# Proof of the Banach-Tarski paradox

# The rotations of 3-space

$\mathrm{SO}(3) := \{\text{rotations in } \mathbb{R}^3 \text{ about an axis through } 0\}$

Every rotation has a unique axis, except the identity $I$.

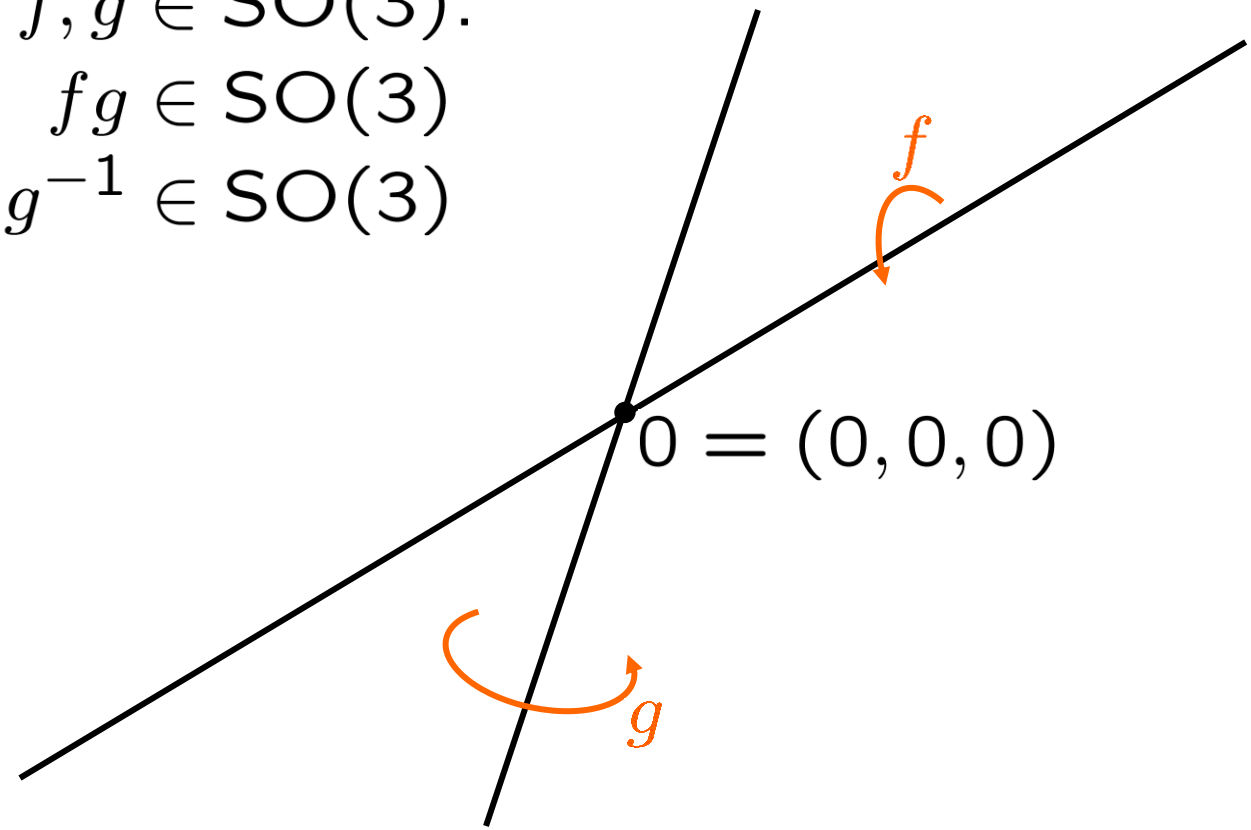
Any composition of rotations is again a rotation.

The inverse of a rotation is again a rotation.

Fix $f, g \in \mathrm{SO}(3)$.

$fg \in \mathrm{SO}(3)$

$f^{-1}, g^{-1} \in \mathrm{SO}(3)$



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Fix $f, g \in \mathsf{SO}(3)$. $\qquad w_{fg} \in \mathsf{SO}(3)$

$$w = t^3 s^{-1} t^{-1} \quad \Rightarrow \quad w_{fg} = g^3 f^{-1} g^{-1}$$

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Fix $f, g \in \mathrm{SO}(3)$.

works for any $f$, $g$ $\quad w = t^3 s^{-1} t^{-1}$

$$\Rightarrow \begin{cases} w_{fg} \in \mathrm{SO}(3) \\ w_{fg} = g^3 f^{-1} g^{-1} \\ \qquad \forall f, g \in \mathrm{SO}(3) \end{cases}$$

works for any $w$ as long as...

$\forall w \in \mathbb{F}_2 \backslash \{1\}$, $\quad w \neq 1$ $\quad (1_{fg} = I)$

$\forall w \in \mathbb{F}_2 \backslash \{1\}$, $\quad w_{f_\circ g_\circ} \neq I$

$w_{fg} \neq I$ a.s.

$$\mathcal{Z}_w := \{(f, g) \in \mathrm{SO}(3) \times \mathrm{SO}(3) \,|\, w_{fg} = I\}$$

"measure zero" in $\mathrm{SO}(3) \times \mathrm{SO}(3)$

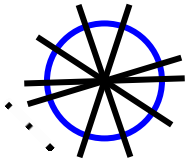
Fix $(f_\circ, g_\circ) \notin \bigcup_{w \in \mathbb{F}_2 \backslash \{1\}} \mathcal{Z}_w$.

# The rotations of 3-space

SO(3):={rotations in $\mathbb{R}^3$ about an axis through 0}

Every rotation has a unique axis, except the identity $I$.

$$\text{SPH} := \{x \in \mathbb{R}^3 \,|\, \text{dist}(x,0) = 1\}$$



countably many axes

$$\text{AXES} := \bigcup_{w \in \mathbb{F}_2 \setminus \{1\}} \text{axis of } w_{f_\circ g_\circ}$$

axis of $w_{f_\circ g_\circ}$
$\|$
{fixpoints of $w_{f_\circ g_\circ}$}

$$\forall w \in \mathbb{F}_2 \setminus \{1\}, \quad w_{f_\circ g_\circ} \neq I$$

$$\forall w \in \mathbb{F}_2 \setminus \{1\},$$
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# The rotations of 3-space

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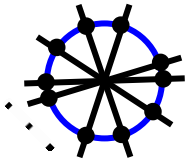

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$$B := \text{AXES} \cap \text{SPH} \qquad \text{countable}$$
$$= \bigcup_{w \in \mathbb{F}_2 \setminus \{1\}} \{\text{fixpoints of } w_{f_\circ g_\circ} : \text{SPH} \to \text{SPH}\}$$

$(f_\circ, g_\circ)$ is free on SPH$\setminus B$, and we can therefore color SPH$\setminus B$ . . .

$$\mathsf{SPH}\backslash B = R \amalg G \amalg P \amalg Y \amalg O$$

$$\mathsf{SPH} = B \amalg R \amalg G \amalg P \amalg Y \amalg O$$

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$$\mathrm{SPH} = B \coprod R \coprod G \coprod P \coprod Y \coprod O$$

$$\left.\begin{aligned} f_\circ^{-1}(R) &= R \coprod G \coprod Y \coprod O \\ g_\circ^{-1}(G) &= R \coprod G \coprod P \coprod O \\ f_\circ(P) &= Y \coprod G \coprod P \coprod O \\ g_\circ(Y) &= Y \coprod R \coprod P \coprod O \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} R &\equiv R \coprod G \coprod Y \coprod O \\ G &\equiv R \coprod G \coprod P \coprod O \\ P &\equiv Y \coprod G \coprod P \coprod O \\ Y &\equiv Y \coprod R \coprod P \coprod O \end{aligned}\right.$$

$$\mathrm{SPH} = B \coprod R \coprod G \coprod P \coprod Y \coprod O$$

$$R \equiv R \coprod G \coprod Y \coprod O$$

$$G \equiv R \coprod G \coprod P \coprod O$$

$$P \equiv Y \coprod G \coprod P \coprod O$$

$$Y \equiv Y \coprod R \coprod P \coprod O$$

$$\boxed{R} \equiv R \coprod G \coprod Y \coprod O$$

$$G \equiv R \coprod G \coprod P \coprod O$$

$$P \equiv Y \coprod G \coprod P \coprod O$$

$$Y \equiv Y \coprod R \coprod P \coprod O$$

$$\mathrm{SPH} \simeq B \amalg R \amalg G \amalg P \amalg Y \amalg O$$

$$\amalg \left[ R \amalg G \amalg Y \amalg O \right] 1$$

10 units along the positive $x$ axis

$$R \equiv R \amalg G \amalg Y \amalg O$$
$$G \equiv R \amalg G \amalg P \amalg O$$
$$P \equiv Y \amalg G \amalg P \amalg O$$
$$Y \equiv Y \amalg R \amalg P \amalg O$$

$$\mathrm{SPH} \sim B\amalg \qquad \boxed{G}\amalg\boxed{P}\amalg\boxed{Y}\amalg O$$

$$\amalg[R\amalg G\amalg Y\amalg O]_1$$

$$\amalg[\boxed{R\amalg G\amalg P}\amalg O]_2 \longleftarrow$$

$$\amalg[Y\amalg G\amalg P\amalg O]_3 \longleftarrow$$

$$\amalg[Y\amalg R\amalg P\amalg O]_4 \longleftarrow$$

$$R \equiv R\amalg G\amalg Y\amalg O$$

$$\boxed{\begin{aligned} G &\equiv R\amalg G\amalg P\amalg O \\ P &\equiv Y\amalg G\amalg P\amalg O \\ Y &\equiv Y\amalg R\amalg P\amalg O \end{aligned}}$$

$$\text{SPH} \sim B \amalg R \amalg G \amalg P \amalg O$$

$$\amalg[R \amalg G \amalg Y \amalg O]_1$$

$$\amalg[R \amalg G \amalg P \amalg O]_2$$

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$$R \equiv R \amalg G \amalg Y \amalg O$$
$$G \equiv R \amalg G \amalg P \amalg O$$
$$P \equiv Y \amalg G \amalg P \amalg O$$
$$Y \equiv Y \amalg R \amalg P \amalg O$$

$$\mathrm{SPH} \sim \boxed{B \amalg R \amalg G \amalg P \amalg Y \amalg O}$$

$$\amalg [R \amalg G \amalg Y \amalg O]_1$$

$$\amalg [\qquad\qquad O]_2$$

$$\amalg [Y \amalg G \amalg P \amalg O]_3$$

$$\amalg [Y \amalg R \amalg P \amalg O]_4$$

SPH $\sim$ SPH

$$\amalg[R\amalg G\amalg Y\amalg O]_1$$

$$\amalg[\qquad\qquad O]_2$$

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# SPH $\sim$ SPH

$$\amalg[R\amalg G\amalg Y\amalg O]_1P]_1$$

$$\amalg[\qquad\qquad O]_2$$

$$\amalg[\quad G\amalg P\amalg O]_3$$

$$\amalg[Y\amalg R\amalg P\amalg O]_4$$

$\mathrm{SPH} \sim \mathrm{SPH}$

$$\amalg [\ \mathrm{SPH} \backslash B\ ]_1$$

$$\amalg [\qquad\qquad O\ ]_2$$

$$\amalg [\qquad G \qquad \amalg O\ ]_3$$

$$\amalg [ Y \amalg R \amalg P \amalg O\ ]_4$$

SPH $\sim$ SPH

$$\amalg\,[\,\mathrm{SPH}\backslash B\,]_1$$

$$\amalg[\qquad\qquad O\,]_2$$

$$\amalg[\quad G\quad \amalg O\,]_3$$

$$\amalg[Y\amalg R\amalg P\amalg O\,]_4 G\,]_4$$

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$$\amalg\, [\, \mathrm{SPH} \backslash B \,]_1$$

$$\amalg [\qquad\qquad O \,]_2$$

$$\amalg [\qquad\qquad O \,]_3$$

$$\amalg [\mathrm{SPH} \backslash B]_4$$

Fix $b, b' \in B$.
works for any $b, b' \in B$

$\forall h \in \mathrm{SO}(3),\ h(b) \in \mathrm{SPH}$
$h(b) \neq b'$ a.s.
$\forall b, b' \in B, h_\circ(b) \neq b'$

$$\forall b, b' \in B,\ \mathcal{Z}_b^{b'} := \{h \in \mathrm{SO}(3) \,|\, h(b) = b'\}$$

"measure zero" in SO(3)

Fix $h_\circ \notin \bigcup_{b,b' \in B} \mathcal{Z}_b^{b'}$.

$$\mathrm{SPH} \sim \mathrm{SPH}$$

$$\amalg\, [\, \mathrm{SPH}\backslash B\, ]_1$$

$$\amalg [\qquad\qquad O\, ]_2$$

$$\amalg [\qquad\qquad O\, ]_3$$

$$\amalg\, [\mathrm{SPH}\backslash B]_4$$

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$$\forall b, b' \in B, h_\circ(b) \neq b'$$

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$h_\circ(B) \subseteq \mathrm{SPH}\backslash B$

$B \subseteq h_\circ^{-1}(\mathrm{SPH}\backslash B)$

Fix $h_\circ \notin \bigcup_{b,b' \in B} \mathcal{Z}_b^{b'}$.

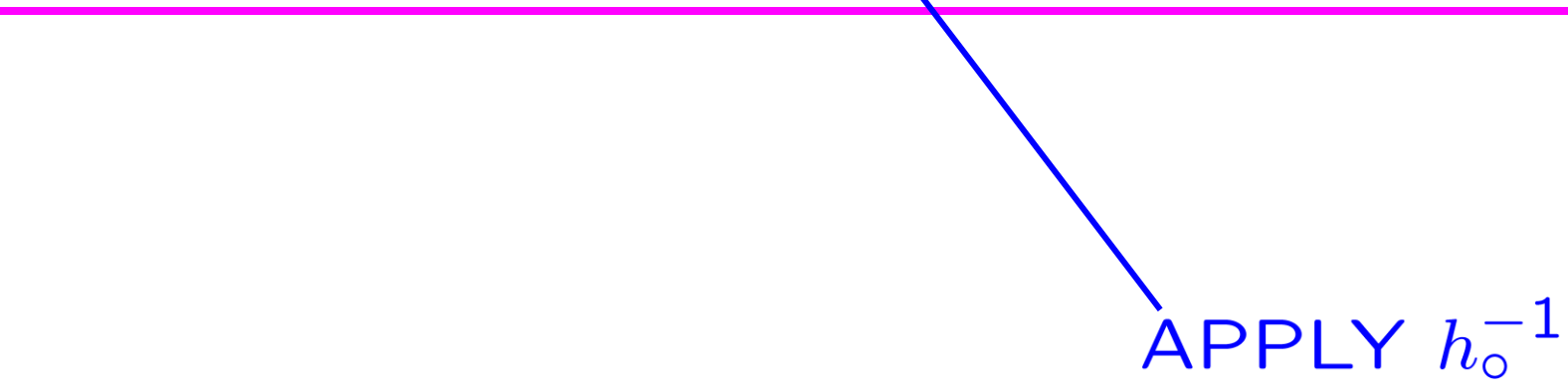
$\mathrm{SPH} \sim \mathrm{SPH}$

$$\amalg\, [\, \mathrm{SPH} \backslash B \,]_1$$

$$\amalg [\qquad\qquad O \,]_2$$

$$\amalg [\qquad\qquad O \,]_3$$

$$\amalg\, [\mathrm{SPH} \backslash B]_4$$




$B \subseteq h_\circ^{-1}(\mathrm{SPH} \backslash B)$

$\mathrm{SPH} \sim \mathrm{SPH}$

$$\amalg\, [\, \mathrm{SPH} \backslash B \,]_1$$

$$\amalg [\qquad\qquad O \,]_2$$

$$\amalg [\qquad\qquad O \,]_3$$

$$\amalg\, [\, h_\circ^{-1}(\mathrm{SPH} \backslash B) \,]_4$$

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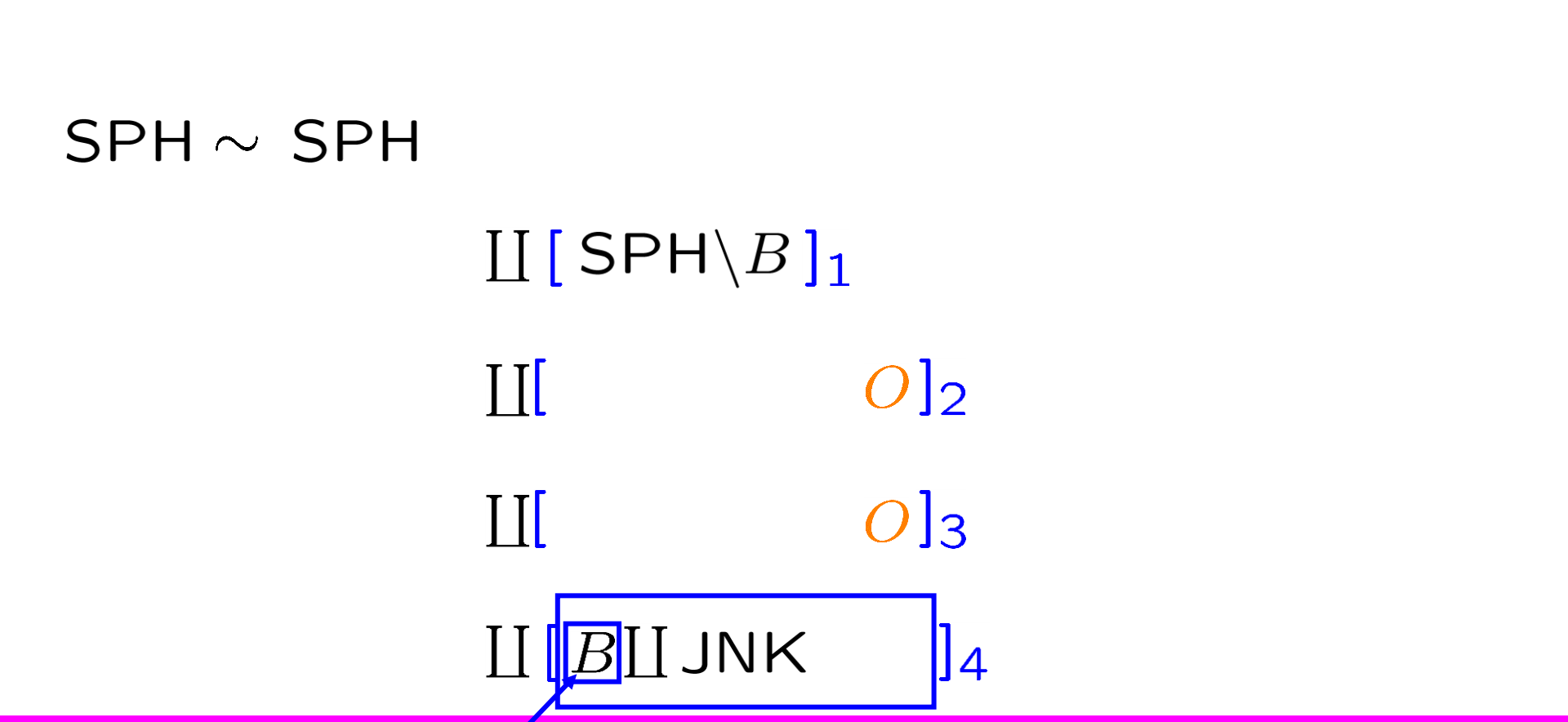
$\mathrm{SPH} \sim \mathrm{SPH}$

$$\amalg\, [\, \mathrm{SPH} \backslash B \,]_1$$

$$\amalg [\qquad\qquad O \,]_2$$

$$\amalg [\qquad\qquad O \,]_3$$

$$\amalg\, [\, B \amalg \mathrm{JNK} \,]_4$$



$$B \subseteq h_\circ^{-1}(\mathrm{SPH} \backslash B)$$

$\mathrm{SPH} \sim \mathrm{SPH}$

$$\amalg\, \boxed{[(\mathrm{SPH} \backslash B) \amalg B]_1}$$

$$\amalg [\qquad\qquad O]_2$$

$$\amalg [\qquad\qquad O]_3$$

$$\amalg\, [\, B \amalg \mathrm{JNK} \qquad ]_4$$

SPH $\sim$ SPH

$$\coprod \boxed{\mathsf{SPH}_1}$$

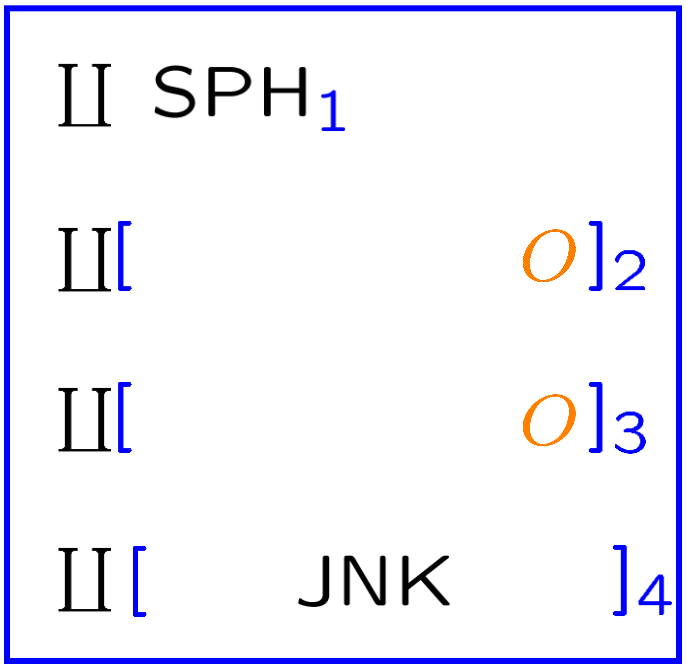
$$\coprod [\qquad\qquad O]_2$$

$$\coprod [\qquad\qquad O]_3$$

$$\coprod [\qquad \mathsf{JNK} \qquad]_4$$

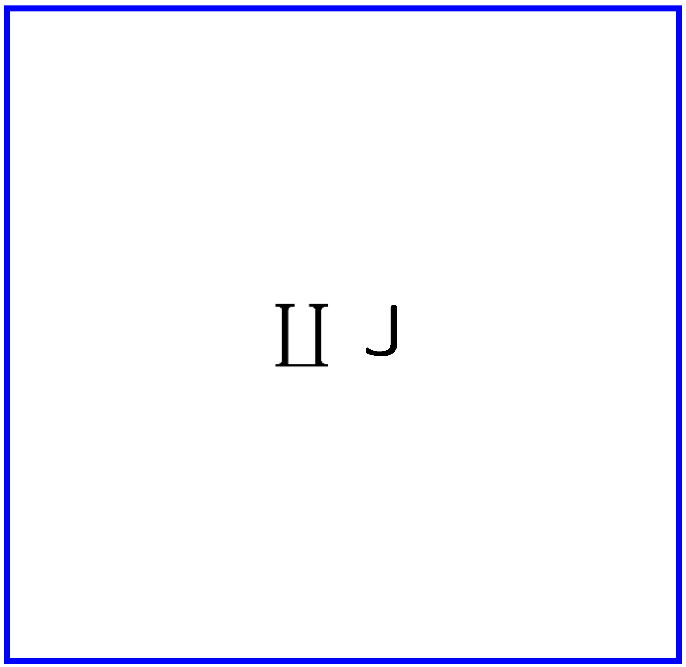
$\widehat{\mathrm{SPH}} \sim \widehat{\mathrm{SPH}}\mathrm{PH} \quad \coprod \quad \mathrm{SPH}_1$

$$\boxed{\begin{array}{l} \coprod \mathrm{SPH}_1 \\ \coprod [\qquad\qquad O]_2 \\ \coprod [\qquad\qquad O]_3 \\ \coprod [\quad \mathrm{JNK} \quad]_4 \end{array}}$$



# Sphere paradox

$$\mathrm{SPH} \sim \mathrm{SPH} \amalg \mathrm{SPH}_1 \amalg J$$



$$\mathrm{SPH} \sim \mathrm{SPH} \amalg \mathrm{SPH}_1 \amalg \mathrm{J}$$

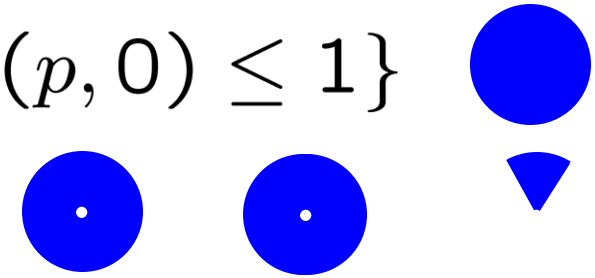
$$\mathrm{XBALL} \sim \mathrm{XBALL} \amalg \mathrm{XBALL}_1 \amalg \mathrm{J'} \qquad a, b \in J'$$

$$= \boxed{\mathrm{XBALL}} \amalg \boxed{\mathrm{XBALL}_1} \amalg \mathrm{J'}\backslash\{a, b\} \amalg \boxed{\{a\}} \amalg \boxed{\{b\}}$$

$$\mathrm{XBALL} \sim \mathrm{BALL} \amalg \mathrm{BALL}_1 \amalg \mathrm{J'}\backslash\{a, b\}$$

$$\boxed{\mathrm{BALL}} := \{p \in \mathbb{R}^3 \,|\, \mathrm{dist}(p, 0) \leq 1\}$$

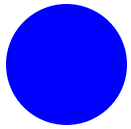
$$\boxed{\mathrm{XBALL}} := \mathrm{BALL}\backslash\{0\}$$



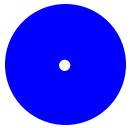
# Schroeder-Bernstein II

$$\begin{array}{ccccc} & & \mathrm{BALL} & & \\ & \subseteq\!\!\!/ & & \subseteq & \\ \mathrm{XBALL} \sim & \mathrm{BALL} & \amalg & \mathrm{BALL}_1 & \amalg\ \mathrm{J'}\backslash\{a,b\} \\ & \subseteq & & \subseteq\!\!\!/ & \\ & \mathrm{BALL} & \amalg & \mathrm{BALL}_1 & \end{array}$$

$\boxed{\mathrm{BALL}} := \{p \in \mathbb{R}^3 \mid \mathrm{dist}(p,0) \leq 1\}$



$\boxed{\mathrm{XBALL}} := \mathrm{BALL}\backslash\{0\}$



# Schroeder-Bernstein II

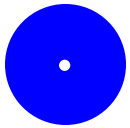
Similar argument works for any radius ball.

$$\begin{array}{ccccc} & & \boxed{\text{BALL}} & & \\ & \swarrow & & \searrow & \\ \text{XBALL} & \sim & \text{BALL} \amalg \text{BALL}_1 \amalg \text{J'}\backslash\{a,b\} & & \\ & \searrow & & \swarrow & \\ & & \boxed{\text{BALL} \amalg \text{BALL}_1} & & \end{array}$$

QED

$\boxed{\text{BALL}} := \{p \in \mathbb{R}^3 \,|\, \text{dist}(p,0) \leq 1\}$

$\boxed{\text{XBALL}} := \text{BALL}\backslash\{0\}$



# Concluding remarks

(1) Modeling is fraught!

(2) No Banach-Tarski paradox in
one or two dimensions.

"amenability"

(3) Banach-Tarski paradox persists in
all dimensions $\geq 3$.

"non-amenability"

Easier in four dimensions –
can make all orbits free,
avoiding the pesky black points.

(4) "Kazhdan's property"
(strengthening of non-amenability)
Ruciewicz problem
expander graphs

THANKS FOR YOUR ATTENTION!