Central Limit Theorem and Finance
Duluth 10 November 2008
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Kyle wants right, but not obligation, to buy 5000 shares of ABC for $5000, 30 days from now.  

Assume:  
Spot price = $1/share.  
Each second, price changes either by a factor of 1.000035616 or by a factor of 0.999964386.  

The one-second risk-free factor is 1.000000001.  

Goal:  
Find the “right” price, i.e., the price that can be used to set up a “perfect hedge”.  

Difficulty: $30 \times 24 \times 60 \times 60$ adjustments  

Salvation: The Central Limit Theorem!
Kyle wants right, but not obligation, to buy 5000 shares of ABC for $5000, Gail, seller 30 days from now. Call option

Assume: Spot price = $1/share.

Each second, price changes either by a factor of 1.000035616 or by a factor of 0.999964386.

\[ 1 + \iota = \rho \]

The one-second risk-free factor \( \iota := 0.000000001 = \rho - 1 \) is 1.000000001.

Goal: Find the “right” price, i.e., the price that can be used to set up a “perfect hedge”.

Payoff function:

\[ f(S) = (5000S - 5000)_+ \]

Exercise: Graph \( f \).

\[ N := 30 \times 24 \times 60 \times 60 = 2,592,000 \]
\[ 1 + \iota = \rho \]

**Expected value:**
- Downside factor: \( 50\% \times \rho S \)
- Upside factor: \( 50\% \times (\rho - 1)S \)

**Expected return:**
- Downside factor: \( \frac{\rho S}{S} \times \iota S \)
- Upside factor: \( \frac{\rho L}{L} \times \iota L \)

**Risk-free factor:**
- \( \rho L - L \)
- \( (\rho - 1)L \)

**Risk-neutral probability:**
- Downside factor: \( 50\% \)
- Upside factor: \( 50\% \)

**Real number line:**
- \( d \)
- \( u \)
ABC share price:

\[ f(S) = (5000S - 5000) + \]

Ending ABC share price:

\[
\begin{align*}
&u^N \\
u^{N-1}d \\
u^{N-2}d^2 \\
 & \vdots \\
d^N
\end{align*}
\]

Contingent claim:

\[
\begin{align*}
f(u^N) \\
f(u^{N-1}d) \\
f(u^{N-2}d^2) \\
 & \vdots \\
f(d^N)
\end{align*}
\]
ABC share price:

\[ f(S) = (5000S - 5000)_{+} \]

\[ P := \text{price of option} \]
\[ = \text{initial value of hedging portfolio} \]
\[ 1 + \nu = \rho \]

\[ \rho^N P = (1 + \nu)^N P = \]
\[ \text{expected final value of hedging portfolio} = \]
\[ \text{expected contingent claim} \]

Contingent claim:

\[ f(u^N) \]
\[ f(u^{N-1}d) \]
\[ f(u^{N-2}d^2) \]
\[ \vdots \]
\[ f(d^N) \]
**Coin-flipping game:** Flip a fair coin \( N \) times. If \( H \) heads and \( T \) tails, pay \( f(u^H d^T) \), 30 days from now.

\[
\begin{align*}
\rho^{-N} P & = \text{expected payout} =: E \\
P & = \rho^{-N} E_{\text{equal}} (1 + \nu)^{-N} E \\
f(S) & = (5000S - 5000)_+ \\
\end{align*}
\]

**Goal**
- \( P \) := price of option
- Initial value of hedging portfolio

\[
\begin{align*}
\rho^N P & = (1 + \nu)^N P = \\
& = \text{expected final value of hedging portfolio} = \\
& = \text{expected contingent claim}
\end{align*}
\]

**Contingent claim:**
\[
\begin{align*}
f(u^N) & \quad \quad \quad N, 0 \\
f(u^{N-1}d) & \quad \quad \quad N - 1, 1 \\
f(u^{N-2}d^2) & \quad \quad \quad N - 2, 2 \\
\vdots & \quad \quad \quad \vdots \\
f(d^N) & \quad \quad \quad 0, N
\end{align*}
\]
Coin-flipping game: Flip a fair coin \( N \) times. If \( H \) heads and \( T \) tails, pay \( f(u^H d^T) \), 30 days from now.

\[
1 + \nu = \rho \\
\rho^N P = \text{expected payout} =: E = ??? \\
P = \rho^{-N} E = (1 + \nu)^{-N} E \\
\text{discounted expected payout}
\]

\[
\begin{array}{l}
P := \text{price of option} \\
\quad = \text{initial value of hedging portfolio} \\
\rho^N P = (1 + \nu)^N P = \text{expected final value of hedging portfolio} = \text{expected contingent claim}
\end{array}
\]

Contingent claim:
\[
\begin{array}{l}
f(u^N) \quad N, \; 0 \\
f(u^{N-1} d) \quad N - 1, \; 1 \\
f(u^{N-2} d^2) \quad N - 2, \; 2 \\
\vdots \\
f(d^N) \quad 0, \; N
\end{array}
\]

Hard problem: expected value problem
Easier problem:

Compute the probability that

\[-\sqrt{N} < H - T < \sqrt{N}.\]

DIVIDE BY $\sqrt{N}$

\[X := (H - T)/\sqrt{N}\]

Easier problem after restatement:

Compute the probability that

\[-1 < X < 1.\]

\[H_1 := \text{number of heads after first flip}\]
\[H_2 := \text{number of heads after second flip}\]
\[\vdots\]
\[H_N := \text{number of heads after } N\text{th flip} = H\]
Easier problem:
Compute the probability that
\[ -\sqrt{N} < H - T < \sqrt{N}. \]

\[ X := (H - T)/\sqrt{N} \]

Easier problem after restatement:
Compute the probability that
\[ -1 < X < 1. \quad X \text{ is hard} \ldots \]

For all integers \( j \in [1, N] \),
\[ H_j := \text{number of heads after } j\text{th flip} \]
\[ T_j := \text{number of tails after } j\text{th flip} \]
\[ D_j := H_j - T_j \quad \text{Easier: } D_1, D_1/7, D_2, D_N \]

\[ H = H_N, \quad T = T_N, \quad X = (H_N - T_N)/\sqrt{N} = D_N/\sqrt{N} \]
$D_1 = H_1 - T_1$:

A random variable whose value is determined by random events.

(probablity) measure of $D_1$

(probablity) distribution of $D_1$

distribution of $T_1 - H_1$ is exactly the same

keep the distribution
forget its origin
\[ e^{it} = \cos t + i \sin t \]

Keep the distribution for \( it = \cos t - i \sin t \)

Repl. \( t \) by \( \frac{t}{7} \)

Inverse Fourier transform

Generating function:

\[ (0.5)z^2 + (0.5)^{-1} \]

Replace \( z \) by \( e^{it} \)

What about \( D_1/7 \)?

D_1 = \( H_1 \)

Cannot recover the Fourier transform of \( D_1 \)

\( \xi \neq \frac{7}{z} \) not time

\[ i = \sqrt{-1} \]

\( z^{-1} \)

\( z \)

Divide by 7

Distribution of \( D_1 \)

Is cost
What about $D_{1/7}$?

Replace $t$ by $t/7$.

\[ i = \sqrt{-1} \]

Replace $z$ by $e^{-it}$

Generating function:

\[ (0.5)z^{1/7} + (0.5)z^{-1/7} \]

Fourier transform:

\[ (0.5)e^{-it/7} + (0.5)e^{it/7} \]

\[ \| \cos(t/7) \]

\[ e^{it/7} = \cos(t/7) + i \sin(t/7) \]

\[ e^{-it/7} = \cos(t/7) - i \sin(t/7) \]
\[ D_2 = H_2 - T_2 : \]

\[
\begin{array}{c|c}
2 & 0.25 \\
0 & 0.25 + 0.25 = 0.5 \\
-2 & 0.25 \\
\end{array}
\]

*forget its origin keep the distribution*
$D_2 = H_2 - T_2$:

\[
\begin{array}{c|cc}
0.25 & 2 & z^2 \\
0.5 & 0 & z^0 = 1 \\
0.25 & -2 & z^{-2}
\end{array}
\]

*forget its origin keep the distribution*

**Generating function:**

\[
(0.25)z^2 + 0.5 + (0.25)z^{-2} = ((0.5)z + (0.5)z^{-1})^2
\]

*the generating function of the distribution of $D_1$*

\[
i = \sqrt{-1}
\]

*Replace $z$ by $e^{-it}$*

**Fourier transform:**

\[(\cos t)^2 = \cos^2 t\]
\[ D_N = \frac{H_N - T_N}{\sqrt{N}} : \]

**Goal:**

What about \( D_N/\sqrt{N} \)? Replace \( t \) by \( t/\sqrt{N} \).

**Generating function:**

\[
= ((0.5)z + (0.5)z^{-1})^N
\]

*the generating function of the distribution of \( D_1 \)*

**Fourier transform:**

\[
(i = \sqrt{-1}) \quad \text{Replace } z \text{ by } e^{-it}
\]

\[
(cos \, t)^N = \cos^N t
\]
$X = D_N / \sqrt{N}$:

Goal: $X \overset{?}{=} D_N / \sqrt{N}$?

Replace $t$ by $t / \sqrt{N}$.

Fourier transform: $\cos^N(t / \sqrt{N})$
\[ X = \frac{D}{\sqrt{N}} \cdot \sqrt{\omega} \]

\[ \cos^N \left( \frac{t}{\sqrt{N}} \right) \]

\[ \text{Generating functions}\]

\[ \text{Fourier transforms} \]
\[ X = \frac{D_N}{\sqrt{N}} : \]

Easier problem after restatement:
Compute the probability that \(-1 < X < 1\).

Exercise: \( \lim_{n \to \infty} \cos^n \left( \frac{3}{\sqrt{n}} \right) = e^{-3^2/2} \)

Fourier transform: \( \cos^N \left( \frac{t}{\sqrt{N}} \right) \approx \lim_{n \to \infty} \cos^n \left( \frac{t}{\sqrt{n}} \right) = e^{-t^2/2} \)

\[ \text{Verify for } t = 3. \]
\[ X = D_N / \sqrt{N} \]

Fourier transform: \( \cos^N(t/\sqrt{N}) \)

\[ \approx \lim_{n \to \infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2} \]

Fourier transform: \( \cos^N(t/\sqrt{N}) \)

\[ \approx \lim_{n \to \infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2} \]
$X = \frac{D_N}{\sqrt{N}}$:

Fourier transform: \[ \cos^N\left(\frac{t}{\sqrt{N}}\right) \]

\[ \approx \lim_{n \to \infty} \cos^n\left(\frac{t}{\sqrt{n}}\right) = e^{-t^2/2} \]

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**Key idea of Central Limit Theorem:**

Let $Z$ have distr. with Fourier transf. $e^{-t^2/2}$. Then $Z$ is “close” to $X$.

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**Easier problem after restatement:**

Compute the probability that $-1 < X < 1$.

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Compute the probability that $-1 < Z < 1$. 
\[ Z: \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \]

\( x \)

Do this for all \( x \in \mathbb{R} \)

\exists \text{ RV } \mathcal{Z} \text{ with this dist.}

**NOTES**

Mistake:

\[ \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi} \]

\( D_2 \in \{2, 0, -2\} \)

distribution supported on three points

\( D_N \in \{-N, -N + 2, \ldots, N - 2, N\} \)

distribution supported on \( N + 1 \) points

By contrast, the distribution of \( Z \)
does not have finite support.
Problem: Compute the probability that $Z = 7$

Solution: $\int_{-\infty}^{7} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = 0$
Problem: Compute the probability that $2 < Z < 3$

Solution: \[ \int_{2}^{3} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = [\Phi(x)]_{x=2}^{x=3} \]
\[ = \Phi(3) - \Phi(2) = 0.0214 \]
\[ = 2.14\% \]
\[ Z: \quad \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \quad \rightarrow \quad x \quad \text{Do this for all } x \in \mathbb{R} \]

Generating function:

\[ \int_{-\infty}^{\infty} z^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \text{Exercise} \]

Fourier transform:

\[ \int_{-\infty}^{\infty} e^{-itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = e^{-t^2/2} \]

Key idea of Central Limit Theorem:

Let \( Z \) have distr. with Fourier transf. \( e^{-t^2/2} \). Then \( Z \) is "close" to \( X \).
\[ Z: \quad \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \quad \text{Do this for all } x \in \mathbb{R} \]

Exercise:
\[ \int_{-\infty}^{\infty} e^{-3ix} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = e^{-3^2/2} \]

Fourier transform:
\[ \int_{-\infty}^{\infty} e^{-itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = e^{-t^2/2} \]

**Verify for** \( t = 3 \).

Key idea of Central Limit Theorem:
Let \( Z \) have distr. with Fourier transf. \( e^{-t^2/2} \).
Then \( Z \) is “close” to \( X \).
Easier problem after restatement:
Compute the probability that $-1 < X < 1$.

Approximately equal to the probability that $-1 < Z < 1$.

Approximate solution:
$$
\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = [\Phi(x)]_{x=-1}^{x=1} = 68.27\%$

Berry-Esseen Theorem
Goal:
Compute the expected value of $f(u^H d^T)$.

Coin-flipping game: Flip a fair coin $N$ times. If $H$ heads and $T$ tails, pay $f(u^H d^T)$, 30 days from now.

$\rho^N P = \text{expected payout} =: E = ???$

$P = \rho^{-N} E = (1 + \nu)^{-N} E$

$\nu := 0.000000001$
\( f(S) = (5000S - 5000)_{+} \)

Goal:
Compute the expected value of \( f(u^H d^T) \).

New easier problem:
Compute the expected value of \( f(D_2) \).

\[ D_2 = H_2 - T_2 \]

\[
\begin{array}{c|c}
0.25 & 2 \\
0.5 & 0 \\
0.25 & -2 \\
\end{array}
\]

\[ f :\rightarrow g \]
works for any function

\[
[0.25][f(2)] + [0.5][f(0)] + [0.25][f(-2)] = 1,250
\]
Define: \( g(S) = 5e^K + S^2 \)

Goal:
Compute the expected value of \( f(u^H d^T) \).

New easier problem:
Compute the expected value of \( g(D_2) \).

\[
D_2 = H_2 - T_2 : \\
\begin{array}{ccc}
0.25 & 2 & g(2) \\
0.5 & 0 & g(0) \\
0.25 & -2 & g(-2) \\
\end{array}
\]

\[
[0.25][g(2)] + [0.5][g(0)] + [0.25][g(-2)] = \text{Exercise}
\]
Recall: \( f(S) = (5000S - 5000)_+ \)

Goal:
Compute the expected value of \( f(u^H d^T) \).

New easier problem:
Compute the expected value of \( f(Z) \).

\[ Z: \quad \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \quad \int_{-\infty}^{\infty} [f(x)] \, e^{-x^2/2} \, dx \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)] \, e^{-x^2/2} \, dx \quad f(x) = (5000x - 5000)_+ \]

Do this for all \( x \in \mathbb{R} \) = exercise
Recall: \( f(S) = (5000S - 5000)_+ \)

Goal:
Compute the expected value of \( f(u^H d^T) \).

New easier problem:
Compute the expected value of \( f(X) \).

\[
\mathbb{E}_X \left[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \right] f(x) = \text{Do this for all } x \in \mathbb{R}
\]

\[
\int_{-\infty}^{\infty} \left[ f(x) \right] \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = f(x) = (5000x - 5000)_+
\]

Approx. Sol’n: \( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)] e^{-x^2/2} \, dx = \text{exercise}_{32} \)
Recall: \( f(S) = (5000S - 5000) + \)

Goal:
Compute the expected value of

New easier problem:
Compute the expected value of

Approx. Sol’n: \( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} dx \)
Recall: \( f(S) = (5000S - 5000) \)

Goal:
Compute the expected value of \( f(u^H d^T) \).

New easier problem:
Compute the expected value of \( g(X) \).

\[
X = \frac{(H - T)}{\sqrt{N}} \times \sqrt{N} \quad \text{add} \quad H + T = N \quad \text{and} \quad H - T = X \sqrt{N}
\]

\[
2H = N + X \sqrt{N} \quad \text{add} \quad 2T = N - X \sqrt{N}
\]

Approx. Sol’n:
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx
\]
Recall: $f(S) = (5000S - 5000) + X$.

Goal:
Compute the expected value of $f(u^H d^T)$. 

New easier problem:
Compute the expected value of $g(X)$. 

\[
H = N/2 + X \sqrt{N}/2 \\
T = N/2 - X \sqrt{N}/2 \\
u^H = u^{N/2} u X \sqrt{N}/2 \\
d^T = d^{N/2} d^{-X \sqrt{N}/2} \\
u^H d^T = N^2 \Rightarrow 30 \times 24 \times 60 \times 60 \times 60 = 2,592,000 \\
2H = N + X \sqrt{N} \\
2T = N - X \sqrt{N}
\]

Approx. Sol’n: \[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx
\]
Recall: \( f(S) = (5000S - 5000) + \) 

Goal: 
Compute the expected value of \( f(u^H d^T) \).

New easier problem: 
Compute the expected value of \( g(X) \).

\[
\begin{aligned}
H &= N/2 + X\sqrt{N}/2 \\
T &= N/2 - X\sqrt{N}/2 \\
\end{aligned}
\]

\[
\begin{aligned}
u^H &= u^{N/2}u^X\sqrt{N}/2 \\
d^T &= d^{N/2}d^{-X}\sqrt{N}/2 \\
\end{aligned}
\]

\[
\begin{aligned}
u^H d^T &= u^{N/2}d^{N/2} \\
&= (ud)^{N/2} \\
&= C e^{kX} \\
k &= \ln((ud)^{N/2}) \\
&= (u/d)^{\sqrt{N}/2} \\
\end{aligned}
\]

Approx. Sol’n: 
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx = e^{k} = (u/d)^{\sqrt{N}/2}
\]
Recall: \( f(S) = (5000S - 5000)_+ \)

Goal:

Compute the expected value of \( f(u^H d^T) \).

New easier problem:

Compute the expected value of \( g(X) \).

\[
\begin{align*}
 f(u^H d^T) &= f(C e^{kX}) = g(X) \\
 g(x) &:= f(C e^{kx}) \\
 u^H d^T &= u^{N/2} d^{N/2} u^X \sqrt{N}/2 d^{-X} \sqrt{N}/2 \\
 &= (ud)^{N/2} (u/d)^X \sqrt{N}/2 \\
 &= C e^{kX} \\
 C &= (ud)^{N/2} \\
 k &= \ln((u/d)^{\sqrt{N}/2}) \\
\end{align*}
\]

Approx. Sol’n: \[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx
\]
Recall: \( f(S) = (5000S - 5000) \)

Goal:
Compute the expected value of \( f(u^H d^T) \).

Restatement of goal:
Compute the expected value of \( g(X) \).

\[
f(u^H d^T) = f(C e^{kX}) = g(X)
\]

\[
u^H d^T = u^{N/2} d^{N/2} u^X \sqrt{N}/2 d^{-X} \sqrt{N}/2
\]

\[
\quad = (ud)^{N/2} (u/d)^X \sqrt{N}/2 C := (ud)^{N/2}
\]

\[
\quad = C e^{kX} k := \ln((u/d) \sqrt{N}/2)
\]

Approx. Sol’n:
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx
\]
Recall: \[ f(S) = (5000S - 5000)_+ = 5000(S - 1)_+ \]

\[ g(x) := f(Ce^{kx}) = 5000(Ce^{kx} - 1)_+ \]

reasonable??

\[
\left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1) + \int e^{-x^2/2} \, dx \right]Ce^{kx}
\]

\[ \begin{array}{c}
N = 2,592,000 \\
\approx u
\end{array} \]

\[ \begin{array}{c}
1.00010005 \\
0.99989997
\end{array} \]

\[ \begin{array}{c}
\approx d
\end{array} \]

\[ \begin{array}{c}
C := (ud)^{N/2}
\end{array} \]

\[ \begin{array}{c}
k := \ln((u/d)^{\sqrt{N}/2})
\end{array} \]

Approx. Sol’n:
\[
\left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] e^{-x^2/2} \, dx \right]
\]
\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1) + e^{-x^2/2} \, dx \]
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1) + e^{-x^2/2} \, dx$$

$$= \frac{5000}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1) + e^{-x^2/2} \, dx$$

$$= \frac{5000}{\sqrt{2\pi}} \int_{a}^{\infty} (Ce^{kx} - 1) + e^{-x^2/2} \, dx$$

$$Ce^{ka} = 1$$

$$Ce^{ka} = 0$$

$$e^{ka} = 1/C$$

$$ka = \ln(1/C) = -\ln C$$

$$a = -(\ln C)/k$$
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1) + e^{-x^2/2} \, dx
\]

\[
= \frac{5000}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1) + e^{-x^2/2} \, dx
\]

\[
= \frac{5000}{\sqrt{2\pi}} \left[ C \int_{a}^{\infty} e^{kx} e^{-x^2/2} \, dx - \int_{a}^{\infty} e^{-x^2/2} \, dx \right]
\]

\[
= \frac{5000}{\sqrt{2\pi}} \left[ C \int_{a}^{\infty} e^{kx} e^{-x^2/2} \, dx - \int_{a}^{\infty} e^{-x^2/2} \, dx \right]
\]

\[
a = -(\ln C')/k
\]
\[
\frac{5000}{\sqrt{2\pi}} \left[ C \int_a^\infty e^{kx} e^{-x^2/2} \, dx \right.
- \left. \int_a^\infty e^{-x^2/2} \, dx \right]
\]

\[
\frac{\sqrt{2\pi}}{} \Phi(-a)
\]

\[
= 5000 \left[ C \int_a^\infty e^{kx} e^{-x^2/2} \, dx \right. - \left. \int_a^\infty e^{-x^2/2} \, dx \right]
\]

\[
\left. a = -(\ln C')/k \right)
\]
\[
\frac{5000}{\sqrt{2\pi}} \left[ C \int_{a}^{\infty} e^{kx} e^{-x^2/2} \, dx - \int_{a}^{\infty} e^{-x^2/2} \, dx \right]
\]

\[
\int_{a-k}^{\infty} e^{k(x+k)} e^{-(x+k)^2/2} \, dx
\]

\[
\int_{a-k}^{\infty} e^{kx} e^{k^2/2} e^{-x^2/2} e^{-k^2/2} e^{-kx} \, dx
\]

\[
e^{k^2/2} \int_{a-k}^{\infty} e^{-x^2/2} \, dx
\]

\[
\sqrt{2\pi} \Phi(k-a)
\]

\[
a = -\left(\ln C\right)/k
\]
\[
\begin{aligned}
&= \frac{5000}{\sqrt{2\pi}} \left[ C \int_{a}^{\infty} e^{kx} e^{-x^2/2} \, dx - \int_{a}^{\infty} e^{-x^2/2} \, dx \right] \\
&\quad + \int_{a-k}^{\infty} e^{k(x+k)} e^{-(x+k)^2/2} \, dx \\
&\quad - e^{k^2/2} e^{-x^2/2} e^{-k^2/2} \\
&\quad \sqrt{2\pi} \Phi(k - a)
\end{aligned}
\]

\[a = -(\ln C')/k\]
\[
\frac{5000}{\sqrt{2\pi}} \left[ C \int_{a}^{\infty} e^{kx} e^{-x^2/2} \, dx \right] - \left[ \int_{a}^{\infty} e^{-x^2/2} \, dx \right]
\]

\[
e^{k^2/2} \sqrt{2\pi} \Phi(k - a)
\]

\[
a = -(\ln C')/k
\]
\[
\frac{5000}{\sqrt{2\pi}} \left[ C \int_a^\infty e^{kx} e^{-x^2/2} \, dx - \int_a^\infty e^{-x^2/2} \, dx \right] + e^{k^2/2} \sqrt{2\pi} \Phi(k - a)
\]

\[
= 5000 \left[ C e^{k^2/2} \left[ \Phi(k - a) \right] - \left[ \Phi(-a) \right] \right]
\]

\[
= 121.0704439
\]

\[
\begin{align*}
a &= -0.01653528434 \\
k &= 0.0573390439 \\
C &= 1.000948567
\end{align*}
\]
Coin-flipping game: Flip a fair coin $N$ times. If $H$ heads and $T$ tails, pay $f(u^H d^T)$, 30 days from now.

$$\rho^N P = \text{expected payout} =: E = ???$$

\[
P = \rho^{-N} E = (1 + \iota)^{-N} E
\]

= discounted expected payout

\[
P = \rho^{-N} E \approx 120.7570357
\]

\[
\rho^{-N} = 0.997411356
\]

\[
N : = 30 \times 24 \times 60 \times 60 = 2,592,000
\]

\[
E \approx 121.0704439
\]

\[
1.000000001
\]
\[ K = 5000 \]
\[ \mu = 0.003917149457 \]
\[ \sigma = 0.057338217 \]
\[ e^r = 1.002595362 \]
\[ S_0 = 5000 \]

\[
K' := K/e^r = 4987.056782
\]

\[
d_{\pm} := \frac{\ln(S_0/K')}{\sigma} \pm \frac{\sigma}{2}
\]

\[
= \frac{0.002592000333}{0.057338217} \pm 0.028669108
\]

\[
d_+ = 0.073874565 \]
\[
d_- = 0.016536349
\]
\[
\Phi(d_+) = 0.52944 \]
\[
\Phi(d_-) = 0.50660
\]

\[
\textbf{Black-Scholes Option Pricing Formula}
\]

\[
\textbf{Black-Scholes Price} = S_0[\Phi(d_+)] - K'[\Phi(d_-)]
\]

\[
= [5000][0.52944] - [4987.056782][0.5066]
\]

\[
= 120.7570357
\]

\[
\text{drift } (\mu) \text{ unused!}
\]
QUESTIONS?

COMMENTS?