Solutions for MATH 4604 (Advanced Calculus II)
Spring 2020

Homework 6: Due on Wednesday 18 March

6-1. Let $V, W \in \text{ES}$, let $* \in \mathcal{B}_W^{VV}$ and let $\# := \text{Sym}^*$.
Show: $\# \in \mathcal{SB}_W^V$ and $\text{Qd}^\# = \text{Qd}^*$.

Proof: We have: $\forall u, v \in V$,

$$u\#v = \frac{u*v+v*u}{2} = \frac{u*v+v*u}{2} = \frac{v*u+u*v}{2} = v\#u.$$

Then $\# \in \mathcal{SB}_W^V$. Want: $\text{Qd}^\# = \text{Qd}^*$. Want: $\forall u \in V$, $\text{Qd}_u^\# = \text{Qd}_u^*$.
Given $u \in V$. Want: $\text{Qd}_u^\# = \text{Qd}_u^*$. We have:

$$\text{Qd}_u^\# = u\#u = \frac{u*u+u*u}{2} = u*u = \text{Qd}_u^*,$$ as desired. QED

6-2. Let $V, W \in \text{ES}$, let $* \in \mathcal{SB}_W^V$ and let $Q := \text{Qd}^*$.
Show: $\forall u, v \in V$, $u * v = \dfrac{Q_{u+v} - Q_u - Q_v}{2}$.

Proof: Given $u, v \in V$. Want: $u * v = \dfrac{Q_{u+v} - Q_u - Q_v}{2}$.
We have: $Q_{u+v} = \text{Qd}_{u+v}^* = (u+v)*(u+v) = (u*u) + 2\cdot(u*v) + (v*v)$
and $Q_u = \text{Qd}_u^* = u*u$, $Q_v = \text{Qd}_v^* = v*v$,
so $Q_{u+v} - Q_u - Q_v = 2\cdot(u*v)$.
Then $$u*v = \frac{2\cdot(u*v)}{2} = \frac{Q_{u+v} - Q_u - Q_v}{2}.$$ QED

6-3. Let $V := \mathbb{R}^2$ and $W := \mathbb{R}$. Define $Q : V \to W$ by:
$\forall x, y \in \mathbb{R}$, $Q(x, y) = 3x^2 - 4xy + 5y^2$.
Let $B \in \mathcal{SB}_W^V$. Assume: $\text{Qd}^B = Q$. Show: $[B] = \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$.

Proof: Let $u := \varepsilon_1^V$ and $v := \varepsilon_2^V$.
Let $x := B(u, u)$ and $y := B(u, v)$ and $z := B(v, v)$.
Then, as $B \in \mathcal{SB}_W^V$, we get: $y = B(v, u)$. Then $[B] = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$.
We have $u = \varepsilon_1^V = (1, 0)$ and $v = \varepsilon_2^V = (0, 1)$. Then $u + v = (1, 1)$.
We have $Q(u) = Q(1, 0) = 3\cdot 1^2 - 4\cdot 1\cdot 0 + 5\cdot 0^2 = 3$.
We have $Q(v) = Q(0, 1) = 3\cdot 0^2 - 4\cdot 0\cdot 1 + 5\cdot 1^2 = 5$.
We have $Q(u + v) = Q(1, 1) = 3\cdot 1^2 - 4\cdot 1\cdot 1 + 5\cdot 1^2 = 3 - 4 + 5 = 4$.

By HW#2, we have $B(u,v) = \dfrac{[Q(u+v)] - [Q(u)] - [Q(v)]}{2}$.

$$\text{Then} \qquad y = B(u,v) = \frac{4-3-5}{2} = \frac{4-8}{2} = \frac{-4}{2} = -2.$$

Also, $x = B(u,u) = \mathrm{Qd}^B(u) = Q(u) = 3$.
Also, $z = B(v,v) = \mathrm{Qd}^B(v) = Q(v) = 5$.
Then $[B] = \begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$, as desired. QED

6-4. Let $V := \mathbb{R}^2$. Define $Q : V \to \mathbb{R}$ by:

$$\forall x, y \in \mathbb{R}, \qquad Q(x,y) = 2x^2 + 6xy + 5y^2.$$

Show: $Q > 0$ on $V_{0_V}^\times$.

Proof: Want: $\forall u \in V_{0_V}^\times$, $Q_u > 0$.
Given $u \in V_{0_V}^\times$. Want: $Q_u > 0$. Want: $2 \cdot Q_u > 0$.
Since $u \in V_{0_V}^\times = V \backslash \{0_V\}$, we conclude: $u \neq 0_V$.
Since $u \in V_{0_V}^\times \subseteq V = \mathbb{R}^2$, choose $x, y \in \mathbb{R}$ s.t. $u = (x,y)$.
Then $Q_u = Q(u) = Q(x,y) = 2x^2 + 6xy + 5y^2$.
Then $2 \cdot Q_v = 4x^2 + 12xy + 10y^2 = (2x+3y)^2 + y^2$.
Want: $(2x+3y)^2 + y^2 > 0$. Let $w := 2x + 3y$. Want: $w^2 + y^2 > 0$.
Let $s := w^2 + y^2$. Want: $s > 0$.
Assume $s \leqslant 0$. Want: Contradiction.
Since $s \leqslant 0$ and $y^2 \geqslant 0$, we get: $s - y^2 \leqslant 0$.
Since $w^2 = s - y^2 \leqslant 0$ and $w^2 \geqslant 0$, we get: $w^2 = 0$. Then $w = 0$.
Then $s = w^2 + y^2 = 0^2 + y^2 = y^2$, so $s = y^2$.
Since $s \leqslant 0$ and $s = y^2 \geqslant 0$, we get: $s = 0$.
Then $y^2 = s = 0$. Then $y = 0$.
Then $0 = w = 2x + 3y = 2x + 3 \cdot 0 = 2x$. Then $x = 0$.
Then $0_V \neq u = (x,y) = (0,0) = 0_V$. Contradiction. QED

6-5. Let $V := \mathbb{R}^2$. Define $Q : V \to \mathbb{R}$ by:

$$\forall x, y \in \mathbb{R}, \qquad Q(x,y) = 2x^2 + 6xy + 4y^2.$$

Find $x, y \in \mathbb{R}$ s.t. $Q(x,y) < 0$.

Solution: Let $x := -3$ and $y := 2$. Then $Q(x,y) = -2 < 0$.

Homework 5: Due on Tuesday 3 March

5-1. Let $\alpha, \beta \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $W := \mathbb{R}^\beta$, $L \in \mathcal{AL}_W^V$.
Show: $\mathrm{Lin}^{[L]} = L$.

Proof: Let $S := [L]$, $F := \mathrm{Lin}^S$. Then: $F = \mathrm{Lin}^{[L]}$. Want: $F = L$.
Want: $\forall v \in V$, $F_v = L_v$. Given $v \in V$. Want: $F_v = L_v$.
We have: $F_v = \mathrm{Lin}^S(v) = \sum_{k\in\mathcal{I}_W} \sum_{j\in\mathcal{I}_V} v_j \cdot S_{kj} \cdot \varepsilon_k^W = \sum_{j\in\mathcal{I}_V} \sum_{k\in\mathcal{I}_W} v_j \cdot S_{kj} \cdot \varepsilon_k^W$

$$= \sum_{j\in\mathcal{I}_V} v_j \cdot \left[\sum_{k\in\mathcal{I}_W} S_{kj} \cdot \varepsilon_k^W \right].$$

$$\text{Want: } \sum_{j\in\mathcal{I}_V} v_j \cdot \left[\sum_{k\in\mathcal{I}_W} S_{kj} \cdot \varepsilon_k^W \right] = L_v.$$

By the reproducing formula, $v = \sum_{j\in\mathcal{I}_V} v_j \cdot \varepsilon_j^V$.

So, as $L \in \mathcal{AL}_W^V$, we conclude: $L_v = \sum_{j\in\mathcal{I}_V} v_j \cdot [L(\varepsilon_j^V)]$.

$$\text{Want: } \sum_{j\in\mathcal{I}_V} v_j \cdot \left[\sum_{k\in\mathcal{I}_W} S_{kj} \cdot \varepsilon_k^W \right] = \sum_{j\in\mathcal{I}_V} v_j \cdot [L(\varepsilon_j^V)].$$

$$\text{Want: } \forall j \in \mathcal{I}_V, \sum_{k\in\mathcal{I}_W} S_{kj} \cdot \varepsilon_k^W = L(\varepsilon_j^V).$$

Given $j \in \mathcal{I}_V$. Want: $\sum_{k\in\mathcal{I}_W} S_{kj} \cdot \varepsilon_k^W = L(\varepsilon_j^V)$.

Let $w := L(\varepsilon_j^V)$. Want: $\sum_{k\in\mathcal{I}_W} S_{kj} \cdot \varepsilon_k^W = w$.

By the reproducing formula, $w = \sum_{k\in\mathcal{I}_W} w_k \cdot \varepsilon_k^W$.

$$\text{Want: } \sum_{k\in\mathcal{I}_W} S_{kj} \cdot \varepsilon_k^W = \sum_{k\in\mathcal{I}_W} w_k \cdot \varepsilon_k^W.$$

Want: $\forall k \in \mathcal{I}_W$, $S_{kj} = w_k$.
Given $k \in \mathcal{I}_W$. Want: $S_{kj} = w_k$.
Recall that $S = [L]$ and that $w = L(\varepsilon_j^V)$.
Then $S_{kj} = [L]_{kj} = (L(\varepsilon_j^V)) \bullet \varepsilon_k^W = w \bullet \varepsilon_k^W = w_k$, as desired. QED

5-2. Let $\alpha, \beta \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $W := \mathbb{R}^\beta$, $S \in W \otimes V$.
Show: $[\,\mathrm{Lin}^S\,] = S$.

Proof: Since $S \in W \otimes V$, we get: $\mathrm{Lin}^S \in \mathcal{L}_W^V$.
Let $L := \mathrm{Lin}^S$. Want: $[L] = S$.
We have $L = \mathrm{Lin}^S \in \mathcal{L}_W^V$, so $[L] \in W \otimes V$. Also, $S \in W \otimes V$.
Want: $\forall k \in \mathcal{I}_W$, $\forall j \in \mathcal{I}_V$, $[L]_{kj} = S_{kj}$.
Given $k \in \mathcal{I}_W$, $j \in \mathcal{I}_V$. Want: $[L]_{kj} = S_{kj}$.
We have $[L]_{kj} = (L(\varepsilon_j^V)) \bullet \varepsilon_k^W$. Want: $(L(\varepsilon_j^V)) \bullet \varepsilon_k^W = S_{kj}$.

Let $w := L(\varepsilon_j^V)$. Want: $w \bullet \varepsilon_k^W = S_{kj}$. Want: $w_k = S_{kj}$.
Let $v := \varepsilon_j^V$. Then $w = L(v) = \mathrm{Lin}^S(v) = \sum_{K \in \mathcal{I}_W} \sum_{J \in \mathcal{I}_V} v_J \cdot S_{KJ} \cdot \varepsilon_K^W$.
Since $v = \varepsilon_j^V$, we see that: $\forall J \in \mathcal{I}_V$, $v_J = \delta_J^j$.
Then $w = \sum_{K \in \mathcal{I}_W} \sum_{J \in \mathcal{I}_V} \delta_J^j \cdot S_{KJ} \cdot \varepsilon_K^W = \sum_{K \in \mathcal{I}_W} S_{Kj} \cdot \varepsilon_K^W$.
Then $w_k = \sum_{K \in \mathcal{I}_W} S_{Kj} \cdot (\varepsilon_K^W)_k = \sum_{K \in \mathcal{I}_W} S_{Kj} \cdot \delta_K^k = S_{kj}$, as desired. QED

5-3. Let $\alpha, \beta, \gamma \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $W := \mathbb{R}^\beta$, $X := \mathbb{R}^\gamma$.
Let $L \in \mathcal{L}_W^V$ and let $M \in \mathcal{L}_X^W$. Show: $[M \circ L] = [M] \cdot [L]$.

Proof: We have $M \circ L \in \mathcal{L}_X^V$, so $[M \circ L] \in X \otimes V$. Then $[M \circ L] \in \mathbb{R}^{\gamma \times \alpha}$.
Since $M \in \mathcal{L}_X^W$ and $L \in \mathcal{L}_W^V$, we get: $[M] \in X \otimes W$ and $[L] \in W \otimes V$.
Then $[M] \in \mathbb{R}^{\gamma \times \beta}$ and $[L] \in \mathbb{R}^{\beta \times \alpha}$, and so $[M] \cdot [L] \in \mathbb{R}^{\gamma \times \alpha}$.
Want: $\forall k \in [1..\gamma]$, $\forall i \in [1..\alpha]$, $[M \circ L]_{ki} = ([M] \cdot [L])_{ki}$.
Given $k \in [1..\gamma]$, $i \in [1..\alpha]$. Want: $[M \circ L]_{ki} = ([M] \cdot [L])_{ki}$.
By definition of matrix multiplication,

$$\text{we get:} \quad ([M] \cdot [L])_{ki} = \sum_{j=1}^{\beta} [M]_{kj} \cdot [L]_{ji}.$$

Want: $[M \circ L]_{ki} = \sum_{j=1}^{\beta} [M]_{kj} \cdot [L]_{ji}$.
Let $w := L(\varepsilon_i^V)$ and let $x := M(w)$.
Then: $(M \circ L)(\varepsilon_i^V) = M(L(\varepsilon_i^V)) = M(w) = x$.
Then: $[M \circ L]_{ki} = ((M \circ L)(\varepsilon_i^V)) \bullet \varepsilon_k^W = x \bullet \varepsilon_k^W = x_k$.
Want: $x_k = \sum_{j=1}^{\beta} [M]_{kj} \cdot [L]_{ji}$.

By the reproducing formula, $w = \sum_{j=1}^{\beta} w_j \cdot \varepsilon_j^W$.

So, since $M \in \mathcal{L}_X^W \subseteq \mathcal{AL}_X^W$, we get $M(w) = \sum_{j=1}^{\beta} w_j \cdot (M(\varepsilon_j^W))$.

Then $x_k = (M(w))_k = \sum_{j=1}^{\beta} w_j \cdot [(M(\varepsilon_j^W))_k]$.

Want: $\sum_{j=1}^{\beta} w_j \cdot [(M(\varepsilon_j^W))_k] = \sum_{j=1}^{\beta} [M]_{kj} \cdot [L]_{ji}$.

Want: $\forall j \in [1..\beta], \quad w_j \cdot [(M(\varepsilon_j^W))_k] = [M]_{kj} \cdot [L]_{ji}$.
Given $j \in [1..\beta]$. Want: $w_j \cdot [(M(\varepsilon_j^W))_k] = [M]_{kj} \cdot [L]_{ji}$.
We have $[M]_{kj} = (M(\varepsilon_j^W)) \bullet \varepsilon_k^X = (M(\varepsilon_j^W))_k$.
Also, $[L]_{ji} = (L(\varepsilon_i^V)) \bullet \varepsilon_j^W = w \bullet \varepsilon_j^W = w_j$.
Then $w_j \cdot [(M(\varepsilon_j^W))_k] = [L]_{ji} \cdot [M]_{kj} = [M]_{kj} \cdot [L]_{ji}$, as desired. QED

5-4. Let $\alpha, \beta, \gamma \in \mathbb{N}, \quad V := \mathbb{R}^\alpha, \quad W := \mathbb{R}^\beta, \quad X := \mathbb{R}^\gamma$.
Let $B \in \mathcal{B}_X^{VW}$. Show: $\text{Bilin}^{[B]} = B$.

Proof: Let $T := [B]$, $F := \text{Bilin}^T$. Then: $F = \text{Bilin}^{[B]}$.
Want: $F = B$. Want: $\forall v \in V, \forall w \in W, F_{(v,w)} = B_{(v,w)}$.
Given $v \in V$, $w \in W$. Want: $F_{(v,w)} = B_{(v,w)}$.
We have:
$$B_{(v,w)} = \text{Bilin}^T_{(v,w)} = \sum_{k\in\mathcal{I}_X} \sum_{i\in\mathcal{I}_V} \sum_{j\in\mathcal{I}_W} v_i \cdot w_j \cdot T_{kij} \cdot \varepsilon_k^X = \sum_{i\in\mathcal{I}_V} \sum_{j\in\mathcal{I}_W} \sum_{k\in\mathcal{I}_X} v_i \cdot w_j \cdot T_{kij} \cdot \varepsilon_k^X = \sum_{i\in\mathcal{I}_V} \sum_{j\in\mathcal{I}_W} v_i \cdot w_j \cdot \left[\sum_{k\in\mathcal{I}_X} T_{kij} \cdot \varepsilon_k^X\right].$$
Want: $\displaystyle\sum_{i\in\mathcal{I}_V} \sum_{j\in\mathcal{I}_W} v_i \cdot w_j \cdot \left[\sum_{k\in\mathcal{I}_X} T_{kij} \cdot \varepsilon_k^X\right] = B_{(v,w)}$.
By the reproducing formula, $v = \displaystyle\sum_{i\in\mathcal{I}_V} v_i \cdot \varepsilon_i^V$ and $w = \displaystyle\sum_{j\in\mathcal{I}_W} w_j \cdot \varepsilon_j^W$.
So, as $B \in \mathcal{B}_X^{VW}$, we conclude: $B_{(v,w)} = \displaystyle\sum_{i\in\mathcal{I}_V} \sum_{j\in\mathcal{I}_W} v_i \cdot w_j \cdot [B(\varepsilon_i^V, \varepsilon_j^W)]$.
Want: $\displaystyle\sum_{i\in\mathcal{I}_V} \sum_{j\in\mathcal{I}_W} v_i \cdot w_j \cdot \left[\sum_{k\in\mathcal{I}_X} T_{kij} \cdot \varepsilon_k^X\right] = \sum_{i\in\mathcal{I}_V} \sum_{j\in\mathcal{I}_W} v_i \cdot w_j \cdot [B(\varepsilon_i^V, \varepsilon_j^W)]$.
Want: $\forall i \in \mathcal{I}_V, \forall j \in \mathcal{I}_W, \displaystyle\sum_{k\in\mathcal{I}_X} T_{kij} \cdot \varepsilon_k^X = B(\varepsilon_i^V, \varepsilon_j^W)$.
Given $i \in \mathcal{I}_V$, $j \in \mathcal{I}_W$. Want: $\displaystyle\sum_{k\in\mathcal{I}_X} T_{kij} \cdot \varepsilon_k^X = B(\varepsilon_i^V, \varepsilon_j^W)$.
Let $w := B(\varepsilon_i^V, \varepsilon_j^W)$. Want: $\displaystyle\sum_{k\in\mathcal{I}_X} T_{kij} \cdot \varepsilon_k^X = w$.
By the reproducing formula, $w = \displaystyle\sum_{k\in\mathcal{I}_X} w_k \cdot \varepsilon_k^X$.
Want: $\displaystyle\sum_{k\in\mathcal{I}_X} T_{kij} \cdot \varepsilon_k^X = \sum_{k\in\mathcal{I}_X} w_k \cdot \varepsilon_k^X$.
Want: $\forall k \in \mathcal{I}_X, \quad T_{kij} = w_k$.

Given $k \in \mathcal{I}_X$. Want: $T_{kij} = w_k$.
Recall that $T = [B]$ and that $w = B(\varepsilon_i^V, \varepsilon_j^W)$.
Then $T_{kij} = [B]_{kij} = (B(\varepsilon_i^V, \varepsilon_j^W)) \bullet \varepsilon_k^X = w \bullet \varepsilon_k^X = w_k$. QED

5-5. Let $\alpha, \gamma \in \mathbb{N}$, $V := \mathbb{R}^\alpha$, $X := \mathbb{R}^\gamma$, $* \in \mathcal{B}_X^{VV}$.
Define $Q : V \to X$ by: $\forall v \in V$, $Q_v = v * v$. Show: $Q \in \mathcal{Q}_X^V$.

Proof: Let $T := [*]$. Then $T \in X \otimes V \otimes V = \mathbb{R}^{\gamma \times \alpha \times \alpha}$.
For all $i, j \in \mathcal{I}_V$, let $x_{ij} := \sum_{k \in \mathcal{I}_X} T_{kij} \cdot \varepsilon_k^X$ and let $M_{ij} := C_V^{x_{ij}} \cdot \pi_i^V \cdot \pi_j^V$.
Then, $\forall i, j \in \mathcal{I}_V$, we have: $x_{ij} \in X$, so $C_V^{x_{ij}} \in \mathcal{C}_X^V$, so $M_{ij} \in \mathcal{M}_2^{VX}$.
Then $\sum_{i \in \mathcal{I}_V} \sum_{j \in \mathcal{I}_V} M_{ij} \in \mathcal{H}_2^{VX} = \mathcal{Q}_X^V$. Want: $Q = \sum_{i \in \mathcal{I}_V} \sum_{j \in \mathcal{I}_V} M_{ij}$.

Want: $\forall v \in V$, $Q(v) = \left(\sum_{i \in \mathcal{I}_V} \sum_{j \in \mathcal{I}_V} M_{ij} \right)(v)$.

Given $v \in V$. Want: $Q(v) = \left(\sum_{i \in \mathcal{I}_V} \sum_{j \in \mathcal{I}_V} M_{ij} \right)(v)$.

Want: $Q_v = \sum_{i \in \mathcal{I}_V} \sum_{j \in \mathcal{I}_V} (M_{ij})_v$.

We have: $\forall i, j \in \mathcal{I}_V$,
$$\begin{aligned}(M_{ij})_v &= (C_V^{x_{ij}} \cdot \pi_i^V \cdot \pi_j^V)_v \\ &= (C_V^{x_{ij}})_v \cdot (\pi_i^V)_v \cdot (\pi_j^V)_v \\ &= x_{ij} \cdot v_i \cdot v_j = v_i \cdot v_j \cdot x_{ij}.\end{aligned}$$

Want: $Q_v = \sum_{i \in \mathcal{I}_V} \sum_{j \in \mathcal{I}_V} v_i \cdot v_j \cdot x_{ij}$.

By the reproducing formula, $v = \sum_{i \in \mathcal{I}_V} v_i \cdot \varepsilon_i^V$ and $v = \sum_{j \in \mathcal{I}_V} v_j \cdot \varepsilon_j^V$.

Then: $Q_v = v * v = \left[\sum_{i \in \mathcal{I}_V} v_i \cdot \varepsilon_i^V \right] * \left[\sum_{j \in \mathcal{I}_V} v_j \cdot \varepsilon_j^V \right]$
$= \sum_{i \in \mathcal{I}_V} \sum_{j \in \mathcal{I}_V} v_i \cdot v_j \cdot [\varepsilon_j^V * \varepsilon_j^V]$.

Want: $\sum_{i \in \mathcal{I}_V} \sum_{j \in \mathcal{I}_V} v_i \cdot v_j \cdot [\varepsilon_j^V * \varepsilon_j^V] = \sum_{i \in \mathcal{I}_V} \sum_{j \in \mathcal{I}_V} v_i \cdot v_j \cdot x_{ij}$.

Want: $\forall i, j \in \mathcal{I}_V$, $\varepsilon_i^V * \varepsilon_j^V = x_{ij}$.
Given $i, j \in \mathcal{I}_V$. Want: $\varepsilon_i^V * \varepsilon_j^V = x_{ij}$.
Want: $\forall \ell \in \mathcal{I}_X$, $(\varepsilon_i^V * \varepsilon_j^V)_\ell = (x_{ij})_\ell$.
Given $\ell \in \mathcal{I}_X$. Want: $(\varepsilon_i^V * \varepsilon_j^V)_\ell = (x_{ij})_\ell$.

Since $x_{ij} = \sum_{k \in \mathcal{I}_X} T_{kij} \cdot \varepsilon_k^X$, we get $(x_{ij})_\ell = \sum_{k \in \mathcal{I}_X} T_{kij} \cdot (\varepsilon_k^X)_\ell$.

Then: $\qquad (x_{ij})_\ell = \sum_{k\in\mathcal{I}_X} T_{kij}\cdot\delta^k_\ell \;=\; T_{\ell ij}.$

Since $T = [*]$, we get $T_{\ell ij} = (\varepsilon^V_i * \varepsilon^V_j)\bullet\varepsilon^X_\ell$.
Then $(\varepsilon^V_i * \varepsilon^V_j)_\ell = (\varepsilon^V_i * \varepsilon^V_j)\bullet\varepsilon^X_\ell = T_{\ell ij} = (x_{ij})_\ell$, as desired. QED

Homework 4: Due on Tuesday 25 February

4-1. Define $a\in\mathbb{R}^{\mathbb{N}_0}$ by: $\forall j\in\mathbb{N}_0$, $a_j = 1/(j!)$. Show: $\mathrm{RC}_a = \infty$.

Proof: Let $T := \{s\geqslant 0\,|\,TS^a_s \text{ is bounded in } \mathbb{R}\}$.
Then $\mathrm{RC}_a = \sup T$. Want: $\sup T = \infty$. Want: $T = [0;\infty)$.
By definition of T, we have $T\subseteq[0;\infty)$. Want: $[0;\infty)\subseteq T$.
Want: $\forall s\in[0;\infty)$, $s\in T$. Given $s\in[0;\infty)$. Want: $s\in T$.
Since $s\geqslant 0$, it suffices to show: TS^a_s is bounded in $\mathbb{R}$.
Want: $\{a_0, a_1 s, a_2 s^2, a_3 s^3,\ldots\}$ is bounded in $\mathbb{R}$.
Define $b\in\mathbb{R}^{\mathbb{N}_0}$ by: $\forall j\in\mathbb{N}_0$, $b_j = a_j s^j$.
Then $\mathbb{I}_b = \{b_0, b_1, b_2, b_3,\ldots\} = \{a_0, a_1 s, a_2 s^2, a_3 s^3,\ldots\}$.
Want: $\mathbb{I}_b$ is bounded in $\mathbb{R}$. Want: b is convergent in $\mathbb{R}$.
Want: $b\to 0$ in $\mathbb{R}$. Let $c := 1/2$. Then $c\in(0;1)$.
By Theorem 4.14.3, it suffices to show:
$\exists k\in\mathbb{N}_0$ s.t. $\forall j\in[k..\infty)$, $b_{j+1}/b_j\leqslant c$.
By the Archimedean Principle, choose $k\in\mathbb{N}$ s.t. $k > 2s-1$.
Then $k\in\mathbb{N}\subseteq\mathbb{N}_0$. Want: $\forall j\in[k..\infty)$, $b_{j+1}/b_j\leqslant c$.
Given $j\in[k..\infty)$. Want: $b_{j+1}/b_j\leqslant c$.
We have $b_{j+1} = a_{j+1}s^{j+1} = s^{j+1}/((j+1)!) = [s^j/(j!)]\cdot[s/(j+1)]$,
$= b_j\cdot[s/(j+1)]$, so $b_{j+1}/b_j = s/(j+1)$.
Since $j\in[k..\infty)\geqslant k > 2s-1$, we get $j > 2s-1$, so $j+1 > 2s$,
so $(1/2)\cdot(j+1) > s$, and so $1/2 > s/(j+1)$.
Then $b_{j+1}/b_j = s/(j+1) < 1/2 = c$, as desired. QED

4-2. Let $a\in\mathbb{R}^{\mathbb{N}_0}$, $x, r\in\mathbb{R}$, $i, j\in\mathbb{N}_0$, $b := P_\bullet SS^a_x$, $c := P_\bullet SS^{|a|}_r$.
Assume: $|x|\leqslant r$. Show: $|b_j - b_i|\leqslant|c_j - c_i|$.

Proof: Let $k := \min\{i,j\}$ and $\ell := \max\{i,j\}$.
Then $|b_j - b_i| = |b_\ell - b_k|$ and $|c_j - c_i| = |c_\ell - c_k|$.
Want: $|b_\ell - b_k|\leqslant|c_\ell - c_k|$. We have $k\leqslant\ell$. Let $Q := (k..\ell]$.

We have: $\qquad \forall n\in\mathbb{N}_0,\quad b_n = P_n SS^a_x = \sum_{m=1}^{n} a_m\cdot x^m.$

Since $\quad b_k = \sum_{m=1}^{k} a_m \cdot x^m \quad$ and $\quad b_\ell = \sum_{m=1}^{\ell} a_m \cdot x^m,$

it follows that: $\quad b_\ell - b_k = \sum_{m \in Q} a_m \cdot x^m.$

We have: $\quad \forall n \in \mathbb{N}_0, \quad c_n = P_n SS_r^{|a|} = \sum_{m=1}^{n} |a|_m \cdot r^m = \sum_{m=1}^{n} |a_m| \cdot r^m.$

Since $\quad c_k = \sum_{m=1}^{k} |a_m| \cdot r^m \quad$ and $\quad c_\ell = \sum_{m=1}^{\ell} |a_m| \cdot r^m,$

it follows that: $\quad c_\ell - c_k = \sum_{m \in Q} |a_m| \cdot r^m.$

Since $0 \leqslant |x| \leqslant r$, we get $r \geqslant 0$, and so $|r| = r$.

Then $|c_\ell - c_k| = \left| \sum_{m \in Q} |a_m| \cdot r^m \right| \leqslant \sum_{m \in Q} |\, |a|_m \cdot r^m|$

$$= \sum_{m \in Q} |a_m| \cdot |r^m| = \sum_{m \in Q} |a_m| \cdot |r|^m = \sum_{m \in Q} |a_m| \cdot r^m.$$

Since $|x| \leqslant r$, we get: $\quad \forall m \in \mathbb{N}_0, \quad |a_m| \cdot |x|^m \leqslant |a_m| \cdot r^m.$

Then: $\quad \sum_{m \in Q} |a_m| \cdot |x|^m \leqslant \sum_{m \in Q} |a_m| \cdot r^m.$

Then $|b_\ell - b_k| = \left| \sum_{m \in Q} a_m \cdot x^m \right| \leqslant \sum_{m \in Q} |a_m \cdot x^m| = \sum_{m \in Q} |a_m| \cdot |x^m|$

$$= \sum_{m \in Q} |a_m| \cdot |x|^m \leqslant \sum_{m \in Q} |a_m| \cdot r^m = |c_\ell - c_k|.$$ QED

4-3. Let $\alpha \in [0;\infty)^{\mathbb{N}_0}$, $\quad r \in [0; \mathrm{RC}_\alpha)$, $\quad c := P_\bullet SS_r^\alpha$.

Show: c is convergent in $\mathbb{R}$.

Proof: We have: $\quad \forall n \in \mathbb{N}_0, \quad c_n = P_n SS_r^\alpha = \sum_{m=1}^{n} \alpha_m \cdot r^m.$

Claim: c is semi-increasing.

Proof of claim: Want: $\forall i \in \mathbb{N}, \quad c_i \geqslant c_{i-1}$.

Given $i \in \mathbb{N}$. Want: $c_i \geqslant c_{i-1}$.

Since $\quad c_{i-1} = \sum_{m=1}^{i-1} \alpha_m \cdot r^m \quad$ and $\quad c_i = \sum_{m=1}^{i} \alpha_m \cdot r^m,$

it follows that: $\quad c_i - c_{i-1} = \alpha_i \cdot r^i.$

Since $\alpha \in [0;\infty)^{\mathbb{N}_0}$, $\quad$ we get: $\alpha_i \in [0;\infty)$.

Since $\alpha_i \in [0;\infty) \geqslant 0$ and $r \in [0;\infty) > 0$, it follows that $\alpha_i \cdot r^i \geqslant 0$.

Then $c_i - c_{i-1} = \alpha_i \cdot r^i \geqslant 0$, and so $c_i \geqslant c_{i-1}$, as desired.

End of proof of claim.

By Theorem 3.11.2, $\forall \tau \in \mathbb{R}^{\mathbb{N}_0}$,

$$[\, (\, \mathbb{I}_\tau \text{ is bounded in } \mathbb{R} \,) \, \& \, (\, \tau \text{ is semi-increasing }) \,] \Rightarrow [\, \tau \text{ is convergent in } \mathbb{R} \,].$$

So, by the claim, it suffices to show: $\mathbb{I}_c$ is bounded in $\mathbb{R}$.
Since $r \in [0; \mathrm{RC}_\alpha), \mathrm{RC}_\alpha$, by Theorem 4.16.16,
choose $s > r$ s.t. TS_s^α is bounded in $\mathbb{R}$.
Choose $B \in \mathcal{B}_\mathbb{R}(0)$ s.t. $\mathrm{TS}_s^\alpha \subseteq B$.
Choose $M > 0$ s.t. $B = B_\mathbb{R}(0, M)$.
We have $r \in [0; \mathrm{RC}_\alpha) \geqslant 0$. Then $0 \leqslant r < s$.
Let $u := r/s$. Then $0 \leqslant u < 1$. Then $u \in [0; 1)$. Also $r = su$.
We have $\mathrm{TS}_s^\alpha = \{\alpha_0, \alpha_1 s^1, \alpha_2 s^2, \alpha_3 s^3, \ldots\}$.
Then: $\forall j \in \mathbb{N}_0$, $\alpha_j s^j \in \mathrm{TS}_s^\alpha$.
Then: $\forall j \in \mathbb{N}_0$, $\alpha_j s^j \in \mathrm{TS}_s^\alpha \subseteq B = B_\mathbb{R}(0, M) < M$.
Then: $\forall j \in \mathbb{N}_0$, $\alpha_j s^j < M$.
Then, since $u \geqslant 0$, we get: $\forall j \in \mathbb{N}_0$, $\alpha_j s^j u^j \leqslant M u^j$.
Then: $\forall j \in \mathbb{N}_0$, $\alpha_j r^j = \alpha_j \cdot (su)^j = \alpha_j s^j u^j \leqslant M u^j$.
Let $R := M/(1 - u)$. As $M > 0$ and $1 - u > 0$, we get: $R > 0$.
Any ball in a metric space is bounded, so $B_\mathbb{R}(0, R)$ is bounded.
It therefore suffices to show: $\mathbb{I}_c \subseteq B_\mathbb{R}(0, R)$.
Want: $\forall y \in \mathbb{I}_c$, $y \in B_\mathbb{R}(0, R)$.
Given $y \in \mathbb{I}_c$. Want: $y \in B_\mathbb{R}(0, R)$. Want: $y \in (-R; R)$.
Want: $-R < y < R$. Want: $0 \leqslant y < R$.
Since $y \in \mathbb{I}_c$ and $\mathbb{D}_c = \mathbb{N}_0$, choose $k \in \mathbb{N}_0$ s.t. $y = c_k$.
Then $y = \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3 + \cdots + \alpha_k r^k$.
So, since $\alpha_0, \ldots, \alpha_k \in \mathbb{I}_\alpha \subseteq [0; \infty) \geqslant 0$ and since $r \in [0; \mathrm{RC}_\alpha) \geqslant 0$,
we conclude: $y \geqslant 0$. It remains to show that: $y < R$.
Recall: $\forall j \in \mathbb{N}_0$, $\alpha_j r^j \leqslant M u^j$. Then: $\displaystyle\sum_{j=0}^{k} \alpha_j r^j \leqslant \sum_{j=0}^{k} M u^j$.
By a theorem proved in class, $1 + u + u^2 + u^3 + \cdots + u^k < 1/(1 - u)$.
So, as $M > 0$, we get: $M \cdot (1 + u + u^2 + u^3 + \cdots + u^k) < M/(1 - u)$.
Then

$$\begin{aligned} y &= \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3 + \cdots + \alpha_k r^k \\ &= \sum_{j=0}^{k} \alpha_j r^j \leqslant \sum_{j=0}^{k} M u^j = M \cdot \sum_{j=0}^{k} u^j \\ &= M \cdot (1 + u + u^2 + u^3 + \cdots + u^k) \\ &< M/(1 - u) = R, \qquad \text{as desired.} \qquad \text{QED} \end{aligned}$$

4-4. Let $a \in \mathbb{R}^{\mathbb{N}_0}$ and let $\alpha := |a|$. Show: $\mathrm{RC}_a = \mathrm{RC}_\alpha$.

Proof: We have: $\forall s \geqslant 0, \quad |s| = s$.
Also: $\forall s \geqslant 0, \quad \mathrm{TS}_s^a = \{a_j s^j \text{ s.t. } j \in \mathbb{N}_0\}$.
Then: $\forall s \geqslant 0$,

$$\begin{aligned} |\mathrm{TS}_s^a| &= \{|a_j s^j| \text{ s.t. } j \in \mathbb{N}_0\} \\ &= \{|a_j| \cdot |s^j| \text{ s.t. } j \in \mathbb{N}_0\} \\ &= \{|a|_j \cdot |s|^j \text{ s.t. } j \in \mathbb{N}_0\} \\ &= \{\alpha_j \cdot s^j \text{ s.t. } j \in \mathbb{N}_0\} = \mathrm{TS}_s^\alpha. \end{aligned}$$

By a theorem from class, we know: $\forall s \geqslant 0$,

$$(\ TS_s^a \text{ is bounded in } \mathbb{R}\ ) \ \Leftrightarrow \ (\ |TS_s^a| \text{ is bounded in } \mathbb{R}\ ).$$

Then $\{s \geqslant 0 \text{ s.t. } \mathrm{TS}_s^a \text{ is bounded in } \mathbb{R}\}$

$$\begin{aligned} &= \{s \geqslant 0 \text{ s.t. } |\mathrm{TS}_s^a| \text{ is bounded in } \mathbb{R}\} \\ &= \{s \geqslant 0 \text{ s.t. } \mathrm{TS}_s^\alpha \text{ is bounded in } \mathbb{R}\}. \end{aligned}$$

Then $\mathrm{RC}_a = \sup\{s \geqslant 0 \text{ s.t. } \mathrm{TS}_s^a \text{ is bounded in } \mathbb{R}\}$

$$= \sup\{s \geqslant 0 \text{ s.t. } \mathrm{TS}_s^\alpha \text{ is bounded in } \mathbb{R}\} = \mathrm{RC}_\alpha. \quad \text{QED}$$

4-5. Let $a \in \mathbb{R}^{\mathbb{N}_0}$. Show: $\mathrm{RC}_a = \mathrm{RC}_{a^*}$.

Proof: Let $b := a^*$. Want: $\mathrm{RC}_a = \mathrm{RC}_b$.
Want: (1) $\mathrm{RC}_a \geqslant \mathrm{RC}_b$ and (2) $\mathrm{RC}_b \geqslant \mathrm{RC}_a$.

Proof of (1): Assume: $\mathrm{RC}_a < \mathrm{RC}_b$. Want: Contradiction.
Since $\mathrm{RC}_a < \mathrm{RC}_b$, choose $s > \mathrm{RC}_a$ s.t. TS_s^b is bounded in $\mathbb{R}$.
Since $s > \mathrm{RC}_a \geqslant 0$, we get $s > 0$. Then $|s| = s$.
Choose $C \in \mathcal{B}_\mathbb{R}(0)$ s.t. $\mathrm{TS}_s^b \subseteq C$.
Choose $M > 0$ s.t. $C = B_\mathbb{R}(0, M)$.
We have $\mathrm{TS}_s^b = \{b_0, b_1 s, b_2 s^2, b_3 s^3, \ldots\} = \{a_1, 2a_2 s, 3a_3 s^2, 4a_4 s^3, \ldots\}$.
Then $\{a_1, 2a_2 s, 3a_3 s^2, 4a_4 s^3, \ldots\} = TS_s^b \subseteq C = B_\mathbb{R}(0, M)$.
Then: $\forall j \in \mathbb{N}, \quad |j a_j s^{j-1}| < M$.
Then: $\forall j \in \mathbb{N}, \quad |a_j s^j| = |s| \cdot |j a_j s^{j-1}|/|j|$

$$= s \cdot |j a_j s^{j-1}|/j < s \cdot M/j \leqslant s \cdot M.$$

Then $\{a_1 s, a_2 s^2, a_3 s^3, a_4 s^4, \ldots\} \subseteq B_\mathbb{R}(0, sM)$.
Then $\{a_1 s, a_2 s^2, a_3 s^3, a_4 s^4, \ldots\}$ is bounded in $\mathbb{R}$.
So, since $\{a_0\}$ is bounded in $\mathbb{R}$,

we get: $\{a_0\} \bigcup \{a_1 s, a_2 s^2, a_3 s^3, a_4 s^4, \ldots\}$ is bounded in $\mathbb{R}$.

Then $\{a_0, a_1 s, a_2 s^2, a_3 s^3, a_4 s^4, \ldots\}$ is bounded in $\mathbb{R}$.
We have $\mathrm{TS}_s^a = \{a_0, a_1 s, a_2 s^2, a_3 s^3, \ldots\}$.

Then TS_s^a is bounded in $\mathbb{R}$. Recall: $s > 0$.
Then $s \in \{r \geqslant 0 \,|\, \mathrm{TS}_r^a \text{ is bounded in } \mathbb{R}\}$.
Then $s \leqslant \sup\{r \geqslant 0 \,|\, \mathrm{TS}_r^a \text{ is bounded in } \mathbb{R}\}$. Recall that $s > \mathrm{RC}_a$.
Then $\mathrm{RC}_a < s \leqslant \sup\{r \geqslant 0 \,|\, \mathrm{TS}_r^a \text{ is bounded in } \mathbb{R}\} = \mathrm{RC}_a$.
Then $\mathrm{RC}_a < \mathrm{RC}_a$. Contradiction.
End of proof of (1).

Proof of (2): Assume: $\mathrm{RC}_b < \mathrm{RC}_a$. Want: Contradiction.
Since $\mathrm{RC}_b < \mathrm{RC}_a$, choose $t > \mathrm{RC}_b$ s.t. TS_t^a is bounded in $\mathbb{R}$.
Since $t > \mathrm{RC}_b \geqslant 0$, we get $t > 0$. Then $t \geqslant 0$.
So, since TS_t^a is bounded in $\mathbb{R}$, we get $t \in \{r \geqslant 0 \,|\, TS_r^a \text{ is bounded in } \mathbb{R}\}$.
Then $t \leqslant \sup\{r \geqslant 0 \,|\, TS_r^a \text{ is bounded in } \mathbb{R}\} = \mathrm{RC}_a$.
We have $\mathrm{RC}_b < t$. Let $s := (\mathrm{RC}_b + t)/2$. Then $\mathrm{RC}_b < s < t$.
Then $0 \leqslant \mathrm{RC}_b < s < t \leqslant \mathrm{RC}_a$.
Let $u := s/t$. Then $0 < u < 1$ and $s = tu$.
Since TS_t^a is bounded in $\mathbb{R}$, choose $C \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\mathrm{TS}_t^a \subseteq C$.
Choose $M > 0$ s.t. $C = B_{\mathbb{R}}(0, M)$.
Since $s > \mathrm{RC}_b = \sup\{r \geqslant 0 \,|\, \mathrm{TS}_r^b \text{ is bounded in } \mathbb{R}\}$,
we see that $s \notin \{r \geqslant 0 \,|\, \mathrm{TS}_r^b \text{ is bounded in } \mathbb{R}\}$,
and so TS_s^b is not bounded in $\mathbb{R}$.
Define $\beta \in \mathbb{R}^{\mathbb{N}_0}$ by: $\forall j \in \mathbb{N}_0,\ \ \beta_j = b_j s^j$.
Then $\mathrm{TS}_s^b = \{\beta_0, \beta_1, \beta_2, \ldots\} = \mathbb{I}_\beta$.
Then $\mathbb{I}_\beta$ is not bounded in $\mathbb{R}$,
so β is not convergent in $\mathbb{R}$,
so $\neg(\beta \to 0 \text{ in } \mathbb{R})$.
Choose $\varepsilon > 0$ s.t., $\forall \ell \in \mathbb{N}_0$, $\exists j \in [\ell..\infty)$ s.t. $d_{\mathbb{R}}(\beta_j, 0) \geqslant \varepsilon$.
Define $\gamma \in \mathbb{R}^{\mathbb{N}_0}$ by: $\forall j \in \mathbb{N}_0,\ \ \gamma_j = (j+1) \cdot u^j$.
By the Archimedean Principle, choose $k \in \mathbb{N}$ such that $k > 2u/(1-u)$.
Recall: $u < 1$. Let $c := (u+1)/2$. Then $u < c < 1$.
So, since $0 \leqslant u$, we get $0 < c < 1$, and so $c \in (0;1)$.
We have: $\forall j \in \mathbb{N}_0,\ \ \gamma_{j+1}/\gamma_j = [(j+1)/j] \cdot u = u + [u/j]$.
We have $u + [(1-u)/2] = (1 - u + 2u)/2 = (u+1)/2 = c$.
We have: $\forall j \in [k..\infty)$, $j \geqslant k > 2u/(1-u)$, so $1 - u > 2u/j$,
so $(1-u)/2 > u/j$, so $u/j < (1-u)/2$,
so $u + [u/j] < u + [(1-u)/2]$, so $\gamma_{j+1}/\gamma_j < c$.
Then, by Theorem 4.14.3, we get $\gamma \to 0$ in $\mathbb{R}$.
Recall: $t > 0$ and $M > 0$ and $\varepsilon > 0$. Then $(t/M) \cdot \varepsilon > 0$.
Choose $\ell \in \mathbb{N}_0$ s.t., $\forall j \in [\ell..\infty)$, $d_{\mathbb{R}}(\gamma_j, 0) < (t/M) \cdot \varepsilon$.

By choice of ε, choose $j \in [\ell..\infty)$ s.t. $d_{\mathbb{R}}(\beta_j, 0) \geqslant \varepsilon$.
Since $j \in [\ell..\infty)$, by choice of ℓ, we have $d_{\mathbb{R}}(\gamma_j, 0) < (t/M) \cdot \varepsilon$.
Then $|\gamma_j| = |\gamma_j - 0| = d_{\mathbb{R}}(\gamma_j, 0) < (t/M) \cdot \varepsilon$, so $(M/t) \cdot |\gamma_j| < \varepsilon$.
We have: $a_{j+1}t^{j+1} \in \mathrm{TS}_t^a \subseteq C \subseteq B_{\mathbb{R}}(0, M)$, so $d_{\mathbb{R}}(a_{j+1}t^{j+1}, 0) < M$.
Then: $|a_{j+1}| \cdot t^{j+1} = |a_{j+1}t^{j+1}| = |a_{j+1}t^{j+1} - 0| = d_{\mathbb{R}}(a_{j+1}t^{j+1}, 0) < M$.
Then: $|a_{j+1}| \cdot t^j = |a_{j+1}| \cdot t^{j+1}/t < M/t$.
Then: $|a_{j+1}| \cdot t^j \cdot |\gamma_j| \leqslant (M/t) \cdot |\gamma_j|$. Recall: $s = tu$.
We have: $\beta_j = b_j s^j = b_j (tu)^j = a_j^* t^j u^j = (j+1)a_{j+1}t^j u^j = a_{j+1}t^j \cdot \gamma_j$,
so $|\beta_j| = |a_{j+1}| \cdot t^j \cdot |\gamma_j| \leqslant (M/t) \cdot |\gamma_j|$.
Then $\varepsilon \leqslant d_{\mathbb{R}}(\beta_j, 0) = |\beta_j - 0| = |\beta_j| \leqslant (M/t) \cdot |\gamma_j| < \varepsilon$.
Then $\varepsilon < \varepsilon$. Contradiction.
End of proof of (2). QED

Homework 3: Due on Tuesday 18 February

3-1. Let X be a metric space, $s \in X^{\mathbb{N}}$, $q \in X$, $m \in \mathbb{N}$.
Define $t \in X^{\mathbb{N}}$ by: $\forall j \in \mathbb{N}$, $t_j = s_{j+m}$.
Assume: $t \to q$ in X. Show: $s \to q$ in X.

Proof: Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$[\ j \geqslant K\ ] \Rightarrow [\ d(s_j, q) < \varepsilon\ ]$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall j \in \mathbb{N}$,
$[\ j \geqslant K\ ] \Rightarrow [\ d(s_j, q) < \varepsilon\ ]$.
Since $t \to q$, choose $L \in \mathbb{N}$ s.t., $\forall i \in \mathbb{N}$,
$[\ i \geqslant L\ ] \Rightarrow [\ d(t_i, q) < \varepsilon\ ]$.
Since $L, m \in \mathbb{N}$, we conclude: $L + m \in \mathbb{N}$.
Let $K := L + m$. Then $K \in \mathbb{N}$.
Want: $\forall j \in \mathbb{N}$, $[\ j \geqslant K\ ] \Rightarrow [\ d(s_j, q) < \varepsilon\ ]$.
Given $j \in \mathbb{N}$. Want: $[\ j \geqslant K\ ] \Rightarrow [\ d(s_j, q) < \varepsilon\ ]$.
Assume: $j \geqslant K$. Want: $d(s_j, q) < \varepsilon$.
Let $i := j - m$. Then $i + m = j$. Then $s_{i+m} = s_j$.
Since $i + m = j \geqslant K = L + m$, we get: $i \geqslant L$.
Since $j, m \in \mathbb{N} \subseteq \mathbb{Z}$, we get: $j - m \in \mathbb{Z}$.
Since $i = j - m \in \mathbb{Z}$ and $i \geqslant L \in \mathbb{N} > 0$, we conclude: $i \in \mathbb{N}$.
Then, by definition of t, we have $t_i = s_{i+m}$.
Since $i \in \mathbb{N}$ and $i \geqslant L$, by choice of L, we get: $d(t_i, q) < \varepsilon$.
So, since $t_i = s_{i+m} = s_j$, we get: $d(s_j, q) < \varepsilon$, as desired. QED

3-2. Let X and Y be metric spaces, $s \in X^{\mathbb{N}}$, $t \in Y^{\mathbb{N}}$.
Show: $[\ (s,t) \text{ is Cauchy in } X \times Y\ ]$
$\Leftrightarrow [\ (\ s \text{ is Cauchy in } X\ ) \ \&\ (\ t \text{ is Cauchy in } Y\ )\ ]$.

Proof: Proof of $\Rightarrow$*:*
Assume: (s,t) is Cauchy in $X \times Y$.
Want: (1) s is Cauchy in X and (2) t is Cauchy in Y.

Proof of (1):
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$[\ i,j \geqslant K\ ] \Rightarrow [\ d(s_i,s_j) < \varepsilon\ ]$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$[\ i,j \geqslant K\ ] \Rightarrow [\ d(s_i,s_j) < \varepsilon\ ]$.
Since (s,t) is Cauchy in $X \times Y$, choose $K \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$[\ i,j \geqslant K\ ] \Rightarrow [\ d(\,(s,t)_i\,,\,(s,t)_j\,) < \varepsilon\ ]$.
Then $K \in \mathbb{N}$.
Want: $\forall i,j \in \mathbb{N}$, $[\ i,j \geqslant K\ ] \Rightarrow [\ d(s_i,s_j) < \varepsilon\ ]$.
Given $i,j \in \mathbb{N}$.
Want: $[\ i,j \geqslant K\ ] \Rightarrow [\ d(s_i,s_j) < \varepsilon\ ]$.
Assume: $i,j \geqslant K$. Want: $d(s_i,s_j) < \varepsilon$.
Since $i,j \geqslant K$, by choice of K, we get: $d(\,(s,t)_i\,,\,(s,t)_j\,) < \varepsilon$.
Let $a := d(s_i,s_j)$ and let $b := d(t_i,t_j)$.
Then $d(\,(s_i,t_i)\,,\,(s_j,t_j)\,) = \sqrt{a^2+b^2}$.
Then $\sqrt{a^2+b^2} = d(\,(s_i,t_i)\,,\,(s_j,t_j)\,) = d(\,(s,t)_i\,,\,(s,t)_j\,) < \varepsilon$.
Since $a = d(s_i,s_j) \geqslant 0$, we get: $a = \sqrt{a^2}$.
Since $0 \leqslant a^2 \leqslant a^2+b^2$, we get $\sqrt{a^2} \leqslant \sqrt{a^2+b^2}$.
Then $d(s_i,s_j) = a = \sqrt{a^2} \leqslant \sqrt{a^2+b^2} < \varepsilon$, as desired.
End of proof of (1).

Proof of (2):
Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$[\ i,j \geqslant K\ ] \Rightarrow [\ d(t_i,t_j) < \varepsilon\ ]$.
Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$[\ i,j \geqslant K\ ] \Rightarrow [\ d(t_i,t_j) < \varepsilon\ ]$.
Since (s,t) is Cauchy in $X \times Y$, choose $K \in \mathbb{N}$ s.t., $\forall i,j \in \mathbb{N}$,
$[\ i,j \geqslant K\ ] \Rightarrow [\ d(\,(s,t)_i\,,\,(s,t)_j\,) < \varepsilon\ ]$.
Then $K \in \mathbb{N}$.
Want: $\forall i,j \in \mathbb{N}$, $[\ i,j \geqslant K\ ] \Rightarrow [\ d(t_i,t_j) < \varepsilon\ ]$.

Given $i, j \in \mathbb{N}$.

Want: $[\ i, j \geqslant K\ ] \ \Rightarrow\ [\ d(t_i, t_j) < \varepsilon\ ]$.

Assume: $i, j \geqslant K$. Want: $d(t_i, t_j) < \varepsilon$.

Since $i, j \geqslant K$, by choice of K, we get: $d(\,(s,t)_i\,,\,(s,t)_j\,)\,) < \varepsilon$.

Let $a := d(s_i, s_j)$ and let $b := d(t_i, t_j)$.

Then $d(\,(s_i, t_i)\,,\,(s_j, t_j)\,) = \sqrt{a^2 + b^2}$.

Then $\sqrt{a^2 + b^2} = d(\,(s_i, t_i)\,,\,(s_j, t_j)\,) = d(\,(s,t)_i\,,\,(s,t)_j\,) < \varepsilon$.

Since $b = d(t_i, t_j) \geqslant 0$, we get: $b = \sqrt{b^2}$.

Since $0 \leqslant b^2 \leqslant a^2 + b^2$, we get $\sqrt{b^2} \leqslant \sqrt{a^2 + b^2}$.

Then $d(t_i, t_j) = b = \sqrt{b^2} \leqslant \sqrt{a^2 + b^2} < \varepsilon$, as desired.

End of proof of (2). *End of proof of* $\Rightarrow$.

Proof of $\Leftarrow$:

Assume: (s is Cauchy in X) & (t is Cauchy in Y).

Want: (s, t) is Cauchy in $X \times Y$.

Want: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,

$$[\ i, j \geqslant K\ ] \ \Rightarrow\ [\ d(\,(s,t)_i\,,\,(s,t)_j\,) < \varepsilon\ ].$$

Given $\varepsilon > 0$. Want: $\exists K \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,

$$[\ i, j \geqslant K\ ] \ \Rightarrow\ [\ d(\,(s,t)_i\,,\,(s,t)_j\,) < \varepsilon\ ].$$

Let $\sigma := \varepsilon/\sqrt{2}$. Then $\sigma > 0$.

Since s is Cauchy in X, choose $L \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,

$$[\ i, j \geqslant L\ ] \ \Rightarrow\ [\ d(s_i, s_j) < \sigma\ ].$$

Since t is Cauchy in X, choose $M \in \mathbb{N}$ s.t., $\forall i, j \in \mathbb{N}$,

$$[\ i, j \geqslant M\ ] \ \Rightarrow\ [\ d(t_i, t_j) < \sigma\ ].$$

Let $K := \max\{L, M\}$. Then $K \in \mathbb{N}$.

Want: $\forall i, j \in \mathbb{N}$, $[\ i, j \geqslant K\ ] \ \Rightarrow\ [\ d(\,(s,t)_i\,,\,(s,t)_j\,) < \varepsilon\ ]$.

Given $i, j \in \mathbb{N}$. Want: $[\ i, j \geqslant K\ ] \ \Rightarrow\ [\ d(\,(s,t)_i\,,\,(s,t)_j\,) < \varepsilon\ ]$.

Assume: $i, j \geqslant K$. Want: $d(\,(s,t)_i\,,\,(s,t)_j\,) < \varepsilon$.

Since $i, j \geqslant K \geqslant L$, by choice of L, we get: $d(s_i, s_j) < \sigma$.

Since $i, j \geqslant K \geqslant M$, by choice of M, we get: $d(t_i, t_j) < \sigma$.

Let $a := d(s_i, s_j)$ and let $b := d(t_i, t_j)$. Then $a < \sigma$ and $b < \sigma$.

Also, $d(\,(s_i, t_i)\,,\,(s_j, t_j)\,) = \sqrt{a^2 + b^2}$.

We have $a = d(s_i, s_j) \geqslant 0$ and $b = d(t_i, t_j) \geqslant 0$.

Since $0 \leqslant a < \sigma$, we get $0 \leqslant a^2 < \sigma^2$.

Since $0 \leqslant b < \sigma$, we get $0 \leqslant b^2 < \sigma^2$.

Then $0 \leqslant a^2 + b^2 < 2 \cdot \sigma^2$. Then $\sqrt{a^2 + b^2} < \sqrt{2 \cdot \sigma^2}$.

Since $\sigma > 0$, we get $\sqrt{\sigma^2} = \sigma$. Since $\sigma = \varepsilon/\sqrt{2}$, we get $\sqrt{2} \cdot \sigma = \varepsilon$.

Then $d(\,(s,t)_i\,,\,(s,t)_j\,) = d(\,(s_i, t_i)\,,\,(s_j, t_j)\,) = \sqrt{a^2 + b^2} < \sqrt{2 \cdot \sigma^2}$

$$= \sqrt{2} \cdot \sqrt{\sigma^2} = \sqrt{2} \cdot \sigma = \varepsilon, \quad \text{as desired.}$$

End of proof of $\Leftarrow$. QED

3-3. Let X and Y be complete metric spaces.
Show $X \times Y$ is complete.

Proof: Let $Z : X \times Y$. Want Z is complete.
Want: $\forall u \in Z^{\mathbb{N}}$, [u is Cauchy in Z] $\Rightarrow$ [u is convergent in Z].
Given $u \in Z^{\mathbb{N}}$. Want: [u is Cauchy in Z] $\Rightarrow$ [u is convergent in Z].
Assume: u is Cauchy in Z. Want: u is convergent in Z.
Define $p : Z \to X$ by: $\forall z \in Z$, $p_z = z_1$.
Define $q : Z \to Y$ by: $\forall z \in Z$, $q_z = z_2$.
We have: $\forall z \in Z$, $z = (z_1, z_2) = (p_z, q_z)$.
Let $s := p \circ u$ and let $t := q \circ u$.
Then we have: $\forall j \in \mathbb{N}$,

$$u_j = (p_{u_j}, q_{u_j}) = ((p \circ u)_j, (q \circ u)_j) = (s_j, t_j) = (s, t)_j.$$

Then $u = (s, t)$. So, as u is Cauchy in Z and $Z = X \times Y$, we get:

(s, t) is Cauchy in $X \times Y$.

Then, by HW#3-2, s is Cauchy in X and t is Cauchy in Y.
So, since X and Y are both complete,
s is convergent in X and t is convergent in Y.
Choose $a \in X$ and $b \in Y$ s.t. $s \to a$ in X and $t \to b$ in Y.
Then, by a theorem from class, we have: $(s, t) \to (a, b)$ in $X \times Y$.
So, since $u = (s, t)$ and $Z = X \times Y$, we get: $u \to (a, b)$ in Z.
Then u is convergent in Z, as desired. QED

3-4. Let $p, q, r \in \mathbb{N}$, $A \in \mathbb{R}^{p \times q}$, $B \in \mathbb{R}^{q \times r}$. Show: $|AB| \leqslant |A| \cdot |B|$.

Proof: Since $|AB| \geqslant 0$, we get $|AB| = \sqrt{|AB|^2}$.
Since $|A| \cdot |B| > 0$, we get $|A| \cdot |B| = \sqrt{(|A| \cdot |B|)^2}$.
Want: $\sqrt{|AB|^2} \leqslant \sqrt{(|A| \cdot |B|)^2}$.
Want: $0 \leqslant |AB|^2 \leqslant (|A| \cdot |B|)^2$.
We have $0 \leqslant |AB|^2$. Want: $|AB|^2 \leqslant (|A| \cdot |B|)^2$.

We have $|AB|^2 = \sum_{i=1}^{p} \sum_{k=1}^{r} ((AB)_{ik})^2 = \sum_{i=1}^{p} \sum_{k=1}^{r} \left(\sum_{j=1}^{q} A_{ij} \cdot B_{jk} \right)^2$.

Also, $(|A| \cdot |B|)^2 = |A|^2 \cdot |B|^2 = \left(\sum_{i=1}^{p} \sum_{\ell=1}^{q} A_{i\ell}^2 \right) \cdot \left(\sum_{m=1}^{q} \sum_{k=1}^{r} B_{mk} \right)$

$$= \sum_{i=1}^{p} \sum_{\ell=1}^{q} \sum_{m=1}^{q} \sum_{k=1}^{r} A_{i\ell}^2 \cdot B_{mk}^2 = \sum_{i=1}^{p} \sum_{k=1}^{r} \left(\sum_{\ell=1}^{q} \sum_{m=1}^{q} A_{i\ell}^2 \cdot B_{mk}^2 \right).$$

Want: $\displaystyle\sum_{i=1}^{p} \sum_{k=1}^{r} \left(\sum_{j=1}^{q} A_{ij} \cdot B_{jk} \right)^2 \leqslant \sum_{i=1}^{p} \sum_{k=1}^{r} \left(\sum_{\ell=1}^{q} \sum_{m=1}^{q} A_{i\ell}^2 \cdot B_{mk}^2 \right).$

Want: $\forall i \in [1..p], \forall k \in [1..r], \quad \displaystyle\left(\sum_{j=1}^{q} A_{ij} \cdot B_{jk} \right)^2 \leqslant \sum_{\ell=1}^{q} \sum_{m=1}^{q} A_{i\ell}^2 \cdot B_{mk}^2.$

Given $i \in [1..p]$ and $k \in [1..r]$.

Want: $\displaystyle\left(\sum_{j=1}^{q} A_{ij} \cdot B_{jk} \right)^2 \leqslant \sum_{\ell=1}^{q} \sum_{m=1}^{q} A_{i\ell}^2 \cdot B_{mk}^2.$

Let $v := (A_{i1}, \ldots, A_{iq})$ and let $w := (B_{1k}, \ldots, B_{qk})$.

Then $v \bullet w = \displaystyle\sum_{j=1}^{q} A_{ij} \cdot B_{jk}$. Want: $(v \bullet w)^2 \leqslant \displaystyle\sum_{\ell=1}^{q} \sum_{m=1}^{q} A_{i\ell}^2 \cdot B_{mk}^2.$

We have: $|v|^2 = \displaystyle\sum_{\ell=1}^{q} A_{i\ell}^2$ and $|w|^2 = \displaystyle\sum_{m=1}^{q} B_{mk}^2.$

Then: $|v|^2 \cdot |w|^2 = \displaystyle\left(\sum_{\ell=1}^{q} A_{i\ell}^2 \right) \cdot \left(\sum_{m=1}^{q} B_{mk}^2 \right) = \sum_{\ell=1}^{q} \sum_{m=1}^{q} (A_{i\ell}^2 \cdot B_{mk}^2).$

Want: $(v \bullet w)^2 \leqslant |v|^2 \cdot |w|^2$.

By Cauchy-Schwarz, we have: $|v \bullet w| \leqslant |v| \cdot |w|$.

Since $0 \leqslant |v \bullet w| \leqslant |v| \cdot |w|$, we get: $|v \bullet w|^2 \leqslant (|v| \cdot |w|)^2$.

Then $(v \bullet w)^2 = |v \bullet w|^2 \leqslant (|v| \cdot |w|)^2 = |v|^2 \cdot |w|^2$. QED

3-5. Let $V \in \mathrm{ES}$. Show:

(a) $|\bullet|_V$ is Lipschitz-1 from V to $\mathbb{R}$

and (b) $\|\bullet\|_V$ is Lipschitz-$\sqrt{\#\mathcal{I}_V}$ from V to $\mathbb{R}$.

Proof:

Proof of (a):

Let $f := |\bullet|_V$. Want: f is Lipschitz-1 from V to $\mathbb{R}$.

Want: $\forall x, y \in V, \quad d_{\mathbb{R}}(f_x, f_y) \leqslant 1 \cdot (d_V(x, y))$.

Given $x, y \in V$. Want: $d_{\mathbb{R}}(f_x, f_y) \leqslant 1 \cdot (d_V(x, y))$.

Want: $|f_x - f_y| \leqslant |x - y|$.

Want: $(f_x - f_y \leqslant |x - y|)$ & $(f_y - f_x \leqslant |x - y|)$.

We have: $\forall z \in V, \quad f_z = (|\bullet|_V)_z = |z|$.

Want: $(|x| - |y| \leqslant |x - y|)$ & $(|y| - |x| \leqslant |x - y|)$.

We have: $|x| = |x - y + y| \leqslant |x - y| + |y|$.

Then: $|x| - |y| \leqslant |\,x - y\,|$. Want: $|y| - |x| \leqslant |\,x - y\,|$.
We have: $|y| = |\,y - x + x\,| \leqslant |\,y - x\,| + |x|$.
Then: $|y| - |x| \leqslant |\,y - x\,| = |\,x - y\,|$, as desired.
End of proof of (a).

Proof of (b):
Let $L := \sqrt{\#\mathcal{I}_V}$ and let $f := \|\bullet\|_V$.
Want: f is Lipschitz-L from V to $\mathbb{R}$.
Want: $\forall x, y \in V$, $d_{\mathbb{R}}(f_x, f_y) \leqslant L \cdot (d_V(x, y))$.
Given $x, y \in V$. Want: $d_{\mathbb{R}}(f_x, f_y) \leqslant L \cdot (d_V(x, y))$.
Want: $|f_x - f_y| \leqslant L \cdot |x - y|$.
By a theorem from class, we have:

$\forall z \in V$, $\|z\| \leqslant L \cdot |z|$.

Then: $\|\,x - y\,\| \leqslant L \cdot |x - y|$.

Want: $|f_x - f_y| \leqslant \|\,x - y\,\|$.
We have: $\forall z \in V$, $f_z = (\|\bullet\|_V)_z = \|z\|$.
Want: $(\ f_x - f_y \leqslant \|\,x - y\,\|\ )$ & $(\ f_y - f_x \leqslant \|\,x - y\,\|\ )$.
Want: $(\ \|x\| - \|y\| \leqslant \|\,x - y\,\|\ )$ & $(\ \|y\| - \|x\| \leqslant \|\,x - y\,\|\ )$.
We have: $\|x\| = \|\,x - y + y\,\| \leqslant \|\,x - y\,\| + \|y\|$.
Then: $\|x\| - \|y\| \leqslant \|\,x - y\,\|$. Want: $\|y\| - \|x\| \leqslant \|\,x - y\,\|$.
We have: $\|y\| = \|\,y - x + x\,\| \leqslant \|\,y - x\,\| + \|x\|$.
Then: $\|y\| - \|x\| \leqslant \|\,y - x\,\| = \|\,x - y\,\|$, as desired.
End of proof of (b). QED

Homework 2: Due on Tuesday 11 February

2-1. Show: $\forall \ell \in \mathbb{N}$, $((\bullet)^\ell)' = \ell \cdot (\bullet)^{\ell-1}$.

Proof: Let $S := \{\ell \in \mathbb{N} \,|\, ((\bullet)^\ell)' = \ell \cdot (\bullet)^{\ell-1}\}$. Want: $S = \mathbb{N}$.
By a result from class, $\forall m \in \mathbb{R}$, $(m \cdot (\bullet))' = C_{\mathbb{R}}^m$. Then $(1 \cdot (\bullet))' = C_{\mathbb{R}}^1$.
Then $((\bullet)^1)' = (1 \cdot (\bullet))' = C_{\mathbb{R}}^1 = (\bullet)^0 = 1 \cdot (\bullet)^0 = 1 \cdot (\bullet)^{1-1}$, so $1 \in S$.
Want: $\forall \ell \in S$, $\ell + 1 \in S$. Given $\ell \in S$. Want: $\ell + 1 \in S$.
Since $\ell \in S$, we have $((\bullet)^\ell)' = \ell \cdot (\bullet)^{\ell-1}$. Want: $((\bullet)^{\ell+1})' = (\ell+1) \cdot (\bullet)^\ell$.
Want: $\forall x \in \mathbb{R}$, $((\bullet)^{\ell+1})'_x = ((\ell + 1) \cdot (\bullet)^\ell)_x$.
Given $x \in \mathbb{R}$. Want: $((\bullet)^{\ell+1})'_x = ((\ell + 1) \cdot (\bullet)^\ell)_x$.
By the Product Rule, $((\bullet) \cdot (\bullet)^\ell)'_x \stackrel{*}{=} (\bullet)'_x \cdot ((\bullet)^\ell)_x + (\bullet)_x \cdot ((\bullet)^\ell)'_x$.
Recall: $(1 \cdot (\bullet))' = C_{\mathbb{R}}^1$. Then $(\bullet)' = C_{\mathbb{R}}^1$. Then $(\bullet)'_x = (C_{\mathbb{R}}^1)_x = 1$.
Recall: $((\bullet)^\ell)' = \ell \cdot (\bullet)^{\ell-1}$. Then $((\bullet)^\ell)'_x = (\ell \cdot (\bullet)^{\ell-1})_x = \ell \cdot x^{\ell-1}$.

$$
\begin{aligned}
\text{Then}\quad ((\bullet)\cdot(\bullet)^{\ell})'_x &=^* (\bullet)'_x \cdot ((\bullet)^{\ell})_x + (\bullet)_x \cdot ((\bullet)^{\ell})'_x \\
&= 1 \cdot x^{\ell} + x \cdot \ell \cdot x^{\ell-1} \\
&= x^{\ell} + \ell \cdot x \cdot x^{\ell-1} \\
&= x^{\ell} + \ell \cdot x^{\ell} = (1+\ell)\cdot x^{\ell} \neq ☹.
\end{aligned}
$$

Then $((\bullet)\cdot(\bullet)^{\ell})'_x = (1+\ell)\cdot x^{\ell} = (\ell+1)\cdot x^{\ell} = ((\ell+1)\cdot(\bullet)^{\ell})_x$, as desired. QED

2-2. Let $m \in \mathbb{R}$, $L : \mathbb{R} \to \mathbb{R}$. Assume: $\forall x \in \mathbb{R},\ L_x = m \cdot x$.
Show: ($\forall x \in \mathbb{R},\ L'_x = m$) & ($\forall x \in \mathbb{R},\ L''_x = 0$).

Proof: By a result from class, $(m \cdot (\bullet))' = C^m_{\mathbb{R}}$.
We have: $\forall x \in \mathbb{R}$, $L_x = m \cdot x = (m \cdot (\bullet))_x$. Then $L = m \cdot (\bullet)$.
Then $L' = (m \cdot (\bullet))' = C^m_{\mathbb{R}}$. Then: $\forall x \in \mathbb{R}$, $L'_x = (C^m_{\mathbb{R}})_x = m$.
Then: $\forall x \in \mathbb{R}$, $L'_x = m$. Want: $\forall x \in \mathbb{R},\ L''_x = 0$.
As $L' = C^m_{\mathbb{R}}$, we get: $L'' = (C^m_{\mathbb{R}})'$. By a result from class, $(C^m_{\mathbb{R}})' = \mathbf{0}$.
Then $L'' = (C^m_{\mathbb{R}})' = \mathbf{0}$. Then: $\forall x \in \mathbb{R}$, $L''_x = \mathbf{0}_x = 0$.
Then: $\forall x \in \mathbb{R}$, $L''_x = 0$, as desired. QED

2-3. Let $c \in \mathbb{R}$, $Q : \mathbb{R} \to \mathbb{R}$. Assume: $\forall x \in \mathbb{R},\ Q_x = c \cdot x^2$.
Show: ($\forall x \in \mathbb{R},\ Q'_x = 2 \cdot c \cdot x$) & ($\forall x \in \mathbb{R},\ Q''_x = 2 \cdot c$).

Proof: We have: $\forall x \in \mathbb{R},\ Q_x = c \cdot x^2 = (c \cdot (\bullet)^2)_x$. Then $Q = c \cdot (\bullet)^2$.
Want: (1) $\forall x \in \mathbb{R},\ Q'_x = 2 \cdot c \cdot x$ and (2) $\forall x \in \mathbb{R},\ Q''_x = 2 \cdot c$.

Proof of (1):
Given $x \in \mathbb{R}$. Want: $Q'_x = 2 \cdot c \cdot x$.
By HW#2-1, we have: $((\bullet)^2)' = 2 \cdot (\bullet)^{2-1}$.
Then: $((\bullet)^2)'_x = (2 \cdot (\bullet)^{2-1})_x = 2 \cdot x^{2-1} = 2 \cdot x$.
By linearity of derivative, $(c \cdot (\bullet)^2)'_x =^* c \cdot ((\bullet)^2)'_x$.
Then $Q'_x = (c \cdot (\bullet)^2)'_x =^* c \cdot ((\bullet)^2)'_x = c \cdot 2 \cdot x \neq ☹$.
Then $Q'_x = c \cdot 2 \cdot x$. $Q'_x = c \cdot 2 \cdot x = 2 \cdot c \cdot x$, as desired.
End of proof of (1).

Proof of (2):
By (1), we have: $\forall x \in \mathbb{R}$, $Q'_x = 2 \cdot c \cdot x$.
Let $m := 2 \cdot c$ We have: $\forall x \in \mathbb{R}$, $Q'_x = m \cdot x$.
Then: $\forall x \in \mathbb{R}$, $Q'_x = m \cdot x = (m \cdot (\bullet))_x$. Then $Q' = m \cdot (\bullet)$.
Let $L := m \cdot (\bullet)$. Then $Q' = L$.
We have $\forall x \in \mathbb{R}$, $L_x = (m \cdot (\bullet))_x = m \cdot x$.

Then, by HW#2-2, we get: $\forall x \in \mathbb{R}$, $L'_x = m$.
Then: $\forall x \in \mathbb{R}$, $Q''_x = L'_x = m = 2 \cdot c$. Then: $\forall x \in \mathbb{R}$, $Q''_x = 2 \cdot c$.
End of proof of (2). QED

2-4. Let $\alpha, \beta : \mathbb{R} \dashrightarrow \mathbb{R}$. Assume: $\forall x \in \mathbb{R}$, $\beta_x \;=^*\; \alpha_x$.
Show: $\forall x \in \mathbb{R}$, $\beta'_x \;=^*\; \alpha'_x$.

Proof: Given $x \in \mathbb{R}$. Want: $\beta'_x \;=^*\; \alpha'_x$.
Want: $(\ \alpha'_x \neq ☹\ ) \Rightarrow (\ \beta'_x = \alpha'_x\ )$.
Assume $\alpha'_x \neq ☹$. Want: $\beta'_x = \alpha'_x$.
By a theorem from class, it suffices to show: $\beta = \alpha$ near x in $\mathbb{R}$.
Want: $\exists B \in \mathcal{B}_{\mathbb{R}}(x)$ s.t. $\beta = \alpha$ on B.
Since $\alpha'_x \neq ☹$, we get $x \in \mathbb{D}_{\alpha'}$, and so, by a theorem from class,
we conclude: α is defined near x in $\mathbb{R}$.
Choose $B \in \mathcal{B}_{\mathbb{R}}(x)$ s.t. $B \subseteq \mathbb{D}_\alpha$.
Then $B \in \mathcal{B}_{\mathbb{R}}(x)$. Want: $\beta = \alpha$ on B.
Want: $\forall t \in B$, $\beta_t = \alpha_t$. Given $t \in B$. Want: $\beta_t = \alpha_t$.
Since $t \in B \subseteq \mathbb{D}_\alpha$, we get $\alpha_t \neq ☹$. Since $B \in \mathcal{B}_{\mathbb{R}}(x)$, we get $B \subseteq \mathbb{R}$.
Since $t \in B \subseteq \mathbb{R}$, by hypothesis, we have: $\beta_t \;=^*\; \alpha_t$.
Since $\beta_t \;=^*\; \alpha_t \neq ☹$, we conclude $\beta_t = \alpha_t$, as desired. QED

2-5. Let $a \in \mathbb{R}$. Define $T : \mathbb{R} \to \mathbb{R}$ by: $\forall h \in \mathbb{R}$, $T_h = h + a$.
Show: $T' = C^1_{\mathbb{R}}$.

Proof: As $\forall h \in \mathbb{R}$, $T_h = h + a = ((\bullet) + C^a_{\mathbb{R}})_h$, we get: $T = (\bullet) + C^a_{\mathbb{R}}$.
By a theorem from class, we have $\forall m \in \mathbb{R}$, $(m \cdot (\bullet))' = C^m_{\mathbb{R}}$.
Then $(1 \cdot (\bullet))' = C^1_{\mathbb{R}}$. Then $(\bullet)' = (1 \cdot (\bullet))' = C^1_{\mathbb{R}}$.
By a theorem from class, $(C^a_{\mathbb{R}})' = \mathbf{0}$.
Want: $\forall h \in \mathbb{R}$, $T'_h = (C^1_{\mathbb{R}})_h$. Given $h \in \mathbb{R}$. Want: $T'_h = (C^1_{\mathbb{R}})_h$.
We have $(\bullet)'_h = (C^1_{\mathbb{R}})_h = 1$ and $(C^a_{\mathbb{R}})'_h = \mathbf{0}_h = 0$.
By linearity of the derivative, $(\,(\bullet) + C^a_{\mathbb{R}}\,)'_h \;=^*\; (\bullet)'_h + (C^a_{\mathbb{R}})'_h$.
Then $T'_h = (\,(\bullet) + C^a_{\mathbb{R}}\,)'_h \;=^*\; (\bullet)'_h + (C^a_{\mathbb{R}})'_h = 1 + 0 = 1 \neq ☹$.
Then $T'_h = 1 = (C^1_{\mathbb{R}})_h$, as desired. QED

Homework 1: Due on Tuesday 4 February

1-1. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $p \in \mathbb{D}_f$.
Show: $(\ f$ has a global strict-maximum at $p\ )$
$\Leftrightarrow (\ f^{\mathbb{T}}_p$ has a global strict-maximum at $0\ )$.

Proof: Since $p \in \mathbb{D}_f$, we get $(f_p^{\mathbb{T}})_0 = 0$. Let $g := f_p^{\mathbb{T}}$. Then $g_0 = 0$.
Since $p \in \mathbb{D}_f$, we get: $f_p \in \mathbb{I}_f$. Want: (f has a global strict-maximum at p) $\Leftrightarrow$ (g has a global strict-maximum at 0).

Proof of $\Rightarrow$*:*
Assume: f has a global strict-maximum at p
Want: g has a global strict-maximum at 0.
Want: $\forall h \in (\mathbb{D}_g)_0^\times$, $g_h < g_0$. Given $h \in (\mathbb{D}_g)_0^\times$. Want: $g_h < g_0$.
Since $h \in (\mathbb{D}_g)_0^\times \subseteq \mathbb{D}_g$, we get: $g_h \neq ☹$.
We have: $f_{p+h} - f_p = (f_p^{\mathbb{T}})_h = g_h$.
Then $f_{p+h} - f_p = g_h \neq ☹$, so $f_{p+h} \neq ☹$, so $p + h \in \mathbb{D}_f$.
Since $h \in (\mathbb{D}_g)_0^\times \subseteq \mathbb{R}_0^\times$, we get: $h + p \in \mathbb{R}_p^\times$, so $h + p \neq p$.
Since $h + p \in \mathbb{D}_f$ and $h + p \neq p$, we get: $h + p \in (\mathbb{D}_f)_p^\times$.
As f has a global strict-maximum at p, we get: $\forall x \in (\mathbb{D}_f)_p^\times$, $f_x < f_p$.
So, since $h + p \in (\mathbb{D}_f)_p^\times$, we see that: $f_{h+p} < f_p$. Then $f_{h+p} - f_p < 0$.
Then $g_h = f_{h+p} - f_p < 0 = g_0$, as desired.
End of proof of $\Rightarrow$*.*

Proof of $\Leftarrow$*:*
Assume: g has a global strict-maximum at 0.
Want: f has a global strict-maximum at p
Want: $\forall x \in (\mathbb{D}_f)_p^\times$, $f_x < f_p$.
Given $x \in (\mathbb{D}_f)_p^\times$. Want: $f_x < f_p$.
Since $x \in (\mathbb{D}_f)_p^\times \subseteq \mathbb{D}_f$, we get: $f_x \in \mathbb{I}_f$. Since $p \in \mathbb{D}_f$, we get: $f_p \in \mathbb{I}_f$.
Since $f_x, f_p \in \mathbb{I}_f \subseteq \mathbb{R}$, we get $f_x - f_p \in \mathbb{R}$. Want: $f_x - f_p < 0$.
We have: $g_{x-p} = (f_p^{\mathbb{T}})_{x-p} = f_{p+x-p} - f_p = f_x - f_p$.
Then $g_{x-p} = f_x - f_p \in \mathbb{R}$, so $g_{x-p} \neq ☹$, so $x - p \in \mathbb{D}_g$.
Since $x \in (\mathbb{D}_f)_p^\times \subseteq \mathbb{R}_p^\times$, we get $x - p \in \mathbb{R}_0^\times$, so $x - p \neq 0$.
Since $x - p \in \mathbb{D}_g$ and $x - p \neq 0$, we get: $x - p \in (\mathbb{D}_g)_0^\times$.
As g has a global strict-maximum at 0, we get: $\forall h \in (\mathbb{D}_g)_0^\times$, $g_h < g_0$.
So, since $x - p \in (\mathbb{D}_g)_0^\times$, it follows that: $g_{x-p} < g_0$.
Then $f_x - f_p = g_{x-p} < g_0 = 0$, as desired.
End of proof of $\Leftarrow$*.* QED

1-2. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, let $c \in \mathbb{R}$, let $L \in \mathrm{LINS}_c f$ and let $\varepsilon > 0$.
Show: $L - \varepsilon \cdot |\bullet| \leqslant f_c^{\mathbb{T}} \leqslant L + \varepsilon \cdot |\bullet|$ near 0 in $\mathbb{R}$.

Proof: Want: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $L - \varepsilon \cdot |\bullet| \leqslant f_c^{\mathbb{T}} \leqslant L + \varepsilon \cdot |\bullet|$ on B.
Since $L \in \mathrm{LINS}_c f$, we get: $f_c^{\mathbb{T}} - L \in \mathcal{O}_1$.

Let $\rho := f_c^{\mathbb{T}} - L$. Then $\rho \in \mathcal{O}_1$.
Choose $\delta > 0$ s.t., $\forall h \in \mathbb{R}$, $(\ |h| < \delta\ ) \Rightarrow (\ |\rho_h| < \varepsilon \cdot |h|^1\ )$.
Let $B := B_{\mathbb{R}}(0, \delta)$. Then $B \in \mathcal{B}_{\mathbb{R}}(0)$.
Want: $L - \varepsilon \cdot |\bullet| \leqslant f_c^{\mathbb{T}} \leqslant L + \varepsilon \cdot |\bullet|$ on B.
Want: $\forall h \in B$, $(L - \varepsilon \cdot |\bullet|)_h \leqslant (f_c^{\mathbb{T}})_h \leqslant (L + \varepsilon \cdot |\bullet|)_h$.
Given $h \in B$. Want: $(L - \varepsilon \cdot |\bullet|)_h \leqslant (f_c^{\mathbb{T}})_h \leqslant (L + \varepsilon \cdot |\bullet|)_h$.
Want: $L_h - \varepsilon \cdot |h| \leqslant (f_c^{\mathbb{T}})_h \leqslant L_h + \varepsilon \cdot |h|$.
Let $s := L_h$ and $t := (f_c^{\mathbb{T}})_h$ and $u := \varepsilon \cdot |h|$. Want: $s - u < t < s + u$.
Since $h \in B = B_{\mathbb{R}}(0, \delta)$, we get: $d(h, 0) < \delta$.
Then $|h| = |h - 0| = d(h, 0) < \delta$. Then, by choice of δ, $|\rho_h| < \varepsilon \cdot |h|^1$.
We have $\rho_h = (f_c^{\mathbb{T}} - L)_h = (f_c^{\mathbb{T}})_h - L_h = t - s$.
Then $|t - s| = |\rho_h| < \varepsilon \cdot |h|^1 = \varepsilon \cdot |h| = u$.
Then $|t - s| < u$, and so $s - u < t < s + u$, as desired. QED

1-3. Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, let $c \in \mathbb{R}$ and let $L \in \mathrm{LINS}_c f$.
Assume: f has a local semi-maximum at c in $\mathbb{R}$.
Show: $L = \mathbf{0}$.

Proof: Since $L \in \mathrm{LINS}_c f$, we get $D_c f = L$ and $L \in \mathcal{L}$.
Since $L \in \mathcal{L}$, we get $L : \mathbb{R} \to \mathbb{R}$. Then $\mathbb{D}_L = \mathbb{R}$.
Since $1 \in \mathbb{R} = \mathbb{D}_L$, we conclude that: $L_1 \neq ☹$.
So, since $L_1 = (D_c f)_1 = f'_c$, we get $f'_c \neq ☹$. Then $c \in \mathbb{D}_{f'}$.
Then $c \in \mathbb{D}_{f'} \subseteq D_f$. Then $(f_c^{\mathbb{T}})_0 = 0$.
Since $L \in \mathrm{LINS}_c f$, we get $f_c^{\mathbb{T}} - L \in \mathcal{O}_1$.
Since f has a local semi-maximum at c in $\mathbb{R}$,
it follows that $f_c^{\mathbb{T}}$ has a local semi-maximum at 0 in $\mathbb{R}$.
Let $g := f_c^{\mathbb{T}}$. Then $g_0 = 0$ and $g - L \in \mathcal{O}_1$.
Also, g has a local semi-maximum at 0 in $\mathbb{R}$.
Choose $A \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $g \leqslant g_0$ on A. Then $g \leqslant 0$ on A.
Since $g - L \in \mathcal{O}_1$ and $L \in \mathcal{L} = \mathcal{H}_1$, choose $\delta > 0$ s.t., $\forall h \in B_{\mathbb{R}}(0, \delta)$,
$[\ (\ L_h < 0\ ) \Rightarrow (\ g_h < 0\ )\ ]\ \&\ [\ (\ L_h > 0\ ) \Rightarrow (\ g_h > 0\ )\ ]$.
Choose $\gamma > 0$ s.t. $A = B_{\mathbb{R}}(0, \gamma)$. Let $\beta := \min\{\gamma/2, \delta/2\}$.
Then $\beta \leqslant \gamma/2 < \gamma$ and $\beta \leqslant \delta/2 < \delta$.
Also, since $\gamma/2, \delta/2 > 0$, we conclude that $\beta > 0$. Then $|\beta| = \beta$.
Assume $L \neq \mathbf{0}$. Want: Contradiction.
Since $L \in \mathcal{L}$, choose $m \in \mathbb{R}$ s.t. $L = m \cdot (\bullet)$.
Since $L \neq \mathbf{0}$, we get $m \neq 0$. Then $m \in \mathbb{R}_0^\times$.
Since $m \in \mathbb{R}_0^\times$, it follows that $|m| > 0$. Let $h := (m/|m|) \cdot \beta$.
Then $d(h, 0) = |h - 0| = |h| = (|m|/|m|) \cdot |\beta| = 1 \cdot \beta = \beta$.

Since $d(h,0) = \beta < \gamma$, we get $h \in B_{\mathbb{R}}(0,\gamma)$.
Since $d(h,0) = \beta < \delta$, we get: $h \in B_{\mathbb{R}}(0,\delta)$.
Since $h \in B_{\mathbb{R}}(0,\gamma) = A$ and since $g \leqslant 0$ on A, we conclude: $g_h \leqslant 0$.
Since $m \in \mathbb{R}_0^\times$, it follows that $m^2 > 0$.
We have $L_h = m \cdot h = m \cdot (m/|m|) \cdot (\delta/2) = (m^2 \cdot \delta)/(2 \cdot |m|)$.
So, since $m^2 > 0$ and $\delta > 0$ and $|m| > 0$, we get: $L_h > 0$.
So, since $h \in B_{\mathbb{R}}(0,\delta)$, by choice of δ, we conclude that $g_h > 0$.
Then $0 < g_h \leqslant 0$, so $0 < 0$. Contradiction. QED

1-4. Let $\phi : \mathbb{R} \dashrightarrow \mathbb{R}$, let $a > 0$ and let $Q := a \cdot (\bullet)^2$.
Assume: $\phi - Q \in \mathcal{O}_2$.
Show: ϕ has a local strict-minimum at 0 in $\mathbb{R}$.

Proof: We have $Q = a \cdot (\bullet)^2 \in \mathcal{H}_2$.
Since $\phi - Q \in \mathcal{O}_2$ and $Q \in \mathcal{H}_2$, choose $\delta > 0$ s.t., $\forall h \in B_{\mathbb{R}}(0,\delta)$,
$[\ (\ Q_h < 0\ ) \Rightarrow (\ \phi_h < 0\ )\ ]\ \&\ [\ (\ Q_h > 0\ ) \Rightarrow (\ \phi_h > 0\ )\ ]$.
Want: $\exists B \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $\phi > \phi_0$ on $B_0^\times$.
Let $B := B_{\mathbb{R}}(0,\delta)$. Then $B \in \mathcal{B}_{\mathbb{R}}(0)$. Want: $\phi > \phi_0$ on $B_0^\times$.
Want: $\forall h \in B_0^\times$, $\phi_h \geqslant \phi_0$. Given $h \in B_0^\times$. Want: $\phi_h > \phi_0$.
Since $\phi - Q \in \mathcal{O}_2 \subseteq \mathcal{O}_0 = \mathrm{CVZ}$, we get: $(\phi - Q)_0 = 0$.
We have $Q_0 = (a \cdot (\bullet))_0 = a \cdot 0^2 = 0$.
Then $\phi_0 = \phi_0 - 0 = \phi_0 - Q_0 = (\phi - Q)_0 = 0$.
Since $h \in \mathbb{R}_0^\times$, we get $h^2 > 0$. So, since $a > 0$, we get $a \cdot h^2 > 0$.
Then $Q_h = (a \cdot (\bullet)^2)_h = a \cdot h^2 > 0$.
So, since $h \in B = B_{\mathbb{R}}(0,\delta)$, by choice of δ, we get: $\phi_h > 0$.
Then $\phi_h > 0 = \phi_0$, as desired. QED

1-5. Let $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$ and let $c \in \mathbb{R}$.
Assume: $f = g$ near c in $\mathbb{R}$.
Show: $\mathrm{LINS}_c f \subseteq \mathrm{LINS}_c g$.

Proof: Want: $\forall L \in \mathrm{LINS}_c f$, $L \in \mathrm{LINS}_c g$.
Given $L \in \mathrm{LINS}_c f$. Want: $L \in \mathrm{LINS}_c g$.
Know: $f_c^{\mathbb{T}} - L \in \mathcal{O}_1$. Want: $g_c^{\mathbb{T}} - L \in \mathcal{O}_1$.
Want: $f_c^{\mathbb{T}} - L = g_c^{\mathbb{T}} - L$ near 0 in $\mathbb{R}$.
Want: $\exists A \in \mathcal{B}_{\mathbb{R}}(0)$ s.t. $f_c^{\mathbb{T}} - L = g_c^{\mathbb{T}} - L$ on A.
Since $f = g$ near c in $\mathbb{R}$, choose $B \in \mathcal{B}_{\mathbb{R}}(c)$ s.t. $f = g$ on B.
Since $B \in \mathcal{B}_{\mathbb{R}}(c)$, we get $c \in B$.

So, since $f = g$ on B, we get $f_c = g_c$.
Choose $\delta > 0$ s.t. $B = B_{\mathbb{R}}(c, \delta)$.
Let $A := B_{\mathbb{R}}(0, \delta)$. Then $A \in \mathcal{B}_{\mathbb{R}}(0)$.
Want: $f_c^{\mathbb{T}} - L = g_c^{\mathbb{T}} - L$ on A.
Want: $\forall h \in A$, $(f_c^{\mathbb{T}} - L)_h = (g_c^{\mathbb{T}} - L)_h$.
Given $h \in A$. Want: $(f_c^{\mathbb{T}} - L)_h = (g_c^{\mathbb{T}} - L)_h$.
Since $h \in A = B_{\mathbb{R}}(0, \delta)$, we get $d(h, 0) < \delta$.
Since $c, h \in \mathbb{R}$, we get $(c + h) - c = h$ and $h = h - 0$.
Then $d(c + h, c) = |(c + h) - c| = |h| = |h - 0| = d(h, 0) < \delta$.
As $d(c+h, c) < \delta$, we get: $c+h \in B_{\mathbb{R}}(c, \delta)$. Then $c+h \in B_{\mathbb{R}}(c, \delta) = B$.
So, since $f = g$ on B, we get $f_{c+h} = g_{c+h}$.
Recall that $f_c = g_c$. Then $(f_c^{\mathbb{T}})_h = f_{c+h} - f_c = g_{c+h} - g_c = (g_c^{\mathbb{T}})_h$.
Then $(f_c^{\mathbb{T}} - L)_h = (f_c^{\mathbb{T}})_h - L_h = (g_c^{\mathbb{T}})_h - L_h = (g_c^{\mathbb{T}} - L)_h$. QED