Homework for MATH 4604 (Advanced Calculus II)
Spring 2020

Homework 8: Due on Tuesday 31 March

8-1. Let $V, W \in \mathrm{ES}$, $f : V \dashrightarrow W$, $q, u \in V$.
Assume: $f = 0_W$ near q in V.
Show: $\nabla_u f = 0_W$ near q in V.

8-2. Let $V, W \in \mathrm{ES}$, $f : V \dashrightarrow W$, $q \in V$.
Assume: $f = 0_W$ near q in V.
Show: $\forall k \in \mathbb{N}$, $\forall u_1, \ldots, u_k \in V$,
$\nabla_{u_k} \cdots \nabla_{u_1} f = 0_W$ near q in V.

8-3. Let $W \in \mathrm{ES}$, $\phi : \mathbb{R} \dashrightarrow W$, $t \in \mathbb{R}$.
Assume: $\phi = 0$ near t in $\mathbb{R}$.
Show: $\phi'_t = 0$.

8-4. Let $V := \mathbb{R}^2$, $f : V \dashrightarrow \mathbb{R}$.
Let $S := \{(x, y) \in \mathbb{R}^2 \mid x^2 \leqslant y \leqslant 3x^2\}$.
Assume: $f = 0$ on $V \backslash S$.
Show: $\forall k \in \mathbb{N}$, $\forall u_1, \ldots, u_k \in V$,
$\nabla_{u_k} \cdots \nabla_{u_1} f = 0$ on $V \backslash S$.

8-5. Let $V := \mathbb{R}^2$, $f : V \dashrightarrow \mathbb{R}$, $u \in V$.
Let $S := \{(x, y) \in \mathbb{R}^2 \mid x^2 \leqslant y \leqslant 3x^2\}$.
Assume: $f = 0$ on $V \backslash S$. Assume: $f_{0_V} = 0$.
Show: $(\nabla_u f)_{0_V} = 0$.

Homework 7: Due on Wednesday 25 March

7-1. Let $V \in \mathrm{ES}$, $x, u \in V$, $s \in \mathbb{R}$.
Show: $(\alpha_x^u)_s^{\mathbb{T}} = \alpha_{0_V}^u$.

7-2. Let $V \in \mathrm{ES}$, $x, u \in V$, $s \in \mathbb{R}$.
Show: $(\alpha_x^u)'_s = u$.

7-3. Let $V \in \mathrm{ES}$, $f : V \dashrightarrow W$, $q, u \in V$.
Define $S : V \to V$ by: $\forall x \in V$, $S_x = x + q$.
Show: $\nabla_u (f \circ S) = (\nabla_u f) \circ S$.

7-4. Let $V, W \in \mathrm{ES}$, $C \in \mathcal{C}_W^V$, $u \in V$.
Show: $\nabla_u C = \mathbf{0}_W^V$.

7-5. Let $V, W \in \mathrm{ES}$, $f, g : V \dashrightarrow W$, $x, u \in V$.
Assume: $f = g$ near x in V.
Show: $f \circ \alpha_x^u = g \circ \alpha_x^u$ near 0 in $\mathbb{R}$.

Homework 6: Due on Wednesday 18 March

6-1. Let $V, W \in \mathrm{ES}$, let $* \in \mathcal{B}_W^{VV}$ and let $\# := \mathrm{Sym}^*$.
Show: $\# \in \mathcal{SB}_W^V$ and $\mathrm{Qd}^\# = \mathrm{Qd}^*$.

6-2. Let $V, W \in \mathrm{ES}$, let $* \in \mathcal{SB}_W^V$ and let $Q := \mathrm{Qd}^*$.

$$\text{Show:} \quad \forall u, v \in V, \quad u * v = \frac{Q_{u+v} - Q_u - Q_v}{2}.$$

6-3. Let $V := \mathbb{R}^2$ and $W := \mathbb{R}$. Define $Q : V \to W$ by:

$$\forall x, y \in \mathbb{R}, \qquad Q(x, y) = 3x^2 - 4xy + 5y^2.$$

Let $B \in \mathcal{SB}_W^V$. Assume: $\mathrm{Qd}^B = Q$. Show: $[B] = \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$.

6-4. Let $V := \mathbb{R}^2$. Define $Q : V \to \mathbb{R}$ by:

$$\forall x, y \in \mathbb{R}, \qquad Q(x, y) = 2x^2 + 6xy + 5y^2.$$

Show: $Q > 0$ on $V_{0_V}^\times$.

6-5. Let $V := \mathbb{R}^2$. Define $Q : V \to \mathbb{R}$ by:

$$\forall x, y \in \mathbb{R}, \qquad Q(x, y) = 2x^2 + 6xy + 4y^2.$$

Find $x, y \in \mathbb{R}$ s.t. $Q(x, y) < 0$.

Homework 5: Due on Tuesday 3 March

5-1. Let $\alpha, \beta \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $W := \mathbb{R}^\beta$, $L \in \mathcal{AL}_W^V$.
Show: $\mathrm{Lin}^{[L]} = L$.

5-2. Let $\alpha, \beta \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $W := \mathbb{R}^\beta$, $S \in W \otimes V$.
Show: $[\,\mathrm{Lin}^S\,] = S$.

5-3. Let $\alpha, \beta, \gamma \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $W := \mathbb{R}^\beta$, $X := \mathbb{R}^\gamma$.
Let $L \in \mathcal{L}_W^V$ and let $M \in \mathcal{L}_X^W$. Show: $[M \circ L] = [M] \cdot [L]$.

5-4. Let $\alpha, \beta, \gamma \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $W := \mathbb{R}^\beta$, $X := \mathbb{R}^\gamma$.
Let $B \in \mathcal{B}_X^{VW}$. Show: $\mathrm{Bilin}^{[B]} = B$.

5-5. Let $\alpha, \gamma \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $X := \mathbb{R}^\gamma$, $* \in \mathcal{B}_X^{VV}$.
Define $Q : V \to X$ by: $\forall v \in V$, $Q_v = v * v$. Show: $Q \in \mathcal{Q}_X^V$.

Homework 4: Due on Tuesday 25 February

4-1. Define $a \in \mathbb{R}^{\mathbb{N}_0}$ by: $\forall j \in \mathbb{N}_0,\ a_j = 1/(j!)$. Show: $\mathrm{RC}_a = \infty$.

4-2. Let $a \in \mathbb{R}^{\mathbb{N}_0}$, $x, r \in \mathbb{R}$, $i, j \in \mathbb{N}_0$, $b := P_\bullet SS_x^a$, $c := P_\bullet SS_r^{|a|}$.
Assume: $|x| \leqslant r$. Show: $|b_j - b_i| \leqslant |c_j - c_i|$.

4-3. Let $\alpha \in [0;\infty)^{\mathbb{N}_0}$, $r \in [0; \mathrm{RC}_\alpha)$, $c := P_\bullet SS_r^\alpha$.
Show: c is convergent in $\mathbb{R}$.

4-4. Let $a \in \mathbb{R}^{\mathbb{N}_0}$ and let $\alpha := |a|$. Show: $\mathrm{RC}_a = \mathrm{RC}_\alpha$.

4-5. Let $a \in \mathbb{R}^{\mathbb{N}_0}$. Show: $\mathrm{RC}_a = \mathrm{RC}_{a*}$.

Homework 3: Due on Tuesday 18 February

3-1. Let X be a metric space, $s \in X^{\mathbb{N}}$, $q \in X$, $m \in \mathbb{N}$.
Define $t \in X^{\mathbb{N}}$ by: $\forall j \in \mathbb{N},\ t_j = s_{j+m}$.
Assume: $t \to q$ in X. Show: $s \to q$ in X.

3-2. Let X and Y be metric spaces, $s \in X^{\mathbb{N}}$, $t \in Y^{\mathbb{N}}$.
Show: [(s,t) is Cauchy in $X \times Y$]
$\Leftrightarrow$ [(s is Cauchy in X) & (t is Cauchy in Y)].

3-3. Let X and Y be complete metric spaces.
Show $X \times Y$ is complete.

3-4. Let $p, q, r \in \mathbb{N}$, $A \in \mathbb{R}^{p \times q}$, $B \in \mathbb{R}^{q \times r}$. Show: $|AB| \leqslant |A| \cdot |B|$.

3-5. Let $V \in \mathrm{ES}$. Show:
(a) $|\bullet|_V$ is Lipschitz-1 from V to $\mathbb{R}$
and (b) $\|\bullet\|_V$ is Lipschitz-$\sqrt{\#\mathcal{I}_V}$ from V to $\mathbb{R}$.

Homework 2: Due on Tuesday 11 February

2-1. Show: $\forall \ell \in \mathbb{N},\ ((\bullet)^\ell)' = \ell \cdot (\bullet)^{\ell-1}$.

2-2. Let $m \in \mathbb{R}$, $L : \mathbb{R} \to \mathbb{R}$. Assume: $\forall x \in \mathbb{R},\ L_x = m \cdot x$.
Show: ($\forall x \in \mathbb{R},\ L'_x = m$) & ($\forall x \in \mathbb{R},\ L''_x = 0$).

2-3. Let $c \in \mathbb{R}$, $Q : \mathbb{R} \to \mathbb{R}$. Assume: $\forall x \in \mathbb{R},\ Q_x = c \cdot x^2$.
Show: ($\forall x \in \mathbb{R},\ Q'_x = 2 \cdot c \cdot x$) & ($\forall x \in \mathbb{R},\ Q''_x = 2 \cdot c$).

2-4. Let $\alpha, \beta : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume: $\forall x \in \mathbb{R},\ \beta_x =^* \alpha_x$.
Show: $\forall x \in \mathbb{R},\ \beta'_x =^* \alpha'_x$.

2-5. Let $a \in \mathbb{R}$. Define $T : \mathbb{R} \to \mathbb{R}$ by: $\forall h \in \mathbb{R},\ T_h = h + a$.
Show: $T' = C^1_{\mathbb{R}}$.

Homework 1: Due on Tuesday 4 February

1-1. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{D}_f$.
Show: (f has a global strict-maximum at p)
$\Leftrightarrow$ ($f_p^{\mathbb{T}}$ has a global strict-maximum at 0).

1-2. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, let $c \in \mathbb{R}$, let $L \in \mathrm{LINS}_c f$ and let $\varepsilon > 0$.
Show: $L - \varepsilon \cdot |\bullet| \leqslant f_c^{\mathbb{T}} \leqslant L + \varepsilon \cdot |\bullet|$ near 0 in $\mathbb{R}$.

1-3. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, let $c \in \mathbb{R}$ and let $L \in \mathrm{LINS}_c f$.
Assume: f has a local semi-maximum at c in $\mathbb{R}$.
Show: $L = \mathbf{0}$.

1-4. Let $\phi : \mathbb{R} \dashrightarrow \mathbb{R}$, let $a > 0$ and let $Q := a \cdot (\bullet)^2$.
Assume: $\phi - Q \in \mathcal{O}_2$.
Show: ϕ has a local strict-minimum at 0 in $\mathbb{R}$.

1-5. Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $c \in \mathbb{R}$.
Assume: $f = g$ near c in $\mathbb{R}$.
Show: $\mathrm{LINS}_c f \subseteq \mathrm{LINS}_c g$.