**Exposition Handout:
General remarks on binding variables and
on exposition in proofs**

I should say, first of all, that, in learning to use any language (*e.g.*, the language of mathematical proofs), there is substantial variation in the way that different people use the various rules of syntax and grammar. In an initial attempt to learn the language, many rigid rules are put forth to which the student is expected to adhere without exception. However, as one progresses, one notices that native speakers will frequently bend and break these rules, and that different situations require different levels of care.

Here, I set down here a few rules, and will expect you to follow these rules closely in your written work. As you move to other courses, however, you may find that you have more leeway. View all this as part of the process of becoming fluent.

First, at the beginning of the statement of a theorem, all variables are unbound (or free), in the sense that they cannot be used except in

(1) an assignment statement, *e.g.*, "Let $x := 1$" or "Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$",
(2) a statement that begins with the word "Given",
(3) a statement that begins with the word "Choose",
(4) a set definition, as in "$\{a^i h \mid i \in \mathbb{Z} \text{ and } h \in H\}$",
(5) a clause that starts with "For all" or "There exists" or
(6) certain statements involving limits, suprema and infima, maxima and minima.

Each of these items takes an unbound variable and "binds" it to something. For example, if the statement "Choose $x \in S$" appears in the middle of a proof, then the variable x is unbound before the statement and becomes bound to an (unspecified) element of S after the statement. As long as the variable is bound, it cannot be bound to anything new, so it is important to understand the circumstances under which a bound variable becomes "unbound" or "free". This is often a challenge, since many mathematicians do not follow a precise pattern, expecting the reader to understand, from context, the binding and freeing of variables. In this course, we will try to be more regulated, as follows.

For (1), (2) and (3) above: If the proof is not broken up into sections, then, once a variable is bound, it stays bound until the end of the proof.

On the other hand, if you are reading inside a section of a proof (*e.g.*, if you are inside of the proof of some claim), and if a variable becomes bound inside of that section, then it will stay bound until that section reaches its end. After that, the variable is free.

For (4) above: The binding only lasts until the end brace at the end of the set definition. So, for exmple, if a statement includes the text "$\{a^i h \,|\, i \in \mathbb{Z} \text{ and } h \in H\}$", then the variable a must have been bound somewhere above, and continues to be bound following the set definition. By contrast, i and h are only bound from "{" to "}".

For (5) above: These types of statements are called "quantified statements". In them, the variable becomes bound immediately after the quantified clause (*i.e.*, the clause that starts with "for all" or "there exists"). It becomes free at the end of the following clause. For example, in "For all $\varepsilon > 0$, there exists $\delta > 0$ such that $(2\delta)^2 + (2\delta) < \varepsilon$", the variable ε becomes bound immediately after "For all ε", while δ becomes bound immediately after "there exists δ". These two bindings are held only until the sentence comes to an end, after which ε and δ become free variables again.

When following best practices, quantified clauses should appear *before* the variable is used. For example, "$x + y = y + x$, for all $x, y \in \mathbb{R}$" is poorly constructed, and should be changed to "For all $x, y \in \mathbb{R}$, $x + y = y + x$". Humans are fallible, and this rule is often broken, but we will try to be careful about it. Great confusion can arise, if someone writes, for example, "There exists $\delta > 0$ such that $(2\delta)^2 + (2\delta) < \varepsilon$, for all $\varepsilon > 0$", because it's unclear whether δ is allowed to depend on ε.

For (6) above: For example, in "$\lim\limits_{x \to a}[f(x)]$", the variable x must be unbound before "$\lim\limits_{x \to a}$" and becomes temporarily bound starting at "$\lim\limits_{x \to a}$". It becomes free following "$[f(x)]$". For another example, in "$f(x) \to L$, as $x \to a$", the variable x must be unbound before this sentence, is temporarily bound, and becomes unbound after the sentence. It is also UNbound in between "$f(x) \to$" and "as $x \to a$"; that is, "L" cannot be replaced by any expression involving x. So, for example, the statement "$x^2 \to 2x$, as $x \to 2$" is not allowed, because the x in "$2x$" is unbound. Instead of "$f(x) \to L$, as $x \to a$", it might be better, though unconventional, to say "As $x \to a$, $f(x) \to L$", so that one doesn't have to read to the end of the sentence in order to understand how the variable x becomes bound.

There are other similar temporary-binding constructions, using sup, inf, lim sup, lim inf, max and min. For example, in "$\max\limits_{-2\leqslant x\leqslant 1}[x^2] = 4$", the variable x is temporarily bound, starting at "$\max\limits_{-2\leqslant x\leqslant 1}$" and becomes free after "$[x^2]$".

To summarize, a free variable can only be used in statements of types (1) through (6). While a variable is bound, it may be used in all sorts of ways. It may also, later in the proof, become free. However, a variable which is bound at some point in a proof *cannot* be bound again until after it becomes free. So, if a statement of the form "Given $\varepsilon > 0$" appears, then the next statement cannot be "Choose $\varepsilon > 0$", or "Let $\varepsilon := 1$", or "For all $\varepsilon > 0, \ldots$". However, if δ is free, then "Let $\delta := \varepsilon/2$" would be an acceptable continuation.

General rules of argument:

(7) When a proof starts, you should be aware that you are assuming all the hypotheses, and that you want to prove the conclusion. It would be acceptable (if annoying) to start a proof by writing "Know:" followed by all the hypotheses, and then "Want:" followed by the conclusion. However, if you do this, take care not to "rebind" variables; a statement like, "Let n be a positive integer" should not be repeated.

(8) Keep in mind that it's always important to distinguish between what's known and what we want to prove. As the proof proceeds, we sometimes work forward from the hypotheses, and sometimes work backward from the conclusion, and confusion can easily arise if we don't carefully track where we hope to go and where we've already been.

(9) When what we want matches what we know, then the proof ends. Conversely, until what we want matches what we know, the proof should **NOT** end.

Templates for WANTED assertions:

(10) There are often many ways to prove a statement that begins "for all", but the most typical way to proceed is as follows. Parse the statement into "for all $\langle$variable$\rangle\langle$range$\rangle$, $\langle$statement$\rangle$". Then say "Given: $\langle$variable$\rangle\langle$range$\rangle$". Then say "Want: $\langle$statement$\rangle$". So, for example, "Want: For all $\varepsilon > 0$, there exists $\delta > 0$ such that $(2\delta)^2 + (2\delta) < \varepsilon$" would typically be followed by "Given $\varepsilon > 0$. Want: There exists $\delta > 0$ such that $(2\delta)^2 + (2\delta) < \varepsilon$". In this example, $\langle$variable$\rangle$ is "ε", $\langle$range$\rangle$ is "> 0" and $\langle$statement$\rangle$ is "there exists $\delta > 0$ such that $(2\delta)^2 + (2\delta) < \varepsilon$".

(11) There are often many ways to prove a statement that begins "there exists", but the most typical way to proceed is: Parse the statement into "there exists $\langle$variable$\rangle\langle$range$\rangle$ such that $\langle$statement$\rangle$". Then there will be a sequence of statements in which $\langle$variable$\rangle$ is bound. (Figuring out this sequence of statements often difficult. This is where mathematicians make their money!) Following that sequence of statements you should write "Want: [$\langle$variable$\rangle\langle$range$\rangle$] and [$\langle$statement$\rangle$]". Frequently, $\langle$variable$\rangle\langle$range$\rangle$ is obvious from the way $\langle$variable$\rangle$ was bound, in which case, we only say "Want: $\langle$statement$\rangle$". For example: Suppose, at some point in a proof, ε is a bound variable, and $\varepsilon > 0$. Suppose, for some reason, we have just written: "Want: There exists $\delta > 0$ such that $(2\delta)^2 + (2\delta) < \varepsilon$". This might then be followed by: "Let $\eta := \min\{\varepsilon/3, \sqrt{\varepsilon/3}\,\}$. Then $\eta \leqslant \varepsilon/3$. Also, $\eta \leqslant \sqrt{\varepsilon/3}$, so $\eta^2 \leqslant \varepsilon/3$. Let $\delta := \eta/2$." At this point δ has become bound, so we would write: "Want: [$\delta > 0$] and [$(2\delta)^2 + (2\delta) < \varepsilon$]". However, in this case, my judgment is that $\delta > 0$ is clear enough that we would probably omit it, and say only, "Want: $(2\delta)^2 + (2\delta) < \varepsilon$". After this, we need to show that our "strategy" (of $\eta := \min\{\varepsilon/3, \sqrt{\varepsilon/3}\,\}$ and $\delta := \eta/2$) works (*i.e.*, somehow yields: $(2\delta)^2 + (2\delta) < \varepsilon$). This could be accomplished by arguing as follows: "We have $\eta^2 + \eta \leqslant (\varepsilon/3) + (\varepsilon/3) = 2\varepsilon/3 < \varepsilon$. Also, $2\delta = \eta$. Then $(2\delta)^2 + 2\delta = \eta^2 + \eta < \varepsilon$, as desired. QED" A key point: Once what you know matches what you want, you STOP.

(12) If you want to prove a statement of the form "A$\Rightarrow$B", you would typically write: "Assume A. Want B."

(13) If you want to prove a statement B, you may always argue by contradiction. You need to know how to negate B, to end up with a new statement, NOT B, whose truth is equivalent to the falsity of B. You then write "Assume NOT B. Want: Contradiction." You then argue until you have two contradictory statements, after which you write "Contradiction. QED". So, for example, if you want to prove, "For all $a \in \mathbb{R}$, $a^2 \geqslant 0$", you may write "Assume there exists $a \in \mathbb{R}$ such that $a^2 < 0$. Want: Contradiction". More argument is now needed, but the proof ends at "Contradiction. QED". I would discourage, for example, writing: "...Contradiction. This contradiction happened because we were assuming existence of an $a \in \mathbb{R}$ such that $a^2 < 0$. Thus we have proved that that's impossible. That is, we have proved, for all $a \in \mathbb{R}$, that $a^2 \geqslant 0$. QED" Leave out all that extraneous, superfluous, unnecessary, repetitive and reiterative stuff!

(14) Say we want to prove [$P \,\&\, Q$]. Then, if we wish, we can break the proof into two sections. Typically, we would write something like: "Want: (a) P, and (b) Q", followed by a section that starts "Proof of (a):", followed by a sequence of statements, at the end of which P is known. We then write "End of proof of (a)". We would then write a section the starts "Proof of (b):". There then follows a sequence of statements, at the end of which Q is known. We then write "End of proof of (b)".

(15) The last bullet point has analogues for any number of statements, not just two. For example, if [$P \,\&\, Q \,\&\, R$] is a wanted statement, then we can break the proof into three sections.

Other templates:

(16) Let P and Q be two mathematical statements, and suppose that [$P \vee Q$] is a known statement. Then, if we wish, we can break the proof into two cases, as follows: First, write "Case 1: P", followed by a sequence of statements in which P is assumed true. This sequence ends when what we know matches what we want. Then write "Case 2: Q", followed by a sequence of statements in which Q is assumed true. This sequence ends when what we know matches what we want. The end of the second sequence marks the end of the entire proof. Keep in mind: Each of these two sequences of statements is considered to be a section of the proof. Remember that variables that are bound within a section of a proof are freed at the end of the section. Therefore, a variable that's bound during Case 1 cannnot be used in Case 2 (unless, of course, it gets bound again somewhere in Case 2). Variables bound before Case 1 in the main body of the proof (*i.e.*, not in any section of the proof), will stay bound through both Case 1 and Case 2.

(17) The last bullet point has analogues for any number of statements, not just two. For example, if [$P \vee Q \vee R$] is a known statement, then we can break the proof into three cases.

(18) Let P be a mathematical statement that we wish to establish, as part of a proof. Sometimes the proof of P can be complicated and, to "modularize" the proof, we wish to set off those statements that establish the validity of P. Typically, in such a case, we will write "Claim: P. Proof of claim:". There then follows a sequence of statements, at the end of which P is known. We then write "End of proof of claim." After that, we finish the main argument, and, as usual, stop when what we want matches what we know. Keep in mind: The proof of the claim is considered to be a section of the overall proof, and variables bound within a section of a proof are freed at the end of the section. So a variable bound during the proof of the claim cannot be used after the proof of the claim (unless, it gets bound again somewhere after the claim's proof). Variables bound before the claim, in the main body of the proof (*i.e.*, not in any section of the proof), will stay bound all the way through the claim, and, then, all the way to the end of the main proof.

(19) The last bullet point has analogues for any number of claims, not just one. In this case, for ease of later reference, it is common to number the claims as "Claim 1", "Claim 2", *etc.*

(20) The Principle of Mathematical Induction template: Say we wish to prove a sequence of statements $\mathbf{P}_1, \mathbf{P}_2, \ldots$. Proceed as follows: "Let $S := \{j \in \mathbb{N} \,|\, \mathbf{P}_j\}$. Want: $S = \mathbb{N}$. $\cdots\cdots$. Then $\mathbf{P}_1$, so $1 \in S$. By the PMI, want: $\forall j \in S$, $j+1 \in S$. Given $j \in S$. Want: $j+1 \in S$. Know: $\mathbf{P}_j$. Want: $\mathbf{P}_{j+1}$." Then work, as usual, until what you know matches what you want.

Replacement rules for KNOWN assertions with quantifiers:

(21) It is acceptable to replace a quantified variable by another variable that is unbound at the quantifier. So for example, if we have a known statement that reads "$\forall x \geqslant 0$, $\exists y \in \mathbb{R}$ s.t. $y^2 = x$", then, assuming that z is unbound, we can replace every "x" by "z" and obtain: "$\forall z \geqslant 0$, $\exists y \in \mathbb{R}$ s.t. $y^2 = z$". However, in "$\forall x \geqslant 0$, $\exists y \in \mathbb{R}$ s.t. $y^2 = x$", we cannot replace "y" by "x", because, while x is unbound at the start of the statement, it is bound at the "$\exists$" quantifier.

(22) In any known existentially quantified statement, it is acceptable to replace "$\exists$" by "choose". This then binds the variable until the end of the section of the proof in which the statement appears. So, if we know "$\exists y \in \mathbb{R}$ s.t. $y^2 = 7$", we can follow that by "Choose $y \in \mathbb{R}$ s.t. $y^2 = 7$". WARNING: If the existential quantifier follows a universal quantifier in the same sentence, then, invoking the Axiom of Choice, we can still replace "$\exists$" by "choose", but the existentially quantified variable needs to be replaced by an expression that takes into account that many choices are being made. So, for example, in the statement "$\forall x \geqslant 0$, $\exists y \in \mathbb{R}$ s.t. $x = y^2$", it would NOT be correct to simply replace "$\exists$" by choose, and say "$\forall x \geqslant 0$, choose $y \in \mathbb{R}$ s.t. $y^2 = x$", because there is not one y that works for every x. We can replace "$\exists$" by "choose", if we ALSO replace "y" by, say, "y_x". This then yields the statement: "$\forall x \geqslant 0$, choose $y_x \in \mathbb{R}$ s.t. $(y_x)^2 = x$".

(23) In any known universally quantified statement, it is acceptable to eliminate the quantified clause, and then, in the rest of the statement, replace the quantified variable by any bound expression that satisfies any conditions that appear in the quantified clause. (An expression is "bound" if all the variables appearing in it are bound.) So, for example, if we know "For all $x \geqslant 0$, there exists $y \in \mathbb{R}$ such that $y^2 = x$", and if, say, q and r are bound variables, and if we know that $q + r \geqslant 0$, then we can replace x by $q + r$ and we obtain the known statement: "There exists $y \in \mathbb{R}$ such that $y^2 = q + r$". Or, because $7 \geqslant 0$, we can replace x by 7 and we then know "There exists $y \in \mathbb{R}$ such that $y^2 = 7$". NOTE: If, after replacement, we end up with a statement of the form "$P \Rightarrow Q$", and if P is known to be false, then the replacement was useless, and should not have been done. If, after replacement, we end up with a statement of the form "$P \Rightarrow Q$", and if P is known to be true, then we should erase "$\Rightarrow$" and change it to "therefore", obtaining "$P, \text{therefore}\, Q$". (See (26) below.)

The preceding replacement rules do NOT apply to statements preceded by "Want". They only apply to known statements.

Miscellaneous comments about exposition in proofs (comments below *only* apply to proofs):

(24) Don't stop until what you want matches what you know.
(25) Stop when what you want matches what you know. No extraneous, superfluous, unnecessary, repetitive, reiterative stuff!
(26) A statement of the form "$A \Rightarrow B$" should typically not be made when A is known to be true. If we are in a situation where we know that A is true and that, as a consequence, B is also true, then it is best to say: "A, therefore B", or, alternatively, "A, so B", or "A, thus B", or something similar. The problem is that "$A \Rightarrow B$" makes it seem as if: we are unsure about A, but, at some point in the future, we may establish A, and then we will know that B is true. For example, suppose, at some point in some proof, we somehow know that $a > 3$, and we want to

conclude that $a^2 > 9$. Writing "$[\,a > 3\,] \;\Rightarrow\; [\,a^2 > 9\,]$" is not wrong, but conveys the impression that we are unsure whether or not $a > 3$ is true. It's therefore better to say, *e.g.*, "Since $a > 3$, it follows that $a^2 > 9$."

(27) In an addendum to the preceding point, it's of no use to say "$A \Rightarrow B$", when A is known to be false. The value of "$A \Rightarrow B$" occurs only when we are unsure whether A is true or false.

(28) The phrases "Let", "Choose", "Given", "For all" and "There exists" should always precede a variable, not an expression. For example, do **NOT** say "Choose $a_i \in \{a_1, \ldots, a_n\}$". Rather, say "Choose $i \in \{1, \ldots, n\}$". Also, take care that the variable used is *unbound*; if i is bound, we cannot say "Choose $i \in \{1, \ldots, n\}$"; instead we'd look for some other variable to use.

(29) In a proof, the word "let" should be followed by an unbound variable, then ":=", then a bound expression. You may use "=" instead of ":=", but I'd prefer ":=". In the statement of a theorem, it's acceptable to say, *e.g.*, "let $x \in \mathbb{R}$", but not in a proof.

(30) The word "Given" should **never** be used except after a statement of the form "Want: For all . . . , we have $***$". If, after such a statement, you choose to follow the "Want: $\forall$" template, then you should write "Given: . . . ", immediately followed by "Want: $***$". "Given" should not be used as a replacement for "Know". So, if, say, an assumption in some theorem reads "$n \geqslant 1$", then it's **not correct**, in the proof, to write "Given $n \geqslant 1$", in the sense of "It is given in the theorem that $n \geqslant 1$". Instead, you should say something like "By assumption, we know that $n \geqslant 1$". If you use the word "given", be prepared to explain exactly which "Want: $\forall$" statement generated it.

(31) I would not typically use "Choose" with an assignment. For example, I would not say "Choose $\delta := \eta/2$." It's like saying, "Choose a card, any card, as long as it's the ace of spades." (Or, to paraphrase Henry Ford: "Choose any color you want for your Model T, as long as it's black.") Instead of "Choose $\delta := \eta/2$", it's better to say "Let $\delta := \eta/2$".

(32) If you use the word "Want" at the beginning of a statement, and then, again, at the beginning of a later statement, you should be prepared to explain why the later statement implies the earlier one. (This doesn't apply if the second "Want" was the result of applying one of our templates.)

(33) A statement such as "$a^2 = b^8$" is equivalent to "Know: $a^2 = b^8$". In other words, if a statement is not a binding statement and if, in addition, it does not begin with "Want" or "Know", then you may place "Know:" in front of it without affecting its meaning. If you assert that you know some statement (whether you actually use the word "Know" or not), you should be prepared to explain why it follows from earlier known facts. This is in contrast to "Want", for which the earlier wanted statement should follow from the later one (assuming the later one didn't arise from the use of a template).

(34) In this course, we will NOT use the word "where" in proofs. This word is sometimes used to mean "for all" and sometimes used to mean "there exists", and sometimes is used as a catchall for general commentary.

(35) We avoid using "for some". Instead of "for some $x \in \mathbb{R}$, we have $x^3 = 8$" please say "there exists $x \in \mathbb{R}$ such that $x^3 = 8$", or, if you want the binding of x to continue after the sentence, say "Choose $x \in \mathbb{R}$ such that $x^3 = 8$". Keep in mind that the statement "$x^3 = 8$, for some $x \in \mathbb{R}$" is in bad form for two reasons: First, it uses "for some", which we are disallowing. Second the binding of the variable x comes *after* its use.

(36) In this course, we will NOT use "without loss of generality" or "we may assume" in proofs. It is possible to set templates for the use of these phrases, but it would be complicated, and their use can always be avoided. The only advantage of these constructions is that, sometimes, in order to avoid them, we will be forced to introduce extra notation, which can obscure the main ideas of the proof. However, I've made the decision that, for this class, the confusion from creating yet more templates outweighs the benefit of reducing notation.

(37) In this class, "for all" does not take "such that" as part of the quantification clause. Some mathematicians do not follow this rule, and might write, for example: "For all $x \in \mathbb{R}$ such that $-1 < x < 1$, we have $x^2 < 1$." In this class, we will replace this with the more standard form: "For all $x \in \mathbb{R}$, we have: $[\ (-1 < x < 1) \ \Rightarrow \ (x^2 < 1)\ ]$." Note that *after* the "for all" quantification clause, if we have a "there exists" quantification clause, then it's quite common and acceptable follow that second clause with "such that". For example, one often sees: "For all $\varepsilon > 0$, there exists $\delta > 0$ such that ...." Simply stated, "for all" does not take "such that", but "there exists" does.

(38) The word "assume" may be used in the statement of a theorem, but should not be used in a proof, *except*: as part of the "implies" template or as part of the proof by contradiction template. If you use the word "assume" in a proof, you should be prepared to explain exactly which template generated it.

(39) Once you know a statement that is the same as the last wanted statement, you should end the proof, unless you are in the middle of a *section* of a proof that has been broken up, in which case you should end that section and go on to the next. Otherwise, you'll end up with extraneous, superfluous, unnecessary, repetitive and reiterative stuff. That is to say, stuff that can be removed.

(40) The standard way to define a function f with domain A and target B is either by the construction, "Define $f : A \to B$ by ..." or by the construction, "Let $f : A \to B$ be defined by ...". Sometimes "$f : A \to B$" is replaced by "$f \in B^A$". So, if we say, "Define $f : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R}$, $f(x) = x^2$", then, for example we would have: $f(5) = 25$. Even though it results in an unbound variable, it's acceptable to omit "$\forall x \in \mathbb{R}$" in this construction and say "Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$". Similarly, if we asy "Let $a \in \mathbb{R}^{\mathbb{N}}$ be defined by $a_j = 1/j$", then $a_\bullet$ would be the sequence $(1, 1/2, 1/3, 1/4, \ldots)$.

(41) Suppose, in a proof, you have written "Want: $a < d$" and "Know: $a < b$" and "Know: $c < d$". Suppose, later in the proof, you know that $b - 3 < c - 3$. Bad form: "Since $b - 3 < c - 3$, we get $a < b < c < d$. QED". It's much better to write: "Since $b - 3 < c - 3$, we get $b < c$. Then $a < b < c < d$. QED"

Let's talk about how to prove a specific theorem:

Theorem. Let $a, b \in \mathbb{R}^{\mathbb{N}}$. Assume that $a_\bullet \to 3$ and $b_\bullet \to 4$. Then $(a+b)_\bullet \to 7$.

Steps in writing a proof of the theorem above:

(42) I often omit this first step, but it's reasonable to begin by taking all the assumptions, and writing out what each one means, in quantified form. So, we might begin "Know: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ |a_j - 3| < \varepsilon\ )\ ]$. Know: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ |b_j - 4| < \varepsilon\ )\ ]$". I now imagine that I have, at my disposal, two oracles, one of whom I call my a-oracle. Given any $\varepsilon > 0$, she returns
a number $K \in \mathbb{N}$ s.t. $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ |a_j - 3| < \varepsilon\ )\ ]$.
The other oracle is my b-oracle. Given any $\varepsilon > 0$, he returns
a number $K \in \mathbb{N}$ s.t. $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ |b_j - 4| < \varepsilon\ )\ ]$.

(43) Write "Want:" followed by the conclusion, written in quantified form. So write "Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t.,
$\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ |(a+b)_j - 7| < \varepsilon\ )\ ]$."
Keep in mind that, by definition of $(a+b)_\bullet$, for all $j \in \mathbb{N}$, we have $(a+b)_j = a_j + b_j$. Now imagine we are playing a game. Someone give us $\varepsilon > 0$, and we give them $K \in \mathbb{N}$ and they give us $j \in \mathbb{N}$ satisfying $j \geqslant K$. If $|(a+b)_j - 7| < \varepsilon$, then we win. We may consult with our a-oracle and b-oracle as needed.

(44) Structure the proof. This means that we apply the templates to the statement we trying to prove, leaving blanks when needed. So, as we are trying to prove the statement in the last bullet point, we would write: "Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ |(a+b)_j - 7| < \varepsilon\ )\ ]$." Then leave blank space, in which the variable K is to be bound. That blank space is for our "K-strategy". Below it, we write "Want: $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ |(a+b)_j - 7| < \varepsilon\ )\ ]$. Given $j \in \mathbb{N}$. Want: $(\ j \geqslant K\ ) \Rightarrow (\ |(a+b)_j - 7| < \varepsilon\ )$. Assume $j \geqslant K$. Want: $|(a+b)_j - 7| < \varepsilon$." This finishes the structuring of the proof. We now need to work on the K-strategy (which will be placed up above, in the blank space) and the finish (at the end). We stop when (and only when) what "Want" matches "Know".

(45) At this point, we might look over the last wanted statement, which reads, "Want: $|(a+b)_j - 7| < \varepsilon$." We need to think about what requirements on K will ensure that $|(a+b)_j - 7| < \varepsilon$ becomes true. Note that, in the space where the K-strategy goes, we do not yet have j bound, so the K-srategy cannot use anything about j. Also, keep in mind that the a-oracle and b-oracle are at our disposal. If we could get $|a_j - 3| < \varepsilon$ and $|b_j - 4| < \varepsilon$, then we could use the triangle inequality to prove $|(a_j-3)+(b_j-4)| < \varepsilon+\varepsilon$, which is the same as $|(a+b)_j - 7| < 2\varepsilon$. This is almost what we want. We just need to use the oracles choose K in a way that guarantees that, no matter which j is chosen, we will have both $|a_j - 3| < \varepsilon/2$ and $|b_j - 4| < \varepsilon/2$. Then everything will work out fine. As we think about all this, we might write some of this down some notes on scratch paper, but none of it will appear in the actual proof. This line of thinking only helps us to clarify our K-strategy.

(46) Now go back to the blank space for the K-strategy and look at the two known statements at the top of the proof. Using replacement rules, fill in the blank space: "Choose $L \in \mathbb{N}$ s.t.

$\forall j \in \mathbb{N}$, $[\ (\ j \geqslant L\ ) \Rightarrow (\ |a_j - 3| < \varepsilon/2\ )\ ]$.

Choose $M \in \mathbb{N}$ s.t.

$\forall j \in \mathbb{N}$, $[\ (\ j \geqslant M\ ) \Rightarrow (\ |b_j - 4| < \varepsilon/2\ )\ ]$.

Let $K := \max\{L, M\}$." This means that we consult the a-oracle to get an L, then the b-oracle to get an M, and then take the larger of L and M and call it K. That's our K-strategy.

(47) The finish. We write the part after "Want: $|(a+b)_j - 7| < \varepsilon$." This is where we show that our K-strategy always works. Write: "As $j \geqslant K \geqslant L$, by choice of L, we have $|a_j - 3| < \varepsilon/2$. As $j \geqslant K \geqslant M$, by choice of M, we have $|b_j - 4| < \varepsilon/2$. Then

$$\begin{aligned} |\,(a+b)_j - 7\,| &= |\,(a_j + b_j) - (3+4)\,| \\ &= |\,(a_j - 3) + (b_j - 4)\,| \\ &\leqslant |a_j - 3| + |b_j - 4| \\ &< [\varepsilon/2] + [\varepsilon/2] = \varepsilon, \end{aligned}$$

as desired."

(48) Technically, in the last point, before the "as desired", we should say, "so $|(a+b)_j - 7| < \varepsilon$"; then "Want" will match "Know". However, I allow that we can omit a final contraction step.
(49) Add a box or "QED" to signify the end of the proof.
(50) Celebrate . . . responsibly!

Here is the finished product:

Theorem. Let $a, b \in \mathbb{R}^{\mathbb{N}}$. Assume that $a_\bullet \to 3$ and $b_\bullet \to 4$. Then $(a+b)_\bullet \to 7$.

Proof: Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t.,
$\quad \forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ |(a+b)_j - 7| < \varepsilon\ )\ ]$.
Given $\varepsilon > 0$.
Want: $\exists K \in \mathbb{N}$ s.t.,
$\quad \forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ |(a+b)_j - 7| < \varepsilon\ )\ ]$.
Choose $L \in \mathbb{N}$ s.t.
$\quad \forall j \in \mathbb{N}$, $[\ (\ j \geqslant L\ ) \Rightarrow (\ |a_j - 3| < \varepsilon/2\ )\ ]$.
Choose $M \in \mathbb{N}$ s.t.
$\quad \forall j \in \mathbb{N}$, $[\ (\ j \geqslant M\ ) \Rightarrow (\ |b_j - 4| < \varepsilon/2\ )\ ]$.
Let $K := \max\{L, M\}$.
Want: $\forall j \in \mathbb{N}$, $[\ (\ j \geqslant K\ ) \Rightarrow (\ |(a+b)_j - 7| < \varepsilon\ )\ ]$.
Given $j \in \mathbb{N}$.
Want: $[\ (\ j \geqslant K\ ) \Rightarrow (\ |(a+b)_j - 7| < \varepsilon\ )\ ]$.
Assume $j \geqslant K$.
Want: $|(a+b)_j - 7| < \varepsilon$.
As $j \geqslant K \geqslant L$, by choice of L, we have $|a_j - 3| < \varepsilon/2$.
As $j \geqslant K \geqslant M$, by choice of M, we have $|b_j - 4| < \varepsilon/2$.
Then

$$\begin{aligned} |\,(a+b)_j - 7\,| &= |\,(a_j - 3) + (b_j - 4)\,| \\ &\leqslant |a_j - 3| + |b_j - 4| \\ &< [\varepsilon/2] + [\varepsilon/2] = \varepsilon, \end{aligned}$$

as desired.
QED