**Writeup on measures for MATH 4604 (Advanced Calculus II)**
**Spring 2015**

*Definition*: \( \mathbb{N} := \{1, 2, 3, \ldots\} \).

*Definition*: For any set \( X \), by \( 2^X \) we denote the set of all subsets of \( X \).

*Definition*: If \( S \) is a set of sets, then \( S \) is **pairwise-disjoint** means: for all \( S, S' \in S \), we have: \( [ S \neq S' ] \Rightarrow [ S \cap S' = \emptyset ] \).

*Definition*: Let \( n \in \mathbb{N} \) and let \( B \subseteq \mathbb{R}^n \). Then \( B \) is a **box** means: either (there exist intervals \( I_1, \ldots, I_n \subseteq \mathbb{R} \) such that \( B = I_1 \times \cdots \times I_n \)) or \( (B = \emptyset) \).

*Examples*: \([1, 2] \times [3, 4] \) and \((1, 2) \times (3, 4) \) and \([4, 6] \times \{2\} \times (-1, 1)\) and \( \mathbb{R}^4 \) and \( \emptyset \) are boxes.

*Definition*: For all \( n \in \mathbb{N} \), by \( B_n \) we denote the set of all boxes in \( \mathbb{R}^n \).

*Remark*: For any \( n \in \mathbb{N} \), for any \( A, B \in B_n \), we have \( A \cap B \in B_n \).

*Definition*: Let \( X \) be a set and let \( S \subseteq 2^X \). We define

\[
\langle S \rangle^\emptyset_{\text{fin} \cup} := \{ \cup F \mid F \subseteq S \text{ is finite and nonempty} \} \cup \{ \emptyset \},
\]

\[
\langle S \rangle^\emptyset_{\text{fin} \cap} := \{ \cap F \mid F \subseteq S \text{ is finite and nonempty} \} \cup \{ X \},
\]

\[
\langle S \rangle^\emptyset_{\text{fin} \cup} := \{ \emptyset \},
\]

\[
\langle S \rangle^\emptyset_{\text{fin} \cap} := \{ \emptyset \}.
\]

*Remark (cupcap)*: Let \( X \) be a set and let \( S \subseteq 2^X \). Then

\[
\left( \left[ \langle S \rangle^\emptyset_{\text{fin} \cup} = S \right] \Leftrightarrow \left[ (\forall A, B \in S, A \cup B \in S \text{ and } (\emptyset \in S) \right] \right)
\]

and

\[
\left( \left[ \langle S \rangle^\emptyset_{\text{fin} \cap} = S \right] \Leftrightarrow \left[ (\forall A, B \in S, A \cap B \in S \text{ and } (X \in S) \right] \right).
\]

*Definition*: Let \( X \) be a set and let \( S \subseteq 2^X \). Then \( S \) is a **Boolean algebra** (or **algebra of sets** or, simply **algebra**) on \( X \) means: \( \langle S \rangle^\emptyset_{\text{fin} \cup} = \langle S \rangle^X_{\text{fin} \cap} = \langle S \rangle^\emptyset_{\text{fin} \cap} = S \).

*Remark*: Let \( X \) be a set and let \( S \subseteq 2^X \). Assume that \( \langle S \rangle^X_{\text{fin} \cap} \subseteq S \) and that \( \langle S \rangle^\emptyset_{\text{fin} \cap} \subseteq S \). Then \( S \) is an algebra on \( X \).

*Proof*: We have \( X \in \langle S \rangle^X_{\text{fin} \cap} \subseteq S \). Then \( \emptyset = X \setminus X \in \langle S \rangle^\emptyset_{\text{fin} \cap} \subseteq S \). By definition of \( \langle S \rangle^X_{\text{fin} \cap} \), we know that \( \langle S \rangle^X_{\text{fin} \cap} \supseteq S \). Then \( \langle S \rangle^X_{\text{fin} \cap} = S \). Since \( \emptyset \in S \), we see, by definition of \( \langle S \rangle^\emptyset_{\text{fin} \cap} \), that \( \langle S \rangle^\emptyset_{\text{fin} \cap} \supseteq S \). Then \( \langle S \rangle^\emptyset_{\text{fin} \cap} = S \). It remains to show that \( \langle S \rangle^\emptyset_{\text{fin} \cup} = S \). Given \( A, B \in S \). By Remark (cupcap), we wish to show that \( A \cup B \in S \).

We have \( X \setminus A, X \setminus B \in \langle S \rangle^\emptyset_{\text{fin} \cap} = S \). Then \( (X \setminus A) \cap (X \setminus B) \in \langle S \rangle^X_{\text{fin} \cap} = S \). Then \( A \cup B = X \setminus [(X \setminus A) \cap (X \setminus B)] \in \langle S \rangle^\emptyset_{\text{fin} \cup} = S \), as desired. QED
Definition: Let $X$ be a set and let $S \subseteq 2^X$. Then $S$ is a **near algebra** on $S$ means: $\langle S \rangle_{\text{fin}} \cap S \subseteq S$ and $\langle S \rangle \setminus \langle S \rangle_{\text{fin}} \cap S = \emptyset$.

Warning: For all $n \in \mathbb{N}$, $\langle B_n \rangle \setminus B_n$, so $B_n$ is NOT an algebra on $\mathbb{R}^n$. However:

**Fact:** For all $n \in \mathbb{N}$, $B_n$ is a near algebra on $\mathbb{R}^n$.

**Theorem:** Let $X$ be a set, and let $S \subseteq 2^X$ be a near algebra on $X$. Then $\langle S \rangle_{\text{fin}} \cap S$ is an algebra on $X$.

**Proof:** This follows from Homework Problem #51. QED

**Definition:** Let $n \in \mathbb{N}$. Then we define $K_n := \langle B_n \rangle_{\text{fin} \cap S}$. A subset $K \subseteq \mathbb{R}^n$ is called a **kidset** if $K \in K_n$.

**Definition:** Let $n \in \mathbb{N}$, and let $B \in B_n$. Then, for all integers $j \in [1, n]$, we define $B_j := \pi_{j}^{(n)}(B)$.

For any $n \in \mathbb{N}$, for any $B \in B_n$, we have $B_1, \ldots, B_n \in S$ and $B = B_1 \times \cdots \times B_n$.

**Definition:** Let $n \in \mathbb{N}$ and let $B \in B_n$. Then $B$ is **degenerate** means: either $B = \emptyset$ or there exists an integer $j \in [1, n]$ such that $\#(B_j) = 1$.

**Definition:** For all $n \in \mathbb{N}$, we define $v_n^B : B_n \to [0, \infty]$ by

$$v_n(B) = \begin{cases} 0, & \text{if } B \text{ is degenerate} \\ \infty, & \text{if } B \text{ is nondegenerate and unbounded} \\ [\ell(B_1)] \cdots [\ell(B_n)], & \text{if } B \text{ is nondegenerate and bounded.} \end{cases}$$

We will make the following conventions on addition and multiplication in $\mathbb{R} \cup \{\infty\}$:

(i) For all $a \in \mathbb{R}$, $a + \infty = \infty$.

(ii) For all $a > 0$, $a \cdot \infty = \infty$.

The conventions (i) and (ii) will be adopted throughout this writeup. Additionally, we will sometimes make the “volume convention” that $0 \times \infty = 0$. With the volume convention, for any box $B$ in $\mathbb{R}^n$, we have $v_n^B(B) = [\ell(B_1)] \cdots [\ell(B_n)]$.

**Definition:** Let $X$ be a set and let $S_0 \subseteq S \subseteq 2^X$. Then we define:

$$FD_S(S_0) := \{ F \subseteq S_0 | F \text{ is finite and pairwise-disjoint, and } \cup F \in S \}.$$
In the preceding definition, “FD” stands for finite decomposition. An element of \( FD_S(S_0) \) is simply a partition of an element of \( S \) by sets in \( S_0 \). (Recall that a partition of a set is a pairwise-disjoint collection of subsets, whose union is the set.)

**Definition:** Let \( X \) be a set and let \( S \subseteq 2^X \) and let \( m : S \rightarrow [0, \infty] \). Then, for any \( S_0 \subseteq S \), \( m \) is \( S_0 \)-additive means: for all \( F \in FD_S(S_0) \), we have \( m(\cup F) = \sum_{F \in F} [m(F)] \). Also, \( m \) is finitely additive means: \( m \) is \( S \)-additive.

To say that \( m \) is \( S_0 \)-additive is to say: Any time we have a partition of a set in \( S \) by subsets that are in \( S_0 \), then the measure of the original set is the sum of the measures of the partitioning sets. Consequently, to say that \( m \) is finitely additive is to say: Any time we have a partition of a set in \( S \) by subsets that are in \( S \), then the measure of the original set is the sum of the measures of the partitioning sets.

Our next goal is to show, for all \( n \in \mathbb{N} \), that \( v^S_n : B_n \rightarrow [0, \infty] \) is finitely additive. We begin with a lemma that allows for some extension of additivity:

**Lemma (extAdd):** Let \( X \) be a set, let \( S \subseteq 2^X \), let \( S_0 \subseteq S \) and let \( m : S \rightarrow [0, \infty] \) be \( S_0 \)-additive. Define \( S_1 := (\langle S_0 \rangle^0_{\text{fin} \cup}) \cap S \). Then \( m : S \rightarrow [0, \infty] \) is \( S_1 \)-additive.

**Proof:** Given \( F \in FD_S(S_1) \). We wish to show that \( m(\cup F) = \sum_{F \in F} [m(F)] \).

From the definition of \( FD_S(S_1) \), we know that \( F \) is pairwise-disjoint, that \( \cup F \subseteq S \) and that \( F \subseteq S_1 \); then \( F \in S_1 \subseteq S \) and \( F \in S_1 \subseteq \langle S_0 \rangle^0_{\text{fin} \cup} \subseteq \langle S_0 \backslash \{\emptyset\} \rangle^0_{\text{fin} \cup} \). For all \( F \in F \), choose a pairwise-disjoint finite \( G_F \subseteq S_0 \backslash \{\emptyset\} \) such that \( F = \cup G_F \); then

\[
G_F \in FD_F(S_0) \subseteq FD_S(S_0),
\]

so, since \( F = \cup G_F \) and \( m \) is \( S_0 \)-additive, we get \( m(F) = m(\cup G_F) = \sum_{G \in G_F} [m(G)] \).

For all \( F, F' \in F \), we have

\[
[F \neq F'] \quad \Rightarrow \quad [F \cap F' = \emptyset] \quad \Rightarrow \quad [(\cup G_F) \cap (\cup G_{F'}) = \emptyset],
\]

so, since \( \emptyset \notin G_F \) and \( \emptyset \notin G_{F'} \), we conclude that:

\[
[F \neq F'] \quad \Rightarrow \quad [G_F \cap G_{F'} \neq \emptyset].
\]

Let \( G := \bigcup_{F \in F} G_F \). Then \( G \subseteq S_0 \backslash \{\emptyset\} \). By the associative law of addition, we conclude that

\[
\sum_{G \in G} [m(G)] = \sum_{F \in F} \left[ \sum_{G \in G_F} [m(G)] \right].
\]

Recall that: for all \( F \in F \), \( \sum_{G \in G_F} [m(G)] = m(F) \). Therefore,

\[
\sum_{G \in G} [m(G)] = \sum_{F \in F} [m(F)].
\]

By the associative law of unions, we conclude that

\[
\cup G = \bigcup_{F \in F} (\cup G_F). \quad \text{Recall that: for all } F \in F, \cup G_F = F. \quad \text{Therefore, } \cup G = \bigcup_{F \in F} F = \cup F.
\]

**Claim:** \( G \) is pairwise-disjoint. **Proof of claim:** Given \( G, G' \in G \). Assume \( G \neq G' \). We wish to show that \( G \cap G' = \emptyset \). Choose \( F, F' \in F \) such that \( G \in G_F \) and \( G' \in G_{F'} \). If
Given $F = F'$, then because $\mathcal{G}_F$ is pairwise-disjoint, we get $G \cap G' = \emptyset$, as desired. We therefore assume that $F \neq F'$. Then, as $\mathcal{F}$ is pairwise-disjoint, we get $F \cap F' = \emptyset$. We have $G \subseteq \cup \mathcal{G}_F = F$ and $G' \subseteq \cup \mathcal{G}_F' = F'$. Then $G \cap G' \subseteq F \cap F' = \emptyset$. End of proof of claim.

We have $\cup \mathcal{G} = \cup \mathcal{F} \in \mathcal{S}$ and $\mathcal{G} \subseteq \mathcal{S}_0$. So, by the Claim, we see that $\mathcal{G} \in FD_\mathcal{S}(\mathcal{S}_0)$. So, since $m$ is $\mathcal{S}_0$-additive, it follows that $m(\cup \mathcal{G}) = \sum_{G \in \mathcal{G}} [m(G)]$. So, since $\cup \mathcal{G} = \cup \mathcal{F}$ and since $m(\cup \mathcal{G}) = \sum_{G \in \mathcal{G}} [m(G)] = \sum_{F \in \mathcal{F}} [m(F)]$, we get $m(\cup \mathcal{F}) = \sum_{F \in \mathcal{F}} [m(F)]$, as desired. QED

**Corollary (impliesFA):** Let $X$ be a set, let $\mathcal{S} \subseteq 2^X$ and let $m : \mathcal{S} \to [0, \infty]$. Assume, for all finite $\mathcal{F} \subseteq \mathcal{S}$, that there exists $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $m$ is $\mathcal{S}_0$-additive and such that $\mathcal{F} \subseteq \langle \mathcal{S}_0 \rangle^{\emptyset}_{\text{fin}\cup\mathcal{U}}$. Then $m$ is finitely additive.

**Proof:** Given $\mathcal{F} \in FD_\mathcal{S}(\mathcal{S})$. We wish to prove that $m(\cup \mathcal{F}) = \sum_{F \in \mathcal{F}} [m(F)]$.

By assumption, choose $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $m$ is $\mathcal{S}_0$-additive and such that $\mathcal{F} \subseteq \langle \mathcal{S}_0 \rangle^{\emptyset}_{\text{fin}\cup\mathcal{U}}$. Let $\mathcal{S}_1 := \langle \mathcal{S}_0 \rangle^{\emptyset}_{\text{fin}\cup\mathcal{U}} \cap \mathcal{S}$. We have $\mathcal{F} \in FD_\mathcal{S}(\mathcal{S})$ and $\mathcal{F} \subseteq \mathcal{S}_1$, so $\mathcal{F} \in FD_\mathcal{S}(\mathcal{S}_1)$.

By Lemma (extAdd), $m$ is $\mathcal{S}_1$-additive. Then $m(\cup \mathcal{F}) = \sum_{F \in \mathcal{F}} [m(F)]$, as desired. QED

**Definition:** Let $k \in \mathbb{N}$, let $a_1, \ldots, a_k \in \mathbb{R}$ and let $A := \{a_1, \ldots, a_k\}$. Assume that $a_1 < a_2 < \cdots < a_k$. Then we define $\mathcal{I}_A \subseteq \mathcal{I}$ by

$$
\mathcal{I}_A := \{-\infty, a_1\}, \{a_1\}, (a_1, a_2), \{a_2\}, \{a_2, a_3\}, \{a_3\}, \ldots, \{a_{k-1}\}, (a_{k-1}, a_k), \{a_k\}, (a_k, \infty\}.
$$

**Fact:** Let $A \subseteq \mathbb{R}$ be finite and nonempty. Then $\ell$ is $\mathcal{I}_A$-additive.

**Proof:** This is Homework Problem #52. QED

**Remark:** For all finite $\mathcal{F} \subseteq \mathcal{I}$, there exists a nonempty finite $A \subseteq \mathbb{R}$ such that $\mathcal{F} \subseteq \langle \mathcal{I}_A \rangle^{\emptyset}_{\text{fin}\cup\mathcal{U}}$.

**Theorem (lengthFA):** The length function $\ell : \mathcal{I} \to [0, \infty]$ is finitely additive.

**Proof:** Given a finite $\mathcal{F} \subseteq \mathcal{I}$. By Corollary (impliesFA), we wish to prove that there exists $\mathcal{S}_0 \subseteq \mathcal{I}$ such that $\ell$ is $\mathcal{S}_0$-additive and such that $\mathcal{F} \subseteq \langle \mathcal{S}_0 \rangle^{\emptyset}_{\text{fin}\cup\mathcal{U}}$.

By the Remark above, choose a nonempty finite $A \subseteq \mathbb{R}$ such that $\mathcal{F} \subseteq \langle \mathcal{I}_A \rangle^{\emptyset}_{\text{fin}\cup\mathcal{U}}$. Let $\mathcal{S}_0 := \mathcal{I}_A$. Then $\mathcal{F} \subseteq \langle \mathcal{S}_0 \rangle^{\emptyset}_{\text{fin}\cup\mathcal{U}}$, and it remains to show that $\ell$ is $\mathcal{S}_0$-additive. By the Fact above, $\ell$ is $\mathcal{I}_A$-additive, so, since $\mathcal{I}_A = \mathcal{S}_0$, we are done. QED

We have $\mathcal{B}_1 = \mathcal{I} \cup \{\emptyset\}$. Also, for all $I \in \mathcal{I}$, we have $v_1^\mathcal{B}(I) = \ell(I)$. Also, $v_1^\mathcal{B}(\emptyset) = 0$. We then leave it as an unassigned exercise to use Theorem (lengthFA) to show that the function $v_1^\mathcal{B} : \mathcal{B}_1 \to [0, \infty]$ is finitely additive. Our next goal is to show that $v_2^\mathcal{B} : \mathcal{B}_2 \to [0, \infty]$ is also finitely additive.

**Definition:** For all $A, B \subseteq 2^\mathbb{R}$, we define $A \ast B := \{A \times B \mid A \in A, B \in B\}$. 

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Fact: Let $A, B \in \mathbb{R}$ be two finite nonempty sets. Then the function $v_2^B : B_2 \to [0, \infty]$ is \((\mathcal{I}_A * \mathcal{I}_B)\)-additive.

Proof: This is Homework Problem 53. QED

Remark: Let $\mathcal{F} \subseteq B_2$ and assume that $\mathcal{F}$ is finite. Then there exist subsets $A, B \subseteq \mathbb{R}$ such that $A$ and $B$ are both finite and nonempty and such that $\mathcal{F} \subseteq \langle \mathcal{I}_A * \mathcal{I}_B \rangle^\theta_{\text{fin}}$.

Proof: This is Homework Problem 54. QED

Theorem (areaFA): The function $v_2^B : B_2 \to [0, \infty]$ is finitely additive.

Proof: Given a finite $\mathcal{F} \subseteq B_2$. By Corollary (impliesFA), we wish to prove that there exists $S_0 \subseteq B_2$ such that $v_2^B$ is $S_0$-additive and such that $\mathcal{F} \subseteq \langle S_0 \rangle^\theta_{\text{fin}}$.

By the Remark above, choose two nonempty finite subsets $A, B \subseteq \mathbb{R}$ such that we have $\mathcal{F} \subseteq \langle \mathcal{I}_A * \mathcal{I}_B \rangle^\theta_{\text{fin}}$. Let $S_0 := \mathcal{I}_A * \mathcal{I}_B$. Then $\mathcal{F} \subseteq \langle S_0 \rangle^\theta_{\text{fin}}$, and it remains to show that $v_2^B$ is $S_0$-additive.

By the Fact above, $v_2^B$ is $(\mathcal{I}_A * \mathcal{I}_B)$-additive, so, since $\mathcal{I}_A * \mathcal{I}_B = S_0$, we are done. QED

Theorem (volFA): For all $n \in \mathbb{N}$, the function $v_n^B : B_n \to [0, \infty]$ is finitely additive.

This last theorem is proved by arguments similar to those in the proof of Theorem (areaFA). Our next goal is to extend $v_n^B$ to a finitely additive $v_n^K : K_n \to [0, \infty]$. Recall that $K_n$ is the set of kidsets in $\mathbb{R}^n$. We begin with:

Lemma (sumSum): Let $X$ be a set and let $S \subseteq 2^X$. Assume, for all $S, T \in S$, that $S \cap T \in S$. Let $m : S \to [0, \infty]$ be finitely additive. Let $\mathcal{F}, \mathcal{G} \subseteq S$. Assume that $\mathcal{F}$ and $\mathcal{G}$ are both finite and pairwise-disjoint. Assume that $\bigcup \mathcal{F} = \bigcup \mathcal{G}$. Then $\sum_{F \in \mathcal{F}} [m(F)] = \sum_{G \in \mathcal{G}} [m(G)]$.

Proof: For all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, we have $F \cap G \in S$.

For all $F \in \mathcal{F}$, we have $F = \bigcup_{G \in \mathcal{G}} [F \cap G]$. So, since $m$ is finitely additive, we see, for all $F \in \mathcal{F}$, that $m(F) = \sum_{G \in \mathcal{G}} [m(F \cap G)]$. Similarly, for all $G \in \mathcal{G}$, we have $G = \bigcup_{F \in \mathcal{F}} [F \cap G]$, and so $m(G) = \sum_{F \in \mathcal{F}} [m(F \cap G)]$.

Then $\sum_{F \in \mathcal{F}} [m(F)] = \sum_{F \in \mathcal{F}} \left[ \sum_{G \in \mathcal{G}} [m(F \cap G)] \right] = \sum_{G \in \mathcal{G}} \left[ \sum_{F \in \mathcal{F}} [m(F \cap G)] \right] = \sum_{G \in \mathcal{G}} [m(G)]$. QED

Theorem (uniqueExt): Let $X$ be a set and let $S \subseteq 2^X$. Let $A := \langle S \rangle^\theta_{\text{fin}}$. Let the two functions $\mu, \nu : A \to [0, \infty]$ both be $S$-additive. Assume that $\mu|S = \nu|S$. Then $\mu = \nu$.

Proof: Given $A \in A$. We wish to prove that $\mu(A) = \nu(A)$.

Choose $F \subseteq S$ such that $F$ is finite and pairwise disjoint and such that $A = \bigcup F$. Because $\mu$ is $S$-additive, we see that $\mu(A) = \sum_{F \in \mathcal{F}} [\mu(F)]$. Because $\nu$ is $S$-additive, we see
that \( \nu(A) = \sum_{F \in \mathcal{F}} [\nu(F)] \). For all \( F \in \mathcal{F} \), we have \( F \in \mathcal{S} \), so
\[
\mu(F) = (\mu|\mathcal{S})(F) = (\nu|\mathcal{S})(F) = \nu(F).
\]
Then \( \mu(A) = \sum_{F \in \mathcal{F}} [\mu(F)] = \sum_{F \in \mathcal{F}} [\nu(F)] = \nu(A) \), as desired. QED

We will extend \( v_n^{\mathcal{B}} \) to kidsets by the following "Measure Extension Theorem":

**Theorem (msrExt):** Let \( X \) be a set and let \( \mathcal{S} \subseteq 2^X \). Assume, for all \( S, T \in \mathcal{S} \), that \( S \cap T \in \mathcal{S} \). Let \( m : \mathcal{S} \to [0, \infty] \) be finitely additive. Let \( \mathcal{A} := \langle \mathcal{S}\rangle_{\text{fin}}^{\emptyset} \). Then there exists a unique finitely additive function \( \mu : \mathcal{A} \to [0, \infty] \) such that \( \mu|\mathcal{S} = m \).

**Proof:** Uniqueness follows from Theorem (uniqueExt), so we need only prove existence.

For all \( A \in \mathcal{A} \), by definition of \( \mathcal{A} \), choose \( \mathcal{F}_A \subseteq \mathcal{S} \) such that \( \mathcal{F}_A \) is pairwise-disjoint and finite, and such that \( A = \bigcup \mathcal{F}_A \). Let \( \mu : \mathcal{A} \to [0, \infty] \) be defined by \( \mu(A) = \sum_{F \in \mathcal{F}_A} [m(F)] \).

For all \( S \in \mathcal{S} \), because \( m \) is \( \mathcal{S} \)-additive, because \( \mathcal{F}_S \subseteq \mathcal{S} \) and because \( S = \bigcup \mathcal{F}_S \), it follows that \( m(S) = \sum_{F \in \mathcal{F}_S} [m(F)] \), i.e., that \( m(S) = \mu(S) \). Then \( m = \mu|\mathcal{S} \). It remains to show that \( \mu \) is finitely additive, i.e., that \( \mu \) is \( \mathcal{A} \)-additive. By Lemma (extAdd), since \( \mathcal{A} = \langle \mathcal{S}\rangle_{\text{fin}}^{\emptyset} \cap \mathcal{A} \), it suffices to show that \( \mu \) is \( \mathcal{S} \)-additive. Given \( \mathcal{G} \in \mathcal{F}_A(\mathcal{S}) \). We wish to prove that \( \mu(\bigcup \mathcal{G}) = \sum_{G \in \mathcal{G}} [\mu(G)] \).

By definition of \( \mathcal{F}_A(\mathcal{S}) \), we know that \( \mathcal{G} \) is finite and pairwise-disjoint, that \( \mathcal{G} \subseteq \mathcal{S} \) and that \( \bigcup \mathcal{G} \in \mathcal{A} \). Because \( \mathcal{G} \subseteq \mathcal{S} \) and \( m = \mu|\mathcal{S} \), we see, for all \( G \in \mathcal{G} \), that \( m(G) = \mu(G) \). Then \( \sum_{G \in \mathcal{G}} [m(G)] = \sum_{G \in \mathcal{G}} [\mu(G)] \). Let \( A := \bigcup \mathcal{G} \). Then \( A \in \langle \mathcal{S}\rangle_{\text{fin}}^{\emptyset} = \mathcal{A} \). Let \( \mathcal{F} := \mathcal{F}_A \). Then \( \mathcal{F} \) is finite and pairwise-disjoint. Also, \( \mathcal{F} \subseteq \mathcal{S} \) and \( A = \bigcup \mathcal{F} \). Also, \( \mu(A) = \sum_{F \in \mathcal{F}} [m(F)] \).

We have \( \bigcup \mathcal{G} = A = \bigcup \mathcal{F} \), so, by Lemma (sumSum), we have \( \sum_{F \in \mathcal{F}} [m(F)] = \sum_{G \in \mathcal{G}} [m(G)] \).

Then \( \mu(\bigcup \mathcal{G}) = \mu(A) = \sum_{F \in \mathcal{F}} [m(F)] = \sum_{G \in \mathcal{G}} [m(G)] = \sum_{G \in \mathcal{G}} [\mu(G)] \), as desired. QED

**Definition:** Let \( n \in \mathbb{N} \). Let \( v_n^K : \mathcal{K}_n \rightarrow [0, \infty] \) be the (by Theorem (msrExt)) unique finitely additive function such that \( v_n^K|\mathcal{B}_n = v_n^\mathcal{B} \). Let \( \mathcal{D}_n \) denote the collection of all bounded subsets of \( \mathbb{R}^n \). Define \( \underline{v}_n, \overline{v}_n : \mathcal{D}_n \to [0, \infty] \) by
\[
\underline{v}_n(S) = \sup\{v_n^K(K) \mid K \in \mathcal{K}_n, K \subseteq S\}, \quad \overline{v}_n(S) = \inf\{v_n^K(L) \mid L \in \mathcal{K}_n, L \supseteq S\}.
\]
The functions \( \underline{v}_n \) and \( \overline{v}_n \) are sometimes called inner volume and outer volume.

**Fact (innerOutr):** Let \( n \in \mathbb{N} \). Then all of the following are true:

(a) For all \( S \in \mathcal{D}_n \), \( \underline{v}_n(S) \leq \overline{v}_n(S) < \infty \).

(b) For all \( S, T \in \mathcal{D}_n \), \( (S \subseteq T) \Rightarrow ([\underline{v}_n(S) \leq \underline{v}_n(T)] \& [\overline{v}_n(S) \leq \overline{v}_n(T)]) \).

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(c) For all \( K \in \mathcal{D}_n \cap \mathcal{K}_n \), \( \nu_n(K) = \overline{\nu}_n(K) = v_n^K(K) \).
(d) For all \( S, T \in \mathcal{D}_n \), \( \overline{\nu}_n(S \cup T) \leq (\overline{\nu}_n(S)) + (\overline{\nu}_n(T)) \).
(e) For all \( S, T \in \mathcal{D}_n \), \( (S \cap T = \emptyset) \Rightarrow (\overline{\nu}_n(S \cup T) \geq (\overline{\nu}_n(S)) + (\overline{\nu}_n(T))) \).

**Definition:** Let \( n \in \mathbb{N} \). For all \( S \subseteq \mathbb{R}^n \), \( S \) is **contented** means:
both \( [S \text{ is bounded}] \) and \( [\nu_n(S) = \overline{\nu}_n(S)] \).

Let \( \mathcal{C}_n \) denote the set of all contented subsets of \( \mathbb{R}^n \). We define \( \nu_n : \mathcal{C}_n \to [0, \infty] \) by \( \nu_n(S) = \overline{\nu}_n(S) = \overline{\nu}_n(S) \).

**Remark:** For all \( n \in \mathbb{N} \), for all \( K \in \mathcal{D}_n \cap \mathcal{K}_n \), we have: both \( K \in \mathcal{C}_n \) and \( \nu_n(K) = v_n^K(K) \).

**Note:** The last remark follows from (c) of Fact (innerOutr). While kids are often not content, this remark asserts that bounded kidsets are always contented.

**Note:** For all \( n \in \mathbb{N} \), \( \mathbb{R}^n \) is unbounded, so \( \mathbb{R}^n \not\in \mathcal{C}_n \), and thus \( \mathcal{C}_n \) is **NOT** an algebra on \( \mathbb{R}^n \). On the other hand, for all \( C, D \in \mathcal{C}_n \), as we will soon see, we have: \( C \cap D, C \cup D, C \setminus D \in \mathcal{C}_n \). Perhaps a collection of sets with these three closure properties could be referred to as a **topless algebra**, if that’s not too risqué . . .

**Lemma (inOutApprox):** Let \( n \in \mathbb{N} \) and let \( S \in \mathcal{D}_n \). Then: \( S \) is contented \( \iff \) for all \( \varepsilon > 0 \), there exist \( K, L \in \mathcal{K}_n \) such that \( K \subseteq S \subseteq L \) and \( v_n^K(L \setminus K) \leq \varepsilon \).

This follows from Lemma (inOutApprox2) from the handout on integration, but give here a self-contained proof.

**Proof:** *Proof of \( \Rightarrow \):* Assume \( S \in \mathcal{C}_n \). Given \( \varepsilon > 0 \). We wish to prove:
\[ \exists K, L \in \mathcal{K}_n \text{ s.t. } K \subseteq S \subseteq L \text{ and } v_n^K(L \setminus K) \leq \varepsilon. \]

Let \( \alpha := \nu_n(S) \). Since \( \alpha = \overline{\nu}_n(S) = \sup\{v_n^K(K) \mid K \in \mathcal{K}_n, K \subseteq S \} \), choose \( K \in \mathcal{K}_n \) such that \( K \subseteq S \) and \( v_n^K(K) \geq \alpha - (\varepsilon/2) \). Since \( \alpha = \overline{\nu}_n(S) = \inf\{v_n^K(L) \mid L \in \mathcal{K}_n, L \supseteq S \} \), choose \( L \in \mathcal{K}_n \) such that \( L \supseteq S \) and \( v_n^K(L) \leq \alpha + (\varepsilon/2) \). Then \( K \subseteq S \subseteq L \), and it remains to prove that \( v_n^K(L \setminus K) \leq \varepsilon \).

We have \( [v_n^K(L \setminus K)] + [v_n^K(K)] = v_n^K(L) \), so \( [v_n^K(L \setminus K)] + [\alpha - (\varepsilon/2)] \leq \alpha + (\varepsilon/2) \), so \( v_n^K(L \setminus K) \leq \varepsilon. \) _End of proof of \( \Rightarrow \)._

*Proof of \( \Leftarrow \):* Assume \( (\ast) \). We wish to prove that \( S \in \mathcal{C}_n \). By assumption, \( S \) is bounded. We wish to show that \( \nu_n(S) = \overline{\nu}_n(S) \). Let \( \beta := [\overline{\nu}_n(S)] - [\nu_n(S)] \) Then \( \beta \geq 0 \) and we wish to show that \( \beta \leq 0 \). Given \( \varepsilon > 0 \). We wish to show that \( \beta \leq \varepsilon \).

By \( (\ast) \), choose \( K, L \in \mathcal{K}_n \) such that \( K \subseteq S \subseteq L \) and such that \( v_n^K(L \setminus K) \leq \varepsilon \). Since \( S \in \mathcal{D}_n \), i.e., \( S \) is bounded, choose a bounded box \( B \subseteq \mathbb{R}^n \) such that \( S \subseteq B \). Let \( L_0 := L \cap B \). Then \( L_0 \) is bounded, \( K \subseteq S \subseteq L_0 \) and \( v_n^K(L_0 \setminus K) \leq v_n^K(L \setminus K) \leq \varepsilon \). As \( [v_n^K(K)] + [v_n^K(L_0 \setminus K)] = v_n^K(L_0) \), we get \( [v_n^K(L_0)] - [v_n^K(K)] = v_n^K(L_0 \setminus K) \). Also, we have both \( \nu_n(K) = v_n^K(K) \) and \( \overline{\nu}_n(L_0) = v_n^K(L_0) \). Then \( [\overline{\nu}_n(L_0)] - [\nu_n(K)] = v_n^K(L_0 \setminus K) \).
Because \( K \subseteq S \subseteq L_0 \), we conclude that \( \nu_n(K) \leq \nu_n(S) \) and that \( \overline{\nu}_n(S) \leq \overline{\nu}_n(L_0) \). Then,
\[ \beta = [\overline{\nu}_n(S)] - [\nu_n(S)] \leq [\overline{\nu}_n(L_0)] - [\nu_n(K)] = v_n^K(L_0 \setminus K) \leq \varepsilon, \]
as desired. _End of proof of \( \Leftarrow \). QED

**Lemma (ctDiff):** Let \( n \in \mathbb{N} \). Let \( C, D \in \mathcal{C}_n \). Then \( C \setminus D \in \mathcal{C}_n \).
Proof: We have $C \setminus D \subseteq C \subseteq C_n \subseteq D_n$, so $C \setminus D \in D_n$. Given $\varepsilon > 0$. By Lemma (inOutApprox), we wish to show

$$\exists K, L \in K_n \quad \text{s.t.} \quad K \subseteq C \setminus D \subseteq L \quad \text{and} \quad v_n^K(L \setminus K) \leq \varepsilon.$$  

By Lemma (inOutApprox), choose $C_*, C^*, D_*, D^* \in K_n$ such that

$$C_* \subseteq C \subseteq C^*, \quad D_* \subseteq D \subseteq D^*, \quad v_n^K(C^* \setminus C_*) \leq \varepsilon/2, \quad v_n^K(D^* \setminus D_*) \leq \varepsilon/2.$$  

We define $K := C_* \setminus D^*$ and $L := C^* \setminus D_*$. Then we have $K \subseteq C \setminus D \subseteq L$ and it remains to prove that $v_n^K(L \setminus K) \leq \varepsilon$.

We have $L \setminus K = (C^* \setminus D_*) \setminus (C_* \setminus D^*) \subseteq (C^* \setminus C_*) \cup (D^* \setminus D_*)$. Then

$$v_n^K(L \setminus K) \leq [v_n^K(C^* \setminus C_*)] + [v_n^K(D^* \setminus D_*)] \leq [\varepsilon/2] + [\varepsilon/2] = \varepsilon,$$

as desired. QED

Lemma (ctCapCup): Let $n \in \mathbb{N}$. Let $C, D \in C_n$. Then $C \cap D, C \cup D \in C_n$.

Proof: We have $C, D \in C_n$, so $C$ and $D$ are bounded, so $C \cup D$ is bounded. Choose a bounded $B \in B_n$ such that $C \cup D \subseteq B$. We have $B \in B_n \subseteq K_n$. Then $B$ is a bounded kidset, so $B \in C_n$. Let $C' := B \setminus C$ and let $D' := B \setminus D$. By Lemma (ctDiff), we conclude both that $C' \in C_n$ and that $D' \in C_n$. We have $C \cap D = C \setminus D'$ so, by Lemma (ctDiff), we see that $C \cap D \in C_n$. It remains to show that $C \cup D \in C_n$.

We have $C' \cap D' = C' \setminus D$, so, by Lemma (ctDiff), we see that $C' \cap D' \in C_n$. We have $C \cup D = B \setminus (C' \cap D')$, so, by Lemma (ctDiff), we see that $C \cup D \in C_n$, as desired. QED

Lemma (volFA): Let $n \in \mathbb{N}$. Then $v_n : C_n \to [0, \infty]$ is finitely additive.

Proof: Given $F \in FD_{C_n}(C_n)$. We wish to prove that $v_n(\cup F) = \sum_{F \in \mathcal{F}} [v_n(F)]$.

As $F \subseteq C_n$ and as $F$ is finite and pairwise-disjoint, by Lemma (ctCapCup), we conclude that $\cup F \in C_n$. Then $v_n(\cup F) = \nu_n(\cup F) = \nu_n(\cup F)$. Also, $F \subseteq C_n$, and so, for all $F \in \mathcal{F}$, we have $F \in C_n$, and, therefore, we have $v_n(F) = \nu_n(F) = \nu_n(F)$.

By (d) and (e) of Fact (innerOutr), we have

$$\nu_n(\cup F) \geq \sum_{F \in \mathcal{F}} [v_n(F)] \quad \text{and} \quad \nu_n(\cup F) \leq \sum_{F \in \mathcal{F}} [\nu_n(F)].$$

Then $v_n(\cup F) \geq \sum_{F \in \mathcal{F}} [v_n(F)]$ and $v_n(\cup F) \leq \sum_{F \in \mathcal{F}} [v_n(F)]$, so $v_n(\cup F) = \sum_{F \in \mathcal{F}} [v_n(F)]$. QED

Definition: Let $n \in \mathbb{N}$ and let $Z \subseteq \mathbb{R}^n$. Then $Z$ is negligible means:

- $Z$ is bounded
- $\nu_n(Z) = 0$.

Lemma (ignoreNgbl): Let $n \in \mathbb{N}$ and $S \in D_n$ and let $Z \subseteq \mathbb{R}^n$. Assume that $Z$ is negligible. Then $\nu_n(S \setminus Z) = \nu_n(S) = \nu_n(S \cup Z)$.

Proof: We have $S \setminus Z \subseteq S \subseteq S \cup Z$, so $\nu_n(S \setminus Z) \leq \nu_n(S) \leq \nu_n(S)$. Also, because $S \cup Z = (S \setminus Z) \cup Z$ and $(S \setminus Z) \cap Z = \emptyset$, we get $\nu_n(S \cup Z) \leq \nu_n(S \setminus Z) + [\nu_n(Z)]$. So, because $\nu_n(Z) = 0$, we conclude that $\nu_n(S \cup Z) \leq \nu_n(S \setminus Z)$.
Then \( \overline{v_n(S \setminus Z)} \leq \overline{v_n(S)} \leq \overline{v_n(S \cup Z)} \leq \overline{v_n(S \setminus Z)} \), and the result follows. QED

From Lemma (ignoreNgb), we see, for any \( n \in \mathbb{N} \), for any \( S \in \mathcal{C}_n \), for any \( Z \subseteq \mathbb{R}^n \), that if \( Z \) is negligible, then \( S \setminus Z \in \mathcal{C}_n \) and \( S \cup Z \in \mathcal{C}_n \) and \( v_n(S \setminus Z) = v_n(S) = v_n(S \cup Z) \).

**Definition:** Let \( n \in \mathbb{N} \) and let \( S \subseteq \mathbb{R}^n \). Then we define:

\[
\mathcal{S} := \bigcap\{A \subseteq \mathbb{R}^n \mid \text{A is closed, } A \supseteq S\}, \quad S^\circ := \bigcup\{U \subseteq \mathbb{R}^n \mid \text{U is open, } U \subseteq S\},
\]

and we define \( \partial S := [\mathcal{S}] \setminus [S^\circ] \). These three sets are called, respectively, the closure of \( S \), the interior of \( S \) and the boundary of \( S \).

**Lemma (cvxConn):** Let \( n \in \mathbb{N} \) and let \( A, F \subseteq \mathbb{R}^n \). Assume that \( F \) is convex, that \( F \cap A \neq \emptyset \) and that \( F \setminus A \neq \emptyset \). Then \( F \cap (\partial A) \neq \emptyset \).

**Proof:** Choose \( x \in F \cap A \) and \( y \in F \setminus A \). Define \( \phi : \mathbb{R} \to \mathbb{R}^n \) by \( \phi(t) = (1 - t)x + ty \). Let \( I := [0, 1] \). As \( F \) is convex, we have \( \phi(I) \subseteq F \). Let \( S := \phi^{-1}(A) \). Because \( \phi(0) = x \in A \), we see that 0 \( \in S \). Because \( \phi(1) = y \notin A \), we see that 1 \( \notin S \). Let \( t_0 := \sup(I \cap S) \). Because \( I \) is closed in \( \mathbb{R} \), we have \( I = [0, 1] \). Then \( t_0 \in I \cap S = I \cap \mathcal{S} \). Let \( z := \phi(t_0) \). Then \( z \in \phi(I) \subseteq F \), so it suffices to show that \( z \in \partial A \), i.e., both that \( z \in \overline{A} \) and that \( z \notin \mathring{A} \).

We have \( S = \phi^{-1}(A) \subseteq \phi^{-1}(\mathring{A}) \). Because \( \phi \) is continuous and \( \mathring{A} \) is closed in \( \mathbb{R}^n \), we conclude that \( \phi^{-1}(\mathring{A}) \) is closed in \( \mathbb{R} \). Then \( \mathcal{S} \subseteq \phi^{-1}(\mathring{A}) \). Then \( t_0 \in \mathcal{S} \subseteq \phi^{-1}(\mathring{A}) \). Then \( z = \phi(t_0) \in \mathring{A} \). It remains to show \( z \notin A^0 \). Assume \( z \in A^0 \). We aim for a contradiction.

We have \( \phi(t_0) = z \in A^0 \), so \( t_0 \in \phi^{-1}(A^0) \). Then \( t_0 \in \phi^{-1}(A^0) = S \). So, since 1 \( \notin S \), we see that \( t_0 \neq 1 \). So, since \( t_0 \in [0, 1] \), we see that \( t_0 < 1 \). Because \( \phi \) is continuous and \( A^0 \) is open in \( \mathbb{R}^n \), we conclude that \( \phi^{-1}(A^0) \) is open in \( \mathbb{R} \). So, since \( t_0 \in \phi^{-1}(A^0) \), choose \( \delta > 0 \) such that \( (t_0 - \delta, t_0 + \delta) \subseteq \phi^{-1}(A^0) \). Since \( t_0 < 1 \), choose \( t_1 \in (t_0, t_0 + \delta) \) such that \( t_1 \leq 1 \). Then \( 0 < t_0 < t_1 \leq 1 \). Then \( t_1 \in [0, 1] = I \). Also,

\[
t_1 \in (t_0, t_0 + \delta) \subseteq (t_0 - \delta, t_0 + \delta) \subseteq \phi^{-1}(A^0) \subseteq \phi^{-1}(A) = S.
\]

Then \( t_1 \in I \cap S \), so \( t_1 \leq \sup(I \cap S) = t_0 \). Then \( t_1 < t_0 \), contradiction. QED

**Lemma (kidCap):** Let \( n \in \mathbb{N} \), let \( A \subseteq \mathbb{R}^n \) and let \( K \in \mathcal{K}_n \). Assume that \( K \cap (\partial A) = \emptyset \). Then \( K \cap A = \emptyset \).

**Proof:** Choose a finite pair-wise disjoint \( \mathcal{F} \subseteq \mathcal{B}_n \) such that \( K = \cup \mathcal{F} \). For all \( F \in \mathcal{F} \), we know both that \( F \) is convex and that \( F \cap (\partial A) = \emptyset \). So, by Lemma (cvxConn), we see, for all \( F \in \mathcal{F} \), that either \( F \subseteq A \) or \( F \cap A = \emptyset \). Let \( \mathcal{F}_0 := \{F \in \mathcal{F} \mid F \subseteq A\} \). Let \( K_0 := \bigcup_{F \in \mathcal{F}_0} F \). Then \( K_0 \in \mathcal{K}_n \), and we will prove that \( K \cap A = K_0 \).

For all \( F \in \mathcal{F}_0 \), we have \( F \cap A = F \). For all \( F \in \mathcal{F} \setminus \mathcal{F}_0 \), we have \( F \cap A = \emptyset \). Then \( \bigcup_{F \in \mathcal{F}} (F \cap A) = \bigcup_{F \in \mathcal{F}_0} F \). Then \( K \cap A = \bigcup_{F \in \mathcal{F}} (F \cap A) = \bigcup_{F \in \mathcal{F}_0} F = K_0 \), as desired. QED

**Theorem (charAtd):** Let \( n \in \mathbb{N} \) and let \( S \subseteq \mathbb{R}^n \) be bounded. Then:

\[
S \text{ is contented } \iff \partial S \text{ is negligible.}
\]
Proof: Proof of $\Rightarrow$: Assume that $S$ is contented. We wish to show that $\partial S$ is negligible. Since $S$ is bounded, $\partial S$ is bounded. It remains to show that $\mathcal{v}_n(\partial S) = 0$. Given $\varepsilon > 0$, we wish to show that $\mathcal{v}_n(\partial S) \leq \varepsilon$.

By Lemma (inOutApprox), choose $K, L \in K_n$ such that $K \subseteq S \subseteq L$ and such that $v_n^K(L \setminus K) \leq \varepsilon$. Choose a bounded $B \in B_n$ such that $S \subseteq B$. Let $L_1 := L \cap B$. Then $L_1$ is bounded and $K \subseteq S \subseteq L_1$ and $v_n^K(L_1 \setminus K) \leq \varepsilon$. Since $L_1 \setminus K$ is a bounded kidset, we know that $L_1 \setminus K$ is contented and that $v_n(L_1 \setminus K) = v_n^K(L_1 \setminus K)$. Since $L_1$ is a bounded kidset, $L_1$ is contented, and so $v_n(\partial L_1) = 0$. Similarly, $v_n(\partial K) = 0$. Then

$$v_n(\overline{L_1}) = v_n(L_1 \cup (\partial L_1)) = v_n(L_1) \quad \text{and} \quad v_n(K^\circ) = v_n(K \setminus (\partial K)) = v_n(K).$$

Then $[v_n(\overline{L_1} \setminus K^\circ)] + [v_n(K^\circ)] = v_n(\overline{L_1}) = v_n(L_1) = [v_n(L_1 \setminus K)] + [v_n(K)]$. So, since $v_n(K^\circ) = v_n(K)$, we get $v_n(\overline{L_1} \setminus K^\circ) = v_n(L_1 \setminus K)$.

We have $K^\circ \subseteq S^\circ \subseteq S \subseteq \overline{L_1}$, and it follows that $\partial S = \overline{S} \setminus S^\circ \subseteq \overline{L_1} \setminus K^\circ$. Then $v_n(\partial S) \leq v_n(\overline{L_1} \setminus K^\circ) = v_n(L_1 \setminus K) = v_n^K(L_1 \setminus K) \leq \varepsilon$, as desired. End of proof of $\Rightarrow$.

Proof of $\Leftarrow$: Assume that $\partial S$ is negligible. We wish to show that $S$ is contented. We know that $S$ is bounded. Given $\varepsilon > 0$. By Lemma (inOutApprox), we wish to show that there exist $K, L \in K_n$ such that $K \subseteq S \subseteq L$ and such that $v_n^K(L \setminus K) \leq \varepsilon$.

As $\mathcal{v}_n(\partial S) = 0 < \varepsilon$, choose $Y \in K_n$ such that $\partial S \subseteq Y$ and $v_n^Y(Y) \leq \varepsilon$. Let $K := S \setminus Y$ and $L := K \cup Y$. We wish to prove: $K, L \in K_n$ and $K \subseteq S \subseteq L$ and $v_n^K(L \setminus K) \leq \varepsilon$.

Because $\mathbb{R}^n \in B_n \subseteq K_n$ and $Y \in K_n$, we see that $\mathbb{R}^n \setminus Y \in K_n$. Also, since $\partial S \subseteq Y$, it follows that $(\mathbb{R}^n \setminus Y) \cap (\partial S) = \emptyset$. Then, by Lemma (kidCap), we get $(\mathbb{R}^n \setminus Y) \cap S \in K_n$. Then $K = S \setminus Y = (\mathbb{R}^n \setminus Y) \cap S \in K_n$. Since $K, Y \in K_n$, we get $L = K \cup Y \in K_n$. We have

$$K = S \setminus Y \subseteq S \subseteq (S \setminus Y) \cup Y = K \cup Y = L,$$

so $K \subseteq S \subseteq L$. It remains to prove that $v_n^K(L \setminus K) \leq \varepsilon$.

As $L = K \cup Y$, we get $L \setminus K \subseteq Y$, so $v_n^K(L \setminus K) \leq v_n^Y(Y) \leq \varepsilon$. End of proof of $\Leftarrow$. QED

Remark: Let $n \in \mathbb{N}$ and let $S \subseteq \mathbb{R}^n$. Then:

$$\partial(S^\circ) \subseteq \partial S \quad \text{and} \quad \partial(\overline{S}) \subseteq \partial S \quad \text{and} \quad S \setminus (\partial S) = S^\circ \quad \text{and} \quad S \cup (\partial S) = \overline{S}.$$

Proof: Because $S^\circ \subseteq S$, we get $\overline{S^\circ} \subseteq \overline{S}$. Also, $(S^\circ)^\circ = S^\circ$. Then

$$\partial(S^\circ) = [\overline{S^\circ}] \setminus [(S^\circ)^\circ] \subseteq [\overline{S}] \setminus [S^\circ] = \partial S.$$}

We have $\overline{S} = \overline{S}$. Also, because $\overline{S} \supseteq S$, we get $(\overline{S})^\circ \subseteq S^\circ$. Then

$$\partial(\overline{S}) = [\overline{S}] \setminus [(\overline{S})^\circ] \subseteq [\overline{S}] \setminus [S^\circ] = \partial S.$$}

Since $S \subseteq \overline{S}$, we get $S \setminus ([\overline{S}] \setminus [S^\circ]) = S^\circ$. That is, $S \setminus (\partial S) = S^\circ$.

Since $\overline{S} \supseteq S^\circ$, we get $S \cup ([\overline{S}] \setminus [S^\circ]) = \overline{S}$. That is, $S \cup (\partial S) = \overline{S}$. QED

Theorem (volBdClInt): Let $n \in \mathbb{N}$ and let $S \subseteq \mathbb{R}^n$ be contented. Then $\partial S$ and $S^\circ$ and $\overline{S}$ are all three contented. Moreover, $v_n(\partial S) = 0$ and $v_n(S^\circ) = v_n(S) = v_n(\overline{S})$.

Proof: This follows from the preceding remark, together with Theorem (charCtd). QED