MATH 4604 Spring 2020, Midterm #1
Handout date: Thursday 27 February 2020
Instructor: Scot Adams

PRINT YOUR NAME:

SOLUTIONS

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One hour exam.

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Turn off all mobile electronic devices.

I. Definitions

A. (5 pts) BNZ := $\cdots$.

$\{f : \mathbb{R} \dashrightarrow \mathbb{R} \mid f$ bdd near 0 $\mathbb{R}$ to $\mathbb{R}\}$

B. (5 pts) Let $k \in \mathbb{N}_0$. Then $\widehat{O}_k := \cdots$

$\mathrm{BNZ} \cdot |\bullet|^k$

C. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $x \in \mathbb{R}$. Then $\mathrm{LINS}_x f := \cdots$.

$\{L \in \mathcal{L} \mid f_x^T - L \in \mathcal{O}_1\}$

D. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x \in \mathbb{R}$. Then $f'_x := \cdots$.

$$(D_x f)_1$$

E. (5 pts) Let X be a metric space. Let $f : X \dashrightarrow \mathbb{R}$, $q \in X$.
By f has a **local strict-maximum** at q in X, we mean ...

$\exists B \in \mathcal{B}_X(q)$ s.t. $f < f_q$ on B_q^X

F. (5 pts) Let $V \in \mathrm{ES}$ and $x, y \in V$. Then $x \bullet y := \cdots$

$$\sum_{j \in J_V} x_j \cdot y_j$$

II. True or false (no partial credit):

a. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Assume $\mathrm{dom}[f'] = \mathbb{R}$.
Assume that f is strictly-increasing on $\mathbb{R}$. Then $f' \geqslant 0$ on $\mathbb{R}$.

True

Even semi-increasing is enough
See Th 4.12.20

b. (5 pts) $\forall a, b \in \mathbb{R}$, $|a| + |b| \leqslant \sqrt{a^2 + b^2}$.

False

$|1| + |1| > \sqrt{1^2 + 1^2}$

c. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$.
Assume both $f(0) = 0$ and $f' \in \mathcal{O}_4$. Then $f \in \mathcal{O}_5$.

True

Th 4.13.7

d. (5 pts) Let $V, W \in \mathrm{ES}$ and let $L \in \mathcal{L}_W^V$.
Then L is Lipschitz from V to W.

True

Th 5.6.15

e. (5 pts) Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $x \in \mathbb{D}_f$.
Then: $\forall h \in \mathbb{R}$, $(f_x^{\mathrm{T}})_h' = f_{x+h}' - f_x'$.

False

Th 4.13.11: $(f_x^{T})_h' = f_{x+h}'$

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I. A,B,C

I. D,E,F

II. a,b,c,d,e

III. 1

III. 2

III. 3

III. 4

III. Hand-graded problems. Show work.

1. (10 pts) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by: $\quad \forall x, y \in \mathbb{R}, \quad f(x, y) = 3x + 4y$.
   Show that f is Lipschitz-5.

<u>Pf</u>: Let $v := (3, 4)$. Then $|v| = \sqrt{3^2 + 4^2} = 5$

$\forall q \in \mathbb{R}^2,$

$$f_q = f(q_1, q_2) = 3q_1 + 4q_2 = (3,4) \odot (q_1, q_2) = v \odot q$$

<u>Want</u>: $\forall p, q \in \mathbb{R}^2, \quad |f_q - f_p| \leq 5 \cdot |q - p|$

Given $p, q \in \mathbb{R}^2$. <u>Want</u>: $|f_q - f_p| \leq 5 \cdot |q - p|$

By Cauchy-Schwarz, $|v \odot (q - p)| \leq |v| \cdot |q - p|$

$$\begin{aligned} |f_q - f_p| &= |v \odot q - v \odot p| \\ &= |v \odot (q - p)| \\ &\leq |v| \cdot |q - p| = 5 \cdot |q - p| \end{aligned}$$

$\blacksquare$

2. (10 pts) Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$.
   Assume: $f_0 = f_0' = g_0 = g_0' = 0$.
   Show: $f \cdot g \in \mathcal{O}_2$.

Pf: $f_0 = f_0' = 0 \quad \therefore \quad f \in \mathcal{O}_1$

$g_0 = g_0' = 0 \quad \therefore \quad g \in \mathcal{O}_1$

$f \cdot g \in \mathcal{O}_1 \cdot \mathcal{O}_1 \subseteq \mathcal{O}_{1+1} = \mathcal{O}_2$

$\blacksquare$

3. (10 pts) Let $a \in [0;\infty)^{\mathbb{N}}$. Assume: $\forall j \in \mathbb{N}$, $a_{j+1} \leqslant a_j/2$.
Using Mathematical Induction, show: $\forall j \in \mathbb{N}$, $a_j \leqslant a_1/2^{j-1}$.

Pf: Let $S := \{ j \in \mathbb{N} \mid a_j \leq \frac{a_1}{2^{j-1}} \}$. Want: $S = \mathbb{N}$

$a_1 = \frac{a_1}{2^{1-1}} \quad \therefore \quad a_1 \leq \frac{a_1}{2^{1-1}} \quad \therefore \quad 1 \in S$

Want: $\forall j \in S$, $j+1 \in S$.

Given $j \in S$. Want: $j+1 \in S$

$a_j \leq \frac{a_1}{2^{j-1}}$ Want: $a_{j+1} \leq \frac{a_1}{2^{(j+1)-1}}$

By hypothesis, $a_{j+1} \leq \frac{a_j}{2}$

$a_j \leq \frac{a_j}{2} = \frac{1}{2} \cdot a_j \leq \frac{1}{2} \cdot \frac{a_1}{2^{j-1}} = \frac{a_1}{2^{(j+1)-1}}$

$\blacksquare$

4. (15 pts.) Let $a \in [0;\infty)^{\mathbb{N}}$. Assume: $\forall j \in \mathbb{N},\ a_{j+1} \leqslant a_j/2$.
Define $b \in [0;\infty)^{\mathbb{N}}$ by: $\forall j \in \mathbb{N},\ b_j = a_1 + a_2 + \cdots + a_j$.
Show that b is convergent in $\mathbb{R}$.

Hint: Use Problem 3, and remember that $\mathbb{R}$ is complete.

Pf: Want: b is Cauchy in $\mathbb{R}$.

Want: $\forall \varepsilon > 0,\ \exists K \in \mathbb{N}$ s.t. $\forall i \in [K..\infty),\ \forall j \in (i..\infty),\ |b_j - b_i| < \varepsilon$

Given $\varepsilon > 0$, Want: $\exists K \in \mathbb{N}$ s.t. $\forall i \in [K..\infty),\ \forall j \in (i..\infty),\ |b_j - b_i| < \varepsilon$

By the Archimedean Principle, choose $K \in \mathbb{N}$ s.t. $K > 2a_1/\varepsilon$

$K \in \mathbb{N}$. Want: $\forall i \in [K..\infty)\ \forall j \in (i..\infty)\ |b_j - b_i| < \varepsilon$

Given $i \in [K..\infty)$, $j \in (i..\infty)$. Want: $|b_j - b_i| < \varepsilon$

By Problem 3, $\forall k \in \mathbb{N}$, $a_k \leqslant \dfrac{a_1}{2^{k-1}}$

$$|b_j - b_i| = |(a_1 + \cdots + a_j) - (a_1 + \cdots + a_i)| = |a_{i+1} + \cdots + a_j|$$

$$= a_{i+1} + \cdots + a_j \leqslant \frac{a_1}{2^i} + \cdots + \frac{a_1}{2^{j-1}}$$

$$= \frac{a_1}{2^{i-1}} \cdot \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{j-i}} \right)$$

$$\leqslant \frac{a_1}{2^{i-1}} \cdot 1 = \frac{2a_1}{2^i} \leqslant \frac{2a_1}{i} \leqslant \frac{2a_1}{K} < \varepsilon$$