f has a local semi-extremum at c in $\mathbb{R}$.
We have: $c \in (a;b) \subseteq \mathbb{D}_{f'}$.
Then by Fermat's Theorem (Theorem 4.8.5), we get $f'_c = 0$. □

4.12. Mean Value Theorem.

DEFINITION 4.12.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$.*

$$\text{Then:} \qquad \mathrm{DQ}^f_{ab} := \frac{f_b - f_a}{b - a}.$$

THEOREM 4.12.2. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, h \in \mathbb{R}$, $b := a + h$.*

$$\text{Then:} \qquad \mathrm{DQ}^f_{ab} = \frac{(f^{\mathbb{T}}_a)_h}{h}.$$

THEOREM 4.12.3. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\quad a, b \in \mathbb{D}_f$.
Assume $a \neq b$. $\quad$ *Let* $m := \mathrm{DQ}_{ab}^f$.
Let $\quad L := m \cdot (\bullet)$, $\quad g := f - L$.
Then: $\quad g_a = g_b$.

Proof. We have $\left(\dfrac{f_b - f_a}{b - a} \right) \cdot (b - a) = f_b - f_a$.
So, since $m = \mathrm{DQ}_{ab}^f = \dfrac{f_b - f_a}{b - a}$, we get $m \cdot (b - a) = f_b - f_a$.
Then $L_b - L_a = m \cdot b - m \cdot a = m \cdot (b - a) = f_b - f_a$.
Since $L_b - L_a = f_b - f_a$, we get: $f_a - L_a = f_b - L_b$.
Then $g_a = (f - L)_a = f_a - L_a = f_b - L_b = (f - L)_b = g_b$, as desired. ☐

DEFINITION 4.12.4. *Let* $L : \mathbb{R} \to \mathbb{R}$.
By L *is* **algebraically linear**, *we mean:*
$$\forall s, t \in \mathbb{R}, \quad L_{s+t} = L_s + L_t$$
and
$$\forall c \in \mathbb{R},\ \forall t \in \mathbb{R}, \quad L_{c \cdot t} = c \cdot L_t.$$

THEOREM 4.12.5. *Let* $L : \mathbb{R} \to \mathbb{R}$.
Assume L *is algebraically linear.*
Let $m := L_1$. $\quad$ *Then* $L = m \cdot (\bullet)$.

Proof. Want: $\forall x \in \mathbb{R}$, $L_x = (m \cdot (\bullet))_x$.
Given $x \in \mathbb{R}$. $\quad$ Want: $L_x = (m \cdot (\bullet))_x$.
We have $L_x = L_{x \cdot 1} = x \cdot L_1 = x \cdot m = m \cdot x = (m \cdot (\bullet))_x$, as desired. ☐

THEOREM 4.12.6. *Let* $L : \mathbb{R} \to \mathbb{R}$.
Then: $\quad (L \in \mathcal{L}) \Leftrightarrow (L$ *is algebraically linear* $)$.

Proof. Unassigned HW. ☐

THEOREM 4.12.7. *Let* $L \in \mathcal{L}$ *and* $x \in \mathbb{R}$. $\quad$ *Then* $L_x^{\mathbb{T}} = L$.

Proof. Want: $\quad \forall h \in \mathbb{R}$, $(L_x^{\mathbb{T}})_h = L_h$.
Given $h \in \mathbb{R}$. $\quad$ Want: $(L_x^{\mathbb{T}})_h = L_h$.
Since L is algebraically linear, we get: $\quad L_{x+h} = L_x + L_h$.
We have $(L_x^{\mathbb{T}})_h = L_{x+h} - L_x = L_x + L_h - L_x = L_h$, as desired. ☐

THEOREM 4.12.8. *Let* $L \in \mathcal{L}$ *and* $x \in \mathbb{R}$. $\quad$ *Then* $D_x L = L$.

Proof. By Theorem 4.12.7, $L_x^{\mathbb{T}} = L$. $\quad$ Then $L_x^{\mathbb{T}} - L = \mathbf{0}$.
Since $L_x^{\mathbb{T}} - L = \mathbf{0} \in \mathcal{O}_1$, we conclude: $\quad L \in \mathrm{LINS}_x L$.
Then, by Theorem 4.4.3, we get $D_x L = L$, as desired. ☐

THEOREM 4.12.9. *Let $m \in \mathbb{R}$ and $L := m \cdot (\bullet)$. Then $L' = C_{\mathbb{R}}^m$.*

Proof. Want: $\forall x \in \mathbb{R}$, $L'_x = (C_{\mathbb{R}}^m)_x$.
Given $x \in \mathbb{R}$. Want: $L'_x = (C_{\mathbb{R}}^m)_x$. By Theorem 4.12.8, $D_x L = L$.
Then $L'_x = (D_x L)_1 = L_1 = m \cdot 1 = m = (C_{\mathbb{R}}^m)_x$, as desired. □

The next theorem is the **Mean Value Theorem**:

THEOREM 4.12.10. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, let $b \in \mathbb{R}$, let $a < b$. Assume f is c/d on $[a; b]$. Then $\exists c \in (a; b)$ s.t. $f'_c = \mathrm{DQ}_{ab}^f$.*

Proof. Let $m := \mathrm{DQ}_{ab}^f$, $L := m \cdot (\bullet)$, $B := C_{\mathbb{R}}^m$.
By Theorem 4.12.9, we have: $L' = B$.
Then $\mathbb{D}_{L'} = \mathbb{D}_B = \mathbb{R}$. Then $[a; b] \subseteq \mathbb{D}_{L'}$. Then L is c/d on $[a; b]$.
So, since f is c/d on $[a; b]$, it follows that $f - L$ is c/d on $[a; b]$.
Let $g := f - L$. Then g is c/d on $[a; b]$.
By Theorem 4.12.3, we get: $g_a = g_b$.
By Rolle's Theorem (Theorem 4.11.10), choose $c \in (a; b)$ s.t. $g'_c = 0$.
Then $c \in (a; b)$. Want: $f'_c = \mathrm{DQ}_{ab}^f$.
Since f is c/d on $[a; b]$, we get $\mathrm{Int}_{\mathbb{R}}[a; b] \subseteq \mathbb{D}_{f'}$.
We have $c \in (a; b) = \mathrm{Int}_{\mathbb{R}}[a; b] \subseteq \mathbb{D}_{f'}$, so $c \in \mathbb{D}_{f'}$.
Since $\mathbb{D}_{f'} \subseteq \mathbb{R}$, we get: $\mathbb{D}_{f'} \bigcap \mathbb{R} = \mathbb{D}_{f'}$.
Then $c \in \mathbb{D}_{f'} = \mathbb{D}_{f'} \bigcap \mathbb{R} = \mathbb{D}_{f'} \bigcap \mathbb{D}_{L'}$.
Then, by linearity of differentiation, we see that $(f - L)'_c = f'_c - L'_c$.
So, since $f - L = g$ and $L' = B$, we get $g'_c = f'_c - B_c$.
So, since $g'_c = 0$ and $B_c = (C_{\mathbb{R}}^m)_c = m$, we get $0 = f'_c - m$, so $m = f'_c$.
Then $f'_c = m = \mathrm{DQ}_{ab}^f$, as desired. □

DEFINITION 4.12.11. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathbb{D}_f$.*
Then $\mathrm{DQ}_S^f := \{\mathrm{DQ}_{ab}^f \,|\, (\, a, b \in S\,) \,\&\, (\, a \neq b\,)\}$.

DEFINITION 4.12.12. *Let f be a function.*
By f is **constant**, *we mean: $\exists y$ s.t., $\forall x \in \mathbb{D}_f$, $f_x = y$.*

THEOREM 4.12.13. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $S \subseteq \mathbb{D}_f$.*
Then: (1) $(\, 0 \notin \mathrm{DQ}_S^f \,) \Leftrightarrow (\, f|S$ *is one-to-one* $)$
and (2) $(\, \mathrm{DQ}_S^f \subseteq \{0\} \,) \Leftrightarrow (\, f|S$ *is constant* $)$
and (3) $(\, \mathrm{DQ}_S^f > 0 \,) \Leftrightarrow (\, f|S$ *is strictly-increasing* $)$
and (4) $(\, \mathrm{DQ}_S^f \geqslant 0 \,) \Leftrightarrow (\, f|S$ *is semi-increasing* $)$
and (5) $(\, \mathrm{DQ}_S^f < 0 \,) \Leftrightarrow (\, f|S$ *is strictly-decreasing* $)$
and (6) $(\, \mathrm{DQ}_S^f \leqslant 0 \,) \Leftrightarrow (\, f|S$ *is semi-decreasing* $)$.

Proof. Unassigned HW. □

DEFINITION 4.12.14. *Let* $S \subseteq \mathbb{R}$.

By S *is an* **interval**, *we mean:* $\forall a, b \in S,\quad [a; b] \subseteq S$.

Let $p, q \in \mathbb{R}$. Then $(p; q)$, $[p; q)$, $(p; q]$ and $[p, q]$ are all intervals.

Let $p \in \mathbb{R}$. Then $(-\infty; p)$, $(-\infty, p]$, (p, ∞) and $[p, \infty)$ are all intervals.

Also, $(-\infty; \infty)$ is an interval. Note that $\mathbb{R} = (-\infty; \infty)$.

THEOREM 4.12.15. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $S \subseteq \mathbb{D}_{f'}$.

Assume that S *is an interval.* *Then* $\mathrm{DQ}^f_S \subseteq f'_* S$.

Proof. Want: $\forall m \in \mathrm{DQ}^f_s,\ m \in f'_* S$.
Given $m \in \mathrm{DQ}^f_s$. Want: $m \in f'_* S$.
Since $m \in \mathrm{DQ}^f_s$, choose $a, b \in S$ s.t. $(\ a \neq b\ )\ \&\ (\ m = \mathrm{DQ}^f_{ab}\ )$.
Since S is an interval and $a, b \in S$, we get: $[a; b] \in S$.
By hypothesis, $S \subseteq \mathbb{D}_{f'}$. Then $[a; b] \subseteq S \subseteq \mathbb{D}_{f'}$.
Since $[a; b] \subseteq \mathbb{D}_{f'}$, we conclude: f is c/d on $[a; b]$.
By the Mean Value Theorem (Theorem 4.12.10),
choose $c \in (a; b)$ s.t. $f'_c = \mathrm{DQ}^f_{ab}$.
We have $c \in (a; b) \subseteq [a; b] \subseteq S$, so $c \in S$.
So, since $c \in S \subseteq \mathbb{D}_{f'}$, we get $c \in S \cap \mathbb{D}_{f'}$. Then $f'_c \in f'_* S$.
Then $m = \mathrm{DQ}^f_{ab} = f'_c \in f'_* S$, as desired. □

THEOREM 4.12.16. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $S \subseteq \mathbb{D}_{f'}$.

Assume that S *is an interval.*

Then: (1) $(\ 0 \notin f'_* S\ ) \Rightarrow (\ f|S$ *is one-to-one* $)$
and (2) $(\ f'_* S \subseteq \{0\}\ ) \Rightarrow (\ f|S$ *is constant* $)$
and (3) $(\ f'_* S > 0\ ) \Rightarrow (\ f|S$ *is strictly-increasing* $)$
and (4) $(\ f'_* S \geqslant 0\ ) \Rightarrow (\ f|S$ *is semi-increasing* $)$
and (5) $(\ f'_* S < 0\ ) \Rightarrow (\ f|S$ *is strictly-decreasing* $)$
and (6) $(\ f'_* S \leqslant 0\ ) \Rightarrow (\ f|S$ *is semi-decreasing* $)$.

Proof. Unassigned HW.
Hint: Combine Theorem 4.12.15 with Theorem 4.12.13. □

The converses of (1) and (3) of Theorem 4.12.16 both fail because of:

Let $f := (\bullet)^3$ and let $S := \mathbb{R}$.

Here, f is 1-1 and strictly-increasing, but $0 \in f'_* S$ and $\neg(f'_* S > 0)$.

The converse of (5) of Theorem 4.12.16 fails because of:

Let $f := -(\bullet)^3$ and let $S := \mathbb{R}$.

Here f is strictly-decreasing, but $\neg(f'_* S < 0)$.

Converses of (2), (4) and (6) of Theorem 4.12.16 happen only when $\#S = 1$. See Theorem 4.12.20, below.

DEFINITION 4.12.17. *We define* $\mathrm{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ *by:*

$$\forall x \in \mathbb{R}, \qquad \mathrm{sgn}_x = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases}$$

THEOREM 4.12.18. *Let* $a, b \in \mathbb{R}$.

Assume: $|b - a| \leqslant |a|/2$. *Then:* $\mathrm{sgn}_b = \mathrm{sgn}_a$.

Proof. Let $\varepsilon := |a|/2$. Then $|b - a| \leqslant \varepsilon$, so $a - \varepsilon \leqslant b \leqslant a + \varepsilon$.
Exactly one of the following holds:

(1) $a > 0$ or (2) $a = 0$ or (3) $a < 0$.

Case (1): Since $a > 0$, we get: $|a| = a$ and $a/2 > 0$.
Then $b \geqslant a - \varepsilon = a - (|a|/2) = a - (a/2) = a/2 > 0$, so $b > 0$.
Since $a > 0$, we get $\mathrm{sgn}_a = 1$. Since $b > 0$, we get $\mathrm{sgn}_b = 1$.
Then $\mathrm{sgn}_b = 1 = \mathrm{sgn}_a$, as desired.
End of Case (1).

Case (2): Since $a = 0$ and $\varepsilon = |a|$, we get: $\varepsilon = 0$.
Then $|b - a| \leqslant \varepsilon = 0$, so, since $|b - a| \geqslant 0$, we get $|b - a| = 0$.
Then $b = a$. Then $\mathrm{sgn}_b = \mathrm{sgn}_a$, as desired. *End of Case (2).*

Case (3): Since $a < 0$, we get: $|a| = -a$ and $a/2 < 0$.
Then $b \leqslant a + \varepsilon = a + (|a|/2) = a + (-a/2) = a/2 < 0$, so $b < 0$.
Since $a < 0$, we get $\mathrm{sgn}_a = -1$. Since $b < 0$, we get $\mathrm{sgn}_b = -1$.
Then $\mathrm{sgn}_b = -1 = \mathrm{sgn}_a$, as desired.
End of Case (3). □

We express Theorem 4.12.18 by saying:

if $|b - a| \leqslant |a|/2$, then b "mimics" a.

THEOREM 4.12.19. *Let* $\phi : \mathbb{R} \dashrightarrow \mathbb{R}$, $k \in \mathbb{N}_0$, $H \in \mathcal{H}_k \backslash \{\mathbf{0}\}$.
Assume: $\phi - H \in \mathcal{O}_k$.
Then: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ *s.t.,* $\forall x \in B$, $\mathrm{sgn}_{\phi_x} = \mathrm{sgn}_{H_x}$.

Proof. Since $H \in \mathcal{H}_k$, we have $H : \mathbb{R} \to \mathbb{R}$, so $\forall x \in \mathbb{R}$, $H_x \in \mathbb{R}$.
Then, by Theorem 4.12.18, it suffices to show:

$\exists \delta > 0$ s.t., $\forall x \in B_{\mathbb{R}}(0, \delta)$, $|\phi_x - H_x| \leqslant |H_x|/2$.

Since $H \in \mathcal{H}_k$, choose $c \in \mathbb{R}$ s.t. $H = c \cdot (\bullet)^k$.
As $H \neq \mathbf{0}$, we get $c \neq 0$, so $|c| > 0$. Let $\varepsilon := |c|/2$. Then $\varepsilon > 0$.
By hypothesis, $\phi - H \in \mathcal{O}_k$. Let $R := \phi - H$. Then $R \in \mathcal{O}_k$.
Choose $\delta > 0$ s.t., $\forall x \in \mathbb{R}$, $(\ |x| < \delta\ ) \Rightarrow (\ |R_x| < \varepsilon \cdot |x|^k\ )$.
Then $\delta > 0$. Want: $\forall x \in B_{\mathbb{R}}(0, \delta)$, $|\phi_x - H_x| \leqslant |H_x|/2$.

Given $x \in B_{\mathbb{R}}(0, \delta)$. Want: $|\phi_x - H_x| \leqslant |H_x|/2$.

Since $x \in B_{\mathbb{R}}(0, \delta)$, we get: $d_{\mathbb{R}}(x, 0) < \delta$.
Then $|x| = |x - 0| = d_{\mathbb{R}}(x, 0) < \delta$, so,

by choice of δ, $|R_x| < \varepsilon \cdot |x|^k$.

We have $H_x = (c \cdot (\bullet)^k)_x = c \cdot x^k$, so $|H_x| = |c| \cdot |x|^k$.

Then: $|H_x|/2 = (|c|/2) \cdot |x|^k = \varepsilon \cdot |x|^k$.

Then: $|\phi_x - H_x| = |(\phi - H)_x| = |R_x| \leqslant \varepsilon \cdot |x|^k = |H_x|/2$, as desired. □

We express Theorem 4.12.19 by saying:

if $H \in \mathcal{H}_k$ and $\phi - H \in \mathcal{O}_k$, then ϕ "mimics" H near 0.

That is,

if a function is approximated by a homogeneous k-polynomial,
with sub-k remainder,
then the function "mimics" the polynomial near 0.

In class, we drew some graphs to indicate this geometrically.

THEOREM 4.12.20. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{R}$. *Assume:* $\#S \neq 1$.
Then: *(1)* $(\ f|S$ *is one-to-one* $) \not\Rightarrow (\ 0 \notin f'_* S\ )$
and *(2)* $(\ f|S$ *is constant* $) \Rightarrow (\ f'_* S \subseteq \{0\}\ )$
and *(3)* $(\ f|S$ *is strictly-increasing* $) \not\Rightarrow (\ f'_* S > 0\ )$
and *(4)* $(\ f|S$ *is semi-increasing* $) \Rightarrow (\ f'_* S \geqslant 0\ )$
and *(5)* $(\ f|S$ *is strictly-decreasing* $) \not\Rightarrow (\ f'_* S < 0\ )$
and *(6)* $(\ f|S$ *is semi-decreasing* $) \Rightarrow (\ f'_* S \leqslant 0\ )$.

Proof. *Proof of (1)-(5):* Unassigned HW. *End of proof of (1)-(5).*

Proof of (6): Want: $\forall m \in f'_* S$, $m \leqslant 0$.
Given $m \in f'_* S$. Want: $m \leqslant 0$.
Assume $m > 0$. Want: Contradiction.
Since $m \in f'_* S$, choose $x \in S \cap \mathbb{D}_{f'}$ s.t. $m = f'_x$.
Let $L := D_x f$. Then $L_1 = f'_x = m$, so $L_1 = m$.
Since $m \in f'_* S$, we get $m \neq ☹$, so $L_1 \neq ☹$, so $L \neq ☹$.

Then $L \in \mathrm{LINS}_x f$. Then $L \in \mathcal{L}$ and $f_x^{\mathbb{T}} - L \in \mathcal{O}_1$.
Let $\phi := f_x^{\mathbb{T}}$. Then $\phi - L \in \mathcal{O}_1$.
Since $L \in \mathcal{L} = \mathcal{H}_1$ and since $\phi - L \in \mathcal{O}_1$,
by Theorem 4.12.19, choose $\delta > 0$ s.t., $\forall h \in B_{\mathbb{R}}(0, \delta)$,

$$[\ (\ L_h > 0\ ) \ \Rightarrow\ (\ \phi_h > 0\ )\ ]$$
$$\text{and} \qquad [\ (\ L_h < 0\ ) \ \Rightarrow\ (\ \phi_h < 0\ )\ ].$$

Since $(\ x \in S \cap \mathbb{D}_{f'} \subseteq S\ )\ \&\ (\ \#S \neq 1\ )$,
we get $S_x^\times \neq \varnothing$, so choose $z \in S_x^\times$.
Since $z \neq x$, exactly one of the following must hold:
(A) $z > x$ or (B) $z < x$.

Case (A):
Let $\gamma := z - x$. Then $\gamma > 0$ and $x + \gamma = z$.
Then $x + \gamma = z \in S_x^\times \subseteq S$. Recall: $x \in S$.
Since $x, x + \gamma \in S$ and since S is an interval,
it follows that $[x; x + \gamma] \subseteq S$.
Let $h := \min\{\gamma, \delta/2\}$. Then: $(\ h > 0\ )\ \&\ (\ h \leqslant \gamma\ )\ \&\ (\ h < \delta\ )$.
Recall: $L_1 = m$. Then $L_h = L_{h \cdot 1} = h \cdot L_1 = h \cdot m$.
So, since $h, m > 0$, we conclude that $L_h > 0$.
We have $h > 0 > -\delta$ and $h < \delta$, so $-\delta < h < \delta$, so $h \in (-\delta; \delta)$.
Since $h \in (-\delta; \delta) = B_{\mathbb{R}}(0, \delta)$ and since $L_h > 0$,
by choice of δ, we get: $\phi_h > 0$.
Since $0 < h \leqslant \gamma$, we get $x < x + h \leqslant x + \gamma$, so $x + h \in (x; x + \gamma]$.
Then $x + h \in (x; x + \gamma] \subseteq [x; x + \gamma] \subseteq S$.
Since $x, x + h \in S$, since $x \leqslant x + h$ and since $f|S$ is semi-decreasing,
we conclude that: $f_x \geqslant f_{x+h}$. Then $f_{x+h} - f_x \leqslant 0$.
Then $0 < \phi_h = (f_x^{\mathbb{T}})_h = f_{x+h} - f_x \leqslant 0$, so $0 < 0$. Contradiction.
End of Case (A).

Case (B):
Let $\gamma := x - z$. Then $\gamma > 0$ and $x - \gamma = z$.
Then $x - \gamma = z \in S_x^\times \subseteq S$. Recall: $x \in S$.
Since $x - \gamma, x \in S$ and since S is an interval,
it follows that $[x - \gamma; x] \subseteq S$.
Let $h := \min\{\gamma, \delta/2\}$. Then: $(\ h > 0\ )\ \&\ (\ h \leqslant \gamma\ )\ \&\ (\ h < \delta\ )$.
Recall: $L_1 = m$. Then $L_{-h} = L_{-h \cdot 1} = -h \cdot L_1 = -h \cdot m$.
So, since $h, m > 0$, we conclude that $L_h < 0$.
We have $-h < 0 < \delta$ and $-h > -\delta$, so $-\delta < -h < \delta$, so $-h \in (-\delta; \delta)$.

Since $-h \in (-\delta;\delta) = B_{\mathbb{R}}(0,\delta)$ and since $L_{-h} < 0$,
by choice of δ, we get: $\phi_{-h} < 0$.
Since $-\gamma \leqslant -h < 0$, we get $x-\gamma \leqslant x-h < x$, so $x-h \in [x-\gamma;x)$.
Then $x-h \in [x-\gamma;x) \subseteq [x-\gamma;x] \subseteq S$.
Since $x-h, x \in S$, since $x-h \leqslant x$ and since $f|S$ is semi-decreasing,
we conclude that: $f_{x-h} \geqslant f_x$. Then $f_{x-h} - f_x \geqslant 0$.
Then $0 > \phi_{-h} = (f_x^{\mathbb{T}})_{-h} = f_{x-h} - f_x \geqslant 0$, so $0 > 0$. Contradiction.
End of Case (B).

End of proof of (6). □

4.13. **Taylor's Theorem, second order.**

THEOREM 4.13.1. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$. *Assume* $f_0 = 0$.
Then: $f_0^{\mathbb{T}} = f$.

Proof. Want: $\forall h \in \mathbb{R}$, $(f_0^{\mathbb{T}})_h = f_h$.
Given $h \in \mathbb{R}$. Want: $(f_0^{\mathbb{T}})_h = f_h$.
We have $(f_0^{\mathbb{T}})_h = f_{0+h} - f_0 = f_h - 0 = f_h$. □

THEOREM 4.13.2. *Let* $L \in \mathcal{L}$. *Assume* $L_1 = 0$. *Then* $L = \mathbf{0}$.

Proof. $\forall x \in \mathbb{R}$, $L_x = L_{x\cdot 1} = x \cdot L_1 = x\cdot 0 = 0 = \mathbf{0}_x$. Then $L = \mathbf{0}$. □

THEOREM 4.13.3. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$. *Assume:* $f_0 = f'_0 = 0$.
Then: $f \in \mathcal{O}_1$.

Proof. Since $f_0 = 0$, by Theorem 4.13.1, we conclude that $f_0^{\mathbb{T}} = f$.
Let $L := D_0 f$. Then $L = \mathrm{UE}(\mathrm{LINS}_0 f)$ and $L_1 = f'_0 = 0$.
Then $L_1 = 0 \neq ☹$, so $L \neq ☹$, so $L \in \mathrm{LINS}_0 f$.
Then $L \in \mathcal{L}$ and $f_0^{\mathbb{T}} - L \in \mathcal{O}_1$.
Since $L \in \mathcal{L}$ and $L_1 = 0$, by Theorem 4.13.2, we see that $L = \mathbf{0}$.
Then $f = f - \mathbf{0} = f_0^{\mathbb{T}} - L \in \mathcal{O}_1$, as desired. □

THEOREM 4.13.4. $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall a, b \in \mathbb{R}$, $\mathrm{DQ}_{ab}^f = \mathrm{DQ}_{ba}^f$.

THEOREM 4.13.5. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $a, b \in \mathbb{R}$.
Let $\alpha := \min\{a,b\}$ *and let* $\beta := \max\{a,b\}$. *Then* $\mathrm{DQ}_{ab}^f = \mathrm{DQ}_{\alpha\beta}^f$.

THEOREM 4.13.6. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $a, b \in \mathbb{R}$.
Assume $[a|b] \subseteq \mathbb{D}_{f'}$. *Then:* $\exists c \in [a|b]$ *s.t.* $f'_c \cdot (b-a) = f_b - f_a$.

Proof. We know: (1) $a = b$ or (2) $a \neq b$.

Case (1):
Let $c := a$. Then $c = a \in [a|b]$. Want: $f'_c \cdot (b - a) = f_b - f_a$.
Since $a, b \in [a|b] \subseteq \mathbb{D}_{f'} \subseteq \mathbb{D}_f$, we get $f_a, f_b \in \mathbb{I}_f$.
Since $c = a \in [a|b] \subseteq \mathbb{D}_{f'}$, we get $f'_c \in \mathbb{I}_{f'}$.
Then $f_a, f_b \in \mathbb{I}_f \subseteq \mathbb{R}$ and $f'_c \in \mathbb{I}_{f'} \subseteq \mathbb{R}$, so $f_a, f_b, f'_c \in \mathbb{R}$.
Since $a = b$, we get $b - a = 0$.
Since $a = b$, we get $f_a = f_b$, so $f_b - f_a = 0$.
Then $f'_c \cdot (b - a) = f'_c \cdot 0 = 0 = f_b - f_a$.
End of Case (1).

Case (2):
Let $\alpha := \min\{a, b\}$ and let $\beta := \max\{a, b\}$. Then $\alpha \leqslant \beta$.
Since $a \neq b$, we get $\alpha \neq \beta$. So, as $\alpha \leqslant \beta$, we get: $\alpha < \beta$.
Since $[\alpha;\beta] = [a|b] \subseteq \mathbb{D}_{f'}$, we get: f is c/d on $[\alpha;\beta]$.
By the Mean Value Theorem, choose $c \in (\alpha;\beta)$ s.t. $f'_c = \mathrm{DQ}^f_{\alpha\beta}$.
Then $c \in (\alpha;\beta) \subseteq [\alpha;\beta] = [a|b]$. Want: $f'_c \cdot (b - a) = f_b - f_a$.
By Theorem 4.13.5, we conclude: $\mathrm{DQ}^f_{ab} = \mathrm{DQ}^f_{\alpha\beta}$.
Since $a, b \in \mathbb{R}$ and $a \neq b$, we get $b - a \in \mathbb{R}_0^\times$, so $\dfrac{b-a}{b-a} = 1$.
We have $f'_c = \mathrm{DQ}^f_{\alpha\beta} = \mathrm{DQ}^f_{\alpha\beta} = \mathrm{DQ}^f_{ab} = \dfrac{f_b - f_a}{b - a}$.
Then $f'_c \cdot (b - a) = (f_b - f_a) \cdot \dfrac{b-a}{b-a} = (f_b - f_a) \cdot 1 = f_b - f_a$.
End of Case (2). □

THEOREM 4.13.7. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $k \in \mathbb{N}_0$.*
Assume: ($f' \in \mathcal{O}_k$) & ($f_0 = 0$). Then: $f \in \mathcal{O}_{k+1}$.

Proof. Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\ |x| < \delta\ ) \Rightarrow (\ |f_x| \leqslant \varepsilon \cdot |x|^{k+1}\ ).$$
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\ |x| < \delta\ ) \Rightarrow (\ |f_x| \leqslant \varepsilon \cdot |x|^{k+1}\ ).$$
Since $f' \in \mathcal{O}_k$, choose $\delta > 0$ s.t., $\forall x \in \mathbb{R}$,
$$(\ |x| < \delta\ ) \Rightarrow (\ |f'_x| \leqslant \varepsilon \cdot |x|^{k}\ ).$$
Then $\delta > 0$. Want: $\forall x \in \mathbb{R}$,
$$(\ |x| < \delta\ ) \Rightarrow (\ |f_x| \leqslant \varepsilon \cdot |x|^{k+1}\ ).$$
Given $x \in \mathbb{R}$. Want: $(\ |x| < \delta\ ) \Rightarrow (\ |f_x| \leqslant \varepsilon \cdot |x|^{k+1}\ )$.
Assume: $|x| < \delta$. Want: $|f_x| \leqslant \varepsilon \cdot |x|^{k+1}$.

Since $f_0 = 0$, it follows that: $f_x - f_0 = 0$.
By Theorem 4.13.6, choose $c \in [0|x]$ s.t. $f'_c \cdot (x - 0) = f_x - f_0$.
So, since $x - 0 = x$ and since $f_x - f_0 = 0$, we get: $f'_c \cdot x = f_x$.
Since $c \in [0|x]$, it follows that $|c| \leqslant |x|$.
Since $|c| \leqslant |x| < \delta$, by choice of δ, we get: $|f'_c| \leqslant \varepsilon \cdot |c|^k$.
Then $|f_x| = |f'_c \cdot x| = |f'_c| \cdot |x| \leqslant \varepsilon \cdot |c|^k \cdot |x| \leqslant \varepsilon \cdot |x|^k \cdot |x| = \varepsilon \cdot |x|^{k+1}$. □

THEOREM 4.13.8. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: $f_0 = f'_0 = f''_0 = 0$. *Then* $f \in \mathcal{O}_2$.

Proof. Let $g := f'$. Then $g_0 = f'_0 = 0$ and $g'_0 = f''_0 = 0$.
Since $g_0 = g'_0 = 0$, by Theorem 4.13.3, we get: $g \in \mathcal{O}_1$.
Since $f' = g \in \mathcal{O}_1$ and $f_0 = 0$, by Theorem 4.13.7, we get $f \in \mathcal{O}_2$. □

THEOREM 4.13.9. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $x \in \mathbb{D}_f$ *and* $h \in \mathbb{R}$.
Then: $(f_x^{\mathbb{T}})_h^{\mathbb{T}} = f_{x+h}^{\mathbb{T}}$.

Proof. Let $g := f_x^{\mathbb{T}}$. Want: $g_h^{\mathbb{T}} = f_{x+h}^{\mathbb{T}}$.
Want: $\forall s \in \mathbb{R}$, $(g_h^{\mathbb{T}})_s = (f_{x+h}^{\mathbb{T}})_s$.
Given $s \in \mathbb{R}$. Want: $(g_h^{\mathbb{T}})_s = (f_{x+h}^{\mathbb{T}})_s$.
Since $x \in \mathbb{D}_f$, we get $f_x \in \mathbb{I}_f$.
Since $f_x \in \mathbb{I}_f \subseteq \mathbb{R}$, we get: $f_x - f_x = 0$.
Then: $(f_{x+h+s} - f_x) - (f_{x+h} - f_x) = f_{x+h+s} - f_{x+h}$.

$$\begin{aligned} \text{Then } (g_h^{\mathbb{T}})_s &= g_{h+s} - g_h = (f_x^{\mathbb{T}})_{h+s} - (f_x^{\mathbb{T}})_h \\ &= (f_{x+h+s} - f_x) - (f_{x+h} - f_x) = f_{x+h+s} - f_{x+h} \\ &= (f_{x+h}^{\mathbb{T}})_s, \quad \text{as desired.} \end{aligned}$$ □

THEOREM 4.13.10. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $x \in \mathbb{D}_f$, $g := f_x^{\mathbb{T}}$, $h \in \mathbb{R}$.
Then: $\mathrm{LINS}_x g = \mathrm{LINS}_{x+h} f$.

Proof. By Theorem 4.13.9, we have $(f_x^{\mathbb{T}})_h^{\mathbb{T}} = f_{x+h}^{\mathbb{T}}$. Then $g_h^{\mathbb{T}} = f_{x+h}^{\mathbb{T}}$.

$$\begin{aligned} \text{Then } \mathrm{LINS}_h g &= \{L \in \mathcal{L} \,|\, g_h^{\mathbb{T}} - L \in \mathcal{O}_1\} \\ &= \{L \in \mathcal{L} \,|\, f_{x+h}^{\mathbb{T}} - L \in \mathcal{O}_1\} = \mathrm{LINS}_{x+h} f. \end{aligned}$$ □

THEOREM 4.13.11. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $x \in \mathbb{D}_f$, $g := f_x^{\mathbb{T}}$, $h \in \mathbb{R}$.
Then: $D_h g = D_{x+h} f$ *and* $g'_h = f'_{x+h}$.

Proof. By Theorem 4.13.10, $\mathrm{LINS}_x g = \mathrm{LINS}_{x+h} f$.
Then $D_h g = \mathrm{UE}(\mathrm{LINS}_x g) = \mathrm{UE}(\mathrm{LINS}_{x+h} f) = D_{x+h} f$.
It remains to show that: $g'_h = f'_{x+h}$.
We have $g'_h = (D_h g)_1 = (D_{x+h} f)_1 = f'_{x+h}$, as desired. □

DEFINITION 4.13.12. *Let a and b be objects.*

Then $a \;=^*\; b$ *means:* $(\, b \neq ☹ \,) \Rightarrow (\, a = b \,)$.
Also, $a \;{}^*{=}\; b$ *means:* $(\, a \neq ☹ \,) \Rightarrow (\, a = b \,)$.

We read "$a \;=^*\; b$" as
"a is contingent equal to b, with the contingency on b".
We read "$a \;{}^*{=}\; b$" as
"a is contingent equal to b, with the contingency on a".

The next two theorems restate linearity of derivatives, the product rule and the chain rule, all in the language of contingent equalities.

THEOREM 4.13.13. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ *and* $x \in \mathbb{R}$.
Then: $(f + g)'_x \;=^*\; f'_x + g'_x$
and $(f \cdot g)'_x \;=^*\; f'_x \cdot g_x + f_x \cdot g'_x$
and $(g \circ f)'_x \;=^*\; g'_{f_x} \cdot f'_x$.

THEOREM 4.13.14. *Let* $c \in \mathbb{R}$, $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and* $x \in \mathbb{R}$.
Then: $(c \cdot f)'_x \;=^*\; c \cdot f'_x$.

We can sometimes "remove the contingency":

THEOREM 4.13.15. $\forall a, b$, $[\ (\ a \;=^*\; b \neq ☹\ ) \Rightarrow (\ a = b\ )\ ]$
$\&\ [\ (\ a \;=^*\; b \neq ☹\ ) \Rightarrow (\ a = b\ )\ ]$.

If we have contingencies simlutaneously on both sides, then the contingency is removable:

THEOREM 4.13.16. *Let a and b be objects.*
Then: $[\ (\ a \;=^*\; b\ )\ \&\ (\ a \;{}^*{=}\; b\ )\ ] \Rightarrow [\ a = b\ ]$.

Idea of proof: at least one of the following must hold:
$a = ☹ = b$ or $a \neq ☹$ or $b \neq ☹$.
In the first case, $a = b$ by contraction.
In the second, removing the contingency from $a \;{}^*{=}\; b$, we get $a = b$.
In the third, removing the contingency from $a \;=^*\; b$, we get $a = b$.
QED

The following captures that the derivative commutes with scalar multiplication, in the language of contingent equality:

THEOREM 4.13.17. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $a, x \in \mathbb{R}$.
Then $(a \cdot f)'_x \;=^*\; a \cdot f'_x$.

Theorem 4.13.17 follows from Theorem 4.7.2.

The following captures additivity of the derivative, along with the product and chain rules, all in the language of contingent equality:

THEOREM 4.13.18. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ *and let* $x \in \mathbb{R}$.

$$\begin{array}{lrcl} \textit{Then} & (f+g)'_x & =^* & f'_x + g'_x \\ \textit{and} & (f \cdot g)'_x & =^* & f'_x \cdot g_x + f_x \cdot g'_x \\ \textit{and} & (g \circ f)'_x & =^* & g'_{f_x} \cdot f'_x. \end{array}$$

Theorem 4.13.18 follows from Theorem 4.7.1 and Theorem 4.7.4.

THEOREM 4.13.19. *Let* $a \in \mathbb{R}_0^\times, \quad f : \mathbb{R} \dashrightarrow \mathbb{R}, \quad x \in \mathbb{R}$.
Then $(a \cdot f)'_x \;=\; a \cdot f'_x$

Proof. We have $(a \cdot f)'_x \;=^*\; a \cdot f'_x$.
It therefore suffices to show: $a \cdot f'_x \;=^*\; (a \cdot f)'_x$.
Let $g := a \cdot f$. Want: $a \cdot f'_x \;=^*\; g'_x$.
Let $b := 1/a$. Then: $b \cdot g = b \cdot a \cdot f = 1 \cdot f = f$.
Then $f = b \cdot g$. Also, $(b \cdot g)'_x \;=^*\; b \cdot g'_x$.
Then $a \cdot f'_x \;=\; a \cdot (b \cdot g)'_x \;=^*\; a \cdot b \cdot g'_x \;=\; 1 \cdot g'_x \;=\; g'_x$, as desired. □

THEOREM 4.13.20. *Let* $f, g : \mathbb{R} \to \mathbb{R}$.
Then: $\forall x \in \mathbb{R}, \quad (f+g)''_x \;=^*\; f''_x + g''_x$.

Proof. By Theorem 4.13.18, we have:
$\forall x \in \mathbb{R}, \quad (f+g)'_x \;=^*\; f'_x + g'_x$.
Then: $\forall x \in \mathbb{R}, \quad (f+g)'_x \;=^*\; (f'+g')_x$.
Let $\alpha := f' + g'$ and $\beta := (f+g)'$. Then: $\forall x \in \mathbb{R}, \quad \beta_x \;=^*\; \alpha_x$.
Then, by HW#2-4, we get: $\forall x \in \mathbb{R}, \quad \beta'_x \;=^*\; \alpha'_x$.
Therefore: $\forall x \in \mathbb{R}, \quad (f+g)''_x \;=^*\; (f'+g')'_x$.
By Theorem 4.13.18 (with f replaced by f' and g by g'), we have:
$\forall x \in \mathbb{R}, \quad (f'+g')'_x \;=^*\; f''_x + g''_x$.
Then: $\forall x \in \mathbb{R}, \quad (f+g)''_x \;=^*\; (f'+g')'_x \;=^*\; f''_x + g''_x$.
Then: $\forall x \in \mathbb{R}, \quad (f+g)''_x \;=^*\; f''_x + g''_x$, as desired. □

THEOREM 4.13.21. *Let* $m, c \in \mathbb{R}$ *and let* $\psi : \mathbb{R} \to \mathbb{R}$.
Assume: $\forall x \in \mathbb{R}, \psi_x = mx + cx^2$.
Then: $\forall x \in \mathbb{R}, [\; (\; \psi'_x = m + 2cx \;) \;\&\; (\; \psi''_x = 2c \;) \;]$.

Proof. Define $L, Q : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}, (\; L_x = mx \;) \;\&\; (\; Q_x = cx^2 \;)$.
We have: $\forall x \in \mathbb{R}, \quad \psi_x = mx + cx^2 = L_x + Q_x = (L+Q)_x$.

Then $\forall x \in \mathbb{R}$, $\psi_x = (L+Q)_x$. Then $\psi = L + Q$.
By HW#2-2, we know: $\forall x \in \mathbb{R}$, $(L'_x = m) \,\&\, (L''_x = 0)$.
By HW#2-3, we know: $\forall x \in \mathbb{R}$, $(Q'_x = 2cx) \,\&\, (Q''_x = 2c)$.
Given $x \in \mathbb{R}$. Want: $(\psi'_x = m + 2cx)$ & $(\psi''_x = 2c)$.
We have: $\psi'_x = (L+Q)'_x =^* L'_x + Q'_x = m + 2cx \neq ☹$,
and so $\psi'_x = m + 2cx$. Want: $\psi''_x = 2c$.
We have: $\psi''_x = (L+Q)''_x =^* L''_x + Q''_x = 2c \neq ☹$,
and so: $\psi''_x = 2c$. □

THEOREM 4.13.22. *Let $\ell \in \mathbb{N}$, $f \in \mathcal{H}_\ell$. Then $f' \in \mathcal{H}_{\ell-1}$.*

Proof. Since $f \ni \mathcal{H}_\ell$, choose $c \in \mathbb{R}$ s.t. $f = c \cdot (\bullet)^\ell$.
Since $c \cdot \ell \cdot (\bullet)^{\ell-1} \in \mathcal{H}_{\ell-1}$, it suffices to show: $f' = c \cdot \ell \cdot (\bullet)^{\ell-1}$.
Want: $\forall x \in \mathbb{R}$, $f'_x = (c \cdot \ell \cdot (\bullet)^{\ell-1})_x$.
Given $x \in \mathbb{R}$. Want: $f'_x = (c \cdot \ell \cdot (\bullet)^{\ell-1})_x$.
By HW#2-1, we have: $((\bullet)^\ell)' = \ell \cdot (\bullet)^{\ell-1}$.
Then $f'_x = (c \cdot (\bullet)^\ell)'_x =^* c \cdot ((\bullet)^\ell)'_x = c \cdot (\ell \cdot (\bullet)^{\ell-1})_x = c \cdot \ell \cdot x^{\ell-1} \neq ☹$.
Then $f'_x = c \cdot \ell \cdot x^{\ell-1} = (c \cdot \ell \cdot (\bullet)^{\ell-1})_x$, as desired. □

THEOREM 4.13.23. *Let $\phi : \mathbb{R} \dashrightarrow \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$.*
Assume: $\mathbb{D}_{\psi'} = \mathbb{R}$. Then: $(\phi - \psi)' = \phi' - \psi'$.

Proof. Let $\rho := \phi - \psi$. Want: $\rho' = \phi' - \psi'$.
Want: $\forall x \in \mathbb{R}$, $\rho'_x = (\phi' - \psi')_x$.
Given $x \in \mathbb{R}$. Want: $\rho'_x = (\phi' - \psi')_x$. Want: $\rho'_x = \phi'_x - \psi'_x$.
Since $\rho'_x = (\phi - \psi)'_x =^* \phi'_x - \psi'_x$, it suffices to show: $\rho'_x \,{}^*\!= \phi'_x - \psi'_x$.
Want: $\phi'_x - \psi'_x =^* \rho'_x$. Since $\phi : \mathbb{R} \to \mathbb{R}$, we get: $\psi - \psi = \mathbf{0}$.
Then $\rho + \psi = \phi + \psi - \psi = \phi + \mathbf{0} = \phi$, so $\phi = \rho + \psi$.
We have $x \in \mathbb{R} = \mathbb{D}_{\psi'}$, so $\psi'_x \in \mathbb{I}_{\psi'}$.
Since $\psi'_x \in \mathbb{I}_{\psi'} \subseteq \mathbb{R}$, we get: $\psi'_x - \psi'_x = 0$.
We have $\phi'_x = (\rho + \psi)'_x =^* \rho'_x + \psi'_x$, so $\phi'_x =^* \rho'_x + \psi'_x$.
Then $\phi'_x - \psi'_x =^* \rho'_x + \psi'_x - \psi'_x = \rho'_x + 0 = \rho'_x$, as desired. □

THEOREM 4.13.24. *Let $\phi : \mathbb{R} \dashrightarrow \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$.*
Assume: $\mathbb{D}_{\psi''} = \mathbb{R}$. Then: $(\phi - \psi)'' = \phi'' - \psi''$.

Proof. Since $\mathbb{R} = \mathbb{D}_{\psi''} \subseteq \mathbb{D}_{\psi'}$ and since $\mathbb{D}_{\psi'} \subseteq \mathbb{R}$, we get: $\mathbb{D}_{\psi'} = \mathbb{R}$.
Then, by Theorem 4.13.23, we get: $(\phi - \psi)' = \phi' - \psi'$.
Let $\alpha := \phi'$ and $\beta := \psi'$. Then: $(\phi - \psi)' = \alpha - \beta$.
Then $\alpha' = \phi''$ and $\beta' = \psi''$ and $(\phi - \psi)'' = (\alpha - \beta)'$.
We have $\mathbb{D}_\beta = \mathbb{D}_{\psi'} = \mathbb{R}$ and $\mathbb{I}_\beta = \mathbb{I}_{\psi'} \subseteq \mathbb{R}$, so $\beta : \mathbb{R} \to \mathbb{R}$.

So, since $\mathbb{D}_{\beta'} = \mathbb{D}_{\psi''} = \mathbb{R}$, it follows,
from Theorem 4.13.23, that: $(\alpha - \beta)' = \alpha' - \beta'$.
Then $(\phi - \psi)'' = (\alpha - \beta)' = \alpha' - \beta' = \phi'' - \psi''$. □

THEOREM 4.13.25. *Let $m, c \in \mathbb{R}$ and let $\rho, g : \mathbb{R} \dashrightarrow \mathbb{R}$.*
Assume: $\forall x \in \mathbb{R}$, $\rho_x = g_x - mx - cx^2$.
Then: $\forall x \in \mathbb{R}$, $[(\rho'_x = g'_x - m - 2cx) \& (\rho''_x = g''_x - 2c)]$.

Proof. Define $\psi : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $\psi_x = mx + cx^2$.
By Theorem 4.13.21, $\forall x \in \mathbb{R}$, $\psi''_x = 2c$.
Since $\forall x \in \mathbb{R}$, $\psi''_x = 2c \neq ☹$, we get $\forall x \in \mathbb{R}$, $x \in \mathbb{D}_{\psi''}$, and so $\mathbb{R} \subseteq \mathbb{D}_{\psi''}$.
Since $\mathbb{R} \subseteq \mathbb{D}_{\psi''} \subseteq \mathbb{D}_{\psi'} \subseteq \mathbb{R}$, we get $\mathbb{D}_{\psi'} = \mathbb{R}$ and $\mathbb{D}_{\psi''} = \mathbb{R}$.
Then, by Theorem 4.13.24 and Theorem 4.13.23,
we get: $(g - \psi)' = g' - \psi'$ and $(g - \psi)'' = g'' - \psi''$.
We have: $\forall x \in \mathbb{R}$, $\rho_x = g_x - (mx + cx^2) = g_x - \psi_x = (g - \psi)_x$.
Then: $\forall x \in \mathbb{R}$, $\rho_x = (g - \psi)_x$. Then: $\rho = g - \psi$.
Then: $\rho' = (g - \psi)' = g' - \psi'$ and $\rho'' = (g - \psi)'' = g'' - \psi''$.
Given $x \in \mathbb{R}$. Want: $[(\rho'_x = g'_x - m - 2cx) \& (\rho''_x = g''_x - 2c)]$.
By Theorem 4.13.21, we have: $(\psi'_x = m + 2cx) \& (\psi''_x = 2c)$.
Then $\rho'_x = (g' - \psi')_x = g'_x - \psi'_x = g'_x - m - 2cx$. Want: $\psi''_x = 2c$.
We have $\rho''_x = (g'' - \psi'')_x = g''_x - \psi''_x = g''_x - 2c$, as desired. □

THEOREM 4.13.26. *Let $\lambda, \mu : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $a \in \mathbb{R}$.*
Assume: $\forall h \in \mathbb{R}$, $\mu_h = \lambda_{h+a}$. *Then:* $\forall h \in \mathbb{R}$, $\mu'_h = \lambda'_{h+a}$.

Proof. Define $S, T : \mathbb{R} \to \mathbb{R}$ by: $\forall h \in \mathbb{R}$, $(S_h = h - a) \& (T_h = h + a)$.
By HW#2-5, $S' = C^1_{\mathbb{R}} = T'$. Also, $S \circ T = \mathrm{id}_{\mathbb{R}} = T \circ S$.
We have: $\forall h \in \mathbb{R}$, $\mu_h = \lambda_{h+a} = \lambda_{T_h} = (\lambda \circ T)_h$. Then $\mu = \lambda \circ T$.
Then $\mu \circ S = \lambda \circ T \circ S = \lambda \circ \mathrm{id}_{\mathbb{R}} = \lambda$, so $\mu \circ S = \lambda$.
Given $h \in \mathbb{R}$. Want: $\mu'_h = \lambda'_{h+a}$. Let $k := h + a$. Want: $\mu'_h = \lambda'_k$.
We have: $T_h = h + a = k$ and $S_k = k - a = h$.
Then $\mu'_h = (\lambda \circ T)'_h =^* \lambda'_{T_h} \cdot T'_h = \lambda'_k \cdot 1 = \lambda'_k$,
so $\mu'_h =^* \lambda'_k$.
It suffices to show: $\mu'_h \; ^*\!= \lambda'_k$. Want: $\lambda'_k =^* \mu'_h$.
We have $\lambda'_k = (\mu \circ S)'_k =^* \mu'_{S_k} \cdot S'_k = \mu'_h \cdot 1 = \mu'_h$, as desired. □

THEOREM 4.13.27. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and $a \in \mathbb{R}$ and $g := f^{\mathbb{T}}_a$.*
Then: $\forall h \in \mathbb{R}$, $g''_h = f''_{h+a}$.

Proof. By Theorem 4.13.11, we have: $\forall h \in \mathbb{R}$, $g'_h = f'_{h+a}$.
Let $\mu := g'$ and $\lambda := f'$. Then: $\forall h \in \mathbb{R}$, $\mu_h = \lambda_{h+a}$.

Then, by Theorem 4.13.26, $\quad\forall h\in\mathbb{R},\quad \mu'_h=\lambda'_{h+a}$.
Then: $\quad\forall h\in\mathbb{R},\quad g''_h=\mu'_h=\lambda'_{h+a}=f''_{h+a},\quad$ as desired. □

The next result is called the **Second Order Taylor Theorem**:

THEOREM 4.13.28. *Let $f:\mathbb{R} \dashrightarrow \mathbb{R}$ and $a\in\mathbb{D}_{f''}$.*
Let $m:=f'_a$ and $c:=f''_a/2$ and $L:=m\cdot(\bullet)$ and $Q:=c\cdot(\bullet)^2$.
Then: $\quad f_a^{\mathbb{T}}-L-Q\ \in\ \mathcal{O}_2$.

Proof. Since $a\in\mathbb{D}_f$, it follows that $(f_a^{\mathbb{T}})_0=0$.
Let $g:=f_a^{\mathbb{T}}$. Then $g_0=0$. Want: $g-L-Q\in\mathcal{O}_2$.
Let $\rho:=g-L-Q$. Want: $\rho\in\mathcal{O}_2$.
By Theorem 4.13.8, it suffices to show: $\rho_0=\rho'_0=\rho''_0=0$.
By Theorem 4.13.11, we have: $\quad\forall h\in\mathbb{R},\quad g'_h=f'_{h+a}$.
Then $g'_0=f'_{0+a}=f'_a=m$.
By Theorem 4.13.27, we have: $\quad\forall h\in\mathbb{R},\quad g''_h=f''_{h+a}$.
Since $c=f''_a/2$, we get: $2c=f''_a$.
Then $g''_0=f''_{0+a}=f''_a=2c$.
By Theorem 4.13.25 (with x replaced by h), we conclude:
$$\forall h\in\mathbb{R},\ [\ (\ \rho'_h=g'_h-m-2ch\ )\ \&\ (\ \rho''_h=g''_h-2c\ )\ ].$$
We have: $\quad g_0=0\quad$ and $\quad g'_0=m\quad$ and $\quad g''_0=2c$.
Then $\quad\rho_0=g_0-m\cdot 0-c\cdot 0^2=0-0-0=0$
and $\quad\rho'_0=g'_0-m-2c\cdot 0=m-m-0=0$
and $\quad\rho''_0=g''_0-2c=2c-2c=0$.
We therefore have: $\quad g_0=g'_0=g''_0=0,\quad$ as desired. □

The next result is called the **Second Derivative Test for Maxima**:

THEOREM 4.13.29. *Let $f:\mathbb{R} \dashrightarrow \mathbb{R}$ and let $a\in\mathbb{R}$.*
Assume: $\quad(\ f'_a=0\ )\ \&\ (\ f''_a<0\ )$.
Then: $\quad f$ has a local strict-maximum at a in $\mathbb{R}$.

Proof. Want: $f_a^{\mathbb{T}}$ has a local strict-maximum at 0 in $\mathbb{R}$.
We have $f'_a=0\neq ☹$, so $a\in\mathbb{D}_{f'}$. Then $a\in\mathbb{D}_{f'}\subseteq\mathbb{D}_f$, so $(f_a^{\mathbb{T}})_0=0$.
Let $g:=f_a^{\mathbb{T}}$. Want: g has a local strict-maximum at 0 in $\mathbb{R}$.
Want: $\exists B\in\mathcal{B}_{\mathbb{R}}(0)$ s.t. $g<g_0$ on $B_0^\times$.
We have $g_0=(f_a^{\mathbb{T}})_0=0$. Want: $\exists B\in\mathcal{B}_{\mathbb{R}}(0)$ s.t. $g<0$ on $B_0^\times$.
Let $m:=f'_a$ and $c:=f''_a/2$ and $L:=m\cdot(\bullet)$ and $Q:=c\cdot(\bullet)^2$.
By Theorem 4.13.28, we have: $f_a^{\mathbb{T}}-L-Q\ \in\ \mathcal{O}_2$.
Since $m=f'_a=0$, we get $m\cdot(\bullet)=\mathbf{0}$. Then: $L=m\cdot(\bullet)=\mathbf{0}$.
Then: $\quad g-Q\ =\ g-\mathbf{0}-Q\ =\ f_a^{\mathbb{T}}-L-Q\ \in\ \mathcal{O}_2$.

So, by Theorem 4.12.19, choose $B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t., $\forall h \in B$, $\mathrm{sgn}_{g_h} = \mathrm{sgn}_{Q_h}$.
Then $B \in \mathcal{B}_{\mathbb{R}}(0)$. Want: $g < 0$ on $B_0^\times$.
Want: $\forall h \in B_0^\times$, $g_h < 0$. Given $h \in B_0^\times$. Want: $g_h < 0$.
Since $f_a'' < 0$ and $c = f_a''/2$, we get: $c < 0$.
Since $h \in \mathbb{B}_0^\times \subseteq \mathbb{R}_0^\times$, we get: $h^2 > 0$.
So, since $c < 0$, we get $ch^2 < 0$. Then $Q_h = (c \cdot (\bullet)^2)_h = ch^2 < 0$.
Since $Q_h < 0$, we get: $\mathrm{sgn}_{Q_h} = -1$. Then $\mathrm{sgn}_{g_h} = \mathrm{sgn}_{Q_h} = -1$.
Since $\mathrm{sgn}_{g_h} = -1$, we conclude that $g_h < 0$, as desired. □

4.14. Some basic limit theorems.

THEOREM 4.14.1. *Let $u > 1$, $a \in \mathbb{R}$. Then $\exists j \in \mathbb{N}$ s.t. $u^j > a$.*

Proof. Assume: $\forall j \in \mathbb{N}$, $u^j \leqslant a$. Want: Contradiction.
Define $s \in \mathbb{R}^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $s_j = u^j$.
Then: ($\forall j \in \mathbb{N}$, $s_{j+1} = u \cdot s_j$) & ($s_j > 1$).
Since $u > 1$, we know: $\forall j \in \mathbb{N}$, $s_j \cdot u > s_j \cdot 1$.
Then: $\forall j \in \mathbb{N}$, $s_{j+1} = u \cdot s_j = s_j \cdot u > s_j \cdot 1 = s_j$.
Then: $\forall j \in \mathbb{N}$, $s_{j+1} > s_j$, so s is strictly-increasing.
We have: $\forall j \in \mathbb{N}$, $0 < 1 < s_j$ and $s_j = u^j \leqslant a$, so $0 < s_j \leqslant a$.
Then $\mathbb{I}_s \subseteq (0; a]$, so $\mathbb{I}_s$ is bounded in $\mathbb{R}$.
Since s is strictly-increasing and since $\mathbb{I}_s$ is bounded in $\mathbb{R}$,
it follows (from Theorem 3.11.4) that s is convergent in $\mathbb{R}$.
Choose $q \in \mathbb{R}$ s.t. $s \to q$ in $\mathbb{R}$. Define $\ell \in \mathbb{N}^{\mathbb{N}}$ by $\ell_j = j + 1$.
We have: $\forall j \in \mathbb{N}$, $\ell_{j+1} = j + 2 > j + 1 = \ell_j$,
so ℓ is strictly-increasing.
Then $s \circ \ell$ is a subsequence of s.
So, since $s \to q$ in $\mathbb{R}$, we get: $s \circ \ell \to q$ in $\mathbb{R}$.
We have: $\forall j \in \mathbb{N}$, $(u \cdot s)_j = u \cdot s_j = s_{j+1} = s_{\ell_j} = (s \circ \ell)_j$,
so $u \cdot s = u \circ \ell$.
Since $u \cdot s = u \circ \ell$ and since $u \circ \ell \to q$ in $\mathbb{R}$
we get: $u \cdot s \to q$ in $\mathbb{R}$.
On the other hand, since $s \to q$ in $\mathbb{R}$, we conclude $u \cdot s \to u \cdot q$ in $\mathbb{R}$.
Since both $u \cdot s \to q$ in $\mathbb{R}$ and $u \cdot s \to u \cdot q$ in $\mathbb{R}$,
we get: $q = u \cdot q$. Then $(u - 1) \cdot q = 0$.
Since $u > 1$, we get $u - 1 > 0$, so $u - 1 \neq 0$.
So, since $(u - 1) \cdot q = 0$, it follows that $q = 0$.
Since $\forall j \in \mathbb{N}$, $s_j > 1$ and since $s \to q$ in $\mathbb{R}$,
we conclude: $q \geqslant 1$.
Then $0 = q \geqslant 1 > 0$, so $0 > 0$. Contradiction. □

THEOREM 4.14.2. *Let* $c \in (0;1)$.

Define $z \in \mathbb{R}^{\mathbb{N}_0}$ *by:* $\forall j \in \mathbb{N}_0, \quad z_j = c^j$.

Then $z \to 0$ *in* $\mathbb{R}$.

Proof. Want: $\forall \varepsilon > 0, \exists K \in \mathbb{N}_0$ s.t., $\forall j \in [K..\infty), \quad |z_j| < \varepsilon$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}_0$ s.t., $\forall j \in [K..\infty), \quad |z_j| < \varepsilon$.
Let $u := 1/c$. By Theorem 4.14.1, choose $K \in \mathbb{N}$ s.t. $u^K > 1/\varepsilon$.
Then $K \in \mathbb{N} \subseteq \mathbb{N}_0$. Want: $\forall j \in [K..\infty), \quad |z_j| < \varepsilon$.
Given $j \in [K..\infty)$. Want: $|z_j| < \varepsilon$.
Since $u^K > 1/\varepsilon > 0$, we conclude: $1/(u^K) < 1/(1/\varepsilon)$.
Then $c^K = (1/u)^K = 1/(u^K) < 1/(1/\varepsilon) = \varepsilon$, so $c^K < \varepsilon$.
Given $j \in [K..\infty)$. Want: $|z_j| < \varepsilon$.
Since $c > 0$, we get $c^K > 0$. Since $0 < c < 1$, we get $c^{j-K} < 1$.
Since $c^K > 0$ and since $c^{j-K} < 1$,
we conclude: $c^K \cdot c^{j-K} < c^K \cdot 1$.
Since $c > 0$, it follows that $c^j > 0$. Then $z_j = c^j > 0$, so $|z_j| = z_j$.
Then: $|z_j| = z_j = c^j = c^K \cdot c^{j-K} < c^K \cdot 1 = c^K < \varepsilon$. □

THEOREM 4.14.3. *Let* $b \in (0;\infty)^{\mathbb{N}_0}$ *and* $c \in (0;1)$ *and* $k \in \mathbb{N}_0$.
Assume: $\forall j \in [k..\infty), \quad b_{j+1}/b_j \leqslant c$. *Then:* $b \to 0$ *in* $\mathbb{R}$.

Proof. Want: $\forall \varepsilon > 0, \exists K \in \mathbb{N}_0$ s.t., $\forall j \in [K..\infty), |b_j| < \varepsilon$.
Given $\varepsilon > 0$. Want: $\exists L \in \mathbb{N}_0$ s.t., $\forall j \in [L..\infty), |b_j| < \varepsilon$.
Define $r \in (0;\infty)^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}, \quad r_j = b_{k+j}/b_{k+j-1}$.
Then, $\forall j \in \mathbb{N}$, we have: $r_j \leqslant c$.
Then, $\forall i \in \mathbb{N}$, we have: $r_i \cdot r_{i-1} \cdot r_{i-2} \cdots r_1 \leqslant c^i$.
Also, $\forall i \in \mathbb{N}$, we have: $r_i \cdot r_{i-1} \cdot r_{i-2} \cdots r_1$

$$= \frac{b_{k+i}}{b_{k+i-1}} \cdot \frac{b_{k+i-1}}{b_{k+i-2}} \cdot \frac{b_{k+i-2}}{b_{k+i-3}} \cdots \frac{b_{k+1}}{b_k} = \frac{b_{k+i}}{b_k}.$$

We conclude: $\forall i \in \mathbb{N}, \quad b_{k+i}/b_k \leqslant c^i$.
Let $a := b_k$. Then: $\forall i \in \mathbb{N}, \quad b_{k+i}/a \leqslant c^i$.
Also, since $b \in (0;\infty)^{\mathbb{N}_0}$, we get $a \in (0;\infty)$. Then $a > 0$.
Define $z \in \mathbb{R}^{\mathbb{N}_0}$ by: $\forall j \in \mathbb{N}_0, \quad z_j = c^j$.
By Theorem 4.14.2, we have: $z \to 0$ in $\mathbb{R}$.
Choose $R \in \mathbb{N}_0$ s.t., $\forall i \in [R..\infty), \quad |z_i| < \varepsilon/a$.
Let $M := R + 1$. Then $M \in \mathbb{N}$. Also, $[M..\infty) \subseteq [R..\infty)$.
Let $L := M + k$. Then $L \in \mathbb{N}_0$. Want: $\forall j \in [L..\infty), |b_j| < \varepsilon$.
Given $j \in [L..\infty)$. Want: $|b_j| < \varepsilon$.
Since $j \in [L..\infty)$ and since $L - k = M$, we get $j - k \in [M..\infty)$.
Recall: $\forall i \in \mathbb{N}, b_{k+i}/a \leqslant c^i$. Let $i := j - k$. Then: $i \in [M..\infty)$.

Also, $k+i=j$ Since $i \in [M..\infty)$ and since $M \in \mathbb{N}$, we get $i \in \mathbb{N}$.
Then: $b_j = b_{k+i}/a \leqslant c^i$. So, since $a>0$, we get: $b_j \leqslant a \cdot c^i$.
Since $i \in [M..\infty) \subseteq [R..\infty)$, by choice of R, we have $|z_i| < \varepsilon/a$.
Since $c \in (0;1) > 0$, we get $c^i > 0$. Then $|c^i| = c^i$.
By definition of z, we have $z_i = c^i$. Then $c^i = |c^i| = |z_i| < \varepsilon/a$.
Since $c^i < \varepsilon/a$ and since $a > 0$, we get: $a \cdot c^i < \varepsilon$.
Then $b_j \leqslant a \cdot c^i < \varepsilon$, as desired. □

THEOREM 4.14.4. *Let $\alpha, \beta \in \mathbb{R}$. Then:*
$$(\alpha \leqslant \beta) \Leftrightarrow (\forall \eta > 0, \alpha \leqslant \beta + \eta).$$

Proof. Proof of $\Rightarrow$:
Assume: $\alpha \leqslant \beta$. Want: $\forall \eta > 0$, $\alpha \leqslant \beta + \eta$.
Given $\eta > 0$. Want: $\alpha \leqslant \beta + \eta$.
Since $\eta > 0$, it follows that $\eta \geqslant 0$. Then $\beta + \eta \geqslant \beta + 0$.
Then $\alpha \leqslant \beta = \beta + 0 \leqslant \beta + \eta$, as desired.
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$:
Assume: $\forall \eta > 0$, $\alpha \leqslant \beta + \eta$. Want: $\alpha \leqslant \beta$.
Assume $\alpha > \beta$. Want: Contradiction.
We have $\alpha - \beta > 0$. Let $\eta := (\alpha - \beta)/2$. Then $\eta > 0$.
Then, by our assumption, $\alpha \leqslant \beta + \eta$. Then $\alpha - \beta \leqslant \eta$.
Since $\eta = (\alpha - \beta)/2$, we get $2\eta = \alpha - \beta$.
Then $2\eta = \alpha - \beta \leqslant \eta$, so $2\eta \leqslant \eta$, so $2\eta - \eta \leqslant \eta - \eta$.
Then $\eta = 2\eta - \eta \leqslant \eta - \eta = 0$, so $\eta \leqslant 0$.
Then $0 \geqslant \eta > 0$, so $0 > 0$. Contradiction.
End of proof of $\Leftarrow$. □

THEOREM 4.14.5. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x, m \in \mathbb{R}$.*
Then:
$$(f'_x = m) \Leftrightarrow (\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}, \forall h \in (-\delta;\delta), |f_{x+h} - f_x - m \cdot h| \leqslant \varepsilon \cdot |h|).$$

Proof. Define $L \in \mathcal{L}$ by: $\forall h \in \mathbb{R}$, $L_h = m \cdot h$.
We have: $\forall h \in \mathbb{R}$, $(f_x^{\mathbb{T}})_h = f_{x+h} - f_x$. Let $\rho := f_x^{\mathbb{T}} - L$.
Then: $\forall h \in \mathbb{R}$, $\rho_h = (f_x^T - L)_h = (f_x^{\mathbb{T}})_h - L_h = f_{x+h} - f_x - m \cdot h$.
Then: $\forall h \in \mathbb{R}$, $\rho_h = f_{x+h} - f_x - m \cdot h$.
Want: $(f'_x = m) \Leftrightarrow (\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}, \forall h \in (-\delta;\delta), |\rho_h| \leqslant \varepsilon \cdot |h|)$.
Know: $(\rho \in \mathcal{O}_1) \Leftrightarrow (\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}, \forall h \in (-\delta;\delta), |\rho_h| \leqslant \varepsilon \cdot |h|)$.
Want: $(f'_x = m) \Leftrightarrow (\rho \in \mathcal{O}_1)$.

By definition of L, we conclude: $(\ f'_x = m\ ) \Leftrightarrow (\ D_x f = L\ )$.
By Theorem 4.4.3, we conclude: $(\ D_x f = L\ ) \Leftrightarrow (\ f_x^{\mathbb{T}} - L \in \mathcal{O}_1\ )$.
Then: $(\ f'_x = m\ ) \Leftrightarrow (\ D_x f = L\ )$
$\Leftrightarrow (\ f_x^{\mathbb{T}} - L \in \mathcal{O}_1\ ) \Leftrightarrow (\ \rho \in \mathcal{O}_1\ )$. □

THEOREM 4.14.6. *Let $s \in \mathbb{R}^{\mathbb{N}_0}$. Assume s is convergent in $\mathbb{R}$. Then $\{s_{j+1} - s_j \,|\, j \in \mathbb{N}_0\}$ is bounded in $\mathbb{R}$.*

Proof. Define $t \in \mathbb{R}^{\mathbb{N}_0}$ by: $\forall j \in \mathbb{N}_0$, $t_j = s_{j+1}$.
Let $u := t - s$. Then: $\forall j \in \mathbb{N}_0$, $u_j = (t-s)_j = t_j - s_j = s_{j+1} - s_j$, so $\mathbb{I}_u = \{s_{j+1} - s_j \,|\, j \in \mathbb{N}_0\}$. Want: $\mathbb{I}_u$ is bounded in $\mathbb{R}$.
Since s is convergent in $\mathbb{R}$, choose $q \in \mathbb{R}$ s.t. $s \to q$ in $\mathbb{R}$.
By HW#3-1, $t \to q$ in $\mathbb{R}$. Then $t - s \to q - q$ in $\mathbb{R}$.
So, since $t - s = u$ and $q - q = 0$, we get: $u \to 0$ in $\mathbb{R}$.
Then u is convergent in $\mathbb{R}$, so $\mathbb{I}_u$ is bounded in $\mathbb{R}$, as desired. □

4.15. **Differentiation commutes with uniform limit.**

We illustrated in class that: $\exists \phi, \psi \in (\mathbb{R}^{\mathbb{R}})^{\mathbb{N}}$, $\exists f, g \in \mathbb{R}^{\mathbb{R}}$
such that $\phi \to f$ pointwise from $\mathbb{R}$ to $\mathbb{R}$
and $\psi \to g$ pointwise from $\mathbb{R}$ to $\mathbb{R}$
and $\forall j \in \mathbb{N}$, $(\phi_j)' = \psi_j$
and $f' \neq g$.
That is, differentiation does not commute with pointwise limit.
In this section, we will show:
differentiation *DOES* commute with *UNIFORM* limit.

DEFINITION 4.15.1. *Let X be a set and let $S \subseteq X$. Let Y be a metric space. Let $f \in (\mathrm{PF}_Y^X)^{\mathbb{N}}$ and let $g \in \mathrm{PF}_Y^X$. By $f \to g$ pointwise on S from X to Y, we mean:*
$\forall p \in S$, $f_\bullet(p) \to g(p)$ *in* Y.
By $f \to g$ uniformly on S from X to Y, we mean:
$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ *s.t.*, $\forall j \in \mathbb{N}$, $\forall p \in S$,
$(\ j \geqslant K\ ) \Rightarrow (\ d(\ f_j(p)\ ,\ g(p)\ ) < \varepsilon\ )$.

Let X be a set and let $S \subseteq X$. Let Y be a metric space.
Let $f \in (\mathrm{PF}_Y^X)^{\mathbb{N}}$ and let $g \in \mathrm{PF}_Y^X$.
Note that, if $f \to g$ uniformly on S from X to Y,
then $f \to g$ pointwise on S from X to Y.
Also, note that, if $f \to g$ pointwise on S from X to Y,
then $S \subseteq \mathbb{D}_g \bigcap \mathbb{D}_{f_1} \bigcap \mathbb{D}_{f_2} \bigcap \mathbb{D}_{f_3} \bigcap \cdots$.

THEOREM 4.15.2. *Let $\phi, \psi \in (\mathrm{PF}_{\mathbb{R}}^{\mathbb{R}})^{\mathbb{N}}$ and let $f, g \in \mathrm{PF}_{\mathbb{R}}^{\mathbb{R}}$.*
Let U be an open subset of $\mathbb{R}$.

Assume: $\phi \to f$ *pointwise on U from $\mathbb{R}$ to $\mathbb{R}$*
and $\psi \to g$ *uniformly on U from $\mathbb{R}$ to $\mathbb{R}$*
and $\forall j \in \mathbb{N}, \quad (\phi_j)' = \psi_j$.
Then: $f' = g$.

Proof. Want: $\forall x \in U$, $f'_x = g_x$. Given $x \in U$. Want: $f'_x = g_x$.
By Theorem 4.14.5, we wish to show:

$$\forall \varepsilon > 0, \quad \exists \delta > 0 \text{ s.t.}, \forall h \in (-\delta; \delta), |f_{x+h} - f_x - g_x \cdot h| \leqslant \varepsilon \cdot |h|.$$

Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall h \in (-\delta; \delta)$, $|f_{x+h} - f_x - m \cdot h| \leqslant \varepsilon \cdot |h|$.
Since U is an open subset of $\mathbb{R}$, we know that $U = \mathrm{Int}_{\mathbb{R}} U$.
Since $x \in U = \mathrm{Int}_{\mathbb{R}} U$, choose $S \in \mathcal{B}_{\mathbb{R}}(U)$ s.t. $S \subseteq U$.
Since $S \in \mathcal{B}_{\mathbb{R}}(U)$, choose $\alpha > 0$ s.t. $S = B_{\mathbb{R}}(x, \alpha)$.
Then $(x - \alpha; x + \alpha) = B_{\mathbb{R}}(x, \alpha) = S \subseteq U$.
Since $\psi \to g$ uniformly on U from $\mathbb{R}$ to $\mathbb{R}$,
choose $k \in \mathbb{N}$ s.t., $\forall j \in [k..\infty)$, $|g - \psi_j| < \varepsilon/8$ on U.
Since $(\phi_k)' = \psi_k$ on U and since $x \in U$,
we conclude: $(\phi_k)'_x = (\psi_k)_x$.
Let $\Phi := \phi_k$ and $\Psi := \psi_k$. Then: $\Phi'_x = \Psi$.
Since $\Phi'_x = \Psi$, by Theorem 4.14.5, choose $\beta > 0$ s.t.,

$$\forall h \in (-\beta; \beta), \qquad |\Phi_{x+h} - \Phi_x - \Psi_x \cdot h| \leqslant (\varepsilon/2) \cdot |h|.$$

Let $\delta := \min\{\alpha, \beta\}$. Then $\delta > 0$.
Want: $\forall h \in (-\delta; \delta)$, $|f_{x+h} - f_x - m \cdot h| \leqslant \varepsilon \cdot |h|$.
Given $h \in (-\delta; \delta)$. Want: $|f_{x+h} - f_x - m \cdot h| \leqslant \varepsilon \cdot |h|$.
Since $0 < \delta \leqslant \alpha$ and $0 < \delta \leqslant \beta$, we get:
both $(-\delta; \delta) \subseteq (-\alpha; \alpha)$ and $(-\delta; \delta) \subseteq (-\beta; \beta)$.
Then $h \in (-\delta; \delta) \subseteq (-\alpha; \alpha)$ and $h \in (-\delta; \delta) \subseteq (-\beta; \beta)$.
Since $h \in (-\beta; \beta)$, by choice of β, we get: $|\Phi_{x+h} - \Phi_x - \Psi_x \cdot h| \leqslant (\varepsilon/2) \cdot |h|$.
Let $A := \Phi_{x+h} - \Phi_x - \Psi_x \cdot h$ and let $B := f_{x+h} - f_x - g_x \cdot h$.
Then $|A| \leqslant (\varepsilon/2) \cdot |h|$. Want: $|B| \leqslant \varepsilon \cdot |h|$.
Since $|B| \leqslant |B - A| + |A| \leqslant |B - A| + (\varepsilon/2) \cdot |h|$,
it suffices to show: $|B - A| \leqslant (\varepsilon/2) \cdot |h|$.
By Theorem 4.14.4, it suffices to prove: $\forall \eta > 0$,

$$|B - A| \leqslant (\varepsilon/2) \cdot |h| + \eta.$$

Given $\eta > 0$. Want: $|B - A| \leqslant (\varepsilon/2) \cdot |h| + \eta$.
Since $h \in (-\alpha; \alpha)$, we get: $x + h \in (x - \alpha; x + \alpha)$.

Then $x + h \in (x - \alpha; x + \alpha) \subseteq U$, so $x + h \in U$.
Since $x, x + h \in U$ and $\phi \to f$ pointwise on U from $\mathbb{R}$ to $\mathbb{R}$, we get:
both $\phi_\bullet(x) \to f(x)$ in $\mathbb{R}$ and $\phi_\bullet(x + h) \to f(x + h)$ in $\mathbb{R}$.
Since $\psi \to g$ uniformly on U from $\mathbb{R}$ to $\mathbb{R}$,
and since uniform convergence implies pointwise convergence,
we conclude: $\psi \to g$ pointwise on U from $\mathbb{R}$ to $\mathbb{R}$.
So, since $x \in U$, we get: $\psi_\bullet(x) \to g(x)$ in $\mathbb{R}$.
We have proved: $\phi_\bullet(x + h) \to f(x + h)$ in $\mathbb{R}$
and $\phi_\bullet(x) \to f(x)$ in $\mathbb{R}$
and $\psi_\bullet(x) \to g(x)$ in $\mathbb{R}$.
So, since $B = f_{x+h} - f_x - g_x \cdot h = [f(x + h)] - [f(x)] - [g(x)] \cdot h$,
we get: $[\phi_\bullet(x + h)] - [\phi_\bullet(x)] - [\psi_\bullet(x)] \cdot h \to B$ in $\mathbb{R}$.
Choose $\ell \in \mathbb{N}$ such that, $\forall j \in [\ell..\infty)$,
$$| \; ([\phi_\bullet(x + h)] - [\phi_\bullet(x)] - [\psi_\bullet(x)] \cdot h)_j \; - \; B \; | \; < \; \eta.$$
Let $m := \max\{k, \ell\}$. Then $m \in [k..\infty)$ and $m \in [\ell..\infty)$.
Since $m \in [\ell..\infty)$, by choice of ℓ, we conclude:
$$| \; ([\phi_\bullet(x + h)] - [\phi_\bullet(x)] - [\psi_\bullet(x)] \cdot h)_m \; - \; B \; | \; < \; \eta.$$
Let $C := ([\phi_\bullet(x + h)] - [\phi_\bullet(x)] - [\psi_\bullet(x)] \cdot h)_m$. Then $|C - B| < \eta$.
Then $|B - A| \leqslant |B - C| + |C - A| = |C - A| + |C - B| \leqslant |C - A| + \eta$.
Then: $|B - A| \leqslant |C - A| + \eta$.
It therefore suffices to prove: $|C - A| \leqslant (\varepsilon/2) \cdot |h|$.
Let $\sigma := \phi_m - \phi_k$ and let $\tau := \psi_m - \psi_k$.
We know: $\forall j \in \mathbb{N}$, $(\phi_j)' = \psi_j$ on U.
We conclude: $\sigma' = \tau$ on U. Then: $U \subseteq \mathbb{D}_{\sigma'}$.
Recall both that $\Phi = \phi_k$ and that $\Psi = \psi_k$.
$$\begin{aligned}\text{We have} \quad A &= \Phi_{x+h} - \Phi_x - \Psi_x \cdot h \\ &= (\phi_k)_{x+h} - (\phi_k)_x - (\psi_k)_x \cdot h.\end{aligned}$$
$$\begin{aligned}\text{We have} \quad C &= ([\phi_\bullet(x + h)] - [\phi_\bullet(x)] - [\psi_\bullet(x)] \cdot h)_m \\ &= [\phi_m(x + h)] - [\phi_m(x)] - [\psi_m(x)] \cdot h \\ &= (\phi_m)_{x+h} - (\phi_m)_x - (\psi_m)_x \cdot h.\end{aligned}$$
$$\begin{aligned}\text{So, since} \quad A &= (\phi_k)_{x+h} - (\phi_k)_x - (\psi_k)_x \cdot h. \\ \text{we get} \quad C - A &= \sigma_{x+h} - \sigma_x - \tau_x \cdot h.\end{aligned}$$
Since $(x - \alpha; x + \alpha)$ is an interval, and since $x, x + h \in (x - \alpha; x + \alpha)$,
we get $[x|x + h] \subseteq (x - \alpha; x + \alpha)$.
So, since $(x - \alpha; x + \alpha) \subseteq U$, we get: $[x|x + h] \subseteq U$.
So, since $U \subseteq \mathbb{D}_{\sigma'}$, we get: $[x|x + h] \subseteq \mathbb{D}_{\sigma'}$.
Then, by Theorem 4.13.6, choose $q \in [x|x + h]$ s.t. $\sigma_{x+h} - \sigma_x = \sigma'_q \cdot h$.
Since $q \in [x|x + h] \subseteq U$, we conclude: $q \in U$.

So, since $\sigma' = \tau$ on U, we conclude: $\sigma'_q = \tau'_q$.
We have: $\sigma_{x+h} - \sigma_x \;=\; \sigma'_q \cdot h \;=\; \tau_q \cdot h$.
Then $C - A = \sigma_{x+h} - \sigma_x - \tau_x \cdot h \;= \tau_q \cdot h - \tau_x \cdot h$.
Then $C - A \;=\; \tau_q \cdot h \;+ (\,-\tau_x \cdot h\,)$.
Then $|C - A| \leqslant |\tau_q \cdot h| + |-\tau_x \cdot h| = |\tau_q| \cdot |h| + |-\tau_x| \cdot |h|$.
$= |\tau_q| \cdot |h| + \;\; |\tau_x| \;\; \cdot |h|$.
Then $|C - A| \leqslant \; |\tau_q| \; \cdot |h| + \;\; |\tau_x| \; \cdot |h|$.
Want: $|C - A| \leqslant (\varepsilon/4) \cdot |h| + (\varepsilon/4) \cdot |h|$.
Want: $|\tau_q| \leqslant \varepsilon/4$ and $|\tau_x| \leqslant \varepsilon/4$.
Since $\forall j \in [k..\infty)$, $|g - \psi_j| < \varepsilon/8$ on U,
snd since $k, m \in [k..\infty)$,
we conclude: $|g - \psi_k| < \varepsilon/8$ on U and $|g - \psi_m| < \varepsilon/8$ on U.
By hypothesis, $\psi \to g$ uniformly on U from $\mathbb{R}$ to $\mathbb{R}$.
Then $U \subseteq \mathbb{D}_g$. Then $|\psi_m - \psi_k| \leqslant |\psi_m - g| + |g - \psi_k|$ on U,
Since $\tau = \psi_m - \psi_k$,
and since $|\psi_m - \psi_k| \leqslant |\psi_m - g| + |g - \psi_k|$ on U,
and since $|\psi_m - g| = |g - \psi_m|$,
we get $|\tau| \leqslant |g - \psi_m| + |g - \psi_k|$ on U.
Then $|\tau| \leqslant \;\; (\varepsilon/8) \;\; + \;\; (\varepsilon/8)$ on U,
and so $|\tau| \leqslant \;\; \varepsilon/4$ on U.
So, since $q, x \in U$, we conclude: $|\tau_q| \leqslant \varepsilon/4$ and $|\tau_x| \leqslant \varepsilon/4$. □

4.16. **Power series.**

A function with domain $\mathbb{N}_0$ will be called a **zero-sequence**.
The theory of 0-sequences is completely parallel to that of sequences.
For example:

DEFINITION 4.16.1. *Let X be a metric space and let $s \in X^{\mathbb{N}_0}$.*
Then s is **Cauchy** *in X means:*
$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}_0$ *s.t.*, $\forall i, j \in [K..\infty)$, $d(s_i, s_j) < \varepsilon$.
Also, $\forall q \in X$, $s \to q$ *in* X *means:*
$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}_0$ *s.t.*, $\forall j \in [K..\infty)$, $d(s_j, q) < \varepsilon$.
Also, s is **convergent** *in X means:* $\exists q \in X$ *s.t.* $s \to q$ *in* X.
Also, $X\text{-}\lim s \;:=\; \mathrm{UE}\{q \in X \,|\, s \to q \text{ in } X\}$.

DEFINITION 4.16.2. *Let s be a zero-sequence.*
Then $s_{\bullet-1}$ is the sequence defined by: $\forall j \in \mathbb{N}$, $(s_{\bullet-1})_j = s_{j-1}$.

THEOREM 4.16.3. *Let X be a complete metric space, $s \in X^{\mathbb{N}_0}$.*
Assume: s is Cauchy in X. Then: s is convergent in X.

Idea of proof:
Since s is Cauchy in X, $s_{\bullet-1}$ is Cauchy in X.
Then, as X is complete, $s_{\bullet-1}$ is convergent in X.
Then s is convergent in X. QED

We have the **Principle of Zero-Induction**:

THEOREM 4.16.4. *Let* $S \subseteq \mathbb{N}_0$.
Assume: $(\, 0 \in S \,) \quad \& \quad (\, \forall j \in S,\, j+1 \in S \,)$.
Then: $S = \mathbb{N}_0$.

Idea of proof:
Since $S \subseteq \mathbb{N}_0$, $S+1 \subseteq \mathbb{N}$.
Since $0 \in S$, $1 \in S+1$.
Since $\forall j \in S,\, j+1 \in S$
we conclude: $\forall j \in S+1, \quad j+1 \in S+1$.
Then, by the Principle of Mathematical Induction, $S+1 = \mathbb{N}$.
Then $S = \mathbb{N} - 1 = \mathbb{N}_0$. QED

The following can be proved by Zero-Induction:

THEOREM 4.16.5. *Let* $u \in \mathbb{R}$. *Then:* $\forall k \in \mathbb{N}_0$,
$$(1-u)(1+u+u^2+\cdots+u^k) = 1-u^{k+1}.$$

THEOREM 4.16.6. *Let* $u \in [0;1)$, $k \in \mathbb{N}_0$.
Then: $1+u+u^2+\cdots+u^k \leqslant 1/(1-u)$.

THEOREM 4.16.7. *Let* $u \in [0;1)$, $k \in \mathbb{N}_0$, $M \geqslant 0$.
Then: $M+Mu+Mu^2+\cdots+Mu^k \leqslant M/(1-u)$.

DEFINITION 4.16.8. *Let* $S \subseteq \mathbb{R}$. *Then* $|S| := \{|x| \text{ s.t. } x \in S\}$.

THEOREM 4.16.9. *Let* $S \subseteq \mathbb{R}$. *Then:*
$$(\, S \text{ is bounded in } \mathbb{R} \,) \Rightarrow (\, |S| \text{ is bounded in } \mathbb{R} \,).$$

Proof. We have: $\forall x \in \mathbb{R}$, $\big|\,|x|\,\big| = |x|$. It follows that:
$\forall x \in \mathbb{R}, \forall K \geqslant 0, \quad (\, x \in B_{\mathbb{R}}(0,K) \,) \Leftrightarrow (\, |x| \in B_{\mathbb{R}}(0,K) \,)$.
Then: $\forall K \geqslant 0, \quad (\, S \subseteq B_{\mathbb{R}}(0,K) \,) \Leftrightarrow (\, |S| \subseteq B_{\mathbb{R}}(0,K) \,)$.
Then:
$$\begin{aligned} (\, S \text{ is bounded in } \mathbb{R} \,) &\Leftrightarrow (\, \exists K \geqslant 0 \text{ s.t. } S \subseteq B_{\mathbb{R}}(0,K) \,) \\ &\Leftrightarrow (\, \exists K \geqslant 0 \text{ s.t. } |S| \subseteq B_{\mathbb{R}}(0,K) \,) \\ &\Leftrightarrow (\, |S| \text{ is bounded in } \mathbb{R} \,). \end{aligned}$$
□

DEFINITION 4.16.10. *Let* $a \in \mathbb{R}^{\mathbb{N}_0}$, $x \in \mathbb{R}$.
Then: $\mathrm{TS}^a_x := \{\, a_0\,,\, a_1 x\,,\, a_2 x^2\,,\, \ldots \}$.

"TS" stands for "Terms of Series".

DEFINITION 4.16.11. *Let* $a \in \mathbb{R}^{\mathbb{N}_0}$, $j \in \mathbb{N}_0$.
Then $P_j SS^a : \mathbb{R} \to \mathbb{R}$ *is defined by:* $\forall x \in \mathbb{R}$,
$$P_j SS^a_x \; = \; a_0 \, + \, a_1 x \, + \, a_2 x^2 \, + \, \cdots \, + \, a_j x^j.$$

"PSS" stands for "Partial Sum of Series".
Note that $P_j SS^a \; = \; a_0 \, + \, a_1(\bullet) \, + \, a_2(\bullet)^2 \, + \, \cdots \, + \, a_j(\bullet)^j$;
this is a (not necessarily homogeneous) polynomial.

DEFINITION 4.16.12. *Let* $a \in \mathbb{R}^{\mathbb{N}_0}$.
Then $P_\bullet SS^a \in (\mathbb{R}^\mathbb{R})^{\mathbb{N}_0}$ *is defined by:* $\forall j \in \mathbb{N}_0$, $\;(P_\bullet SS^a)_j \; = \; P_j SS^a$.

DEFINITION 4.16.13. *Let* $a \in \mathbb{R}^{\mathbb{N}_0}$, $x \in \mathbb{R}$.
Then $P_\bullet SS^a_x \in \mathbb{R}^{\mathbb{N}_0}$ *is defined by:* $\forall j \in \mathbb{N}_0$, $\;(P_\bullet SS^a_x)_j \; = \; P_j SS^a_x$.

DEFINITION 4.16.14. *Let* $a \in \mathbb{R}^{\mathbb{N}_0}$.
$$\text{Then } \mathrm{RC}_a := \sup\{s \geqslant 0 \,|\, \mathrm{TS}^a_s \text{ is bounded in } \mathbb{R}\}.$$

"RC" stands for "Radius of Convergence".

DEFINITION 4.16.15. *Let* $a \in \mathbb{R}^{\mathbb{N}_0}$, $\rho := \mathrm{RC}_a$.
$$\text{Then } \mathrm{IC}_a := (-\rho; \rho).$$

"IC" stands for "Interval of Convergence".

DEFINITION 4.16.16. *Let* X *and* Y *be sets,* $f \in (Y^X)^\mathbb{N}$, $p \in X$.
$$\text{Then } f_\bullet(p) \in Y^\mathbb{N} \text{ is defined by: } \quad \forall j \in \mathbb{N},\ (f_\bullet(p))_j = f_j(p).$$

DEFINITION 4.16.17. *Let* $a \in \mathbb{R}^{\mathbb{N}_0}$. *Then* a *is denoted*
$$_0(a_0, a_1, a_2, \ldots).$$

THEOREM 4.16.18. *Let* $a \in \mathbb{R}^{\mathbb{N}_0}$ *and let* $r < \mathrm{RC}_a$.
Then: $\exists s > r$ *s.t.* TS^a_s *is bounded in* $\mathbb{R}$.

Proof. Let $T := \{s \geqslant 0 \,|\, TS^a_s$ is bounded in $\mathbb{R}\}$.
Since $r < \mathrm{RC}_a = \sup T$, we conclude: $\neg(\sup T \leqslant r)$.
Then $\neg(T \leqslant r)$, so choose $s \in T$ s.t. $\neg(s \leqslant r)$.
Then $s > r$. Want: TS^a_s is bounded in $\mathbb{R}$.
Since $s \in T$, we get: TS^a_s is bounded in $\mathbb{R}$, as desired. □

THEOREM 4.16.19. *Let* $a \in \mathbb{R}^{\mathbb{N}_0}$ *and let* $x \in \mathbb{R}$.
Assume: $|x| > \mathrm{RC}_a$. *Then:* $P_\bullet SS^a_x$ *is not convergent in* $\mathbb{R}$.

Proof. Assume $P_\bullet SS^a_x$ is convergent in $\mathbb{R}$. Want: Contradiction.
Let $u := P_\bullet SS^a_x$. Then u is convergent in $\mathbb{R}$.
By Theorem 4.14.6, $\{u_{j+1} - u_j \,|\, j \in \mathbb{N}_0\}$ is bounded in $\mathbb{R}$.
Let $D := \{u_{j+1} - u_j \,|\, j \in \mathbb{N}_0\}$. Then D is bounded in $\mathbb{R}$.
We have: $\forall j \in \mathbb{N}_0$,

$$\begin{aligned} u_{j+1} - u_j &= P_{j+1}SS^a_x - P_jSS^a_x \\ &= (a_0 + a_1x + \cdots + a_jx^j + a_{j+1}x^{j+1}) \\ &\quad - (a_0 + a_1x + \cdots + a_jx^j) \quad = \quad a_{j+1}x^{j+1}. \end{aligned}$$

Then $D = \{u_{j+1} - u_j \,|\, j \in \mathbb{N}_0\} = \{\, a_1x\,,\, a_2x^2\,,\, a_3x^3\,,\, \ldots\}$.
Since D is bounded in $\mathbb{R}$ and $\{a_0\}$ is bounded in $\mathbb{R}$,
we see that $\{a_0\} \bigcup D$ is bounded in $\mathbb{R}$.
Then, by Theorem 4.16.9, $|\,\{a_0\} \bigcup D\,|$ is bounded in $\mathbb{R}$.
We have $|\,\{a_0\} \bigcup D\,| \;=\; \{\,|a_0|\,,\, |a_1|\cdot|x|\,,\, |a_2|\cdot|x|^2\,,\, |a_3|\cdot|x|^3\,,\, \ldots\}$.
Let $E := \mathrm{TS}^a_{|x|}$. Then $E = \{\, a_0\,,\, a_1\cdot|x|\,,\, a_2\cdot|x|^2\,,\, a_3\cdot|x|^3\,,\, \ldots\}$.
Then $|E| \;=\; \{\,|a_0|\,,\, |a_1|\cdot|x|\,,\, |a_2|\cdot|x|^2\,,\, |a_3|\cdot|x|^3\,,\, \ldots\} \;=\; |\,\{a_0\} \bigcup D\,|$.
So, since $|\,\{a_0\} \bigcup D\,|$ is bounded in $\mathbb{R}$, we get: $|E|$ is bounded in $\mathbb{R}$.
Then, by Theorem 4.16.9, E is bounded in $\mathbb{R}$.
Let $T := \{s \geqslant 0 \,|\, TS^a_s$ is bounded in $\mathbb{R}\}$. Then $\mathrm{RC}_a = \sup T$.
Since $E = \mathrm{TS}^a_{|x|}$ and since E is bounded in $\mathbb{R}$, we get: $|x| \in T$.
By hypothesis, $|x| > \mathrm{RC}_a$. Then $|x| \in T \leqslant \sup T = \mathrm{RC}_a < |x|$.
Then $|x| < |x|$. Contradiction. ∎

THEOREM 4.16.20. *Let $a \in \mathbb{R}^{\mathbb{N}_0}$ and let $x \in \mathrm{IC}_a$.*
Then $P_\bullet SS^a_x$ is convergent in $\mathbb{R}$.

Proof. Let $b := P_\bullet SS^a_x$. Want: b is convergent in $\mathbb{R}$.
Want: b is Cauchy in $\mathbb{R}$.
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}_0$ s.t., $\forall i, j \in [K..\infty)$, $|b_i - b_j| < \varepsilon$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}_0$ s.t., $\forall i, j \in [K..\infty)$, $|b_i - b_j| < \varepsilon$.
Let $\alpha := |a|$ and let $\rho := \mathrm{RC}_a$. By HW#4-4, $\rho = \mathrm{RC}_\alpha$.
Since $x \in \mathrm{IC}_a = (-\rho; \rho)$, we conclude: $|x| < \rho$.
Let $r := |x|$. Since $r = |x|$ and $0 \leqslant |x| < \rho$, we get $r \in [0; \rho)$.
Let $c := P_\bullet SS^\alpha_r$. Since $r \in [0; \rho) = [0; \mathrm{RC}_\alpha)$, by HW#4-3,
we conclude: c is convergent in $\mathbb{R}$.
Then c is Cauchy in $\mathbb{R}$, so choose $K \in \mathbb{N}_0$ s.t.,
$\forall i, j \in [K..\infty)$, $|c_i - c_j| < \varepsilon$.
Then $K \in \mathbb{N}_0$. Want: $\forall i, j \in [K..\infty)$, $|b_i - b_j| < \varepsilon$.
Given $i, j \in [K..\infty)$. Want: $|b_i - b_j| < \varepsilon$.
Since $i, j \in [K..\infty)$, by choice of K, we have: $|c_i - c_j| < \varepsilon$.

By HW#4-2, we conclude that: $|b_i - b_j| \leqslant |c_i - c_j|$.

Then: $|b_i - b_j| \leqslant |c_i - c_j| < \varepsilon$. □

DEFINITION 4.16.21. *Let $a \in \mathbb{R}^{\mathbb{N}_0}$.*
Then $SS^a : \mathrm{IC}_a \to \mathbb{R}$ is defined by:
$$\forall x \in \mathrm{IC}_a, \quad SS^a_x \;=\; \mathbb{R}\text{-}\lim P_\bullet SS^a_x.$$

DEFINITION 4.16.22. *Let $a := {}_0\left(\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots\right)$.*
Then we define: $\exp \;:=\; SS^a$.

By HW#4-1, we have $\mathbb{D}_{\exp} = \mathbb{R}$.
More colloquially, we would say:

"Let $\exp : \mathbb{R} \to \mathbb{R}$ be defined by:
$$\forall x \in \mathbb{R}, \quad \exp_x \;=\; \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^3}{4!} + \cdots."$$

Our main remaining goal in this section is to show that
differentiation of power series
works via term-by-term differentiation.
In particular, we will show that $\exp' = \exp$.

DEFINITION 4.16.23. *Let $a \in \mathbb{R}^{\mathbb{N}_0}$. Then $a^* \in \mathbb{R}^{\mathbb{N}_0}$ is defined by:*
$$\forall j \in \mathbb{N}_0, \quad a^*_j \;:=\; (j+1) \cdot a_{j+1}.$$

We have: $\forall a \in \mathbb{R}^{\mathbb{N}_0}, \quad a^* \;=\; {}_0(\, a_1 \,,\, 2 \cdot a_2 \,,\, 3 \cdot a_3 \,,\, 4 \cdot a_4 \,,\, \ldots)$.

THEOREM 4.16.24. *Let $a := {}_0\left(\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots\right)$.*
Then: $a^* \;=\; a$.

By the preceding theorem, to show $\exp' = \exp$, it suffices to show:
$$\forall a \in \mathbb{R}^{\mathbb{N}_0}, \quad (SS^a)' \;=\; SS^{a^*}.$$
We begin with the partial sum version:

THEOREM 4.16.25. *Let $a \in \mathbb{R}^{\mathbb{N}_0}$ and let $j \in \mathbb{N}$.*
Then: $(P_j SS^a)' \;=\; P_{j-1} SS^{a^*}$.

Proof. Since $P_j SS^a = a_0 + a_1 \cdot (\bullet) + a_2 \cdot (\bullet)^2 + \cdots + a_j \cdot (\bullet)^j$,
we get: $(P_j SS^a)' = a_1 + 2 \cdot a_2 \cdot (\bullet) + \cdots + j \cdot a_j \cdot (\bullet)^{j-1}$.
So, since $P_{j-1} SS^{a^*} = a_1 + 2 \cdot a_2 \cdot (\bullet) + \cdots + j \cdot a_j \cdot (\bullet)^{j-1}$,
we get: $(P_j SS^a)' \;=\; P_{j-1} SS^{a^*}$, as desired. □

DEFINITION 4.16.26. *Let X be a set and let $S \subseteq X$.*
Let Y be a metric space. Let $f \in (\mathrm{PF}^X_Y)^{\mathbb{N}_0}$ and let $g \in \mathrm{PF}^X_Y$.

By $f \to g$ pointwise on S from X to Y, we mean:

$\forall p \in S, \quad f_\bullet(p) \to g(p)$ *in* Y.

By $f \to g$ uniformly on S from X to Y, we mean:

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}_0$ *s.t.*, $\forall j \in \mathbb{N}_0$, $\forall p \in S$,

$$(\ j \geqslant K\ ) \ \Rightarrow\ (\ d(\ f_j(p)\,,\ g(p)\,)\ <\ \varepsilon\ ).$$

Let X be a set and let $S \subseteq X$. Let Y be a metric space.

Let $\quad f \in (\mathrm{PF}_Y^X)^{\mathbb{N}_0} \quad$ and let $\quad g \in \mathrm{PF}_Y^X$.

Note that, if $f \to g$ uniformly on S from X to Y,
then $f \to g$ pointwise on S from X to Y.

Also, note that, if $f \to g$ pointwise on S from X to Y,
then $S \subseteq \mathbb{D}_g \bigcap \mathbb{D}_{f_0} \bigcap \mathbb{D}_{f_1} \bigcap \mathbb{D}_{f_2} \bigcap \cdots$.

DEFINITION 4.16.27. *Let X and Y be sets, $f \in (Y^X)^{\mathbb{N}_0}$, $p \in X$.*
Then $f_\bullet(p) \in Y^{\mathbb{N}_0}$ is defined by: $\quad \forall j \in \mathbb{N}_0$, $(f_\bullet(p))_j = f_j(p)$.

THEOREM 4.16.28. *Let $a \in \mathbb{R}^{\mathbb{N}_0}$ and let $r \in [0, \mathrm{RC}_a)$.*
Then: $\quad P_\bullet SS^a$ *to* $SS^a \quad$ *uniformly on* $[-r; r]$ *from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. Let $f := P_\bullet SS^a$, $g := SS^a$, $T := [-r; r]$.
Then $T = [-r; r] \subseteq (-\mathrm{RC}_a, \mathrm{RC}_a) = \mathrm{IC}_a$, so $T \subseteq \mathrm{IC}_a$.
We wish to show: $f \to g$ uniformly on T from $\mathbb{R}$ to $\mathbb{R}$.
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}_0$ s.t., $\forall i \in [K..\infty)$, $|g - f_i| < \varepsilon$ on T.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}_0$ s.t., $\forall i \in [K..\infty)$, $|g - f_i| < \varepsilon$ on T.
By HW#4-4, we have: $\mathrm{IC}_a = \mathrm{IC}_{|a|}$.
Since $r \in [-r; r] = T \subseteq \mathrm{IC}_a = \mathrm{IC}_{|a|}$, we get $r \in \mathrm{IC}_{|a|}$.
Then $P_\bullet SS_r^{|a|}$ is convergent in $\mathbb{R}$, and $P_\bullet SS_r^{|a|} \to SS_r^{|a|}$ in $\mathbb{R}$.
Let $c := P_\bullet SS_r^{|a|}$ and let $z := SS_r^{|a|}$. Then $c \to z$ in $\mathbb{R}$.
Choose $K \in \mathbb{N}_0$ s.t., $\forall i \in [K..\infty)$, $|z - c_i| < \varepsilon$.
Then $K \in \mathbb{N}_0$. Want: $\forall i \in [K..\infty)$, $|g - f_i| < \varepsilon$ on T.
Given $i \in [K..\infty)$. Want: $|g - f_i| < \varepsilon$ on T.
Want: $\forall x \in T$, $|g - f_i|_x < \varepsilon$.
Given $x \in T$. Want: $|g - f_i|_x < \varepsilon$.
Since $i \in [K..\infty)$, by choice of K, we get: $|z - c_i| < \varepsilon$.
It therefore suffices to show: $|g - f_i|_x \leqslant |z - c_i|$.
Want: $\forall \eta > 0$, $|g - f_i|_x \leqslant |z - c_i| + \eta$.
Given $\eta > 0$. Want: $|g - f_i|_x \leqslant |z - c_i| + \eta$.
We have $x \in T \subseteq \mathrm{IC}_a$.
Then $P_\bullet SS_x^a$ is convergent in $\mathbb{R}$, and $P_\bullet SS_x^a \to SS_x^a$ in $\mathbb{R}$.
Let $b := P_\bullet SS_x^a$. Recall: $g = SS^a$. Then $b \to g_x$ in $\mathbb{R}$.

Choose $L \in \mathbb{N}_0$ s.t., $\forall j \in [L..\infty)$, $|g_x - b_j| < \eta/2$.
Recall: $c \to z$ in $\mathbb{R}$.
Choose $M \in \mathbb{N}_0$ s.t., $\forall j \in [M..\infty)$, $|z - c_j| < \eta/2$.
Let $j := \max\{L, M\}$. Then $j \in [L..\infty)$ and $j \in [M..\infty)$.
Since $j \in [L..\infty)$, by choice of L, we get: $|g_x - b_j| < \eta/2$.
Since $j \in [M..\infty)$, by choice of M, we get: $|z - c_j| < \eta/2$.
Since $f = P_\bullet SS^a$, we get $f_i = P_i SS^a$, and so $(f_i)_x = P_i SS^a_x$.
Then $b_i = (P_\bullet SS^a_x)_i = P_i SS^a_x = (f_i)_x$.
By HW#4-2, $|b_j - b_i| \leqslant |c_j - c_i|$.

$$
\begin{aligned}
\text{Then: } |g - f_i|_x &= |g_x - (f_i)_x| \\
&= |g_x - b_i| \\
&\leqslant |g_x - b_j| + |b_j - b_i| \\
&< (\eta/2) + |b_j - b_i| \\
&\leqslant (\eta/2) + |c_j - c_i| \\
&\leqslant (\eta/2) + |c_j - z| + |z - c_i| \\
&\leqslant (\eta/2) + |z - c_j| + |z - c_i| \\
&< (\eta/2) + (\eta/2) + |z - c_i| \\
&= |z - c_i| + \eta, \qquad \text{as desired.}
\end{aligned}
$$

□

Unassigned HW:
Let X be a set, let $S \subseteq X$ and let Y be a metric space.
Let $\alpha \in (\mathrm{PF}^X_Y)^{\mathbb{N}_0}$ and let $\omega \in \mathrm{PF}^X_Y$.
Assume that: $\alpha \to \omega$ uniformly on S from X to Y.
Define $\beta \in (\mathrm{PF}^X_Y)^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $\beta_j = \alpha_{j-1}$.
Show that: $\beta \to \omega$ uniformly on S from X to Y.

Unassigned HW:
Let X be a set, let $S \subseteq X$ and let Y be a metric space.
Let $\alpha \in (\mathrm{PF}^X_Y)^{\mathbb{N}_0}$ and let $\omega \in \mathrm{PF}^X_Y$.
Assume that: $\alpha \to \omega$ uniformly on S from X to Y.
Define $\beta \in (\mathrm{PF}^X_Y)^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $\beta_j = \alpha_j$.
Show that: $\beta \to \omega$ uniformly on S from X to Y.

THEOREM 4.16.29. *Let* $a \in \mathbb{R}^{\mathbb{N}_0}$. *Then:* $(SS^a)' = SS^{a^*}$.

Proof. Let $b := a^*$ and $f := SS^a$ and $g := SS^b$. Want: $f' = g$.
Let $U := \mathrm{IC}_a$. By HW#4-5, $\mathrm{RC}_a = \mathrm{RC}_b$.
Then $U = \mathrm{IC}_b$. Then: $\mathbb{D}_{SS^a} = U = \mathbb{D}_{SS^b}$.
Want: $\forall x \in U$, $(SS^a)'_x = SS^b_x$.
Given $x \in U$. Want: $(SS^a)'_x = SS^b_x$.

Let $\rho := \mathrm{RC}_a$. Then $x \in \mathrm{IC}_a = (-\rho;\rho)$, so $|x| < \rho$.
Let $r := \dfrac{|x|+\rho}{2}$. Then $|x| < r < \rho$.
Since $|x| < r$, we get: $x \in (-r;r)$.
By Theorem 4.16.28, we have:

$$P_\bullet SS^a \to SS^a \quad \text{uniformly on } [-r;r] \text{ from } \mathbb{R} \text{ to } \mathbb{R}$$
$$\text{and} \quad P_\bullet SS^b \to SS^b \quad \text{uniformly on } [-r;r] \text{ from } \mathbb{R} \text{ to } \mathbb{R}.$$

Define $\phi, \psi \in (\mathrm{PF}_{\mathbb{R}}^{\mathbb{R}})^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$,

$$\phi_j = P_j SS^a \qquad \text{and} \qquad \psi_j = P_{j-1} SS^b.$$

Then $\phi \to SS^a$ uniformly on $[-r;r]$ from $\mathbb{R}$ to $\mathbb{R}$
and $\psi \to SS^b$ uniformly on $[-r;r]$ from $\mathbb{R}$ to $\mathbb{R}$.
Recall that $f = SS^a$ and that $g = SS^b$.
Then $\phi \to f$ uniformly on $[-r;r]$ from $\mathbb{R}$ to $\mathbb{R}$
and $\psi \to g$ uniformly on $[-r;r]$ from $\mathbb{R}$ to $\mathbb{R}$.
So, as $(-r;r) \subseteq [-r;r]$, we conclude:

$$\phi \to f \quad \text{uniformly on } (-r;r) \text{ from } \mathbb{R} \text{ to } \mathbb{R}$$
$$\text{and} \quad \psi \to g \quad \text{uniformly on } (-r;r) \text{ from } \mathbb{R} \text{ to } \mathbb{R}.$$

Since uniform convergence implies pointwise convergence, we get:

$$\phi \to f \quad \text{pointwise on } (-r;r) \text{ from } \mathbb{R} \text{ to } \mathbb{R}.$$

By Theorem 4.16.25, $\forall j \in \mathbb{N}$, $(P_j SS^a)' = P_{j-1} SS^{a^*}$.
Then, $\forall j \in \mathbb{N}$, $(\phi_j)' = (P_j SS^a)' = P_{j-1} SS^{a^*} = P_{j-1} SS^b = \psi_j$.
Then, by Theorem 4.15.2, $f' = g$, as desired. □

5. Multivariable Differential Calculus

5.1. Euclidean spaces.

THEOREM 5.1.1. $(5) = (1 \mapsto 5) \in \mathbb{R}^1$ *and* $5 \in \mathbb{R}$ *and* $(5) \neq 5$.

THEOREM 5.1.2. *We have:* $\mathbb{R}^1 \bigcap \mathbb{R} = \varnothing$.

DEFINITION 5.1.3. $\mathrm{ES} := \{\mathbb{R}\} \bigcup \{\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots\}$.

An element of ES is called a **Euclidean space**.
We will call $\mathbb{R}$ the **scalar space** or **scalar Euclidean space**.
An element of $\{\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots\}$ will be called a
vector space or **vector Euclidean space**.
Recall that any element of $\mathbb{R}$ is called a scalar.
Elements of $\mathbb{R}^1 \bigcup \mathbb{R}^2 \bigcup \mathbb{R}^3 \bigcup \cdots$ will be called **Euclidean vectors** or, simply, **vectors** in this course.

Note that $\bigcup \mathrm{ES} = \mathbb{R} \cup \mathbb{R}^1 \cup \mathbb{R}^2 \cup \mathbb{R}^3 \cup \cdots$.
Then $\bigcup \mathrm{ES}$ is the set of all objects that are either scalar or vector.
In the next definition, "v/s" is read "vector or scalar":

DEFINITION 5.1.4. *Let x be an object.*
*By x is a **v/s**, we mean:* $x \in \bigcup \mathrm{ES}$.

THEOREM 5.1.5. $(2,3)$ *is a v/s and* 5 *is a v/s and* $(-3,6,5)$ *is a v/s and* $\{-3,6,5\}$ *is NOT a v/s.*

THEOREM 5.1.6. $\forall m \in \mathbb{N}$, $\forall x \in \mathbb{R}^m$, $\forall j \in [1..m]$, $x_j \in \mathbb{R}$.

DEFINITION 5.1.7. *Let* $m \in \mathbb{N}$, $V := \mathbb{R}^m$. *Then* $\mathcal{I}_V := [1..m]$.

THEOREM 5.1.8. $\forall V \in \mathrm{ES} \backslash \{\mathbb{R}\}$, $\forall j \in \mathcal{I}_V$, $x_j \in \mathbb{R}$.

DEFINITION 5.1.9. $\forall x \in \mathbb{R}$, $x_0 := x$.

THEOREM 5.1.10. $7_0 = 7 \in \mathbb{R}$.

THEOREM 5.1.11. $\forall x \in \mathbb{R}$, $x_0 = x \in \mathbb{R}$.

DEFINITION 5.1.12. $\mathcal{I}_{\mathbb{R}} := \{0\}$.

THEOREM 5.1.13. $\forall x \in \mathbb{R}$, $\forall j \in \mathcal{I}_{\mathbb{R}}$, $x_j = x \in \mathbb{R}$.

THEOREM 5.1.14. $\forall V \in \mathrm{ES}$, $\forall j \in \mathcal{I}_V$, $x_j \in \mathbb{R}$.

Let $V \in \mathrm{ES}$. Then $\mathcal{I}_V$ is called the **indexing set** of V.

THEOREM 5.1.15. *Let* $V \in \mathrm{ES}$ *and* $x, y \in V$.
Then: $(\ x = y\ ) \Leftrightarrow (\ \forall j \in \mathcal{I}_V, x_j = y_j\ )$.

DEFINITION 5.1.16. *Let* $V \in \mathrm{ES}$.
Then $0_V \in V$ *is defined by:* $\forall j \in \mathcal{I}_V$, $(0_V)_j = 0$.

THEOREM 5.1.17. $(\ 0_{\mathbb{R}} = 0\ ) \& (\ 0_{\mathbb{R}^1} = (0)\ ) \& (\ 0_{\mathbb{R}^2} = (0,0)\ ) \& (\ 0_{\mathbb{R}^3} = (0,0,0)\ ) \& (\ 0_{\mathbb{R}^4} = (0,0,0,0)\ ) \& (\ 0_{\mathbb{R}^5} = (0,0,0,0,0)\ )$.

DEFINITION 5.1.18. *Let* $V \in \mathrm{ES}$ *and let* $x, y \in V$.
Then
$$x \bullet y := \sum_{j \in \mathcal{I}_V} x_j \cdot y_j.$$

THEOREM 5.1.19. $4 \bullet 2 = 4_0 \cdot 2_0 = 4 \cdot 2 = 8$.

THEOREM 5.1.20. $(4) \bullet (2) = (4)_1 \cdot (2)_1 = 4 \cdot 2 = 8$.

THEOREM 5.1.21. $(3,6) \bullet (8,7) = (3,6)_1 \cdot (8,7)_1 + (3,6)_2 \cdot (8,7)_2 = 3 \cdot 8 + 6 \cdot 7 = 24 + 42 = 66$.

DEFINITION 5.1.22. *Let $V, W \in \mathrm{ES}$ and $x \in V$ and $y \in W$. Assume $V \neq W$. Then $x \bullet y :=$ ☹ and $x + y :=$ ☹.*

Let $x := (5,6)$ and $y := (7,8,9)$. If we think of x and y as functionals,

$$x = \begin{pmatrix} 1 \mapsto 5 \\ 2 \mapsto 6 \end{pmatrix} \qquad \text{and} \qquad y = \begin{pmatrix} 1 \mapsto 7 \\ 2 \mapsto 8 \\ 3 \mapsto 9 \end{pmatrix},$$

then, as $5 + 7 = 12$ and $6 + 8 = 14$, we compute

$$x + y = \begin{pmatrix} 1 \mapsto 12 \\ 2 \mapsto 14 \end{pmatrix} = (12, 14).$$

On the other hand, thinking of x and y as vectors,
since $x \in \mathbb{R}^2$ and since $y \in \mathbb{R}^3$ and since $\mathbb{R}^2 \neq \mathbb{R}^3$,
by the preceding definition, we compuate $x + y =$ ☹.
Each object, has a *type* and the meaning of $x + y$ depends on
whether we type x and y as functionals or as vectors.
We could avoid this confusion
by introducing, say, ${}_{\text{vector}}+_{\text{vector}}$ to indicate vector addition.
Then $x + y = (12, 14)$ and $x \;{}_{\text{vector}}+_{\text{vector}}\; y =$ ☹.
However, ${}_{\text{vector}}+_{\text{vector}}$ is simply hard to read.
We have decided, instead, to use $+$ to denote
both addition of functionals and addition of vectors,
and the reader has to determine, by context, which is meant.

Frownie is infective:

DEFINITION 5.1.23. $\forall x$, *$x \bullet$ ☹ := ☹ and ☹ $\bullet\, x$:= ☹.*

THEOREM 5.1.24. [$4 \bullet (2) =$ ☹] & [$(3,6) \bullet (2,4,7) =$ ☹] & [$(2,8) \bullet$ ☹ = ☹ = ☹ $\bullet\, 6$].

THEOREM 5.1.25. [$3 \cdot (5,8) = (15,24)$] & [$3 \bullet (5,8) =$ ☹].

The next two theorems can be summarized as: "$\bullet$ is *bilinear*".

THEOREM 5.1.26. *Let $V \in \mathrm{ES}$, $x, y, z \in V$. Then $(x + y) \bullet z = (x \bullet z) + (y \bullet z)$ and $x \bullet (y + z) = (x \bullet y) + (x \bullet z)$.*

THEOREM 5.1.27. *Let $V \in \mathrm{ES}$, $c \in \mathbb{R}$, $x, y \in V$. Then $(c \cdot x) \bullet y = c \cdot (x \bullet y)$ and $x \bullet (c \cdot y) = c \cdot (x \bullet y)$.*

The next theorem can be summarized as:
"$\bullet$ is *commutative* (a.k.a. *symmetric*)".

THEOREM 5.1.28. *Let $V \in \mathrm{ES}$, $x, y \in V$. Then $x \bullet y = y \bullet x$.*

DEFINITION 5.1.29. $\forall a,b,\quad \delta^a_b \;:=\; \begin{cases} 1, & \textit{if } a=b \\ 0, & \textit{if } a\neq b. \end{cases}$

DEFINITION 5.1.30. *Let* $V\in \mathrm{ES}$ *and* $j\in\mathcal{I}_V$.
Then $\varepsilon^V_j \in V$ *is defined by:* $\quad \forall i\in\mathcal{I}_V,\quad (\varepsilon^V_j)_i=\delta^i_j$.

THEOREM 5.1.31. *Let* $V:=\mathbb{R}^3$. *Then:*
$[\ \varepsilon^V_1 \ =\ (1,0,0)\ ]\ \&\ [\ \varepsilon^V_2\ =\ (0,1,0)\ ]\ \&\ [\ \varepsilon^V_3\ =\ (0,0,1)\ ]$.

THEOREM 5.1.32. $\varepsilon^{\mathbb{R}}_0=1$.

THEOREM 5.1.33. *Let* $V:=\mathbb{R}^3$ *and* $x:=(-3,6,5)$.
Then: $\quad x\bullet\varepsilon^V_2=(3,6,-5)\bullet(0,1,0)=6=x_2$.

THEOREM 5.1.34. $\forall V\in\mathrm{ES}$, $\forall x\in V$, $\forall j\in\mathcal{I}_V$, $\quad x\bullet\varepsilon^V_j=x_j$.

THEOREM 5.1.35. *Let* $V:=\mathbb{R}^3$ *and* $x:=(-3,6,5)$. *Then:*
$x=(-3,6,5)=-3\varepsilon^V_1+6\varepsilon^V_2+5\varepsilon^V_3=x_1\varepsilon^V_1+x_2\varepsilon^V_2+x_3\varepsilon^V_3$.

THEOREM 5.1.36. *Let* $V\in\mathrm{ES}$ *and* $x\in V$.
Then: $\quad x\ =\ \displaystyle\sum_{j\in\mathcal{I}_V} x_j\varepsilon^V_j$.

DEFINITION 5.1.37. *Let* $V\in\mathrm{ES}$ *and* $j\in\mathcal{I}_V$.
Then $\pi^V_j:V\to\mathbb{R}$ *is defined by:* $\quad\forall x\in V,\quad \pi^V_j(x)=x_j$.

THEOREM 5.1.38. *Let* $V:=\mathbb{R}^3$ *and* $x:=(-3,6,5)$. *Then:*
$[\ \pi^V_1(x)\ =\ -3\ ]\ \&\ [\ \pi^V_2(x)\ =\ 6\ ]\ \&\ [\ \pi^V_3(x)\ =\ 5\ ]$.

THEOREM 5.1.39. $\pi^{\mathbb{R}}_0(4)=4_0=4$.

THEOREM 5.1.40. $\pi^{\mathbb{R}}_0=\mathrm{id}_{\mathbb{R}}$.

THEOREM 5.1.41. $\forall V\in\mathrm{ES},\forall x\in V,\forall j\in\mathcal{I}_V,\ \pi^V_j(x)=x_j=x\bullet\varepsilon^V_j$.

DEFINITION 5.1.42. $\forall V\in\mathrm{ES},\quad \Pi_V\ :=\ \{\pi^V_j\,|\,j\in\mathcal{I}_V\}$.

THEOREM 5.1.43. *Let* $V:=\mathbb{R}^3$. *Then:* $\Pi_V\ =\ \{\,\pi^V_1\,,\,\pi^V_2\,,\,\pi^V_3\,\}$.

THEOREM 5.1.44. $\Pi_{\mathbb{R}}\ =\ \{\,\mathrm{id}_{\mathbb{R}}\,\}$.

DEFINITION 5.1.45. *Let* X *and* Y *be sets.*
Then: $\quad \mathrm{PF}^X_Y\ :=\ \{f\,|\,f:X \dashrightarrow Y\}$.

Let X and Y be sets. Then $\mathrm{PF}^X_Y=\displaystyle\bigcup_{W\subseteq X} Y^W$.

DEFINITION 5.1.46. *Let Y be a set and let f be an object.*
By f is a Y-**function**, *we mean:* (*f is a function*) & ($\mathbb{I}_f \subseteq Y$).

DEFINITION 5.1.47. *Let $V \in \mathrm{ES}$, f a V-function and $j \in \mathcal{I}_V$.*
Then: $\pi_j f \;:=\; \pi_j^V \circ f$.

THEOREM 5.1.48. *Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:*

$$\forall x \in \mathbb{R}^2, \quad f_x \;=\; (\, x_1 x_2 \,,\, x_1^2 + x_2^2 \,,\, x_1^3 - 6x_1x_2 \,).$$

Then, $\forall x \in \mathbb{R}^2$, we have:

$$[\ (\pi_1 f)_x = x_1x_2\ ]\ \&\ [\ (\pi_2 f)_x = x_1^2 + x_2^2\ ]\ \&\ [\ (\pi_3 f)_x = x_1^3 - 6x_1x_2\ ].$$

DEFINITION 5.1.49. *Let $A, B \in \bigcup \mathrm{ES}$.*
Then $A + B := \{x + y \,|\, x \in A,\ y \in B,\ x + y \neq ☹\}$.
Also, $A - B := \{x - y \,|\, x \in A,\ y \in B,\ x - y \neq ☹\}$.
Also, $A \bullet B := \{x \bullet y \,|\, x \in A,\ y \in B,\ x \bullet y \neq ☹\}$.

DEFINITION 5.1.50. *Let $x \in \bigcup \mathrm{ES}$, $A \subseteq \bigcup \mathrm{ES}$.*
Then $x + A := \{x\} + A$. *Also,* $A + x := A + \{x\}$.
Also, $x - A := \{x\} - A$. *Also,* $A - x := A - \{x\}$.
Also, $x \bullet A := \{x\} \bullet A$. *Also,* $A \bullet x := A \bullet \{x\}$.

DEFINITION 5.1.51. *Let $A \subseteq \mathbb{R}$ and $B \subseteq \bigcup \mathrm{ES}$.*
Then $A \cdot B := \{c \cdot y \,|\, c \in A,\ y \in B\}$. *Also,* $B \cdot A := \{y \cdot c \,|\, c \in A,\ y \in B\}$.

DEFINITION 5.1.52. *Let $c \in \mathbb{R}$, $B \subseteq \mathrm{ES}$.*
Then $c \cdot B := \{c\} \cdot B$. *Also,* $B \cdot c := B \cdot \{c\}$.

DEFINITION 5.1.53. *Let $A \subseteq \mathbb{R}$, $y \in \bigcup \mathrm{ES}$.*
Then $A \cdot y := A \cdot \{y\}$. *Also,* $y \cdot A := \{y\} \cdot A$.

In the last three definitions, "·" is sometimes omitted.

DEFINITION 5.1.54. *Let f be an object.*
By f is a **v/s-function**, *we mean: f is a* ($\bigcup \mathrm{ES}$)*-function.*

That is, by a "v/s-function", we mean a function whose image consists of vectors and scalars.

THEOREM 5.1.55. *Let $V \in \mathrm{ES}$ and let f be a V-function.*
Then f is a v/s-function.

Recall that a "functional" is a scalar-valued function, or a real-valued function, or an $\mathbb{R}$-function, or an $\mathbb{R}$-valued function; in other words, a function whose image consists of scalars.

THEOREM 5.1.56. *Let f be a functional. Then f is a v/s-function.*

DEFINITION 5.1.57. *Let $c \in \mathbb{R}$, f a v/s-function.*
Then $c \cdot f$ and $f \cdot c$ are the v/s-functions defined by:
$$\forall x, \qquad [\ (c \cdot f)_x = c \cdot f_x\ ]\ \&\ [\ (f \cdot c)_x = f_x \cdot c\ ].$$

Again, $\cdot$ is often omitted, but no $\bullet$ or $+$ or $-$.

DEFINITION 5.1.58. *Let f and g be v/s-functions.*
Then $f + g$ is the v/s-function defined by:
$$\forall x, \qquad (f+g)_x = f_x + g_x.$$
Also, $f - g$ is the v/s-function defined by:
$$\forall x, \qquad (f-g)_x = f_x - g_x.$$
Also, $f \bullet g$ is the v/s-function defined by:
$$\forall x, \qquad (f \bullet g)_x = f_x \bullet g_x.$$

THEOREM 5.1.59. *Define $f, g : \mathbb{R}^2 \to \mathbb{R}^3$ by: $\quad \forall x \in \mathbb{R}^2$,*
$$f_x \;=\; (\, x_1x_2\, ,\, x_1^2 + x_2^2\, ,\, x_1^3 - 6x_1x_2\,) \quad \textit{and} \quad g_x \;=\; (\, 0\, ,\, 1\, ,\, 8x_2\,).$$
Then, $\forall x \in \mathbb{R}^2$,
$$(f+g)_x \;=\; (\, x_1x_2\, ,\, x_2^2 + x_2^2 + 1\, ,\, x_1^3 - 6x_1x_2 + 8x_2\,)$$
$$\& \quad (f \bullet g)_x \;=\; x_1^2 + x_2^2 + [x_1^3 + 6x_1x_2] \cdot [8x_2].$$

DEFINITION 5.1.60. *Let f be a functional, g a v/s-function.*
Then $f \cdot g$ and $g \cdot f$ are the v/s-functions defined by:
$$\forall x, \qquad [\ (f \cdot g)_x = f_x \cdot g_x\ ]\ \&\ [\ (g \cdot f)_x = g_x \cdot f_x\ ].$$

Again, $\cdot$ is often omitted.

THEOREM 5.1.61. *Define $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$ by:*
$$\forall x \in \mathbb{R}^2, \quad [\, f_x \;=\; x_1^5 - 9x_2^4\,]\,\&\,[\, g_x \;=\; (\, 0\, ,\, 1\, ,\, 8x_1x_2\,)\,].$$
Then, $\forall x \in \mathbb{R}^2$,
$$(f \cdot g)_x \;=\; (\, 0\, ,\, x_1^5 - 9x_2^4\, ,\, [x_1^5 - 9x_2^4] \cdot [8x_1x_2]\,) \;=\; (g \cdot f)_x.$$

DEFINITION 5.1.62. *Let f be a v/s-function, $v \in \bigcup \mathrm{ES}$.*
Then $f + v$ and $v + f$ are the v/s-functions defined by: $\forall x$,
$$(f+v)_x = f_x + v \qquad \textit{and} \qquad (v+f)_x = v + f_x.$$
Also, $f - v$ and $v - f$ are the v/s-functions defined by: $\forall x$,
$$(f-v)_x = f_x - v \qquad \textit{and} \qquad (v-f)_x = v - f_x.$$
Also, $f \bullet v$ and $v \bullet f$ are the v/s-functions defined by: $\forall x$,
$$(f \bullet v)_x = f_x \bullet v \qquad \textit{and} \qquad (v \bullet f)_x = v \bullet f_x.$$

Again, no $\bullet$ or $+$ or $-$ is ever omitted.

DEFINITION 5.1.63. *Let F and G be sets of v/s-functions.*
Then $F + G \;:=\; \{f + g \,|\, f \in F,\, g \in G\}$.

Also, $F - G \ := \ \{f - g \,|\, f \in F,\, g \in G\}$.
Also, $F \bullet G \ := \ \{f \bullet g \,|\, f \in F,\, g \in G\}$.

In the preceding definition, $F + G$, $F - G$ and $F \bullet G$ are all sets of v/s-functions.

DEFINITION 5.1.64. *Let f be a v/s-function, G a set of v/s-functions.*
Then $f + G \ := \ \{f\} + G$. *Also,* $G + f \ := \ G + \{f\}$.
Also, $f - G \ := \ \{f\} - G$. *Also,* $G - f \ := \ G - \{f\}$.
Also, $f \bullet G \ := \ \{f\} \bullet G$. *Also,* $G \bullet f \ := \ G \bullet \{f\}$.

DEFINITION 5.1.65. *Let F be a set of v/s-functions,* $B \subseteq \bigcup \mathrm{ES}$.
Then $F + B \ := \ \{f + v \,|\, f \in F,\, v \in B\}$.
Also, $B + F \ := \ \{v + f \,|\, f \in F,\, v \in B\}$.
Also, $F - B \ := \ \{f - v \,|\, f \in F,\, v \in B\}$.
Also, $B - F \ := \ \{v - f \,|\, f \in F,\, v \in B\}$.
Also, $F \bullet B \ := \ \{f \bullet v \,|\, f \in F,\, v \in B\}$.
Also, $B \bullet F \ := \ \{v \bullet f \,|\, f \in F,\, v \in B\}$.

DEFINITION 5.1.66. *Let f be a v/s-function,* $B \subseteq \bigcup \mathrm{ES}$.
Then $f + B \ := \ \{f\} + B$. *Also,* $B + f \ := \ B + \{f\}$.
Also, $f - B \ := \ \{f\} - B$. *Also,* $B - f \ := \ B - \{f\}$.
Also, $f \bullet B \ := \ \{f\} \bullet B$. *Also,* $B \bullet f \ := \ B \bullet \{f\}$.

DEFINITION 5.1.67. *Let F be a set of v/s-functions,* $v \in \bigcup \mathrm{ES}$.
Then $F + v \ := \ F + \{v\}$. *Also,* $v + F \ := \ \{v\} + F$.
Also, $F - v \ := \ F - \{v\}$. *Also,* $v - F \ := \ \{v\} - F$.
Also, $F \bullet v \ := \ F \bullet \{v\}$. *Also,* $v \bullet F \ := \ \{v\} \bullet F$.

DEFINITION 5.1.68. *Let F be a set of functionals.*
Let G be a set of v/s-functions.
Then $F \cdot G \ := \ \{f \cdot g \,|\, f \in A,\, g \in B\}$.
Also $G \cdot F \ := \ \{g \cdot f \,|\, f \in A,\, g \in B\}$.

DEFINITION 5.1.69. *Let f be a functional.*
Let G be a set of v/s-functions.
Then $f \cdot G \ := \ \{f\} \cdot G$. *Also* $G \cdot f \ := \ G \cdot \{f\}$.

DEFINITION 5.1.70. *Let F be a set of v/s-functions.*
Let g be a functional.
Then $F \cdot g \ := \ F \cdot \{g\}$. *Also* $g \cdot F \ := \ \{g\} \cdot F$.

DEFINITION 5.1.71. *Let* $A \subseteq \mathbb{R}$, G *a set of v/s-functions.*
Then $A \cdot G \ := \ \{c \cdot g \,|\, c \in A,\ g \in G\}$.
Also, $G \cdot A \ := \ \{g \cdot c \,|\, c \in A,\ g \in G\}$.

DEFINITION 5.1.72. *Let* $c \in \mathbb{R}$, G *a set of v/s-functions.*
Then $c \cdot G \ := \ \{c\} \cdot G$. *Also,* $G \cdot c \ := \ G \cdot \{c\}$.

DEFINITION 5.1.73. *Let* $A \subseteq \mathbb{R}$, g *a v/s-function.*
Then $A \cdot g \ := \ A \cdot \{g\}$. *Also,* $g \cdot A \ := \ \{g\} \cdot A$.

In all of the preceding defitions, keep in mind that
$\cdot$, $\bullet$ and $\bullet$ are all commutative, but $-$ is not.
So, for example, in the last definition, $g \cdot A = A \cdot g$.
We also have a dot product on any matrix space:

DEFINITION 5.1.74. *Let* $p, q \in \mathbb{N}$ *and let* $A, B \in \mathbb{R}^{p \times q}$.

$$\text{Then } A \bullet B \ := \ \sum_{i=1}^{p} \sum_{j=1}^{q} A_{ij} \cdot B_{ij}.$$

Keep in mind that this dot prouct is scalar-valued.
It is *NOT* the same as matrix multiplication.

5.2. Basics of dot product and the standard norm.

THEOREM 5.2.1. $\forall V \in \mathrm{ES}$, $\forall x \in V$, $\quad x \bullet x = \sum_{j \in \mathcal{I}_V} x_j^2$.

THEOREM 5.2.2. *Let* $V \in \mathrm{ES}$ *and let* $x \in V$.
Then: $[\ x \bullet x \geqslant 0\ ]$ & $[\ (x \bullet x = 0) \ \Leftrightarrow \ (\ x = 0_V\ )\ ]$.

DEFINITION 5.2.3. $\forall x \in \bigcup \mathrm{ES}$, $|x| := \sqrt{x \bullet x}$.

The notation "$|x|$" is read "the norm of x".

THEOREM 5.2.4. *We have:* $|(5, -2, 1)| = \sqrt{25 + 4 + 1} = \sqrt{30}$.

The next theorem is expressed by saying:
$|\bullet|$ separates zero.

THEOREM 5.2.5. *Let* $V \in \mathrm{ES}$, $x \in V$.
Then: $(\ |x| = 0\ ) \ \Leftrightarrow \ (\ x = 0_V\ )$.

THEOREM 5.2.6. *Let* $x \in \bigcup \mathrm{ES}$ *and let* $c \in \mathbb{R}$.
Then: $\quad (c \cdot x) \bullet (c \cdot x) = c^2 \cdot (x \bullet x)$.
Also: $\quad |c \cdot x| = |c| \cdot |x|$.

The formula $|c \cdot x| = |c| \cdot |x|$ is expressed by saying:

$|\bullet|$ is absolute homogeneous.

THEOREM 5.2.7. *Let $V \in \mathrm{ES}$ and let $j \in \mathcal{I}_V$.*
Then: $(\ \varepsilon_j^V \bullet \varepsilon_j^V = 1\ )\ \&\ (\ |\varepsilon_j^V| = 1\ )$.

DEFINITION 5.2.8. *Let $V \in \mathrm{ES}$. Then $S_V := \{x \in V \text{ s.t. } |x| = 1\}$.*

In the last definition, S_V is called the **unit sphere** in V.
Also, the elements of S_V are called **unit vectors**.
Every standard basis vector is a unit vector:

THEOREM 5.2.9. *Let $V \in \mathrm{ES}$, $j \in \mathcal{I}_V$. Then $\varepsilon_j^V \in S_V$.*

THEOREM 5.2.10. *Let $V \in \mathrm{ES}$. Then $S_V \neq \varnothing$.*

The next result asserts that every Euclidean vector can be written as its norm times a unit vector; such a form for the vector is called a "polar form". Unassigned HW: Show that every nonzero Euclidean vector has a unique polar form, but, for zero, the form is not unique.

THEOREM 5.2.11. *Let $V \in \mathrm{ES}$ and let $x \in V$.*
Then $\exists u \in S_V$ s.t. $x = |x| \cdot u$.

Proof. Either (1) $x = 0_V$ or (2) $x \neq 0_V$.

Case (1): Since $x = 0_V$, we get: $|x| = 0$.
Since $S_V \neq \varnothing$, choose $u \in S_V$.
Then $u \in S_V$ and we wish to show: $x = |x| \cdot u$.
We have $x = 0_V = 0 \cdot u = |x| \cdot u$, as desired.
End of Case (1).

Case (2): Since $x \neq 0_V$, we get: $|x| \neq 0$. Then $(1/|x|) \cdot |x| = 1$.
Since $|x| \geqslant 0$ and $|x| \neq 0$, we get $|x| > 0$. Then $1/|x| > 0$.
Let $a := 1/|x|$. Then $a > 0$. Then $|a| = a$. Let $u := a \cdot x$.
We have $a \cdot |x| = (1/|x|) \cdot |x| = 1$. Then $|u| = |a \cdot x| = |a| \cdot |x| = a \cdot |x| = 1$.
Then $u \in S_V$. Want: $x = |x| \cdot u$.
We have $x = 1 \cdot x = (a \cdot |x|) \cdot x = |x| \cdot (a \cdot x) = |x| \cdot u$, as desired.
End of Case (2). □

We also have a standard norm on any matrix space:

DEFINITION 5.2.12. *Let $p, q \in \mathbb{N}$ and let $A \in \mathbb{R}^{p \times q}$.*
Then $|A| := \sqrt{A \bullet A}$.

5.3. **Cauchy-Schwarz.** The next theorem is **Weak Cauchy-Schwarz**:

THEOREM 5.3.1. *Let* $V \in \mathrm{ES}$, $x, y \in V$. *Then* $x \bullet y \leqslant |x| \cdot |y|$.

Proof. Choose $t, u \in S_V$ s.t. $(\, x = |x| \cdot t \,) \,\&\, (\, y = |y| \cdot u \,)$.
Want: $(|x| \cdot t) \bullet (|y| \cdot u) \leqslant |x| \cdot |y|$.
By bilinearity, we have: $(|x| \cdot t) \bullet (|y| \cdot u) = |x| \cdot |y| \cdot (t \bullet u)$.
Want: $|x| \cdot |y| \cdot (t \bullet u) \leqslant |x| \cdot |y|$. Want: $t \bullet u \leqslant 1$.
Since $t, u \in S_V$, we get: $|t| = 1 = |u|$.
Then: $0 \leqslant |t - u|^2 = (t - u) \bullet (t - u) = (t \bullet t) - 2 \cdot (t \bullet u) + (u \bullet u)$
$= |t|^2 - 2 \cdot (t \bullet u) + |u|^2 = 1^2 - 2 \cdot (t \bullet u) + 1^2 = 2 - 2 \cdot (t \bullet u)$.
Then $0 \leqslant 2 - 2 \cdot (t \bullet u)$, so $2 \cdot (t \bullet u) \leqslant 2$, so $t \bullet u \leqslant 1$. □

The next theorem is **Cauchy-Schwarz**:

THEOREM 5.3.2. *Let* $V \in \mathrm{ES}$, $x, y \in V$. *Then* $|x \bullet y| \leqslant |x| \cdot |y|$.

Proof. Since $|x \bullet y| = \max\{\, x \bullet y \,,\, -(x \bullet y) \,\}$,
it suffices to show $(\; x \bullet y \leqslant |x| \cdot |y| \;) \,\&\, (\; -(x \bullet y) \leqslant |x| \cdot |y| \;)$.
By Weak Cauchy-Schwarz, $x \bullet y \leqslant |x| \cdot |y|$. Want: $-(x \bullet y) \leqslant |x| \cdot |y|$.
By Weak Cauchy-Schwarz, $(-x) \bullet y \leqslant |-x| \cdot |y|$.
We have $(-x) \bullet y = ((-1) \cdot x) \bullet y = (-1) \cdot (x \bullet y) = -(x \bullet y)$.
We have $|-x| = |(-1) \cdot x| = |-1| \cdot |x| = 1 \cdot |x| = |x|$.
Then $-(x \bullet y) = (-x) \bullet y \leqslant |-x| \cdot |y| = |x| \cdot |y|$, as desired. □

5.4. **The 1-norm.**

DEFINITION 5.4.1. $\forall V \in \mathrm{ES}$, $\forall x \in V$, $\|x\| := \sum_{j \in \mathcal{I}_V} |x_j|$.

The quantity $\|x\|$ is called the **one-norm** of x.

THEOREM 5.4.2. *Let* $x := (5, -2, 1)$.
Then $|x| = \sqrt{25 + 4 + 1} = \sqrt{30}$ *and* $\|x\| = 5 + 2 + 1 = 8$.

THEOREM 5.4.3. $\forall$*finite set* I, $\forall a \in [0; \infty)^I$, *we have:*

$$\left(\sum_{j \in I} a_j \right)^2 \geqslant \sum_{i \in I} a_i^2 .$$

Proof. We have:

$$\left(\sum_{j \in I} a_j \right)^2 = \left[\sum_{j \in I} a_j \right] \cdot \left[\sum_{k \in I} a_k \right] = \sum_{j \in I} \sum_{k \in I} a_j a_k \geqslant \sum_{i \in I} a_i a_i = \sum_{i \in I} a_i^2 .$$ □

THEOREM 5.4.4. *Let* $V \in \mathrm{ES}$, $m := \# \mathcal{I}_V$, $x \in V$.
Then: $|x| \leqslant \|x\| \leqslant \sqrt{m} \cdot |x|$.

Proof. We have:

$$\|x\|^2 = \left(\sum_{j\in\mathcal{I}_V} |x_j|\right)^2 \geqslant \sum_{j\in\mathcal{I}_V} |x_j|^2 = \sum_{j\in\mathcal{I}_V} x_j^2 = |x|^2.$$

Then $0 \leqslant |x|^2 \leqslant \|x\|^2$, so $\sqrt{|x|^2} \leqslant \sqrt{\|x\|^2}$, so $|x| \leqslant \|x\|$.
It remains to show: $\|x\| \leqslant \sqrt{m} \cdot |x|$.
We have: $\forall j \in \mathcal{I}_V$, $\exists a \in \{-1,1\}$ s.t. $a \cdot x_j = |x_j|$.
For all $j \in \mathcal{I}_V$, let $A_j := \{a \in \{-1,1\}$ s.t. $a \cdot x_j = |x_j|\}$.
Then we have: $\forall j \in \mathcal{I}_V$, $A_j \neq \varnothing$.
Choose $s \in V$ s.t., $\forall j \in \mathcal{I}_V$, $s_j \in A_j$.
Then $\forall j \in \mathcal{I}_V$, we have $s_j \cdot x_j = |x_j|$.
Also, $\forall j \in \mathcal{I}_V$, we have $s_j \in A_j \subseteq \{-1,1\}$, and so $s_j^2 = 1$.
Then $|s|^2 = \sum_{j\in\mathcal{I}_V} s_j^2 = \sum_{j\in\mathcal{I}_V} 1 = \#\mathcal{I}_V = m$. Then $|s| = \sqrt{|s|^2} = \sqrt{m}$.
By Weak Cauchy-Schwarz, $s \bullet x \leqslant |s| \cdot |x|$.
Then: $\|x\| = \sum_{j\in\mathcal{I}_V} |x_j| = \sum_{j\in\mathcal{I}_V} s_j \cdot x_j = s \bullet x \leqslant |s| \cdot |x| = \sqrt{m} \cdot |x|$. □

THEOREM 5.4.5. *Let $a, b \in \mathbb{R}$.*
Then: $\sqrt{a^2+b^2} \leqslant |a| + |b| \leqslant \sqrt{2} \cdot \sqrt{a^2+b^2}$.

Proof. By Theorem 5.4.4,

$$|(a,b)| \leqslant \|(a,b)\| \leqslant \sqrt{2} \cdot |(a,b)|.$$

Then: $\sqrt{a^2+b^2} \leqslant |a| + |b| \leqslant \sqrt{2} \cdot \sqrt{a^2+b^2}$. □

We also have a one-norm on any matrix space:

DEFINITION 5.4.6. *Let $p, q \in \mathbb{N}$ and let $A \in \mathbb{R}^{p\times q}$.*

$$\textit{Then } \|A\| := \sum_{i=1}^{p} \sum_{j=1}^{q} |A_{ij}|.$$

5.5. The standard metrics on a Euclidean space.

THEOREM 5.5.1. *Let $V \in \mathrm{ES}$ and let $x, y \in V$.*
Then $|x+y| \leqslant |x| + |y|$.

Proof. By weak Cauchy-Schwarz, $x \bullet y \leqslant |x| \cdot |y|$.
Then $2 \cdot (x \bullet y) \leqslant 2 \cdot |x| \cdot |y|$.
Then $|x|^2 + 2 \cdot (x \bullet y) + |y|^2 \leqslant |x|^2 + 2 \cdot |x| \cdot |y| + |y|^2$.
Then

$$\begin{aligned} |x+y|^2 &= (x+y) \bullet (x+y) \\ &= (x \bullet x) + 2 \cdot (x \bullet y) + (y \bullet y) \\ &= |x|^2 + 2 \cdot (x \bullet y) + |y|^2 \\ &\leqslant |x|^2 + 2 \cdot |x| \cdot |y| + |y|^2 = (|x| + |y|)^2. \end{aligned}$$

Then $0 \leqslant |x+y|^2 \leqslant (|x|+|y|)^2$, so $\sqrt{|x+y|^2} \leqslant \sqrt{(|x|+|y|)^2}$.
Since $|x+y| \geqslant 0$ and $|x|+|y| \geqslant 0$, we conclude:

$$\sqrt{|x+y|^2} = |x+y| \qquad \text{and} \qquad \sqrt{(|x|+|y|)^2} = |x|+|y|.$$

Then $|x+y| = \sqrt{|x+y|^2} \leqslant \sqrt{(|x|+|y|)^2} = |x|+|y|$, as desired. □

THEOREM 5.5.2. *Let* $V \in \mathrm{ES}$.
Define $d : V \times V \to [0;\infty)$ *by:* $\forall x, y \in V,\ \ d(x,y) = |y-x|$.
Then: $d \in \mathcal{M}(V)$.

Proof. We wish to show:

$$(1)\ \forall x, y \in V,\ \ (\, d(x,y) = 0\,) \Leftrightarrow (\, x = y\,).$$

and $(2)\ \forall x, y \in V,\ \ d(x,y) = d(y,x)$.

and $(3)\ \forall x, y, z \in V,\ \ d(x,z) \leqslant [\, d(x,y)\,] + [\, d(y,z)\,]$.

Proof of (1): Given $x, y \in V$. Want: $(\, d(x,y) = 0\,) \Leftrightarrow (\, x = y\,)$.
We have: $(\, d(x,y) = 0\,) \Leftrightarrow (\, |y-x| = 0\,) \Leftrightarrow (\, y-x = 0_V\,) \Leftrightarrow (\, x = y\,)$.
End of proof of (1).

Proof of (2): Given $x, y \in V$. Want: $d(x,y) = d(y,x)$.
We have: $d(x,y) = |y-x| = |(-1)\cdot(x-y)|$
$= |-1|\cdot|x-y| = 1\cdot[d(y,x)] = d(y,x)$.
End of proof of (2).

Proof of (3): Given $x, y, z \in V$. Want: $d(x,z) \leqslant [\, d(x,y)\,] + [\, d(y,z)\,]$.
We have $d(x,z) = |z-x| = |\,(z-y) + (y-x)\,|$
$\leqslant |z-y| + |y-x| = [\, d(y,z)\,] + [\, d(x,y)\,]$
$= [\, d(x,y)\,] + [\, d(y,z)\,]$.
End of proof of (3). □

Notation: Let $V \in \mathrm{ES}$ and define $d : V \times V \to [0;\infty)$ by:

$$\forall x, y \in V, \quad d(x,y) = |y-x|.$$

Then the metric space (V,d) is denoted V.

THEOREM 5.5.3. *Let* $V \in \mathrm{ES}$ *and let* $x, y \in V$.
Then $d_V(x,y) = |y-x|$.

5.6. Multivariable polynomials.

DEFINITION 5.6.1. *Let* X *and* Y *be sets. Then* $\mathcal{C}_Y^X := \{C_X^z \mid z \in Y\}$.

Then $\mathcal{C}_Y^X$ is the set of constant functions from X to Y.

THEOREM 5.6.2. *Let* $V := \mathbb{R}^2$, $W := \mathbb{R}^3$, $\alpha := \varepsilon_1^W$, $C := C_V^{8\alpha}$.
Then $C \in \mathcal{C}_W^V$.

DEFINITION 5.6.3. *Let* $V, W \in \mathrm{ES}$ *and let* $k \in \mathbb{N}_0$.
Then $\mathcal{M}_k^{VW} := \mathcal{C}_W^V \cdot \Pi_V \cdot \Pi_V \cdot \cdots \cdot \Pi_V$,
with k *copies of* Π_V.

The logic purist would dislike the preceding definition, preferring:

DEFINITION 5.6.4. *Let* $V, W \in \mathrm{ES}$ *and let* $\mathcal{F} := W^V$.
Define $\Phi : 2^{\mathcal{F}} \to 2^{\mathcal{F}}$ *by:* $\forall S \subseteq \mathcal{F}$, $\Phi_S = S \cdot \Pi_V$.
Then, for all $k \in \mathbb{N}_0$, *we define:* $\mathcal{M}_k^{VW} := \Phi_\circ^k(\mathcal{C}_W^V)$.

Elements of $\mathcal{M}_k^{VW}$ are called k-**monomials** from V to W.

THEOREM 5.6.5. *Let* $V := \mathbb{R}^2$ *and let* $W := \mathbb{R}^3$.
Define $f : V \to W$ *by:* $\forall x, y \in \mathbb{R}$, $f(x, y) = (8x^3y, 0, 0)$.
Then $f \in \mathcal{M}_4^{VW}$.

Proof. Let $\alpha := \varepsilon_1^W$, $C := C_V^{8\alpha}$, $X := \pi_1^V$, $Y := \pi_2^V$.
Then $f = C \cdot X^3 \cdot Y = C \cdot X \cdot X \cdot X \cdot Y \in \mathcal{C}_W^V \cdot \pi_V \cdot \pi_V \cdot \pi_V \cdot \pi_V = \mathcal{M}_4^{VW}$. □

DEFINITION 5.6.6. *Let* V *be a set, let* $W \in \mathrm{ES}$ *and let* $S \subseteq W^V$.
By S *is* **v.op.-closed**, *we mean:*
$S + S \subseteq S$ *and* $\mathbb{R} \cdot S \subseteq S$.

That is, S is closed under vector addition and scalar multiplication.

We leave it to the reader to verify that,
$\forall$set V, $\forall W \in \mathrm{ES}$, $\mathcal{C}_W^V$ is v.op.-closed.
In particular, we get closure under scalar multiplication:
$\forall$set V, $\forall W \in \mathrm{ES}$, $\mathbb{R} \cdot \mathcal{C}_W^V \subseteq \mathcal{C}_W^V$.
Let $X := \mathbb{R}^2$. We also leave it to the reader to verify that,
$$\mathcal{M}_1^{X\mathbb{R}} + \mathcal{M}_1^{X\mathbb{R}} \nsubseteq \mathcal{M}_1^{X\mathbb{R}},$$
so $\mathcal{M}_1^{X\mathbb{R}}$ is *NOT* v.op.-closed.
However, $\mathcal{M}_1^{X\mathbb{R}}$ *IS* closed under scalar multiplication:

THEOREM 5.6.7. *Let* $V, W \in \mathrm{ES}$ *and let* $k \in \mathbb{N}_0$.
Then $\mathbb{R} \cdot \mathcal{M}_k^{VW} \subseteq \mathcal{M}_k^{VW}$.

Proof. Let $\mathcal{A} := \Pi_V \cdot \Pi_V \cdot \cdots \cdot \Pi_V$, with k copies of Π_V.
Then $\mathcal{M}_k^{VW} = \mathcal{C}_W^V \cdot \mathcal{A}$. Then $\mathbb{R} \cdot \mathcal{M}_k^{VW} = \mathbb{R} \cdot \mathcal{C}_W^V \cdot \mathcal{A} \subseteq \mathcal{C}_W^V \cdot \mathcal{A} = \mathcal{M}_k^{VW}$. □

DEFINITION 5.6.8. *Let $V, W \in \mathrm{ES}$ and let $k \in \mathbb{N}_0$.*
Then $\mathcal{H}_k^{VW} \;:=\; \{\phi_1 + \cdots + \phi_m \,|\, m \in \mathbb{N},\ \phi_1, \ldots, \phi_m \in \mathcal{M}_k^{VW}\}$.

Elements of $\mathcal{H}_k^{VW}$ are called k-**polynomials** from V to W.
So a k-polynomial is, by definition, a sum of k-monomials.

Another perspective: The function space $\mathcal{H}_k^{VW}$ is exactly
the closure of $\mathcal{M}_k^{VW}$ under vector operations.
Since $\mathcal{M}_k^{VW}$ was already closed under scalar multiplication,
we only needed to close it under vector addition for v.op.-closure.
In particular, we record:

THEOREM 5.6.9. *Let $V, W \in \mathrm{ES}$ and let $k \in \mathbb{N}_0$.*
Then $\mathcal{H}_k^{VW}$ is v.op.-closed.

THEOREM 5.6.10. *Let $V := \mathbb{R}^2$ and $W := \mathbb{R}^3$.*
Define $g : V \to W$ by: $\forall x, y \in \mathbb{R}$,

$$g(x,y) \quad = \quad (\ x^4 + 6x^3y\ ,\ 7x^2y^2 + \sqrt{2}xy^3\ ,\ 9x^4 + y^4\ ).$$

Then: $\quad g \in \mathcal{H}_4^{VW}$.

Proof. Let $\quad \alpha := \varepsilon_1^W, \quad \beta := \varepsilon_2^W, \quad \gamma := \varepsilon_3^W, \quad X := \pi_1^V, \quad Y := \pi_2^V.$
Then

$$\begin{aligned} g \quad = \quad & C_V^{\alpha} \cdot X^4 && + \quad C_V^{3\alpha} \cdot X^3 \cdot Y \\ + \quad & C_V^{7\beta} \cdot X^2 \cdot Y^2 && + \quad C_V^{\sqrt{2}\beta} \cdot X \cdot Y^3 \\ + \quad & C_V^{9\gamma} \cdot X^4 && + \quad C_V^{\gamma} \cdot Y^4, \end{aligned}$$

so $g \in \mathcal{M}_4^{VW} + \mathcal{M}_4^{VW} + \mathcal{M}_4^{VW} + \mathcal{M}_4^{VW} + \mathcal{M}_4^{VW} + \mathcal{M}_4^{VW} \subseteq \mathcal{H}_4^{VW}$. □

THEOREM 5.6.11. *Let $V, W \in \mathrm{ES}$. Then $\mathcal{C}_W^V = \mathcal{M}_0^{VW} = \mathcal{H}_0^{VW}$.*

DEFINITION 5.6.12. *Let $V, W \in \mathrm{ES}$.*
Then: $\quad \mathcal{L}_W^V := \mathcal{H}_1^{VW} \quad$ *and* $\quad \mathcal{Q}_W^V := \mathcal{H}_2^{VW} \quad$ *and* $\quad \mathcal{K}_W^V := \mathcal{H}_3^{VW}$.

Let $V, W \in \mathrm{ES}$. Then:
element of $\mathcal{C}_W^V$ are called constant functions from V to W and
element of $\mathcal{L}_W^V$ are called linear functions from V to W and
element of $\mathcal{K}_W^V$ are called cubic functions from V to W.
We next argue that linear is the same as "algebraically linear":

THEOREM 5.6.13. *Let $V, W \in \mathrm{ES}$ and let $f : V \to W$.*
Then:

$$\begin{aligned} (\ f \in \mathcal{L}_W^V\ ) \quad & \Leftrightarrow \\ (\qquad & [\ \forall c \in \mathbb{R}, \forall x \in V, \quad f_{c \cdot x} = c \cdot f_x\ ] \\ \& \quad & [\ \forall x, y \in V, \qquad f_{x+y} = f_x + f_y\ ] \quad). \end{aligned}$$

Proof. Unassigned HW. □

The next two theorems show that constants and linear functions are all Lipschitz.
Consequently, they are all uniformly continuous, and, therefore, continuous.

THEOREM 5.6.14. *Let V and W be metric spaces, and let $C \in \mathcal{C}_W^V$. Then C is Lipschitz-0 from V to W.*

Proof. Unassigned HW. □

THEOREM 5.6.15. *Let $V, W \in \mathrm{ES}$ and let $L \in \mathcal{L}_W^V$. Then L is Lipschitz from V to W.*

Proof. Want: $\exists K \geqslant 0$ s.t. L is Lipschitz-K from V to W.
Let $A := \max\{\,|L_{\varepsilon_j^V}| \text{ s.t. } j \in \mathcal{I}_V\,\}$, $m := \#\mathcal{I}_V$, $K := A \cdot \sqrt{m}$.
Then $K \geqslant 0$. Want: L is Lipschitz-K from V to W.
Want: $\forall x, y \in V$, $d(L_x, L_y) \leqslant K \cdot [d(x,y)]$.
Given $x, y \in V$. Want: $d(L_x, L_y) \leqslant K \cdot [d(x,y)]$.
Let $I := \mathcal{I}_V$. For all $j \in I$, let $e_j := \varepsilon_j^V$. For all $j \in I$, let $q_j := L_{e_j}$.
Since $x = \sum\limits_{j \in I} x_j \cdot e_j$ and L is algebraically linear, we get: $L_x = \sum\limits_{j \in I} x_j \cdot q_j$.
Since $y = \sum\limits_{j \in I} y_j \cdot e_j$ and L is algebraically linear, we get: $L_y = \sum\limits_{j \in I} y_j \cdot q_j$.
By definition of A, we have:

$$\forall j \in I, \qquad |q_j| \leqslant A.$$

By Cauchy-Schwarz, we have:

$$\forall j \in I, \qquad |(y_j - x_j) \cdot q_j| \leqslant |y_j - x_j| \cdot |q_j|.$$

We have $\|y - x\| = \sum\limits_{j \in I} |y_i - x_i|$ and $\|y - x\| \leqslant \sqrt{m} \cdot |y - x|$.

$$\begin{aligned}
\text{Then } d(L_x, L_y) &= |L_y - L_x| = \left| \left(\sum_{j \in I} y_j q_j \right) - \left(\sum_{j \in I} x_j q_j \right) \right| \\
&= \left| \sum_{j \in I} (y_j q_j - x_j \cdot q_j) \right| \\
&= \left| \sum_{j \in I} \Big((y_j - x_j) \cdot q_j \Big) \right| \\
&\leqslant \sum_{j \in I} |(y_j - x_j) \cdot q_j| \\
&\leqslant \sum_{j \in I} \Big(|y_j - x_j| \cdot |q_j| \Big)
\end{aligned}$$

$$\begin{array}{llll} \leqslant & \displaystyle\sum_{j\in I}\Big(|y_j - x_j|\cdot A\Big) & = & \Big(\displaystyle\sum_{j\in I}|y_j - x_j|\Big)\cdot A \\ = & \|y-x\|\cdot A & \leqslant & \sqrt{m}\cdot|y-x|\cdot A \\ = & A\cdot\sqrt{m}\cdot|y-x| & = & K\cdot[d(x,y)]. \qquad \square \end{array}$$

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5.7. **Bilinear multiplications.**

DEFINITION 5.7.1. *Let U, V and W be sets and let $* : U\times V \to W$. Then, $\forall x\in U$, $\forall y\in V$, we denote $*(x,y)$ by $x*y$.*
Also, $\forall x\in U$, the function $x*\bullet : V\to W$ is defined by:

$$\forall y\in V, \quad (x*\bullet)_y = x*y.$$

Also, $\forall y\in V$, the function $\bullet*y : U\to W$ is defined by:

$$\forall x\in U, \quad (\bullet*y)_x = x*y.$$

DEFINITION 5.7.2. *Let $U,V,W\in\mathrm{ES}$. Then:*

$$\mathcal{B}_W^{UV} \;:=\; \{\; * : U\times V\to W \;\mid\; (\forall x\in U,\ x*\bullet\in\mathcal{L}_W^V) \;\&\; (\forall y\in V,\ \bullet*y\in\mathcal{L}_W^U) \;\}.$$

THEOREM 5.7.3. *Define $* : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by:*

$$\forall a,b\in\mathbb{R},\ a*b = a\cdot b.$$

Then $*\in\mathcal{B}_{\mathbb{R}}^{\mathbb{R}\mathbb{R}}$.

THEOREM 5.7.4. *Let $V\in\mathrm{ES}$. Define $* : \mathbb{R}\times V\to V$ by:*

$$\forall a\in\mathbb{R},\ \forall x\in V,\ a*x = a\cdot x.$$

Then $*\in\mathcal{B}_V^{V\mathbb{R}}$.

THEOREM 5.7.5. *Let $V\in\mathrm{ES}$. Define $* : V\times V\to\mathbb{R}$ by:*

$$\forall x,y\in V,\ x*y = x\bullet y.$$

Then $*\in\mathcal{B}_{\mathbb{R}}^{VV}$.

THEOREM 5.7.6. *Let $U := \mathbb{R}^3$, $V := \mathbb{R}^4$, $W := \mathbb{R}$ and*

$$A := \begin{bmatrix} 0 & 2\pi & 4 & 6 \\ -1 & 3 & -5 & \sqrt{7} \\ -2 & 7 & 8 & 9 \end{bmatrix}.$$

Define $* : U\times V\to W$ by:

$$\forall x\in U,\ \forall y\in V, \quad x*y \;=\; \sum_{j=1}^{3}\sum_{k=1}^{4} x_j\cdot A_{jk}\cdot y_k.$$

Then $*\in\mathcal{B}_W^{UV}$.

Continuing in the theme of the last theorem, there is a map

$$\mathbb{R}^{3\times4} \;\to\; \mathcal{B}_{\mathbb{R}}^{\mathbb{R}^3,\mathbb{R}^4}.$$

It is not hard to show that this map is

algebraically linear and bijective onto $\mathcal{B}_{\mathbb{R}}^{\mathbb{R}^3,\mathbb{R}^4}$.

More generally, $\forall p,q \in \mathbb{N}$, there is an algebraically linear bijection

$\mathbb{R}^{p\times q} \hookrightarrow\!> \mathcal{B}_{\mathbb{R}}^{\mathbb{R}^p,\mathbb{R}^q}$.

Also, $\forall p,q,m \in \mathbb{N}$, there is an algebraically linear bijection

$\mathbb{R}^{p\times q\times m} \hookrightarrow\!> \mathcal{B}_{\mathbb{R}^m}^{\mathbb{R}^p,\mathbb{R}^q}$.

In this course, we will content ourselves with quadratic approximations (the Second Order Taylor Theorem), which means that we will use many bilinar maps. Were we to focus on cubic approximations (Third Order Taylor), we would need to develop the bookkeeping of trilinear maps. Quartic and Higher order polynomial approximations would require quadrilinear and higher order multilinear maps.

Just as we showed that linear maps are Lipschitz (Theorem 5.6.15), we now show that bilinear maps are "doubly Lipschitz":

THEOREM 5.7.7. *Let $U,V,W \in$ ES and let $* \in \mathcal{B}_W^{UV}$.*
Then: $\exists L \geqslant 0$ s.t., $\forall x \in U$, $\forall y \in V$, $|x * y| \leqslant L \cdot |x| \cdot |y|$.

Proof. Let $\ell := \#\mathcal{I}_U$, $m := \#\mathcal{I}_V$, $J := \mathcal{I}_V$, $K := \mathcal{I}_W$.
Let $A := \max\{\,|\varepsilon_j^U * \varepsilon_k^V| \text{ s.t. } j \in J,\, k \in K,\}$.
Let $L := A \cdot \sqrt{\ell} \cdot \sqrt{m}$. Then $L \geqslant 0$.

Want: $\forall x \in U$, $\forall y \in V$, $|x * y| \leqslant L \cdot |x| \cdot |y|$.

Given $x \in U$, $y \in V$. Want: $|x * y| \leqslant L \cdot |x| \cdot |y|$.
By the reproducing formula, $x = \sum\limits_{j\in J} x_j \varepsilon_j^U$ and $y = \sum\limits_{k\in K} y_k \varepsilon_k^V$.

Then: $x * y = \left(\sum\limits_{j\in J} x_j \varepsilon_j^U\right) * \left(\sum\limits_{j\in J} y_k \varepsilon_k^V\right)$.

By bilinearity, $\left(\sum\limits_{j\in J} x_j \varepsilon_j^U\right) * \left(\sum\limits_{j\in J} y_k \varepsilon_k^V\right) = \sum\limits_{j\in J}\sum\limits_{k\in K} \left[x_j \cdot y_k \cdot (\varepsilon_j^U * \varepsilon_k^V)\right]$.

Then: $x * y = \sum\limits_{j\in J}\sum\limits_{k\in K} \left[x_j \cdot y_k \cdot (\varepsilon_j^U * \varepsilon_k^V)\right]$.

Then, by subadditivity, $|x * y| \leqslant \sum\limits_{j\in J}\sum\limits_{k\in K} |x_j \cdot y_k \cdot (\varepsilon_j^U * \varepsilon_k^V)|$.

Then, by absolute homogeneity, $|x * y| \leqslant \sum\limits_{j\in J}\sum\limits_{k\in K} (\,|x_j| \cdot |y_k| \cdot |\varepsilon_j^U * \varepsilon_k^V|\,)$.

Then, by choice of A, $|x * y| \leqslant \sum\limits_{j\in J}\sum\limits_{k\in K} (\,|x_j| \cdot |y_k| \cdot A\,)$.

By the distributive law, $\left(\sum_{j\in J}|x_j|\right)\cdot\left(\sum_{k\in K}|y_k|\right)=\sum_{j\in J}\sum_{k\in K}(\,|x_j|\cdot|y_k|\,)$.
So, by commutativity of multiplication and the distributive law,

$$A\cdot\left(\sum_{j\in J}|x_j|\right)\cdot\left(\sum_{k\in K}|y_k|\right)=\sum_{j\in J}\sum_{k\in K}(\,|x_j|\cdot|y_k|\cdot A\,).$$

So, since $\|x\|=\sum_{j\in J}|x_j|$ and $\|y\|=\sum_{k\in K}|y_k|$, we get

$$A\cdot\|x\|\cdot\|y\|=\sum_{j\in J}\sum_{k\in K}(\,|x_j|\cdot|y_k|\cdot A\,).$$

By Theorem 5.4.4, $\quad\|x\|\leqslant\sqrt{\ell}\cdot|x|\quad$ and $\quad\|y\|\leqslant\sqrt{m}\cdot|y|$.
Then

$$\begin{aligned}|x*y| &\leqslant \sum_{j\in J}\sum_{k\in K}(\,|x_j|\cdot|y_k|\cdot A\,)\\ &= A\cdot\|x\|\cdot\|y\| \quad\leqslant\quad A\cdot\sqrt{\ell}\cdot|x|\cdot\sqrt{m}\cdot|y|\\ &= L\cdot|x|\cdot|y|, \qquad \text{as desired.}\end{aligned}$$

□

DEFINITION 5.7.8. *Let $U,V,W\in\mathrm{ES}$ and let $*\in\mathcal{B}^{UV}_W$.*
Let f be a U-function and let g be a V-function.
Then $f*g$ is the W-function defined by: $\forall t,\ (f*g)_t=f_t*g_t$.

DEFINITION 5.7.9. *Let $U,V,W\in\mathrm{ES}$ and let $*\in\mathcal{B}^{UV}_W$.*
Let F be a set of U-functions and let G be a set of V-functions.
Then $F*G\ :=\ \{\,f*g\,|\,f\in F,\ g\in G\,\}$.

DEFINITION 5.7.10. *Let $U,V,W\in\mathrm{ES}$ and let $*\in\mathcal{B}^{UV}_W$.*
Let f be a U-function and let G be a set of V-functions.
Then $f*G\ :=\ \{\,f*g\,|\,g\in G\,\}$.

DEFINITION 5.7.11. *Let $U,V,W\in\mathrm{ES}$ and let $*\in\mathcal{B}^{UV}_W$.*
Let F be a set of U-functions and let g be a V-function.
Then $F*g\ :=\ \{\,f*g\,|\,f\in F\,\}$.

DEFINITION 5.7.12. *Let $U,V,W\in\mathrm{ES}$ and let $*\in\mathcal{B}^{UV}_W$.*
Let $X\subseteq U$-function and let G be a set of V-functions.
Then $X*G\ :=\ \{\,x*g\,|\,x\in X,\ g\in G\,\}$.

DEFINITION 5.7.13. *Let $U,V,W\in\mathrm{ES}$ and let $*\in\mathcal{B}^{UV}_W$.*
Let $x\in U$ and let g be a V-function.
Then $x*g$ is the W-function defined by: $\forall t,\ (x*g)_t=x*g_t$.

DEFINITION 5.7.14. *Let $U,V,W\in\mathrm{ES}$ and let $*\in\mathcal{B}^{UV}_W$.*
Let f be a U-function and let $y\in V$.
Then $f*y$ is the W-function defined by: $\forall t,\ (f*y)_t=f_t*y$.

DEFINITION 5.7.15. *Let $U, V, W \in \mathrm{ES}$ and let $* \in \mathcal{B}_W^{UV}$.*
Let F be a set of U-functions and let $Y \subseteq V$.
Then $F * Y := \{ f * y \,|\, f \in F,\, y \in Y \}$.

DEFINITION 5.7.16. *Let $U, V, W \in \mathrm{ES}$ and let $* \in \mathcal{B}_W^{UV}$.*
Let $X \subseteq U$ and let $Y \subseteq V$.
Then $X * Y := \{ x * y \,|\, x \in X,\, y \in Y \}$.

DEFINITION 5.7.17. *Let $U, V, W \in \mathrm{ES}$ and let $* \in \mathcal{B}_W^{UV}$.*
Let $x \in U$ and let $Y \subseteq V$.
Then $x * Y := \{ x * y \,|\, y \in Y \}$.

DEFINITION 5.7.18. *Let $U, V, W \in \mathrm{ES}$ and let $* \in \mathcal{B}_W^{UV}$.*
Let $X \subseteq U$ and let $y \in V$.
Then $X * y := \{ x * y \,|\, x \in X \}$.

THEOREM 5.7.19. *Let $S, U, V, W \in \mathrm{ES}$, $k, \ell \in \mathbb{N}_0$ and $* \in \mathcal{B}_W^{UV}$.*
Then: $\mathcal{M}_k^{SU} * \mathcal{M}_\ell^{SV} \subseteq \mathcal{M}_{k+\ell}^{SW}$.

Proof. Want: $\forall h \in \mathcal{M}_k^{SU} * \mathcal{M}_\ell^{SV}$, $h \in \mathcal{M}_{k+\ell}^{SW}$.
Given $h \in \mathcal{M}_k^{SU} * \mathcal{M}_\ell^{SV}$. Want: $h \in \mathcal{M}_{k+\ell}^{SW}$.
Since $h \in \mathcal{M}_k^{SU} * \mathcal{M}_\ell^{SV}$, choose $f \in \mathcal{M}_k^{SU}$ and $g \in \mathcal{M}_\ell^{SV}$ s.t. $h = f * g$.
As $f \in \mathcal{M}_k^{SU}$, choose $C \in \mathcal{C}_U^S$ and $\alpha_1, \ldots, \alpha_k \in \Pi_S$ s.t. $f = C \cdot \alpha_1 \cdots \alpha_k$.
As $g \in \mathcal{M}_\ell^{SV}$, choose $D \in \mathcal{C}_V^S$ and $\beta_1, \ldots, \beta_\ell \in \Pi_S$ s.t. $g = D \cdot \beta_1 \cdots \beta_\ell$.
By bilinearity of $*$, as $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell$ are all functionals, we get:
$$\forall t \in S, \qquad (f * g)_t = ((C * D) \cdot \alpha_1 \cdots \alpha_k \cdot \beta_1 \cdots \beta_\ell)_t.$$
Then $f * g = (C * D) \cdot \alpha_1 \cdots \alpha_k \cdot \beta_1 \cdots \beta_\ell$.
Since $C \in \mathcal{C}_U^S$ and $D \in \mathcal{C}_V^S$ and $* : U \times V \to W$, we conclude: $C * D \in \mathcal{C}_W^S$.
So, since $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \Pi_S$, we see that:
$$(C * D) \cdot \alpha_1 \cdots \alpha_k \cdot \beta_1 \cdots \beta_\ell \in \mathcal{M}_{k+\ell}^{SW}.$$
Then: $h = f * g = (C * D) \cdot \alpha_1 \cdots \alpha_k \cdot \beta_1 \cdots \beta_\ell \in \mathcal{M}_{k+\ell}^{SW}$, as desired. □

THEOREM 5.7.20. *Let $S, U, V, W \in \mathrm{ES}$, $k, \ell \in \mathbb{N}_0$ and $* \in \mathcal{B}_W^{UV}$.*
Then: $\mathcal{H}_k^{SU} * \mathcal{H}_\ell^{SV} \subseteq \mathcal{H}_{k+\ell}^{SW}$.

Proof. Want: $\forall h \in \mathcal{H}_k^{SU} * \mathcal{H}_\ell^{SV}$, $h \in \mathcal{H}_{k+\ell}^{SW}$.
Given $h \in \mathcal{H}_k^{SU} * \mathcal{H}_\ell^{SV}$. Want: $h \in \mathcal{H}_{k+\ell}^{SW}$.
Since $h \in \mathcal{H}_k^{SU} * \mathcal{H}_\ell^{SV}$, choose $f \in \mathcal{H}_k^{SU}$ and $g \in \mathcal{H}_\ell^{SV}$ s.t. $h = f * g$.
As $f \in \mathcal{M}_k^{SU}$, choose $m \in \mathbb{N}$, $\phi_1, \ldots, \phi_m \in \mathcal{M}_k^{SU}$ s.t. $f = \phi_1 + \cdots + \phi_m$.
As $g \in \mathcal{M}_\ell^{SV}$, choose $n \in \mathbb{N}$, $\psi_1, \ldots, \psi_n \in \mathcal{M}_k^{SV}$ s.t. $g = \psi_1 + \cdots + \psi_n$.
By bilinearity, we have: $\left(\sum_{j=1}^m \phi_j \right) * \left(\sum_{k=1}^n \psi_k \right) = \sum_{j=1}^m \sum_{k=1}^n (\phi_j * \psi_k)$,

Then: $h = f * g = \left(\sum_{j=1}^{m} \phi_j \right) * \left(\sum_{k=1}^{n} \psi_k \right) = \sum_{j=1}^{m} \sum_{k=1}^{n} (\phi_j * \psi_k)$.

It therefore suffices to show: $\forall j \in [1..m]$, $\forall k \in [1..n]$, $\phi_j * \psi_k \in \mathcal{M}_{k+\ell}^{SW}$.

Given $j \in [1..m]$, $k \in [1..n]$. Want: $\phi_j * \psi_k \in \mathcal{M}_{k+\ell}^{SW}$.

By Theorem 5.7.19, $\mathcal{M}_k^{SU} * \mathcal{M}_\ell^{SV} \subseteq \mathcal{M}_{k+\ell}^{SW}$.

Then: $\phi_j * \psi_k \in \mathcal{M}_k^{SU} * \mathcal{M}_\ell^{SV} \subseteq \mathcal{M}_{k+\ell}^{SW}$, as desired. □

THEOREM 5.7.21. *Let* $S, U \in \mathrm{ES}$, $k, \ell \in \mathbb{N}_0$.
Then: $\mathcal{H}_k^{SU} \bullet \mathcal{H}_\ell^{SU} \subseteq \mathcal{H}_{k+\ell}^{S\mathbb{R}}$.

Proof. Define $* \in \mathcal{B}_{\mathbb{R}}^{UU}$ by: $\forall x, y \in U$, $x * y = x \bullet y$.

Let $V := U$ and $W := \mathbb{R}$. Then $* \in \mathcal{B}_W^{UV}$.

By Theorem 5.7.20, we have: $\mathcal{H}_k^{SU} * \mathcal{H}_\ell^{SV} \subseteq \mathcal{H}_{k+\ell}^{SW}$.

Then $\mathcal{H}_k^{SU} \bullet \mathcal{H}_\ell^{SV} = \mathcal{H}_k^{SU} * \mathcal{H}_\ell^{SV} \subseteq \mathcal{H}_{k+\ell}^{SW} = \mathcal{H}_{k+\ell}^{S\mathbb{R}}$, as desired. □

THEOREM 5.7.22. *Let* $S \in \mathrm{ES}$, $k, \ell \in \mathbb{N}_0$.
Then: $\mathcal{H}_k^{S\mathbb{R}} \cdot \mathcal{H}_\ell^{S\mathbb{R}} \subseteq \mathcal{H}_{k+\ell}^{S\mathbb{R}}$.

Proof. Define $* \in \mathcal{B}_{\mathbb{R}}^{\mathbb{R}\mathbb{R}}$ by: $\forall a, b \in \mathbb{R}$, $a * b = a \cdot b$.

Let $V := \mathbb{R}$ and $W := \mathbb{R}$. Then $* \in \mathcal{B}_W^{UV}$.

By Theorem 5.7.20, we have: $\mathcal{H}_k^{SU} * \mathcal{H}_\ell^{SV} \subseteq \mathcal{H}_{k+\ell}^{SW}$.

Then $\mathcal{H}_k^{S\mathbb{R}} \cdot \mathcal{H}_\ell^{S\mathbb{R}} = \mathcal{H}_k^{SU} * \mathcal{H}_\ell^{SU} \subseteq \mathcal{H}_{k+\ell}^{SW} = \mathcal{H}_{k+\ell}^{W\mathbb{R}}$, as desired. □

THEOREM 5.7.23. *Let* $S \in \mathrm{ES}$, $k, \ell \in \mathbb{N}_0$. *Then:* $\forall m \in \mathbb{N}$,
$\forall f_1, \ldots, f_m \in \mathcal{H}_k^{S\mathbb{R}}$, $f_1 \cdots f_m \in \mathcal{H}_{km}^{S\mathbb{R}}$.

Proof. Unassigned HW. Hint: Use induction, starting with:
Let $A := \{m \in \mathbb{N} \mid \forall f_1, \ldots, f_m \in \mathcal{H}_k^{S\mathbb{R}},\ f_1 \cdots f_m \in \mathcal{H}_{km}^{S\mathbb{R}}\}$.
Want: $A = \mathbb{N}$. □

THEOREM 5.7.24. *Let* $S, U \in \mathrm{ES}$, $k, \ell \in \mathbb{N}_0$.
Then: $\mathcal{H}_k^{SU} \cdot \mathcal{H}_\ell^{S\mathbb{R}} \subseteq \mathcal{H}_{k+\ell}^{SU}$.

Proof. Define $* \in \mathcal{B}_U^{U\mathbb{R}}$ by: $\forall x \in U$, $\forall a \in \mathbb{R}$, $x * a = x \cdot a$.

Let $V := \mathbb{R}$ and $W := U$. Then $* \in \mathcal{B}_W^{UV}$.

By Theorem 5.7.20, we have: $\mathcal{H}_k^{SU} * \mathcal{H}_\ell^{SV} \subseteq \mathcal{H}_{k+\ell}^{SW}$.

Then $\mathcal{H}_k^{SU} \cdot \mathcal{H}_\ell^{S\mathbb{R}} = \mathcal{H}_k^{SU} * \mathcal{H}_\ell^{SV} \subseteq \mathcal{H}_{k+\ell}^{SW} = \mathcal{H}_{k+\ell}^{SU}$, as desired. □

THEOREM 5.7.25. *Let* $V, W, X \in \mathrm{ES}$, $k \in \mathbb{N}_0$.
Then: $\mathcal{L}_X^W \circ \mathcal{M}_k^{VW} \subseteq \mathcal{M}_k^{VX}$.

Proof. Want: $\forall h \in \mathcal{L}_X^W \circ \mathcal{M}_k^{VW}$, $h \in \mathcal{M}_k^{VX}$.

Given $h \in \mathcal{L}_X^W \circ \mathcal{M}_k^{VW}$. Want: $h \in \mathcal{M}_k^{VX}$.

Since $h \in \mathcal{L}_X^W \circ \mathcal{M}_k^{VW}$, choose $g \in \mathcal{L}_X^W$ and $f \in \mathcal{M}_k^{VW}$ s.t. $h = g \circ f$.
As $f \in \mathcal{M}_k^{VW}$, choose $C \in \mathcal{C}_W^V$ and $\alpha_1, \ldots, \alpha_k \in \Pi_V$ s.t. $f = C \cdot \alpha_1 \cdots \alpha_k$.
Since $g \in \mathcal{L}_X^W$, we know that g is algebrically linear.
Then: $\forall t \in S, \quad (g \circ f)_t = ((g \circ C) \cdot \alpha_1 \cdots \alpha_k)_t$.
Then: $g \circ f = (g \circ C) \cdot \alpha_1 \cdots \alpha_k$.
Since $C \in \mathcal{C}_W^V$ and $g : W \to X$, we get: $g \circ C \in \mathcal{C}_X^V$.
So, since $\alpha_1, \ldots, \alpha_k \in \Pi_V$, we get: $(g \circ C) \cdot \alpha_1 \cdots \alpha_k \in \mathcal{M}_k^{VX}$.
Then $h = g \circ f = (g \circ C) \cdot \alpha_1 \cdots \alpha_k \in \mathcal{M}_k^{VX}$, as desired. □

THEOREM 5.7.26. *Let $V, W, X \in \mathrm{ES}$, $k \in \mathbb{N}_0$.*
Then: $\mathcal{L}_X^W \circ \mathcal{H}_k^{VW} \subseteq \mathcal{H}_k^{VX}$.

Proof. Want: $\forall h \in \mathcal{L}_X^W \circ \mathcal{H}_k^{VW}, h \in \mathcal{H}_k^{VX}$.
Given $h \in \mathcal{L}_X^W \circ \mathcal{H}_k^{VW}$. Want: $h \in \mathcal{H}_k^{VX}$.
Since $h \in \mathcal{L}_X^W \circ \mathcal{H}_k^{VW}$, choose $g \in \mathcal{L}_X^W$ and $f \in \mathcal{H}_k^{VW}$ s.t. $h = g \circ f$.
As $f \in \mathcal{H}_k^{VW}$, choose $m \in \mathbb{N}$, $\phi_1, \ldots, \phi_m \in \mathcal{M}_k^{VW}$ s.t. $f = \phi_1 + \cdots + \phi_m$.
Since $g \in \mathcal{L}_X^W$, we know that g is algebrically linear.
Then: $\forall t \in S, \quad (g \circ f)_t = ((g \circ \phi_1) + \cdots + (g \circ \phi_m))_t$.
Then: $g \circ f = (g \circ \phi_1) + \cdots + (g \circ \phi_m)$.
By Theorem 5.7.25, $\mathcal{L}_X^W \circ \mathcal{M}_k^{VW} \subseteq \mathcal{M}_k^{VX}$.
Then, $\forall j \in [1..m], \quad g \circ \phi_j \in \mathcal{L}_X^W \circ \mathcal{M}_k^{VW} \subseteq \mathcal{M}_k^{VX}$.
Then: $(g \circ \phi_1) + \cdots + (g \circ \phi_m) \in \mathcal{H}_k^{VX}$.
Then $h = g \circ f = (g \circ \phi_1) + \cdots + (g \circ \phi_m) \in \mathcal{H}_k^{VX}$, as desired. □

THEOREM 5.7.27. *Let $V, W, X \in \mathrm{ES}$, $k \in \mathbb{N}_0$.*
Then: $\mathcal{M}_\ell^{WX} \circ \mathcal{H}_k^{VW} \subseteq \mathcal{H}_{k\ell}^{VX}$.

Proof. Want: $\forall h \in \mathcal{M}_\ell^{WX} \circ \mathcal{H}_k^{VW}, h \in \mathcal{H}_k^{VX}$.
Given $h \in \mathcal{M}_\ell^{WX} \circ \mathcal{H}_k^{VW}$. Want: $h \in \mathcal{H}_k^{VX}$.
Since $h \in \mathcal{M}_\ell^{WX} \circ \mathcal{H}_k^{VW}$, choose $g \in \mathcal{M}_\ell^{WX}$ and $f \in \mathcal{H}_k^{VW}$ s.t. $h = g \circ f$.
As $g \in \mathcal{M}_\ell^{WX}$, choose $C \in \mathcal{C}_X^W$ and $\alpha_1, \ldots, \alpha_\ell \in \Pi_W$ s.t. $g = C \cdot \alpha_1 \cdots \alpha_\ell$.
Then: $\forall t \in S, \quad (g \circ f)_t = ((C \circ f) \cdot (\alpha_1 \circ f) \cdots (\alpha_\ell \circ f))_t$.
Then: $g \circ f = (C \circ f) \cdot (\alpha_1 \circ f) \cdots (\alpha_\ell \circ f)$.
Let $\beta := (\alpha_1 \circ f) \cdots (\alpha_\ell \circ f)$. Then $g \circ f = (C \circ f) \cdot \beta$.
Since $f \in \mathcal{H}_k^{VW}$, we conclude that $f : V \to W$.
So, since $C \in \mathcal{C}_X^W$, we get: $C \circ f \in \mathcal{C}_X^V$. Then $C \circ f \in \mathcal{C}_X^V = \mathcal{H}_0^{VX}$.
We have: $\alpha_1, \ldots, \alpha_\ell \in \Pi_W \subseteq \mathcal{L}_{\mathbb{R}}^W$.
By Theorem 5.7.26, we have: $\mathcal{L}_{\mathbb{R}}^W \circ \mathcal{H}_k^{VW} \subseteq \mathcal{H}_k^{V\mathbb{R}}$.
Then: $\alpha_1 \circ f, \ldots, \alpha_\ell \circ f \in \mathcal{L}_{\mathbb{R}}^W \circ \mathcal{H}_k^{VW} \subseteq \mathcal{H}_k^{V\mathbb{R}}$.
Then, by Theorem 5.7.23, $(\alpha_1 \circ f) \cdots (\alpha_\ell \circ f) \in \mathcal{H}_{k\ell}^{V\mathbb{R}}$.

Then: $\beta = (\alpha_1 \circ f) \cdots (\alpha_\ell \circ f) \in \mathcal{H}_{k\ell}^{V\mathbb{R}}$.
By Theorem 5.7.24, we have: $\mathcal{H}_0^{VX} \cdot \mathcal{H}_{k\ell}^{V\mathbb{R}} \subseteq \mathcal{H}_{k\ell}^{VX}$.
Then: $h = g \circ f = (C \circ f) \cdot \beta \in \mathcal{H}_0^{VX} \cdot \mathcal{H}_{k\ell}^{V\mathbb{R}} \subseteq \mathcal{H}_{k\ell}^{VX}$, as desired. □

THEOREM 5.7.28. *Let $V, W, X \in$ ES, $k, \ell \in \mathbb{N}_0$.*
Then: $\mathcal{H}_\ell^{WX} \circ \mathcal{H}_k^{VW} \subseteq \mathcal{H}_{k\ell}^{VX}$.

Proof. Want: $\forall h \in \mathcal{H}_\ell^{WX} \circ \mathcal{H}_k^{VW}$, $h \in \mathcal{H}_k^{VX}$.
Given $h \in \mathcal{H}_\ell^{WX} \circ \mathcal{H}_k^{VW}$. Want: $h \in \mathcal{H}_k^{VX}$.
Since $h \in \mathcal{H}_\ell^{WX} \circ \mathcal{H}_k^{VW}$, choose $g \in \mathcal{H}_\ell^{WX}$ and $f \in \mathcal{H}_k^{VW}$ s.t. $h = g \circ f$.
As $g \in \mathcal{H}_\ell^{WX}$, choose $m \in \mathbb{N}$, $\psi_1, \ldots, \psi_m \in \mathcal{M}_\ell^{WX}$ s.t. $g = \psi_1 + \cdots + \psi_m$.
Then: $\forall t \in S$, $(g \circ f)_t = ((\psi_1 \circ f) + \cdots + (\psi_m \circ f))_t$.
Then: $g \circ f = (\psi_1 \circ f) + \cdots + (\psi_m \circ f)$.
By Theorem 5.7.27, we have: $\mathcal{M}_\ell^{WX} \circ \mathcal{H}_k^{VW} \subseteq \mathcal{H}_{k\ell}^{VX}$.
Then: $\psi_1 \circ f, \ldots, \psi_m \circ f \in \mathcal{M}_\ell^{WX} \circ \mathcal{H}_k^{VW} \subseteq \mathcal{H}_{k\ell}^{VX}$.
So, since $\mathcal{H}_{k\ell}^{VX}$ is v.op.-closed, $(\psi_1 \circ f) + \cdots + (\psi_m \circ f) \in \mathcal{H}_{k\ell}^{VX}$.
Then $h = g \circ f = (\psi_1 \circ f) + \cdots + (\psi_m \circ f) \in \mathcal{H}_{k\ell}^{VX}$, as desired. □

5.8. Continuity of polynomials.

THEOREM 5.8.1. *Let $V, W \in$ ES, $f, g : V \dashrightarrow W$, $t \in V$.*
Assume: f and g are both continuous at t from V to W.
Then $f + g$ is continuous at t from V to W.

Proof. Let $h := f + g$. Want: h is continuous at t from V to W.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_h$,
$[\ d(x,t) < \delta\ ] \Rightarrow [\ d(h_x, h_t) < \varepsilon\ ]$.
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_h$,
$[\ d(x,t) < \delta\ ] \Rightarrow [\ d(h_x, h_t) < \varepsilon\ ]$.
Let $\sigma := \varepsilon/2$. Then $\sigma > 0$.
So, as f is continuous at t from V to W, choose $\lambda > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$[\ d(x,t) < \lambda\ ] \Rightarrow [\ d(f_x, f_t) < \sigma\ ]$.
Also, as g is continuous at t from V to W, choose $\mu > 0$ s.t., $\forall x \in \mathbb{D}_g$,
$[\ d(x,t) < \mu\ ] \Rightarrow [\ d(g_x, g_t) < \sigma\ ]$.
Let $\delta := \min\{\lambda, \mu\}$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_h$, $[\ d(x,t) < \delta\ ] \Rightarrow [\ d(h_x, h_t) < \varepsilon\ ]$.
Given $x \in \mathbb{D}_h$. Want: $[\ d(x,t) < \delta\ ] \Rightarrow [\ d(h_x, h_t) < \varepsilon\ ]$.
Assume: $d(x,t) < \delta$. Want: $d(h_x, h_t) < \varepsilon$.
Since $x \in \mathbb{D}_h = \mathbb{D}_{f+g} = \mathbb{D}_f \bigcap \mathbb{D}_g$, we get: $x \in \mathbb{D}_f$ and $x \in \mathbb{D}_g$.
So, since $d(x,t) < \delta \leqslant \lambda$, by choice of λ, we have: $d(f_x, f_t) < \sigma$.
Also, since $d(x,t) < \delta \leqslant \mu$, by choice of μ, we have: $d(g_x, g_t) < \sigma$.

Since $h = f + g$, we get both $h_x = f_x + g_x$ and $h_t = f_t + g_t$.
Since $\sigma = \varepsilon/2$, we get $\sigma + \sigma = \varepsilon$.
Then: $d(h_x, h_t) = d(f_x + g_x, f_t + g_t) = |(f_x + g_x) - (f_t + g_t)|$
$= |(f_x - f_t) + (g_x - g_t)| \leqslant |f_x - f_t| + |g_x - g_t|$
$= [d(f_x, f_t)] + [d(g_x, g_t)] \leqslant \sigma + \sigma = \varepsilon$. □

THEOREM 5.8.2. *Let $V, W \in$ ES, $t \in V$. Then, $\forall \ell \in \mathbb{N}$,*
$\forall \alpha_1, \ldots, \alpha_\ell : V \dashrightarrow W$,
$[\,\alpha_1, \ldots, \alpha_\ell$ *are all continuous at t from V to W*$\,]$
$\Rightarrow$ $[\,\alpha_1 + \cdots + \alpha_\ell$ *is continuous at t from V to W*$\,]$.

Proof. Unassigned HW. Hint: Use induction, starting with:
Let $A := \{\ell \in \mathbb{N} \,|\, \forall \alpha_1, \ldots, \alpha_\ell : V \dashrightarrow W$,
$[\,\alpha_1, \ldots, \alpha_\ell$ are all continuous at t from V to $W\,]$
$\Rightarrow$ $[\,\alpha_1 + \cdots + \alpha_\ell$ is continuous at t from V to $W\,]$ $\}$.
Want: $A = \mathbb{N}$. □

THEOREM 5.8.3. *Let $S, U, V, W \in$ ES, let $* \in \mathcal{B}_W^{UV}$ and let $t \in S$.*
Let $f : S \dashrightarrow U$ and let $g : S \dashrightarrow V$.
Assume that f is continuous at t from S to U.
Assume that g is continuous at t from S to V.
Then $f * g$ is continuous at t from S to W.

Proof. Let $h := f * g$. Want: h is continuous at t from S to W.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_h$,
$[\, d(x, t) < \delta \,] \Rightarrow [\, d(h_x, h_t) < \varepsilon \,]$.
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall x \in \mathbb{D}_h$,
$[\, d(x, t) < \delta \,] \Rightarrow [\, d(h_x, h_t) < \varepsilon \,]$.
Choose $L \geqslant 0$ s.t., $\forall p \in U$, $\forall q \in V$, $|p * q| \leqslant L \cdot |p| \cdot |q|$.
Let $a := f_t$ and $b := g_t$ and $\sigma := \min\left\{ 1\ ,\ \dfrac{\varepsilon}{(L+1) \cdot (|b| + |a| + 1)} \right\}$.
Then: $\sigma \leqslant 1$ and $L \cdot \sigma \cdot (|b| + |a| + 1) < \varepsilon$.
Also, we have: $\sigma > 0$.
So, as f is continuous at t from S to V, choose $\lambda > 0$ s.t., $\forall x \in \mathbb{D}_f$,
$[\, d(x, t) < \lambda \,] \Rightarrow [\, d(f_x, f_t) < \sigma \,]$.
Also, as g is continuous at t from S to W, choose $\mu > 0$ s.t., $\forall x \in \mathbb{D}_g$,
$[\, d(x, t) < \mu \,] \Rightarrow [\, d(g_x, g_t) < \sigma \,]$.
Let $\delta := \min\{\lambda, \mu\}$. Then $\delta > 0$.
Want: $\forall x \in \mathbb{D}_h$, $[\, d(x, t) < \delta \,] \Rightarrow [\, d(h_x, h_t) < \varepsilon \,]$.
Given $x \in \mathbb{D}_h$. Want: $[\, d(x, t) < \delta \,] \Rightarrow [\, d(h_x, h_t) < \varepsilon \,]$.

Assume: $d(x,t) < \delta$. Want: $d(h_x, h_t) < \varepsilon$.
Since $x \in \mathbb{D}_h = \mathbb{D}_{f*g} = \mathbb{D}_f \bigcap \mathbb{D}_g$, we get: $x \in \mathbb{D}_f$ and $x \in \mathbb{D}_g$.
So, since $d(x,t) < \delta \leqslant \lambda$, by choice of λ, we have: $d(f_x, f_t) < \sigma$.
Also, since $d(x,t) < \delta \leqslant \mu$, by choice of μ, we have: $d(g_x, g_t) < \sigma$.
Recall that $a = f_t$ and $b = g_t$. Then $h_t = (f*g)_t = f_t * g_t = a*b$.
Let $A := f_x$ and $B := g_x$. Then $h_x = (f*g)_x = f_x * g_x = A*B$.
Also, we have: $|A-a| = |f_x - f_t| = d(f_x, f_t) < \sigma$.
Also, we have: $|B-b| = |g_x - g_t| = d(g_x, g_t) < \sigma$.
By bilinearity, we have:

$$[A*B] - [a*b] = [\,(A-a)*b\,] + [\,a*(B-b)\,] + [\,(A-a)*(B-b)\,].$$

Recall: $\sigma \leqslant 1$ and $L\cdot\sigma\cdot(\,|\,b\,| + |\,a\,| + 1\,) < \varepsilon$.
Then:

$$\begin{aligned} d(h_x, h_t) &= d(A*B, a*b) = |\,[A*B] - [a*b]\,| = \\ &= |\,[\,(A-a)*b\,] + [\,a*(B-b)\,] + [\,(A-a)*(B-b)\,]\,| \\ &\leqslant |\,(A-a)*b\,| + |\,a*(B-b)\,| + |\,(A-a)*(B-b)\,| \\ &\leqslant L\cdot|\,A-a\,|\cdot|\,b\,| + L\cdot|\,a\,|\cdot|\,B-b\,| + L\cdot|\,A-a\,|\cdot|\,B-b\,| \\ &\leqslant L\cdot\sigma\cdot|\,b\,| + L\cdot|\,a\,|\cdot\sigma + L\cdot\sigma\cdot\sigma \\ &= L\cdot\sigma\cdot(\,|\,b\,| + |\,a\,| + \sigma\,) \\ &\leqslant L\cdot\sigma\cdot(\,|\,b\,| + |\,a\,| + 1\,) < \varepsilon, \quad \text{as desired.} \end{aligned}$$

□

THEOREM 5.8.4. *Let $S, U \in$ ES, let $f, g : S \dashrightarrow U$ and let $t \in S$. Assume that f and g are both continuous at t from S to U. Then $f \bullet g$ is continuous at t from S to $\mathbb{R}$.*

To use Theorem 5.8.3 to prove the preceeding theorem, define $* \in \mathcal{B}^{UU}_{\mathbb{R}}$ by: $\forall x, y \in U$, $x * y = x \bullet y$.

THEOREM 5.8.5. *Let $S \in$ ES, let $f, g : S \dashrightarrow \mathbb{R}$ and let $t \in S$. Assume that f and g are both continuous at t from S to $\mathbb{R}$. Then $f \cdot g$ is continuous at t from S to $\mathbb{R}$.*

To use Theorem 5.8.3 to prove the preceeding theorem, define $* \in \mathcal{B}^{\mathbb{R}\mathbb{R}}_{\mathbb{R}}$ by: $\forall a, b \in \mathbb{R}$, $a * b = a \cdot b$.
The preceding theorem can be used the prove the following theorem, by induction on k.

THEOREM 5.8.6. *Let $S \in$ ES and let $t \in S$. Then, $\forall k \in \mathbb{N}$, $\forall \alpha_1, \ldots, \alpha_k : S \dashrightarrow \mathbb{R}$,*

$$(\ \alpha_1, \ldots, \alpha_k \text{ are all continuous at } t \text{ from } S \text{ to } \mathbb{R}\ ) \Leftrightarrow (\ \alpha_1 \cdots \alpha_k \text{ is continuous at } t \text{ from } S \text{ to } \mathbb{R}\ ).$$

THEOREM 5.8.7. *Let $S, U \in \mathrm{ES}$, $f : S \dashrightarrow U$, $g : S \dashrightarrow \mathbb{R}$, $t \in S$.*
Assume that f is continuous at t from S to U.
Assume that g is continuous at t from S to $\mathbb{R}$.
Then $f \cdot g$ is continuous at t from S to U.

To use Theorem 5.8.3 to prove the preceeding theorem,
define $* \in \mathcal{B}_U^{U\mathbb{R}}$ by: $\forall x \in U$, $\forall a \in \mathbb{R}$, $x * a = x \cdot a$.

THEOREM 5.8.8. *Let $V, W \in \mathrm{ES}$, $k \in \mathbb{N}_0$, $f \in \mathcal{M}_k^{VW}$.*
Then f is continuous from V to W.

Proof. Since $f \in \mathcal{M}_k^{VW}$, choose $\phi \in \mathcal{C}_W^V$ and $\alpha_1, \ldots, \alpha_k \in \Pi_V$
such that: $f = \phi \cdot \alpha_1 \cdots \alpha_k$.
Since $\alpha_1, \ldots, \alpha_k \in \Pi_V \subseteq \mathcal{L}_W^V$,
we conclude that $\alpha_1, \ldots, \alpha_k$ are all Lipschitz from V to $\mathbb{R}$,
and so $\alpha_1, \ldots, \alpha_k$ are all continuous from V to $\mathbb{R}$.
Then $\alpha_1 \cdots \alpha_k$ is continuous from V to $\mathbb{R}$.
Since $\phi \in \mathcal{C}_W^V$, we conclude that ϕ is Lipschitz-0 from V to W,
and so ϕ is continuous from V to W.
Then $\phi \cdot \alpha_1 \cdots \alpha_k$ is continuous from V to W.
So, since $f = \phi \cdot \alpha_1 \cdots \alpha_k$, we conclude that:
f is continuous from V to W, as desired. □

THEOREM 5.8.9. *Let $V, W \in \mathrm{ES}$, $k \in \mathbb{N}_0$, $f \in \mathcal{H}_k^{VW}$.*
Then f is continuous from V to W.

Proof. Since $f \in \mathcal{H}_k^{VW}$, choose $\ell \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_\ell \in \mathcal{M}_k^{VW}$
such that: $f = \alpha_1 + \cdots + \alpha_k$.
By the preceding theorem,
$\alpha_1, \ldots, \alpha_\ell$ are all continuous from V to $\mathbb{R}$.
Then $\alpha_1 + \cdots + \alpha_\ell$ is continuous from V to $\mathbb{R}$.
So, since $f = \alpha_1 + \cdots + \alpha_\ell$, we conclude that:
f is continuous from V to W, as desired. □

5.9. **Homogeneity of (homogeneous) polynomials.**

THEOREM 5.9.1. *Let $V, W \in \mathrm{ES}$, $k \in \mathbb{N}_0$, $f \in \mathcal{M}_k^{VW}$, $x \in V$, $c \in \mathbb{R}$.*
Then: $f_{c \cdot x} = c^k \cdot f_x$.

Proof. Since $f \in \mathcal{M}_k^{VW}$, choose $\phi \in \mathcal{C}_W^V$ and $\alpha_1, \ldots, \alpha_k \in \Pi_V$
s.t. $f = \phi \cdot \alpha_1 \cdots \alpha_k$.
For all $j \in [1..k]$, as $\alpha_j \in \Pi_V \subseteq \mathcal{L}_{\mathbb{R}}^V$, we get: $(\alpha_j)_{c \cdot x} = c \cdot ((\alpha_j)_x)$.

Since $\phi \in \mathcal{C}_W^V$, we conclude: $\phi_{c\cdot x} = \phi_x$.

Then $f_{c\cdot x} = (\phi \cdot \alpha_1 \cdots \alpha_k)_{c\cdot x}$

$$\begin{aligned}
&= \phi_{c\cdot x} \cdot [(\alpha_1)_{c\cdot x}] \cdots [(\alpha_k)_{c\cdot x}] \\
&= \phi_x \cdot [c \cdot ((\alpha_1)_x)] \cdots [c \cdot ((\alpha_k)_x)] \\
&= c^k \cdot \phi_x \cdot [(\alpha_1)_x] \cdots [(\alpha_k)_x] \\
&= c^k \cdot (\phi \cdot \alpha_1 \cdots \alpha_k)_x \\
&= c^k \cdot f_x, \qquad \text{as desired.}
\end{aligned}$$

□

THEOREM 5.9.2. *Let* $V, W \in \mathrm{ES}$, $k \in \mathbb{N}_0$, $f \in \mathcal{H}_k^{VW}$, $x \in V$, $c \in \mathbb{R}$. *Then:* $f_{c\cdot x} \;=\; c^k \cdot f_x$.

Proof. Since $f \in \mathcal{M}_k^{VW}$, choose $\ell \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_\ell \in \mathcal{M}_k^{VW}$

s.t. $\qquad f = \alpha_1 + \cdots + \alpha_k$.

By the preceding theorem, $\forall j \in [1..\ell]$, we have: $(\alpha_j)_{c\cdot x} = c^k \cdot ((\alpha_j)_x)$.

Then $f_{c\cdot x} = (\alpha_1 + \cdots + \alpha_k)_{c\cdot x}$

$$\begin{aligned}
&= [(\alpha_1)_{c\cdot x}] + \cdots + [(\alpha_k)_{c\cdot x}] \\
&= [c^k \cdot ((\alpha_1)_x)] + \cdots + [c^k \cdot ((\alpha_k)_x)] \\
&= c^k \cdot [((\alpha_1)_x) + \cdots + ((\alpha_k)_x)] \\
&= c^k \cdot [(\alpha_1 + \cdots + \alpha_k)_x] \\
&= c^k \cdot f_x, \qquad \text{as desired.}
\end{aligned}$$

□

5.10. **Tensors of linear and bilinear maps.**

DEFINITION 5.10.1. *Let* $V, W \in \mathrm{ES}$ *and let* $L : V \to W$.
By L *is* **algebraically linear**, *we mean:*

$$\forall u, v \in V, \quad L_{u+v} = L_u + L_v$$

and $\qquad \forall c \in \mathbb{R}, \forall v \in V, \quad L_{c\cdot v} = c \cdot L_v.$

DEFINITION 5.10.2. *Let* $V, W \in \mathrm{ES}$. *Then:*

$$\mathcal{AL}_W^V \;:=\; \{\ f : V \to W \mid f \text{ is algebraically linear } \}.$$

THEOREM 5.10.3. *Let* $V, W \in \mathrm{ES}$. *Then* $\mathcal{M}_1^{VW} \subseteq \mathcal{AL}_W^V$.

Proof. Want: $\forall M \in \mathcal{M}_1^{VW}, \quad M \in \mathcal{AL}_W^V$.
Given $M \in \mathcal{M}_1^{VW}$. $\qquad$ Want: $M \in \mathcal{AL}_W^V$.
Choose $C \in \mathcal{C}_W^V$ and $p \in \Pi_V$ s.t. $M = C \cdot p$.
Unassigned HW: Show that $\Pi_V \subseteq \mathcal{AL}_{\mathbb{R}}^V$. $\qquad$ Then $p \in \mathcal{AL}_{\mathbb{R}}^V$.
Unassigned HW: Show that $\mathcal{C}_W^V \cdot \mathcal{AL}_{\mathbb{R}}^V \subseteq \mathcal{AL}_W^V$.
Then $M = C \cdot p \in \mathcal{C}_W^V \cdot \mathcal{AL}_{\mathbb{R}}^V \subseteq \mathcal{AL}_{\mathbb{R}}^V$. □

THEOREM 5.10.4. *Let* $V, W \in \mathrm{ES}$. *Then* $\mathcal{L}_W^V \subseteq \mathcal{AL}_W^V$.

Proof. Want: $\forall L \in \mathcal{L}_W^V, \quad L \in \mathcal{AL}_W^V$.
Given $L \in \mathcal{L}_W^V$. Want: $L \in \mathcal{AL}_W^V$.
Since $L \in \mathcal{L}_W^V = \mathcal{H}_1^{VW}$, choose $k \in \mathbb{N}$ and $\phi_1, \ldots, \phi_k \in \mathcal{M}_1^{VW}$
such that $L = \phi_1 + \cdots + \phi_k$.
By Theorem 5.10.3, $\mathcal{M}_1^{VW} \subseteq \mathcal{AL}_W^V$.
Unassigned HW: Show $\mathcal{AL}_W^V + \mathcal{AL}_W^V \subseteq \mathcal{AL}_W^V$.
Then, as $\phi_1, \ldots, \phi_k \in \mathcal{M}_1^{VW} \subseteq \mathcal{AL}_W^V$, we get: $\phi_1 + \cdots + \phi_k \in \mathcal{AL}_W^V$.
Then $L = \phi_1 + \cdots + \phi_k \in \mathcal{AL}_W^V$. □

DEFINITION 5.10.5. *We define:*

$$\begin{array}{lrcl} & \forall \alpha, \beta \in \mathbb{N}, \quad \mathbb{R}^\alpha \otimes \mathbb{R}^\beta & := & \mathbb{R}^{\alpha \times \beta} \\ \text{and} & \forall \beta \in \mathbb{N}, \quad \mathbb{R} \otimes \mathbb{R}^\beta & := & \mathbb{R}^\beta \\ \text{and} & \forall \alpha \in \mathbb{N}, \quad \mathbb{R}^\alpha \otimes \mathbb{R} & := & \mathbb{R}^\alpha \\ \text{and} & \mathbb{R} \otimes \mathbb{R} & := & \mathbb{R}. \end{array}$$

DEFINITION 5.10.6. *Let $\alpha, \beta \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $W := \mathbb{R}^\beta$.*
Then: $\forall L \in \mathcal{AL}_W^V, \quad [L] \in W \otimes V$ *is defined by:*

$$\forall j \in [1..\beta], \, \forall i \in [1..\alpha], \quad [L]_{ji} \; = \; (L(\varepsilon_i^V)) \bullet \varepsilon_j^W.$$

Also: $\forall S \in W \otimes V, \quad \mathrm{Lin}_{VW}^S \in \mathcal{L}_W^V$ *is defined by:*

$$\forall v \in V, \quad \mathrm{Lin}_{VW}^S(v) \; = \; \sum_{j=1}^{\beta} \sum_{i=1}^{\alpha} S_{ji} \cdot v_i \cdot \varepsilon_j^W.$$

When the Euclidean spaces are obvious, we sometimes omit V and W, and write "Lin^S", instead of "Lin_{VW}^S".

THEOREM 5.10.7. *Define $L : \mathbb{R}^3 \to \mathbb{R}^2$ by: $\forall x, y, z \in \mathbb{R}$,*

$$L(x, y, z) \; = \; (\, 2x + 3y - 7z \, , \, 4x - y - 9z \,).$$

Then $L \in \mathcal{AL}_{\mathbb{R}^2}^{\mathbb{R}^3}$ and $[L] = \begin{bmatrix} 2 & 3 & -7 \\ 4 & -1 & -9 \end{bmatrix} \in \mathbb{R}^{2 \times 3} = \mathbb{R}^3 \otimes \mathbb{R}^2$.

DEFINITION 5.10.8. *Let $\alpha \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $W := \mathbb{R}$.*
Then: $\forall L \in \mathcal{AL}_W^V, \quad [L] \in W \otimes V$ *is defined by:*

$$\forall i \in [1..\alpha], \quad [L]_i \; = \; (L(\varepsilon_i^V)) \bullet 1.$$

Also: $\forall S \in W \otimes V, \quad \mathrm{Lin}_{VW}^S \in \mathcal{L}_W^V$ *is defined by:*

$$\forall v \in V, \quad \mathrm{Lin}_{VW}^S(v) \; = \; \sum_{i=1}^{\alpha} S_i \cdot v_i \cdot 1.$$

When the Euclidean spaces are obvious, we sometimes omit V and W, and write "Lin^S", instead of "Lin_{VW}^S".

DEFINITION 5.10.9. *Let $\beta \in \mathbb{N}$, $V := \mathbb{R}$, $W := \mathbb{R}^\beta$.*
Then: $\forall L \in \mathcal{AL}_W^V, \quad [L] \in W \otimes V$ *is defined by:*

$$\forall j \in [1..\beta], \quad [L]_j \;=\; (L(1)) \bullet \varepsilon_j^W.$$

Also: $\forall S \in W \otimes V$, $\mathrm{Lin}_{VW}^S \in \mathcal{L}_W^V$ *is defined by:*

$$\forall v \in V, \quad \mathrm{Lin}_{VW}^S(v) \;=\; \sum_{j=1}^{\beta} S_j \cdot v \cdot \varepsilon_j^W.$$

When the Euclidean spaces are obvious, we sometimes omit V and W, and write "Lin^S", instead of "Lin_{VW}^S".

DEFINITION 5.10.10. *Let* $\beta \in \mathbb{N}$, $V := \mathbb{R}$, $W := \mathbb{R}$.
Then: $\forall L \in \mathcal{AL}_W^V$, $[L] \in W \otimes V$ *is defined by:*

$$[L] \;=\; (L(1)) \bullet 1.$$

Also: $\forall S \in W \otimes V$, $\mathrm{Lin}_{VW}^S \in \mathcal{L}_W^V$ *is defined by:*

$$\forall v \in V, \quad \mathrm{Lin}_{VW}^S(v) \;=\; S \cdot v \cdot 1.$$

When the Euclidean spaces are obvious, we sometimes omit V and W, and write "Lin^S", instead of "Lin_{VW}^S".

THEOREM 5.10.11. *Let* $V, W \in \mathrm{ES}$, $L \in \mathcal{AL}_W^V$. *Then* $\mathrm{Lin}^{[L]} = L$.

Proof. Unassigned HW. □

THEOREM 5.10.12. *Let* $V, W \in \mathrm{ES}$. *Then* $\mathcal{AL}_W^V = \mathcal{L}_W^V$.

Proof. By Theorem 5.10.4, we have: $\mathcal{L}_W^V \subseteq \mathcal{AL}_W^V$.
Want: $\mathcal{AL}_W^V \subseteq \mathcal{L}_W^V$. Given $L \in \mathcal{AL}_W^V$. Want: $L \in \mathcal{L}_W^V$.
We have $L = \mathrm{Lin}^{[L]} \in \mathcal{L}_W^V$, as desired. □

THEOREM 5.10.13. *Let* $V, W \in \mathrm{ES}$, $S \in W \otimes V$. *Then* $[\mathrm{Lin}^S] = S$.

Proof. Unassigned HW. □

Let $V, W \in \mathrm{ES}$. Then

$$L \mapsto [L] : \mathcal{L}_W^V \to W \otimes V \quad \text{and} \quad S \mapsto \mathrm{Lin}^S : W \otimes V$$

are bijections are are inverses of one another.

DEFINITION 5.10.14. *We define:*

$$\begin{aligned} & \forall i, j \in \mathbb{N}, & i\|j &:= (i, j) \\ \textit{and} \quad & \forall j \in \mathbb{N}, & 0\|j &:= j \\ \textit{and} \quad & \forall i \in \mathbb{N}, & i\|0 &:= i \\ \textit{and} \quad & & 0\|0 &:= 0. \end{aligned}$$

We can now fold the eight formulas in

Definition 5.10.6, Definition 5.10.8,
Definition 5.10.9 and Definition 5.10.10

into two:

THEOREM 5.10.15. *Let* $V, W \in \mathrm{ES}$.
Then: $\forall L \in \mathcal{AL}_W^V$, $\forall j \in \mathcal{I}_W$, $\forall i \in \mathcal{I}_V$, $\quad [L]_{j\|i} \; = \; (L(\varepsilon_i^V)) \bullet \varepsilon_j^W$.
Also: $\forall S \in W \otimes V$, $\forall v \in V$, $\quad \mathrm{Lin}^S(v) \; = \; \sum_{j \in \mathcal{I}_W}^{\beta} \sum_{i \in \mathcal{I}_V}^{\alpha} S_{j\|i} \cdot v_i \cdot \varepsilon_j^W$.

DEFINITION 5.10.16. *We define:*

$$
\begin{array}{lrcl}
& \forall \alpha, \beta, \gamma \in \mathbb{N}, \quad \mathbb{R}^\alpha \otimes \mathbb{R}^{\beta \times \gamma} & := & \mathbb{R}^{\alpha \times \beta \times \gamma} \\
& \forall \alpha, \beta, \gamma \in \mathbb{N}, \quad \mathbb{R}^{\alpha, \beta} \otimes \mathbb{R}^{\gamma} & := & \mathbb{R}^{\alpha \times \beta \times \gamma} \\
\text{and} & \forall \beta, \gamma \in \mathbb{N}, \quad \mathbb{R} \otimes \mathbb{R}^{\beta \times \gamma} & := & \mathbb{R}^{\beta \times \gamma} \\
\text{and} & \forall \alpha, \beta \in \mathbb{N}, \quad \mathbb{R}^{\alpha \times \beta} \otimes \mathbb{R} & := & \mathbb{R}^{\alpha \times \beta}.
\end{array}
$$

DEFINITION 5.10.17. *We define:*

$$
\begin{array}{lrcl}
& \forall i, j, k \in \mathbb{N}, \quad i\|(j,k) & := & (i,j,k) \\
& \forall i, j, k \in \mathbb{N}, \quad (i,j)\|k & := & (i,j,k) \\
\text{and} & \forall j, k \in \mathbb{N}, \quad 0\|(j,k) & := & (j,k) \\
\text{and} & \forall i, j \in \mathbb{N}, \quad (i,j)\|0 & := & (i,j).
\end{array}
$$

DEFINITION 5.10.18. *Let* $V, W \in \mathrm{ES}$.
Then: $\forall B \in \mathcal{B}_W^V$, $\quad [B] \in X \otimes V \otimes W$ *is defined by:*
$\forall k \in \mathcal{I}_X$, $\forall i \in \mathcal{I}_V$, $\forall j \in \mathcal{I}_W$, $\quad [B]_{k\|i\|j} \; = \; (B(\varepsilon_i^V, \varepsilon_j^W)) \bullet \varepsilon_k^X$.
Also: $\forall T \in X \otimes V \otimes W$, $\quad \mathrm{Bilin}_{VWX}^T \in \mathcal{B}_X^{VW}$ *is defined by:*
$\forall v \in V$, $\forall w \in W$, $\mathrm{Bilin}_{VWX}^T(v,w) = \sum_{k \in \mathcal{I}_X} \sum_{i \in \mathcal{I}_V} \sum_{j \in \mathcal{I}_W} T_{k\|i\|j} \cdot v_i \cdot w_j \cdot \varepsilon_k^X$.

When the Euclidean spaces are obvious, we sometimes omit V and W and X, and write "Bilin^T", instead of "Bilin_{VWX}^T".

5.11. Polarization and diagonal restriction.

DEFINITION 5.11.1. *Let* $V, W \in \mathrm{ES}$ *and* $* \in \mathcal{B}_W^{VV}$.
Then $\mathrm{Qd}^* : V \to W$ *is defined by:* $\forall v \in V$, $\mathrm{Qd}_v^* = v * v$.

Let $V, W \in \mathrm{ES}$. According to HW#5-5, $\mathrm{Qd}^* \in \mathcal{Q}_W^V$.

Let $V, W \in \mathrm{ES}$. The next two results can be summarized as saying:
$* \mapsto \mathrm{Qd}^* : \mathcal{B}_W^{VV} \to \mathcal{Q}_W^V$ is algebraically linear.

THEOREM 5.11.2. *let* $V, W \in \mathrm{ES}$ *and* $B, C \in \mathcal{B}_W^{VV}$.
Then $\mathrm{Qd}^{B+C} = \mathrm{Qd}^B + \mathrm{Qd}^C$.

THEOREM 5.11.3. *let* $V, W \in \mathrm{ES}$ *and* $a \in \mathbb{R}$ *and* $B \in \mathcal{B}_W^{VV}$.
Then $\mathrm{Qd}^{a \cdot B} = a \cdot \mathrm{Qd}^B$.

THEOREM 5.11.4. *Let* $V, W \in \mathrm{ES}$ *and* $F \in \mathcal{M}_2^{VW}$.
Then $\exists * \in \mathcal{B}_W^{VV}$ *s.t.* $\mathrm{Qd}^* = F$.

Proof. Since $F \in \mathcal{M}_2^{VW}$,
choose $C \in \mathcal{C}_W^V$ and $p, q \in \Pi_V$ s.t. $F = C \cdot p \cdot q$.
Since $C \in \mathcal{C}_W^V$, choose $w \in W$ s.t. $C = C_V^w$.
Define $* : V \times V \to W$ by $\forall u, v \in V$, $u * v = w \cdot p_u \cdot q_v$.
Since $p, q \in \Pi_V \subseteq \mathcal{L}_{\mathbb{R}}^V$, it follows that: $* \in \mathcal{B}_W^{VV}$. Want: $\mathrm{Qd}^* = F$.
Want: $\forall v \in V$, $\mathrm{Qd}_v^* = F_v$. Given $v \in V$. Want: $\mathrm{Qd}_v^* = F_v$.
We have $C_v = (C_V^w)_v = w$. Then $w = C_v$.
Then: $\mathrm{Qd}_v^* = v * v = w \cdot p_v \cdot q_v = C_v \cdot p_v \cdot q_v$
$= (C \cdot p \cdot q)_v = F_v$, as desired. □

Let $V, W \in \mathrm{ES}$. The next result can be summarized as saying:
$* \mapsto \mathrm{Qd}^* : \mathcal{B}_W^{VV} \to \mathcal{Q}_W^V$ is surjective.

THEOREM 5.11.5. *Let* $V, W \in \mathrm{ES}$ *and* $F \in \mathcal{Q}_W^V$.
Then $\exists * \in \mathcal{B}_W^{VV}$ *s.t.* $\mathrm{Qd}^* = F$.

Proof. Since $F \in \mathcal{Q}_W^V = \mathcal{H}_2^{VW}$, choose $k \in \mathbb{N}$ and $\psi_1, \ldots, \psi_k \in \mathcal{M}_2^{VW}$
s.t. $F = \psi_1 + \cdots + \psi_k$.
By Theorem 5.11.4, $\forall j \in [1..k]$, choose $B_j \in \mathcal{B}_W^{VV}$ s.t. $\psi_j = \mathrm{Qd}^{B_j}$.
Let $* := B_1 + \cdots + B_k$. Then $* \in \mathcal{B}_W^{VV}$. Want: $\mathrm{Qd}^* = F$.
We have $\mathrm{Qd}^* = \mathrm{Qd}^{B_1 + \cdots + B_k} = \mathrm{Qd}^{B_1} + \cdots + \mathrm{Qd}^{B_k}$
$= \psi_1 + \cdots + \psi_k = F$, as desired. □

5.12. Principal minors and positive definiteness.

DEFINITION 5.12.1. *Let* $V, W \in \mathrm{ES}$. *Then:*

$$\mathcal{SB}_W^V \quad := \quad \{ * \in \mathcal{B}_W^{VV} \mid \forall u, v \in V,\ u * v = v * u \}.$$

DEFINITION 5.12.2. *Let* $V, W \in \mathrm{ES}$ *and let* $* \in \mathcal{B}_W^{VV}$.
Then $\mathrm{Sym}^* : V \times V \to W$ *is defined by:*

$$\forall u, v \in V, \qquad \mathrm{Sym}^*(u, v) \ = \ \frac{u * v \ + \ v * u}{2}.$$

Unassigned HW: Let $V, W \in \mathrm{ES}$ and let $* \in \mathcal{SB}_W^V$. Show: $\mathrm{Sym}^* = *$.

Monastery and convent story about the joys of polarization ...

DEFINITION 5.12.3. *Let* $m \in \mathbb{N}$ *and let* $A \in \mathbb{R}^{m \times m}$.
Assume that A *is symmetric.*
Then PM_A *denotes the set of principal minors of* A.

THEOREM 5.12.4. *Let* $m \in \mathbb{N}$, $V := \mathbb{R}^m$, $B \in \mathcal{SB}_{\mathbb{R}}^V$, $Q := \mathrm{Qd}^B$.
Then: $\quad (Q > 0 \text{ on } V_{0_V}^{\times}) \Leftrightarrow (\mathrm{PM}_{[B]} > 0)$.

Proof. Omitted. Belongs in a linear algebra course. ☐

We described the Multivariable Second Derivative for Minima.
We'll come back to this later.

5.13. **Multivariable** DNZ**,** BNZ**,** CVZ**, big-O and little-o.**

DEFINITION 5.13.1. *Let* $V, W \in \mathrm{ES}$. *Then* $\mathbf{0}_W^V := C_V^{0_W}$.

DEFINITION 5.13.2. *Let* f *be a v/s-function.*
Then $|f|$ *is the functional defined by:*
$$\forall x, \quad |f|_x = |f_x|.$$

DEFINITION 5.13.3. *Let* $V, W \in \mathrm{ES}$. *Then:*
$$\begin{aligned}
\mathrm{DNZ}_W^V &:= \{ f : V \dashrightarrow W \mid f \text{ is defined near } 0_V \text{ in } V \},\\
\mathrm{BNZ}_W^V &:= \{ f : V \dashrightarrow W \mid f \text{ is bounded near } 0_V \text{ from } V \text{ to } W \},\\
\mathrm{CVZ}_W^V &:= \{ f \in \mathrm{DNZ}_W^V \mid (f_{0_V} = 0_W) \ \& \\
&\qquad\qquad (f \text{ is continuous at } 0_V \text{ from } V \text{ to } W) \}.
\end{aligned}$$

THEOREM 5.13.4. *Let* $V, W \in \mathrm{ES}$.
Then DNZ_W^V *and* BNZ_W^V *and* CVZ_W^V *are all v.op-closed.*

THEOREM 5.13.5. *Let* $V, W \in \mathrm{ES}$ *and let* $f, g : V \dashrightarrow W$.
Assume: $\quad g = f$ *near* 0_V *in* V.
$$\begin{aligned}
\text{Then:} \quad & [(f \in \mathrm{DNZ}_W^V) \Leftrightarrow (g \in \mathrm{DNZ}_W^V)]\\
\& \ & [(f \in \mathrm{BNZ}_W^V) \Leftrightarrow (g \in \mathrm{BNZ}_W^V)]\\
\& \ & [(f \in \mathrm{CVZ}_W^V) \Leftrightarrow (g \in \mathrm{CVZ}_W^V)].
\end{aligned}$$

THEOREM 5.13.6. *Let* $V, W \in \mathrm{ES}$ *and let* $f, g : V \dashrightarrow W$. *Then:*
$[f \in \mathrm{BNZ}_W^V] \Leftrightarrow [\exists \delta, L > 0 \text{ s.t.}, \forall x \in V, (|x| < \delta) \Rightarrow (|f_x| \leqslant L)]$
and
$[f \in \mathrm{CVZ}_W^V] \Leftrightarrow [\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}, \forall x \in V, (|x| < \delta) \Rightarrow (|f_x| \leqslant \varepsilon)]$.

DEFINITION 5.13.7. *Let* $V, W \in \mathrm{ES}$ *and let* $k \in \mathbb{N}_0$.
$$\begin{aligned}
\text{Then:} \quad & \widehat{\mathcal{O}}_k^{VW} := (\mathrm{BNZ}_W^V) \cdot (| \bullet |_V^k)\\
\& \quad & \mathcal{o}_k^{VW} := (\mathrm{CVZ}_W^V) \cdot (| \bullet |_V^k).
\end{aligned}$$

THEOREM 5.13.8. *Let* $V, W \in \mathrm{ES}$ *and let* $k \in \mathbb{N}_0$.
Then $\widehat{\mathcal{O}}_k^{VW}$ *and* $\mathcal{o}_k^{VW}$ *are both v.op.-closed.*

THEOREM 5.13.9. *Let* $V, W \in \mathrm{ES}$, $k \in \mathbb{N}_0$ *and* $f, g : V \dashrightarrow W$.

$$\begin{array}{rl} \textit{Assume:} & g = f \textit{ near } 0_V \textit{ in } V. \\ \textit{Then:} & [\ (\ f \in \widehat{\mathcal{O}}_k^{VW}\ ) \ \Leftrightarrow\ (\ g \in \widehat{\mathcal{O}}_k^{VW}\ )\ ] \\ & \&\ [\ (\ f \in o_k^{VW}\ ) \ \Leftrightarrow\ (\ g \in o_k^{VW}\ )\ ]. \end{array}$$

THEOREM 5.13.10. *Let* $V, W \in \mathrm{ES}$, $k \in \mathbb{N}_0$, $f : V \dashrightarrow W$. *Then:*

$[f \in \widehat{\mathcal{O}}_k^{VW}] \Leftrightarrow [\exists \delta, L > 0 \textit{ s.t.}, \forall x \in V, (|x| < \delta) \Rightarrow (|f_x| \leqslant L \cdot |x|^k)]$

and

$[f \in o_k^{VW}] \Leftrightarrow [\forall \varepsilon > 0, \exists \delta > 0 \textit{ s.t.}, \forall x \in V, (|x| < \delta) \Rightarrow (|f_x| \leqslant \varepsilon \cdot |x|^k)]$.

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THEOREM 5.13.11. *Let* $V \in \mathrm{ES}$. *Then* S_V *is compact.*

Proof. Since $S_V \in B_V(0_V, 2)$, we see that S_V is bounded in V.
So, since V is proper, it suffices to show: S_V is closed in V.
Want: $S_V \in \mathcal{T}_V'$. We have $S_V = (|\bullet|_V)^*(\{1\})$.
Because finite sets in metric spaces are closed,
we conclude that $\{1\} \in \mathcal{T}_{\mathbb{R}}'$.
Since $|\bullet|_V$ is Lipschitz-1 from V to $\mathbb{R}$,
it follows that $|\bullet|_V$ is continuous from V to $\mathbb{R}$.
Then, we have: $\forall C \in \mathcal{T}_{\mathbb{R}}'$, $(|\bullet|_V)^*(C) \in \mathcal{T}_V'$.
Then $(|\bullet|_V)^*(\{1\}) \in \mathcal{T}_V'$. Then $S_V = (|\bullet|_V)^*(\{1\}) \in \mathcal{T}_V'$. □

THEOREM 5.13.12. *Let* $V, W \in \mathrm{ES}$ *and let* $k \in \mathbb{N}_0$.
Then: $\mathcal{H}_k^{VW} \subseteq \widehat{\mathcal{O}}_k^{VW}$.

Proof. Want: $\forall f \in \mathcal{H}_k^{VW}$, $f \in \widehat{\mathcal{O}}_k^{VW}$. Given $f \in \mathcal{H}_k^{VW}$. Want: $f \in \widehat{\mathcal{O}}_k^{VW}$.
Want: $\exists \delta, L > 0$ s.t., $\forall x \in V$, $(\ |x| < \delta\ ) \Rightarrow (\ |f_x| \leqslant L \cdot |x|^k\ )$.
We have $f \in \mathcal{H}_k^{VW}$, so f is continuous from V to W.
So, since S_V is compact, we see that f_*S_V is compact as well.
Then f_*S_V is closed and bounded in W.
Since f_*S_V is bounded in W,
choose $L > 0$ s.t. $f_*S_V \subseteq B_W(0_W, L)$.
Let $\delta := 1$. Then $\delta, L > 0$.
Want: $\forall x \in V$, $(\ |x| < \delta\ ) \Rightarrow (\ |f_x| \leqslant L \cdot |x|^k\ )$.
Given $x \in V$. Want: $(\ |x| < \delta\ ) \Rightarrow (\ |f_x| \leqslant L \cdot |x|^k\ )$.
Assume $|x| < \delta$. Want: $|f_x| \leqslant L \cdot |x|^k$.
By polar decomposition, choose $u \in S_V$ s.t. $x = |x| \cdot u$.
Let $c := |x|$. Then $x = c \cdot u$. Also, as $c \geqslant 0$, we get: $|c| = c$.
We have $f_x = f_{cu} = c^k \cdot f_u$, so $|f_x| = |c|^k \cdot |f_u|$.

Then, as $|c| = c$, this gives: $|f_x| = c^k \cdot |f_u|$.
Since $f \in \mathcal{H}_k^{VW}$, we conclude: $\mathbb{D}_f = V$.
Recall: $u \in S_V$. Then $u \in S_V \subseteq V = \mathbb{D}_f$. Then $u \in S_V \bigcap \mathbb{D}_f$.
Then $f_u \in f_* S_V \subseteq B_W(0_W, L)$, so $d_W(f_u, 0_W) < L$.
Then $|f_u| = |f_u - 0| = d_W(f_u, 0_W) < L$, and so $|f_u| \leqslant L$.
So, since $|x|^k \geqslant 0$, we get: $|f_u| \cdot |x|^k \leqslant L \cdot |x|^k$.
Recall that $|f_x| = c^k \cdot |f_u|$ and that $c = |x|$.
Then $|f_x| = c^k \cdot |f_u| = |f_u| \cdot c^k = |f_u| \cdot |x|^k \leqslant L \cdot |x|^k$, as desired. □

THEOREM 5.13.13. *Let* $V, W \in \mathrm{ES}$ *and* $k \in \mathbb{N}_0$ *and* $f \in \mathcal{O}_k^{VW}$.
Let $u \in S_V$ *and* $\varepsilon > 0$. *Then* $\exists t > 0$ *s.t.* $|f_{t\cdot u}| < \varepsilon \cdot t^k$.

Proof. Choose $\delta > 0$ s.t., $\forall x \in V$, $[\ |x| < \delta\ ] \Rightarrow [\ |f_x| \leqslant (\varepsilon/2) \cdot |x|^k\ ]$.
Let $t := \delta/2$. Then $t > 0$. Want: $|f_{t\cdot u}| < \varepsilon \cdot t^k$.
Let $x := t \cdot u$. Want: $|f_x| < \varepsilon \cdot t^k$.
Since $u \in S_V$, we get: $|u| = 1$. Since $t > 0$, we get: $|t| = t$.
Then $|x| = |t \cdot u| = |t| \cdot |u| = t \cdot 1 = t$. Then $(\varepsilon/2) \cdot |x|^k = (\varepsilon/2) \cdot t^k$.
Since $|x| = t = \delta/2 < \delta$, by choice of δ, we get: $|f_x| \leqslant (\varepsilon/2) \cdot |x|^k$.
Since $t > 0$, it follows that $t^k > 0$.
So, since $\varepsilon/2 < \varepsilon$, we get: $(\varepsilon/2) \cdot t^k < \varepsilon \cdot t^k$.
Then: $|f_x| \leqslant (\varepsilon/2) \cdot |x|^k = (\varepsilon/2) \cdot t^k < \varepsilon \cdot t^k$, as desired. □

THEOREM 5.13.14. *Let* $V, W \in \mathrm{ES}$ *and* $k \in \mathbb{N}_0$.
Then: $\mathcal{H}_k^{VW} \bigcap \mathcal{O}_k^{VW} = \{\mathbf{0}_W^V\}$.

Proof. Since $\mathbf{0}_W^V \in \mathcal{H}_k^{VW}$ and $\mathbf{0}_W^V \in \mathcal{O}_k^{VW}$, we get $\mathbf{0}_W^V \in \mathcal{H}_k^{VW} \bigcap \mathcal{O}_k^{VW}$.
Then $\{\mathbf{0}_W^V\} \subseteq \mathcal{H}_k^{VW} \bigcap \mathcal{O}_k^{VW}$. Want: $\mathcal{H}_k^{VW} \bigcap \mathcal{O}_k^{VW} \subseteq \{\mathbf{0}_W^V\}$.
Want: $\forall f \in \mathcal{H}_k^{VW} \bigcap \mathcal{O}_k^{VW}$, $f \in \{\mathbf{0}_W^V\}$.
Given $f \in \mathcal{H}_k^{VW} \bigcap \mathcal{O}_k^{VW}$. Want: $f \in \{\mathbf{0}_W^V\}$. Want: $f = \mathbf{0}_W^V$.
Assume that $f \neq \mathbf{0}_W^V$. Want: Contradiction.
Since $f \in \mathcal{H}_k^{VW}$, we get $\mathbb{D}_f = V$. Also $\mathbb{D}_{\mathbf{0}_W^V} = V$.
So, since $f \neq \mathbf{0}_W^V$, choose $y \in V$ s.t. $f_y \neq (\mathbf{0}_W^V)_y$.
By polar decomposition, choose $u \in S_V$ s.t. $y = |y| \cdot u$.
Let $c := |y|$. Then $y = c \cdot u$.
Since $c^k \cdot f_u = f_{c\cdot u} = f_y \neq (\mathbf{0}_W^V)_y = (C_V^{0_W})_y = 0_W$, we get $f_u \neq 0_W$.
Then $|f_u| > 0$. Let $\varepsilon := |f_u|$. Then $\varepsilon > 0$.
By the preceding theorem, choose $t > 0$ s.t. $|f_{t\cdot u}| < \varepsilon \cdot t^k$.
Then $t^k \cdot \varepsilon = \varepsilon \cdot t^k > |f_{t\cdot u}|$, so $t^k \cdot \varepsilon > |f_{t\cdot u}|$.
Then $t^k \cdot \varepsilon > |f_{t\cdot u}| = |t^k \cdot f_u| = |t|^k \cdot |f_u| = t^k \cdot \varepsilon$,
and so $t^k \cdot \varepsilon > t^k \cdot \varepsilon$. Contradiction. □

THEOREM 5.13.15. *Let* $V, W \in \mathrm{ES}$. *Then* $\mathrm{CVZ}_W^V \subseteq \mathrm{BNZ}_W^V$.

Proof. Unassigned HW. *Hint:* Follow the proof of Theorem 4.2.37. □

THEOREM 5.13.16. *Let* $V, W \in \mathrm{ES}$, $k \in \mathbb{N}_0$. *Then* $\mathcal{O}_k^{VW} \subseteq \widehat{\mathcal{O}}_k^{VW}$.

Proof. We have: $\mathcal{O}_k^{VW} = \mathrm{CVZ}_W^V \cdot |\bullet|^k \subseteq \mathrm{BNZ}_W^V \cdot |\bullet|^k = \widehat{\mathcal{O}}_k^{VW}$. □

5.14. **Bilinear products of function spaces.**

THEOREM 5.14.1. *Let* $S, U, V, W \in \mathrm{ES}$ *and let* $* \in \mathcal{B}_W^{VW}$.
Then $\mathrm{BNZ}_U^S * \mathrm{CVZ}_V^S \subseteq \mathrm{CVZ}_W^S$.

Proof. Want: $\forall h \in \mathrm{BNZ}_U^S * \mathrm{CVZ}_V^S$, $h \in \mathrm{CVZ}_W^S$.
Given $h \in \mathrm{BNZ}_U^S * \mathrm{CVZ}_V^S$. Want: $h \in \mathrm{CVZ}_W^S$.
Want: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t., $\forall t \in S$, $(|t| < \delta) \Rightarrow (|h_t| \leqslant \varepsilon)$.
Given $\varepsilon > 0$. Want: $\exists \delta > 0$ s.t., $\forall t \in S$, $(|t| < \delta) \Rightarrow (|h_t| \leqslant \varepsilon)$.
As $h \in \mathrm{BNZ}_U^S * \mathrm{CVZ}_V^S$, choose $f \in \mathrm{BNZ}_U^S$, $g \in \mathrm{CVZ}_V^S$ s.t. $h = f * g$.
Since $f \in \mathrm{BNZ}_U^S$, choose $K, \alpha > 0$ s.t.,
$$\forall t \in S, \qquad (|t| < \alpha) \Rightarrow (|f_t| \leqslant K).$$
Since $*$ is bilinear, and therefore double-Lipschitz, choose $L \geqslant 0$ s.t.,
$$\forall x \in U,\ \forall y \in V, \qquad |x * y| \leqslant L \cdot |x| \cdot |y|.$$
Since $g \in \mathrm{CVZ}_V^S$, choose $\beta > 0$ s.t.,
$$\forall t \in S, \qquad (|t| < \beta) \Rightarrow \left(|g_t| \leqslant \frac{\varepsilon}{(L+1)\cdot K}\right).$$
Let $\delta := \min\{\alpha, \beta\}$. Then $\delta > 0$.
Want: $\forall t \in S$, $(|t| < \delta) \Rightarrow (|h_t| \leqslant \varepsilon)$.
Given $t \in S$. Want: $(|t| < \delta) \Rightarrow (|h_t| \leqslant \varepsilon)$.
Assume: $|t| < \delta$. Want: $|h_t| \leqslant \varepsilon$.
Since $|t| < \delta \leqslant \alpha$, by choice of α, we get: $|f_t| \leqslant K$.
Since $|t| < \delta \leqslant \beta$, by choice of β, we get: $|g_t| \leqslant \dfrac{\varepsilon}{(L+1)\cdot K}$.
Since $|f_t| \leqslant K$ and $|g_t| \leqslant \dfrac{\varepsilon}{(L+1)\cdot K}$, we get: $f_t \neq ☹ \neq g_t$.
Then $t \in \mathbb{D}_f$ and $t \in \mathbb{D}_g$. Then $f_t \in \mathbb{I}_f \subseteq U$ and $g_t \in \mathbb{I}_g \subseteq V$.
Let $x := f_t$ and $y := g_t$. Then $x \in U$ and $y \in V$.
So, by choice of L, we get: $|x * y| \leqslant L \cdot |x| \cdot |y|$.
We have $|x| = |f_t| \leqslant K$ and $|y| = |g_t| \leqslant \dfrac{\varepsilon}{(L+1)\cdot K}$.
Since $0 \leqslant |x| \leqslant K$ and $0 \leqslant |y| \leqslant \dfrac{\varepsilon}{(L+1)\cdot K}$,
we conclude that $|x| \cdot |y| \leqslant \dfrac{\varepsilon}{L+1}$,
and so, as $L \geqslant 0$, we get $L \cdot |x| \cdot |y| < \varepsilon$.

We have $h_t = (f * g)_t = f_t * g_t = x * y$.
Then $|h_t| = |x * y| \leqslant L \cdot |x| \cdot |y| < \varepsilon$, and so $|h_t| \leqslant \varepsilon$, as desired. □

THEOREM 5.14.2. *Let* $S \in \mathrm{ES}$. *Then* $\mathrm{BNZ}_{\mathbb{R}}^{S} \cdot \mathrm{CVZ}_{\mathbb{R}}^{S} \subseteq \mathrm{CVZ}_{\mathbb{R}}^{S}$.

Proof. Define $* \in \mathcal{B}_{\mathbb{R}}^{\mathbb{R}\mathbb{R}}$ by: $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \quad a * b = a \cdot b$.
Then $\mathrm{BNZ}_{\mathbb{R}}^{S} \cdot \mathrm{CVZ}_{\mathbb{R}}^{S} = \mathrm{BNZ}_{\mathbb{R}}^{S} * \mathrm{CVZ}_{\mathbb{R}}^{S} \subseteq \mathrm{CVZ}_{\mathbb{R}}^{S}$, as desired. □

THEOREM 5.14.3. *Let* $S, U \in \mathrm{ES}$. *Then* $\mathrm{BNZ}_{U}^{S} \cdot \mathrm{CVZ}_{\mathbb{R}}^{S} \subseteq \mathrm{CVZ}_{U}^{S}$.

Proof. Define $* \in \mathcal{B}_{U}^{U\mathbb{R}}$ by: $\forall x \in U, \forall c \in \mathbb{R}, \quad x * c = x \cdot c$.
Then $\mathrm{BNZ}_{U}^{S} \cdot \mathrm{CVZ}_{\mathbb{R}}^{S} = \mathrm{BNZ}_{U}^{S} * \mathrm{CVZ}_{\mathbb{R}}^{S} \subseteq \mathrm{CVZ}_{U}^{S}$, as desired. □

THEOREM 5.14.4. *Let* $S, U \in \mathrm{ES}$. *Then* $\mathrm{BNZ}_{U}^{S} \bullet \mathrm{CVZ}_{U}^{S} \subseteq \mathrm{CVZ}_{\mathbb{R}}^{S}$.

Proof. Define $* \in \mathcal{B}_{\mathbb{R}}^{UU}$ by: $\forall x \in U, \forall y \in U, \quad x * y = x \bullet y$.
Then $\mathrm{BNZ}_{U}^{S} \cdot \mathrm{CVZ}_{U}^{S} = \mathrm{BNZ}_{U}^{S} * \mathrm{CVZ}_{U}^{S} \subseteq \mathrm{CVZ}_{\mathbb{R}}^{S}$, as desired. □

THEOREM 5.14.5. *Let* $S, U, V, W \in \mathrm{ES}$ *and let* $* \in \mathcal{B}_{W}^{UV}$.

$$\begin{aligned} \textit{Then} \quad & \mathrm{BNZ}_{U}^{S} * \mathrm{BNZ}_{V}^{S} \subseteq \mathrm{BNZ}_{W}^{S} \\ \textit{and} \quad & \mathrm{CVZ}_{U}^{S} * \mathrm{BNZ}_{V}^{S} \subseteq \mathrm{CVZ}_{W}^{S} \\ \textit{and} \quad & \mathrm{BNZ}_{U}^{S} * \mathrm{CVZ}_{V}^{S} \subseteq \mathrm{CVZ}_{W}^{S} \\ \textit{and} \quad & \mathrm{CVZ}_{U}^{S} * \mathrm{CVZ}_{V}^{S} \subseteq \mathrm{CVZ}_{W}^{S}. \end{aligned}$$

Proof. Unassigned HW. □

THEOREM 5.14.6. *Let* $S \in \mathrm{ES}$. *Then* $\mathrm{CVZ}_{\mathbb{R}}^{S} \cdot \mathrm{CVZ}_{\mathbb{R}}^{S} \subseteq \mathrm{CVZ}_{U}^{S}$.

Proof. Define $* \in \mathcal{B}_{\mathbb{R}}^{\mathbb{R}\mathbb{R}}$ by: $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \quad a * b = a \cdot b$.
Then $\mathrm{CVZ}_{\mathbb{R}}^{S} \cdot \mathrm{CVZ}_{\mathbb{R}}^{S} = \mathrm{CVZ}_{\mathbb{R}}^{S} * \mathrm{CVZ}_{\mathbb{R}}^{S} \subseteq \mathrm{CVZ}_{\mathbb{R}}^{S}$, as desired. □

THEOREM 5.14.7. *Let* $S, U, V, W \in \mathrm{ES}$ *and* $* \in \mathcal{B}_{W}^{UV}$ *and* $k, \ell \in \mathbb{N}_0$.

$$\begin{aligned} \textit{Then} \quad & \widehat{\mathcal{O}}_{k}^{SU} * \widehat{\mathcal{O}}_{\ell}^{SV} \subseteq \widehat{\mathcal{O}}_{k+\ell}^{SW} \\ \textit{and} \quad & \mathcal{o}_{k}^{SU} * \widehat{\mathcal{O}}_{\ell}^{SV} \subseteq \mathcal{o}_{k+\ell}^{SW} \\ \textit{and} \quad & \widehat{\mathcal{O}}_{k}^{SU} * \mathcal{o}_{\ell}^{SV} \subseteq \mathcal{o}_{k+\ell}^{SW} \\ \textit{and} \quad & \mathcal{o}_{k}^{SU} * \mathcal{o}_{\ell}^{SV} \subseteq \mathcal{o}_{k+\ell}^{SW}. \end{aligned}$$

Proof. Unassigned HW. □

THEOREM 5.14.8. *Let* $V \in \mathrm{ES}$. *Then* $|\bullet|_V \in \mathrm{CVZ}_{\mathbb{R}}^{V}$.

Proof. Since $\mathbb{D}_{|\bullet|_V} = V$ and $\mathbb{I}_{|\bullet|_V} \subseteq \mathbb{R}$, we get: $|\bullet|_V \in \mathrm{DNZ}_{\mathbb{R}}^{V}$.
Also, we have: $(|\bullet|_V)_{0_V} = |0_V| = 0 = 0_{\mathbb{R}}$.
It remains to show: $|\bullet|_V$ is continuous at 0_V from V to $\mathbb{R}$.
By HW#3-5(a), we know that $|\bullet|_V$ is Lipschitz-1 from V to $\mathbb{R}$,
and so $|\bullet|_V$ is continuous from V to $\mathbb{R}$.
Then $|\bullet|_V$ is continuous at 0_V from V to $\mathbb{R}$. □

THEOREM 5.14.9. *Let* $V, W \in \mathrm{ES}$, $k \in \mathbb{N}_0$. *Then* $\widehat{\mathcal{O}}_{k+1}^{VW} \subseteq \mathcal{O}_k^{VW}$.

Proof. We have:
$$\begin{aligned}\widehat{\mathcal{O}}_{k+1}^{VW} &= \mathrm{BNZ}_W^V \cdot \ |\bullet|_V^{k+1} \\ &= \mathrm{BNZ}_W^V \cdot \ |\bullet|_V \ \cdot \ |\bullet|_V^k \\ &\subseteq \mathrm{BNZ}_W^V \cdot \mathrm{CVZ}_W^V \cdot \ |\bullet|_V^k \\ &\subseteq \mathrm{CVZ}_W^V \cdot \ |\bullet|_V^k \ = \ \mathcal{O}_k^{VW}.\end{aligned}$$
□

5.15. **Compositions of function spaces.**

THEOREM 5.15.1. *Let* $V, W, X \in \mathrm{ES}$.
Then $\mathrm{BNZ}_X^W \circ \mathrm{CVZ}_W^V \subseteq \mathrm{BNZ}_X^V$.

Proof. Want: $\forall h \in \mathrm{BNZ}_X^W \circ \mathrm{CVZ}_W^V$, $h \in \mathrm{BNZ}_X^V$.
Given $h \in \mathrm{BNZ}_X^W \circ \mathrm{CVZ}_W^V$. Want: $h \in \mathrm{BNZ}_X^V$.
Want: $\exists \delta, L > 0$ s.t., $\forall x \in V$, $(|x| < \delta) \Rightarrow (|h_x| \leqslant L)$.
As $h \in \mathrm{BNZ}_X^W \circ \mathrm{CVZ}_W^V$, choose $g \in \mathrm{BNZ}_X^W$, $f \in \mathrm{CVZ}_W^V$ s.t. $h = g \circ f$.
Since $g \in \mathrm{BNZ}_X^W$, choose $\varepsilon, L > 0$ s.t.,
$\forall y \in W$, $(|y| < \varepsilon) \Rightarrow (|g_y| \leqslant L)$.
Since $f \in \mathrm{CVZ}_W^V$, choose $\delta > 0$ s.t.,
$\forall x \in V$, $(|x| < \delta) \Rightarrow (|f_x| \leqslant \varepsilon/2)$.
Then $\delta, L > 0$. Want: $\forall x \in V$, $(|x| < \delta) \Rightarrow (|h_x| \leqslant L)$.
Given $x \in V$. Want: $(|x| < \delta) \Rightarrow (|h_x| \leqslant L)$.
Assume: $|x| < \delta$. Want: $|h_x| \leqslant L$.
Since $\varepsilon > 0$, we get: $\varepsilon/2 < \varepsilon$. Let $y := f_x$.
Since $|x| < \delta$, by choice of δ, we get:
$|f_x| \leqslant \varepsilon/2$.
Since $|y| = |f_x| \leqslant \varepsilon/2 < \varepsilon$, by choice of ε, we get: $|g_y| \leqslant L$.
We have $h_x = (g \circ f)_x = g_{f_x} = g_y$. Then $|h_x| = |g_y| \leqslant L$. □

THEOREM 5.15.2. *Let* $V, W, X \in \mathrm{ES}$.
Then $\mathrm{CVZ}_X^W \circ \mathrm{CVZ}_W^V \subseteq \mathrm{CVZ}_X^V$.

Proof. Unassigned HW. □

Warning: As we observed earlier, we have:
$$\mathrm{CVZ}_{\mathbb{R}}^{\mathbb{R}} \circ \mathrm{BNZ}_{\mathbb{R}}^{\mathbb{R}} \ \nsubseteq \ \mathrm{DNZ}_{\mathbb{R}}^{\mathbb{R}}.$$
So, if BNZ or (any larger function space) appears on the RHS of $\circ$,
then all bets are off.

THEOREM 5.15.3. *Let* $V, W \in \mathrm{ES}$, $\phi \in \mathrm{CVZ}_W^V$.
Then: $(\ |\phi| \in \mathrm{CVZ}_{\mathbb{R}}^V\ )$ & $(\ \forall \ell \in \mathbb{N},\ |\phi|^\ell \in \mathrm{CVZ}_{\mathbb{R}}^V\ )$.

Proof. We have $|\phi| = (|\bullet|_W) \circ \phi \in \mathrm{CVZ}^W_{\mathbb{R}} \circ \mathrm{CVZ}^V_W \subseteq \mathrm{CVZ}^V_{\mathbb{R}}$.

Want: $\forall \ell \in \mathbb{N},\ |\phi|^\ell \in \mathrm{CVZ}^V_{\mathbb{R}}$.

As $|\phi| \in \mathrm{CVZ}^V_{\mathbb{R}}$, by Theorem 5.14.6 and induction on ℓ, we get:

$\forall \ell \in \mathbb{N},\ |\phi|^\ell \in \mathrm{CVZ}^V_{\mathbb{R}}$, as desired. □

THEOREM 5.15.4. *Let $V, W, X \in \mathrm{ES}$ and $k, \ell \in \mathbb{N}$.*
Then: $\widehat{\mathcal{O}}^{WX}_\ell \circ \mathcal{O}^{VW}_k \subseteq \mathcal{O}^{VX}_{k\ell}$.

Proof. Want: $\forall h \in \widehat{\mathcal{O}}^{WX}_\ell \circ \mathcal{O}^{VW}_k,\ \ h \in \mathcal{O}^{VX}_{k\ell}$.

Given $h \in \widehat{\mathcal{O}}^{WX}_\ell \circ \mathcal{O}^{VW}_k$. Want: $h \in \mathcal{O}^{VX}_{k\ell}$.

Since $h \in \widehat{\mathcal{O}}^{WX}_\ell \circ \mathcal{O}^{VW}_k$, choose $g \in \widehat{\mathcal{O}}^{WX}_\ell$ and $f \in \mathcal{O}^{VW}_k$ s.t. $h = g \circ f$.
Since $g \in \widehat{\mathcal{O}}^{WX}_\ell$, choose $\psi \in \mathrm{BNZ}^W_X$ s.t. $g = \psi \cdot |\bullet|^\ell_W$.
Since $f \in \mathcal{O}^{VW}_k$, choose $\phi \in \mathrm{CVZ}^V_W$ s.t. $f = \phi \cdot |\bullet|^k_V$.
We have: $\forall t \in V$, $f_t = (\phi \cdot |\bullet|^k_V) = \phi_t \cdot |t|^k$, so

$$\begin{aligned}(g \circ f)_t &= g(f_t) = [\psi(f_t)] \cdot |f_t|^\ell = [(\psi \circ f)_t] \cdot |\,[\phi_t \cdot |t|^k]\,|^\ell \\ &= [(\psi \circ f)_t] \cdot |\phi_t|^\ell \cdot |t|^{k\ell} = ((\psi \circ f) \cdot |\phi|^\ell \cdot |\bullet|^{k\ell}_V)_t.\end{aligned}$$

Then: $g \circ f = (\psi \circ f) \cdot |\phi|^\ell \cdot |\bullet|^{k\ell}_V$.

We have $f \in \mathcal{O}^{VW}_k \subseteq \mathcal{O}^{VW}_0 = \mathrm{CVZ}^V_W$,
so $\psi \circ f \in \mathrm{BNZ}^W_X \circ \mathrm{CVZ}^V_W \subseteq \mathrm{BNZ}^V_X$.

Since $|\phi| = |\bullet|_W \circ \phi \in \mathrm{CVZ}^W_{\mathbb{R}} \circ \mathrm{CVZ}^V_W \subseteq \mathrm{CVZ}^V_{\mathbb{R}}$, we get $|\phi|^\ell \in \mathrm{CVZ}^V_{\mathbb{R}}$.

$$\begin{aligned}\text{Then } g \circ f &= (\psi \circ f) \cdot |\phi|^\ell \cdot |\bullet|^k_V \\ &\in \mathrm{BNZ}^V_X \cdot \mathrm{CVZ}^V_{\mathbb{R}} \cdot |\bullet|^{k\ell}_V \\ &\subseteq \mathrm{CVZ}^V_{\mathbb{R}} \cdot |\bullet|^{k\ell}_V = \mathcal{O}^{VX}_k,\end{aligned}$$

as desired. □

THEOREM 5.15.5. *Let $V, W, X \in \mathrm{ES}$, and $k, \ell \in \mathbb{N}$.*

$$\begin{aligned}\textit{Then}\quad & \widehat{\mathcal{O}}^{WX}_k \circ \widehat{\mathcal{O}}^{VW}_\ell \subseteq \widehat{\mathcal{O}}^{VX}_{k\ell} \\ \textit{and}\quad & \mathcal{O}^{WX}_k \circ \widehat{\mathcal{O}}^{VW}_\ell \subseteq \mathcal{O}^{VX}_{k\ell} \\ \textit{and}\quad & \widehat{\mathcal{O}}^{WX}_k \circ \mathcal{O}^{VW}_\ell \subseteq \mathcal{O}^{VX}_{k\ell} \\ \textit{and}\quad & \mathcal{O}^{WX}_k \circ \mathcal{O}^{VW}_\ell \subseteq \mathcal{O}^{VX}_{k\ell}.\end{aligned}$$

Proof. One of these is the preceding theorem, and the rest are proved similarly and left as unassigned HW. □

5.16. The multi-variable D-derivative.

DEFINITION 5.16.1. *Let $V, W \in \mathrm{ES}$, $f : V \dashrightarrow W$, $q \in V$.*
Then $f^{\mathbb{T}}_q : V \dashrightarrow W$ is defined by: $\forall h \in V$, $(f^{\mathbb{T}}_q)_h = f_{q+h} - f_q$.

DEFINITION 5.16.2. *Let $V, W \in \mathrm{ES}$, $f : V \dashrightarrow W$, $q \in V$.*
Then: $\mathrm{LINS}_q f := \{\, L \in \mathcal{L}^V_W \mid f^{\mathbb{T}}_q - L \in \mathcal{O}^{VW}_1 \,\}$.

THEOREM 5.16.3. *Let $V, W \in \mathrm{ES}$, $f : V \dashrightarrow W$, $q \in V$.*
Assume: $\mathrm{LINS}_q f \neq \varnothing$. *Then:* $f^{\mathbb{T}}_q \in \widehat{\mathcal{O}}^{VW}_1$.

Proof. Choose $L \in \mathrm{LINS}_q f$. Then $L \in \mathcal{L}_W^V$ and $f_q^{\mathbb{T}} - L \in \mathcal{O}_1^{VW}$.
Since $L \in \mathcal{L}_W^V$, we get $L : V \to W$, and so $L - L = \mathbf{0}_W^V$.
Then $f_q^{\mathbb{T}} - L + L = f_q^{\mathbb{T}}$. We have $L \in \mathcal{L}_W^V = \mathcal{H}_1^{VW} \subseteq \widehat{\mathcal{O}}_1^{VW}$.
Then $f_q^{\mathbb{T}} = (f_q^{\mathbb{T}} - L) + L \in \mathcal{O}_1^{VW} + L \subseteq \widehat{\mathcal{O}}_1^{VW} + \widehat{\mathcal{O}}_1^{VW} \subseteq \widehat{\mathcal{O}}_1^{VW}$. □

THEOREM 5.16.4. *Let $V, W \in \mathrm{ES}$, $f : V \dashrightarrow W$, $q \in V$.*
Then: $\#\mathrm{LINS}_q f \leqslant 1$.

Proof. Want: $\forall L, M \in \mathrm{LINS}_q f, \quad L = M$.
Given $L, M \in \mathrm{LINS}_q f$. Want: $L = M$.
We have $f_q^{\mathbb{T}} - L \in \mathcal{O}_1^{VW}$ and $f_q^{\mathbb{T}} - M \in \mathcal{O}_1^{VW}$,
and so $(f_q^{\mathbb{T}} - L) - (f_q^{\mathbb{T}} - M) \in \mathcal{O}_1^{VW} - \mathcal{O}_1^{VW} \subseteq \mathcal{O}_1^{VW}$.
We have $f_q^{\mathbb{T}} \in \widehat{\mathcal{O}}_1^{VW} \subseteq \mathrm{DNZ}_W^V$, so $f_q^{\mathbb{T}} - f_q^{\mathbb{T}} = \mathbf{0}_W^V$ near 0_V in V.
Then $(f_q^{\mathbb{T}} - L) - (f_q^{\mathbb{T}} - M) = M - L$ near 0_V in V.
So, since $(f_q^{\mathbb{T}} - L) - (f_q^{\mathbb{T}} - M) \in \mathcal{O}_1^{VW}$ and since $\mathcal{O}_1^{VW}$ is zero-local,
we conclude: $M - L \in \mathcal{O}_1^{VW}$.
Also, $M - L \in \mathcal{L}_W^V - \mathcal{L}_W^V \subseteq \mathcal{L}_W^V = \mathcal{H}_1^{VW}$.
Then $M - L \in \mathcal{H}_1^{VW} \bigcap \mathcal{O}_1^{VW} = \{\mathbf{0}_W^V\}$.
Then $M - L = \mathbf{0}_W^V$, and so $L = M$, as desired. □

DEFINITION 5.16.5. *Let $V, W \in \mathrm{ES}$, $f : V \dashrightarrow W$, $q \in V$.*
Then: $D_q f \;:=\; \mathrm{UE}(\mathrm{LINS}_q f)$.

THEOREM 5.16.6. *Let $V, W \in \mathrm{ES}$, $f : V \dashrightarrow W$, $q \in V$.*
Assume: $D_q f \neq ☹$.
Then: ($f_q^{\mathbb{T}} \in \widehat{\mathcal{O}}_1^{VW}$)
& (*f is defined near q in V*)
& (*f is continuous at q from V to W*).

Proof. Since $\mathrm{UE}(\mathrm{LINS}_q f) = D_q f \neq ☹$, we get: $\mathrm{LINS}_q f \neq \varnothing$.
Then $f_q^{\mathbb{T}} \in \widehat{\mathcal{O}}_1^{VW}$. Want: (f is defined near q in V)
& (f is continuous at q from V to W).
We have $f_q^{\mathbb{T}} \in \widehat{\mathcal{O}}_1^{VW} \subseteq \mathcal{O}_1^{VW} \subseteq \mathcal{O}_0^{VW} = \mathrm{CVZ}_W^V$.
Then: ($f_q^{\mathbb{T}}$ is defined near 0_V in V)
& ($f_q^{\mathbb{T}}$ is continuous at 0_V from V to W).
Then: (f is defined near q in V)
& (f is continuous at q from V to W),
as desired. □

THEOREM 5.16.7. *Let $V, W \in \mathrm{ES}$, $f : V \dashrightarrow W$, $L \in \mathcal{L}_W^V$, $q \in V$.*
Assume: $L \in \mathrm{LINS}_q f$. *Then:* $D_q f = L$.

Proof. Since $L \in \mathrm{LINS}_q f$ and $\#\mathrm{LINS}_q f \leqslant 1$, we get: $\mathrm{LINS}_q f = \{L\}$. Then $D_q f = \mathrm{UE}(\mathrm{LINS}_q f) = \mathrm{UE}\{L\} = L$, as desired. □

THEOREM 5.16.8. *Let $V := \mathbb{R}^2$ and $q := (1, -1)$.*
Define $f : V \to \mathbb{R}$ by: $\forall x, y \in \mathbb{R}$, $f(x, y) = x^3 + 9xy + 3y^2$.
Define $L \in \mathcal{L}_{\mathbb{R}}^V$ by: $\forall s, t \in \mathbb{R}$, $L(s, t) = -6s + 3t$. Then: $D_q f = L$.

Proof. Want: $L \in \mathrm{LINS}_q f$. Want: $f_q^{\mathbb{T}} - L \in \mathcal{O}_1^{V\mathbb{R}}$.
We have: $\forall s, t \in \mathbb{R}$,

$$\begin{aligned}
f_q^{\mathbb{T}}(s,t) &= f_{q+(s,t)} - f_q = f_{(s+1,t-1)} - f_{(1,-1)} \\
&= [(s+1)^3 + 9\cdot(s+1)\cdot(t-1) + 3\cdot(t-1)^2] \\
&\quad - [\ 1^3 + 9 \cdot 1 \cdot (-1) + 3 \cdot (-1)^2\ ] \\
&= (s^3 + 3s^2 + 3s) + (9st - 9s + 9t) + (3t^2 - 6t) \\
&= (-6s + 3t) + (3s^2 + 9st + 3t^2) + s^3 \\
&= (\,L(s,t)\,) + (3s^2 + 9st + 3t^2) + s^3.
\end{aligned}$$

Let $S := \pi_1^V$ and $T := \pi_2^V$. Then: $\forall s, t \in \mathbb{R}$,

$$\begin{aligned}
(f_q^{\mathbb{T}} - L)(s,t) &= (\ f_q^T(s,t)\ ) - (\ L(s,t)\ ) \\
&= (\,3s^2 + 9st + 3t^2\,) + s^3 \\
&= (\ (3S^2 + 9ST + 3T^2) + S^3\ )(s,t).
\end{aligned}$$

Then $f_q^{\mathbb{T}} - L = (\,3S^2 + 9ST + 3T^2\,) + S^3$.
Then

$$\begin{aligned}
f_q^T - L &\in (\mathcal{M}_2^{V\mathbb{R}} + \mathcal{M}_2^{V\mathbb{R}} + \mathcal{M}_2^{V\mathbb{R}}) + \mathcal{M}_3^{V\mathbb{R}} \\
&\subseteq \mathcal{H}_2^{V\mathbb{R}} + \mathcal{H}_3^{V\mathbb{R}} \subseteq \widehat{\mathcal{O}}_2^{V\mathbb{R}} + \widehat{\mathcal{O}}_3^{V\mathbb{R}} \\
&\subseteq \mathcal{O}_1^{V\mathbb{R}} + \mathcal{O}_1^{V\mathbb{R}} \subseteq \mathcal{O}_1^{V\mathbb{R}}, \quad \text{as desired.}
\end{aligned}$$

□

5.17. **Miscellaneous.** The following is the D-derivative **chain rule**.

THEOREM 5.17.1. *Let $V, W, X \in \mathrm{ES}$.*
Let $f : V \dashrightarrow W$, let $g : W \dashrightarrow X$ and let $q \in X$.
Then: $D_q(g \circ f) =^* (D_{f_q} g) \circ (D_q f)$.

Proof. I owe you. □

Here is a definition we needed in class:

DEFINITION 5.17.2. *Let $V, W \in \mathrm{ES}$.*
Let $f : V \dashrightarrow W$ and let $j \in \mathcal{I}_V$.
Then: $\partial_j f := \nabla_{\varepsilon_j^V} f$.

The notation "$\partial_j f$" is read "the jth **partial derivative** of f".

FOR OFFICE USE ONLY: Theorem 4.13.11