Central Limit Theorem, Fourier Analysis and Finance
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Applied Coin-Flipping

\[ N = 10^{10^{100}} \]

\[ N \text{ coin flips} \quad \frac{H}{T}\text{ heads tails} \]

Male height (inches): \(69 + 5 \frac{H - T}{\sqrt{N}}\)

Probability that: \(69 - 5 \leq \text{ht} \leq 69 + 5?\)

\[ 69 - 5 \leq 69 + 5 \frac{H - T}{\sqrt{N}} \leq 69 + 5 \]

\[ -5 \leq 5 \frac{H - T}{\sqrt{N}} \leq 5 \]

DIVIDE BY 5

\[ -1 \leq \frac{H - T}{\sqrt{N}} \leq 1 \]
$N = 10^{10^{100}}$  

$N$ coin flips  

$H$ heads  

$T$ tails

Male height (inches): $69 + 5 \frac{H - T}{\sqrt{N}}$

Probability that: $69 - 5 \leq \text{ht} \leq 69 + 5$?

Probability that: $-1 \leq \frac{H - T}{\sqrt{N}} \leq 1$?
$N = 10^{10^{100}} \quad N \text{ coin flips} \quad \frac{H - T}{\sqrt{N}} \quad \text{heads tails}$

Male height (inches): $69 + 5 \cdot \frac{H - T}{\sqrt{N}}$

Probability that: $69 - 5 \leq \text{ht} \leq 69 + 5$?

Probability that: $-1 \leq \frac{H - T}{\sqrt{N}} \leq 1$? Answer: $\approx 68\%$

Grav accel (ft/sec$^2$): $32 + 10^6 \cdot \frac{H - T}{N}$

Probability that: $32 - \frac{10^9}{\sqrt{N}} \leq \text{acc} \leq 32 + \frac{10^9}{\sqrt{N}}$?

EXTREMELY small

Probability that: $-1000 \leq \frac{H - T}{\sqrt{N}} \leq 1000$?

EXTREMELY close to 100%
Applied Coin-Flipping

\[ N = \text{number of seconds in 30 days} \]

Current stock price: 1 USD

\[
x_+ := \begin{cases} 
  x, & \text{if } x \geq 0 \\
  0, & \text{if } x \leq 0 
\end{cases}
\]

\[ S := \text{stock price} \]

\[ 30 \text{ days from now} \]

Contract pays: \((S - 1)_+\) USD, \(30 \text{ days from now}\)

Expected payout?

Each second, price changes either by a factor of 1.0000035616 or by a factor of 0.9999964386.

50% chance of uptick,
50% chance of downtick.
Applied Coin-Flipping

Coin-flipping game: Flip a fair coin $N$ times. If $H$ heads and $T$ tails, pay $(u^H d^T - 1)_+$, 30 days from now.

Contract pays: $(S - 1)_+$ USD, 30 days from now

Expected payout?

Each second, price changes either by a factor of 1.0000035616 or by a factor of 0.999964386. 50% chance of uptick, 50% chance of downtick.
Applied Coin-Flipping

Coin-flipping game: Flip a fair coin \( N \) times. If \( H \) heads and \( T \) tails, pay \((u^H d^T - 1)_{+}\), 30 days from now.

Expected payout?

Computing probabilities is relatively easy, computing expected values is generally harder.
Compute the probability that
\[-1 < \frac{H - T}{\sqrt{N}} < 1.\]

\[X := (H - T)/\sqrt{N}\]

Compute the probability that
\[-1 < X < 1.\]

\[H_1 := \text{number of heads after first flip}\]
\[H_2 := \text{number of heads after second flip}\]
\[\vdots\]
\[H_N := \text{number of heads after } N\text{th flip} = H\]
Compute the probability that

\[-1 < \frac{H - T}{\sqrt{N}} < 1.\]

\[X := \frac{H - T}{\sqrt{N}}\]

Compute the probability that

\[-1 < X < 1.\]

\(X\) is hard . . .

For all integers \(j \in [1, N]\),

\[H_j := \text{number of heads after } j\text{th flip}\]
\[T_j := \text{number of tails after } j\text{th flip}\]
\[D_j := H_j - T_j\]

Easier: \(D_1, D_1/7, D_2, D_N\)

\[H = H_N, \quad T = T_N, \quad X = \frac{H_N - T_N}{\sqrt{N}} = \frac{D_N}{\sqrt{N}}\]
$D_1 = \begin{pmatrix} 0 \\ H_1 - T_1 \end{pmatrix}$: random variable

a variable whose value is determined by random events

distribution of $D_1$

distribution of $T_1 - H_1$

is exactly the same

keep the distribution

forget its origin
$D_1 = H_1 - T_1$

Cannot recover the random variable! Only its distribution.

What about $D_1/7$?

Fourier transform of the distribution of $D_1$ is $\cos t$

$i = \sqrt{-1}$

Replace $z$ by $e^{-\xi t}$

$T_1 - H_1$ has the same distribution.

Generating function:

Fourier transform:

$0.5 \times \begin{bmatrix} e^{it} \end{bmatrix} = \cos t + i \sin t$

$0.5 \times \begin{bmatrix} e^{-it} \end{bmatrix} = \cos t - i \sin t$

Keep the distribution, forget its origin

Repl. $t$ by $t/7$

Inverse Fourier transform
$D_{1/7}$:

- $0.5 \rightarrow 1/7 \rightarrow z^{1/7}$
- $0.5 \rightarrow -1/7 \rightarrow z^{-1/7}$

Generating function:

- $i = \sqrt{-1}$
- Replace $z$ by $e^{-it}$

Fourier transform:

- $(0.5)z^{1/7} + (0.5)z^{-1/7}$
- $(0.5)e^{-it/7} + (0.5)e^{it/7}$
- $\cos(t/7)$

- $e^{it/7} = \cos(t/7) + i \sin(t/7)$
- $e^{-it/7} = \cos(t/7) - i \sin(t/7)$
\[ D_2 = H_2 - T_2 \]

- \[ 2 \rightarrow 2 \rightarrow 0 \rightarrow -2 \]
- \[ 2 \rightarrow 0 \rightarrow 0.25 + 0.25 = 0.5 \]
- \[ 2 \rightarrow -2 \rightarrow 0.25 \]

*forget its origin* keep the distribution
\( D_2 = H_2 - T_2 \):

\[
\begin{array}{c|c|c}
0.25 & 2 & z^2 \\
0.5 & 0 & z^0 = 1 \\
0.25 & -2 & z^{-2}
\end{array}
\]

*forget its origin keep the distribution*

**Generating function:**

\[
(0.25)z^2 + 0.5 + (0.25)z^{-2}
\]

\[= ((0.5)z + (0.5)z^{-1})^2\]

the generating function of the distribution of \( D_1 \)

\[i = \sqrt{-1}\]

Replace \( z \) by \( e^{-it} \)

**Fourier transform:**

\[(\cos t)^2 = \cos^2 t\]
$$D_N = H_N^T T_N$$

divide by $\sqrt{N}$

**Goal:**

What about $D_N/\sqrt{N}$?
Replace $t$ by $t/\sqrt{N}$.

Generating function:

**NO WAY!!**

$$= (\frac{1}{2}z + \frac{1}{2}z^{-1})^N$$

the generating function of the distribution of $D_1$

Fourier transform:

$$i = \sqrt{-1}$$

Replace $z$ by $e^{-it}$

$$\cos t)^N = \cos^N t$$
$X = \frac{D_N}{\sqrt{N}}$:

**Goal:** $X \approx \frac{D_N}{\sqrt{N}}$?
Replace $t$ by $t/\sqrt{N}$.

**NO WAY!!**

Fourier transform: $\cos^N(t/\sqrt{N})$
\[ X = \frac{D_N}{\sqrt{N}} : \]

Generating functions
Fourier transforms

\[ \text{Fourier transform: } \cos^N(t/\sqrt{N}) \]

\[ \text{Fourier transform: } \cos^N(t/\sqrt{N}) \]
\[ X = D_N / \sqrt{N} : \]

**The problem:**
Compute the probability that 
\(-1 < X < 1.\)

**Exercise:** \( \lim_{n \to \infty} \cos^n \left( \frac{3}{\sqrt{n}} \right) = e^{-3^2/2} \)

**Fourier transform:** 
\( \cos^N \left( \frac{t}{\sqrt{N}} \right) \)

\( \approx \lim_{n \to \infty} \cos^n \left( \frac{t}{\sqrt{n}} \right) \equiv e^{-t^2/2} \)

Verify for \( t = 3. \)
\[ X = \frac{D_N}{\sqrt{N}} : \]

Fourier transform: \[ \cos^N(t/\sqrt{N}) \]

\[ \approx \lim_{n \to \infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2} \]

Fourier transform: \[ \cos^N(t/\sqrt{N}) \]

\[ \approx \lim_{n \to \infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2} \]
\[ X = \frac{D_N}{\sqrt{N}} : \]

Fourier transform: \( \cos^N(t/\sqrt{N}) \)

\[ \approx \lim_{n \to \infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2} \]

Key idea of Central Limit Theorem:
Let \( Z \) have distr. with Fourier transf. \( e^{-t^2/2} \).
Then \( Z \) is “close” to \( X \).

The problem:
Compute the probability that \( -1 < X < 1 \).

Approximately equal to the probability that \( -1 < Z < 1 \).
NOTES
Mistake:
\[
\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi}
\]

\[D_2 \in \{2, 0, -2\}\]
distribution supported on three points

\[D_N \in \{-N, -N + 2, \ldots, N - 2, N\}\]
distribution supported on \(N + 1\) points

By contrast, the distribution of \(Z\) does not have finite support.
\[ Z: \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \quad \text{for all } x \in \mathbb{R} \]

Problem: Compute the probability that \( Z = 7 \)

Solution: \[ \int_{-7}^{7} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = 0 \]
Z:
\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} \, dx \]

Do this for all \( x \in \mathbb{R} \)

Problem: Compute the probability that \( 2 < Z < 3 \)

Solution:
\[ \int_{2}^{3} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = [\Phi(x)]_{x=2}^{x=3} \]
\[ = \Phi(3) - \Phi(2) = 0.0214 \]
\[ = 2.14\% \]
Z:
\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \]
\[ x \quad z^x \]
Do this for all \( x \in \mathbb{R} \)

Generating function:
\[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \text{Exercise} \]

Fourier transform:
\[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-itx} e^{-x^2/2} \, dx = e^{-t^2/2} \]

Verify for \( t = 3i \).

Key idea of Central Limit Theorem:
Let \( Z \) have distr. with Fourier transf. \( e^{-t^2/2} \).
Then \( Z \) is “close” to \( X \).
\[ Z: \quad \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \]

Do this for all \( x \in \mathbb{R} \)

Exercise:
\[
\int_{-\infty}^{\infty} e^{3x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = e^{3^2/2}
\]

Fourier transform:
\[
\int_{-\infty}^{\infty} e^{-itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = e^{-t^2/2}
\]

Verify for \( t = 3i \).

Key idea of Central Limit Theorem:
Let \( Z \) have distr. with Fourier transf. \( e^{-t^2/2} \).
Then \( Z \) is “close” to \( X \).
The problem:
Compute the probability that
\(-1 < X < 1.\)

Approximately equal to the probability that
\(-1 < Z < 1.\)

Approximate solution:
\[
\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \left[ \Phi(x) \right]_{x=-1}^{x=1} = 68.27\%.
\]
Goal:
Compute the expected value of $f(u^H d^T)$.

Coin-flipping game: Flip a fair coin $N$ times. If $H$ heads and $T$ tails, pay $(u^H d^T - 1)_+$, 30 days from now.

$$f(x) = (x - 1)_+$$
\[ f(x) = (x - 1)_+ \]

**Goal:**
Compute the expected value of \( f(u^H d^T) \).

**New easier problem:**
Compute the expected value of \( f(D_2) \).

\[ D_2 = H_2 - T_2 : \]

\[
\begin{array}{c|ccc}
 & 2 & f(2) & f(2) \\
\hline
0.25 & 0 & f(0) & f(0) \\
0.5 & -2 & f(-2) & f(-2) \\
0.25 & & & \\
\end{array}
\]

\[
[0.25][f(2)] + [0.5][f(0)] + [0.25][f(-2)] = 0.25
\]
Define: \( g(x) = 5e^x + x^2 \)

Goal:
Compute the expected value of \( f(u^H d^T) \).

New easier problem:
Compute the expected value of \( g(D_2) \).

\[
D_2 = H_2 - T_2 : \\
\begin{array}{c|c|c}
0.25 & 2 & g(2) \\
0.5 & 0 & g(0) \\
0.25 & -2 & g(-2) \\
\end{array}
\]

\[
[0.25][g(2)] + [0.5][g(0)] + [0.25][g(-2)] = \text{Exercise}
\]
Recall: \[ f(x) = (x - 1)_+ \]

Goal:
Compute the expected value of \( f(u^H d^T) \).

New easier problem:
Compute the expected value of \( f(Z) \).

\[ Z: \quad \frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} \, dx \quad \left| \begin{array}{c} x \quad f(x) \end{array} \right. \quad \text{Do this for all } x \in \mathbb{R} \]

\[ \int_\infty^{-\infty} [f(x)] \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)] e^{-x^2/2} \, dx = \text{exercise} \]
Recall: \[ f(x) = (x - 1)_+ \]

Goal:
Compute the expected value of \( f(u^H d^T) \).

New easier problem:
Compute the expected value of \( f(X) \).

\[
\begin{align*}
\mathbb{E} \left[ \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right] & \bigg| x \quad f(x) \\
\int_{-\infty}^{\infty} \left[ f(x) \right] \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx & = \text{exercise}
\end{align*}
\]
Recall: \[ f(x) = (x - 1)_+ \]

Goal:

Compute the expected value of \( f(u^H d^T) \).

New easier problem:

Compute the expected value of \( g(X) \).

Approx. Sol’n: \[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx \]
Recall: \[ f(x) = (x - 1) \]  
Goal: 
Compute the expected value of \( f(u^H d^T) \).

New easier problem:  
Compute the expected value of \( g(X) \).

\[
X = \frac{(H - T)}{\sqrt{N}} \times \sqrt{N}
\]

\[
H + T = N
\]

\[
H - T = \frac{X}{\sqrt{N}}
\]

\[
2H = N + X \sqrt{N}
\]

\[
2T = N - X \sqrt{N}
\]

Approx. Sol’n: \[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx
\]
Recall: \( f(x) = (x - 1)_+ \)

Goal:
Computing the expected value of \( f(u^H d^T) \).

New easier problem:
Computing the expected value of \( g(X) \).

\[
H = \frac{N}{2} + X \sqrt{N} / 2 \\
T = \frac{N}{2} - X \sqrt{N} / 2 \\
\]

\[
u^H = u^{N/2} u X \sqrt{N} / 2 \\
d^T = d^{N/2} d^{-X \sqrt{N} / 2} \\
\]

\[
u^H d^T = \sum_{u} 30 \times 24 \times \sum_{u} 60 \times 60 = 2,592,000 \\
2H = N + X \sqrt{N} \\
2T = N - X \sqrt{N} \\
\]

Approx. Sol’n:
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx
\]
Recall: \( f(x) = (x - 1)_+ \)

Goal:

Compute the expected value of \( f(u^H d^T) \).

New easier problem:

Compute the expected value of \( g(X) \).

\[
\begin{align*}
H &= N/2 + X \sqrt{N}/2 \\
T &= N/2 - X \sqrt{N}/2 \\

u^H &= u^{N/2} u^{X \sqrt{N}/2} \\
d^T &= d^{N/2} d^{-X \sqrt{N}/2} \\

u^H d^T &= (u d)^{N/2} \\
&= (u/d)^{X \sqrt{N}/2} \\
C &:= (u/d)^{N/2} \\
k &:= \ln((u/d)^{\sqrt{N}/2}) \\

\text{Approx. Sol'n:} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx
\end{align*}
\]
Recall: \[ f(x) = (x - 1)_+ \]

Goal:

Compute the expected value of \( f(u^H d^T) \).

New easier problem:

Compute the expected value of \( g(X) \).

\[
\begin{align*}
    f(u^H d^T) &= f(C e^{kX}) = g(X) \\
    u^H d^T &= u^{N/2} d^{N/2} u X \sqrt{N}/2 d^{-X} \sqrt{N}/2 \\
    &= (ud)^{N/2} (u/d) X \sqrt{N}/2 \\
    &= C e^{kX} \\
    C &:= (ud)^{N/2} \\
    k &:= \ln((u/d) \sqrt{N}/2)
\end{align*}
\]

Approx. Sol’n:

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx
\]
Recall: \( f(x) = (x - 1)_+ \)

Goal:
Compute the expected value of \( f(u^H d^T) \).

Restatement of goal:
Compute the expected value of \( g(X) \).

\[
\begin{align*}
  f(u^H d^T) &= f(C e^{kX}) = g(X) \\
  u^H d^T &= u^{N/2} d^{N/2} u^{X \sqrt{N}/2} d^{-X \sqrt{N}/2} \\
  &= (ud)^{N/2} (u/d)^{X \sqrt{N}/2} C := (ud)^{N/2} \\
  &= C e^{kX} \\
  k &:= \ln((u/d)^{\sqrt{N}/2})
\end{align*}
\]

Approx. Sol’n:
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx
\]
Recall: \( f(x) = (x - 1)_+ \)

\[
g(x) := f(Ce^{kx}) = (Ce^{kx} - 1)_+
\]

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1)_+ e^{-x^2/2} \, dx
\]

\[
N = 2,592,000
\]

\[
C := (ud)^{N/2}
\]

\[
k := \ln\left(\frac{u}{d}\right)\sqrt{\frac{N}{2}}
\]

Approx. Sol’n:

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx
\]
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1) + e^{-x^2/2} \, dx
\]
\[
{1 \over \sqrt{2\pi}} \int_{-\infty}^{\infty} \left( C e^{kx} - 1 \right) e^{-x^2/2} \, dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (C e^{kx} - 1) e^{-x^2/2} \, dx
\]

\[
Ce^{ka} - 1 = 0
\]

\[
Ce^{ka} = 1
\]

\[
e^{ka} = 1/C
\]

\[
ka = \ln(1/C) = -\ln C
\]

\[
a = -(\ln C)/k
\]
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1) + e^{-x^2/2} \, dx
\]

\[= \frac{1}{\sqrt{2\pi}} \left[ \int_{a}^{\infty} (Ce^{kx} - 1) \, dx + \int_{a}^{\infty} e^{-x^2/2} \, dx \right]
\]

\[= \frac{1}{\sqrt{2\pi}} \left[ C \int_{a}^{\infty} e^{kx} \, dx - \int_{a}^{\infty} e^{-x^2/2} \, dx \right]
\]

\[a = -(\ln C)/k\]
\[
\frac{1}{\sqrt{2\pi}} \left[ C \int_{a}^{\infty} e^{kx} e^{-x^2/2} \, dx - \int_{a}^{\infty} e^{-x^2/2} \, dx \right]
\]

\[
\sqrt{2\pi} \left[ 1 - \Phi(a) \right]
\]

\[
= \frac{1}{\sqrt{2\pi}} \left[ C \int_{a}^{\infty} e^{kx} e^{-x^2/2} \, dx - \int_{a}^{\infty} e^{-x^2/2} \, dx \right]
\]

\[
a = -\left( \ln C \right)/k
\]
\[
= \frac{1}{\sqrt{2\pi}} \left[ C \int_a^\infty e^{kx} e^{-x^2/2} \, dx - \int_a^\infty e^{-x^2/2} \, dx \right]
\]

\[
= \int_{a-k}^\infty e^{k(x+k)} e^{-(x+k)^2/2} \, dx
\]

\[
e^{k^2/2} \int_{a-k}^\infty e^{-x^2/2} e^{-k^2/2} \, dx
\]

\[
\sqrt{2\pi} \Phi(-a)
\]

\[
a = -(\ln C)/k
\]
\[
= \frac{1}{\sqrt{2\pi}} \left[ C \int_a^{\infty} e^{kx} e^{-x^2/2} \, dx - \int_a^{\infty} e^{-x^2/2} \, dx \right]
\]

\[
= \int_a^{-a - k} e^{k(x+k)} e^{-(x+k)^2/2} \, dx \quad \left( e^{k^2/2} e^{-x^2/2} e^{-k^2/2} \right)
\]

\[
= \sqrt{2\pi} \Phi(-a)
\]

\[
a = -(\ln C)/k
\]
\[
= \frac{1}{\sqrt{2\pi}} \left[ C \int_{a}^{\infty} e^{kx} e^{-x^2/2} \, dx - \int_{a}^{\infty} e^{-x^2/2} \, dx \right] + \sqrt{2\pi} \Phi(-a)
\]

\[
e^{k^2/2} \sqrt{2\pi} \Phi(k - a)
\]

\[
a = -(\ln C)/k
\]
\[ = \frac{1}{\sqrt{2\pi}} \left[ C \int_a^\infty e^{kx} e^{-x^2/2} \, dx \right. \left. - \int_a^\infty e^{-x^2/2} \, dx \right] \]
\[ = e^{k^2/2} \sqrt{2\pi} \Phi(k-a) - \sqrt{2\pi} \Phi(-a) \]
\[ = 0.024214088 \]

\[ a = -0.01653528434 \]
\[ k = 0.0573390439 \]
\[ C = 1.000948567 \]
\[ a = -(\ln C)/k \]
SUMMARY:
Coin flipping problems are tractable via CLT, and useful in many applied settings, in particular, finance.