1. Let $f(x) = (3x^2 + 5x - 6)^3$. Then $f'(1)$ is equal to

(A) 12
(B) $3(3+5-6)^2$
(C) $(3+5-6)^4$
(D) $3(3+5-6)^2(6+5)$
(E) 144

$f'(x) = 3\ (3x^2 + 5x - 6)^2\ (6x+5)$

2. The tangent line to the curve $y = x^3 - 2x^2 + 2x + 1$ at the point (2,5) has equation

(A) $y - 5 = (3x^2 - 4x + 2)(x - 2)$
(B) $y = 5x/2$
(C) $y - 5 = (12 - 8 + 2)(x - 2)$
(D) $x - 2 = (12 - 8 + 2)(y - 5)$
(E) $y - 5 = -6(x - 2)$

$f'(x) = 3x^2 - 4x + 2$

$f'(2) = 12 - 8 + 2$

3. $\lim_{x \to 2} \dfrac{x^2 - 4}{x^3 - 8}$ is equal to

(A) 0
(B) 1/2
(C) 1/3
(D) 2/5
(E) 2/3

$$\lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)} = \frac{4}{4+4+4}$$

4. Let $f(x)$ be defined by

$$f(x) = \begin{cases} |x-2|, & \text{if } x < 3 \\ (x-2)^2, & \text{if } 3 \le x \le 4 \\ x-4, & \text{if } x > 4. \end{cases}$$

Then f is continuous

(A) except at $x = 2$;

(B) except at $x = 3$;

(C) except at $x = 4$;

(D) except at $x = 3$ and $x = 4$;

(E) except at $x = 2$, $x = 3$ and $x = 4$.

$$\lim_{x \to 3^-} f(x) = 1$$

$$\lim_{x \to 3^+} f(x) = 1$$

$$\lim_{x \to 4^-} f(x) = 4$$

$$\lim_{x \to 4^+} f(x) = 0$$

5. Let $f(x) = 2x^3 - 3x^2 - 12x$. Then

$$f'(x) = 6(x-2)(x+1) \text{ and } f''(x) = 6(2x-1).$$

Then the absolute maximum of $f(x)$ on the interval $[-2, 2]$ occurs

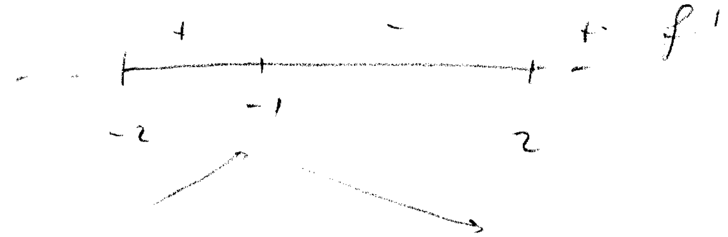
(A) at $x = -2$

(B) at $x = -1$

(C) at $x = 0$

(D) at $x = 2$

(E) nowhere



6. The equation $7x^2y^3 - 5xy^2 - 4y = 7$ defines y implicitly as a function of x. Find dy/dx.

(A) $\dfrac{14xy^3 + 5y^2}{4 - 21x^2y^2 - 10xy}$

(B) $\dfrac{5y^2 - 14xy^3}{21x^2y^2 - 10xy - 4}$

(C) $\dfrac{5y^2 + 14xy^3}{21x^2y^2 - 10xy - 4}$

(D) $(7x^2y^3 - 5xy^2)/4$

(E) 0

$14xy^3 + 21x^2y^2y'$

$-5y^2 - 10xyy' - 4y' = 0$

$y' = \dfrac{5y^2 - 14xy^3}{21x^2y^2 - 10xy - 4}$

7. Suppose that $f(x)$ is a function with first derivative $f'(x) = \dfrac{x^2 - 3x}{(x-1)^2}$. Then $f(x)$ is increasing on

(A) $(-\infty, 1)$ and $[3, \infty)$

(B) $[0, 1)$ and $[3, \infty)$

(C) $(-\infty, 0]$ and $(1, 3]$

(D) $(-\infty, 0]$ and $(1, \infty)$

(E) $(-\infty, 0]$ and $[3, \infty)$

f': $+$ | 0 | $-$ | 1 | $-$ | 3 | $+$

8. Let $f(x) = (x+2)e^x$. Then, using the Mean Value Theorem, we can conclude that there is at least one number c between 1 and 4 such that $f'(c)$ is equal to

(A) $2e^4 - e$

(B) $3e^4 - (3/2)e$

(C) $3e^4 + (3/2)e$

(D) $6e^4 - 3e$

(E) $6e^4$

$$f'(c) = \frac{f(4) - f(1)}{3} = \frac{6e^4 - 3e}{3} = 2e^4 - e$$

9. $\int \frac{x^{1/2}+x}{x^{5/2}}dx =$

$\int (x^{-2} + x^{-3/2})\, dx$

$= -x^{-1} + 2x^{-1/2} + C$

(A) $-\frac{1}{x} - \frac{2}{\sqrt{x}} + C$

(B) $\frac{\frac{3}{2}x^{3/2}+\frac{1}{2}x^2}{\frac{7}{2}x^{7/2}+C}$

(C) $\frac{3}{x^3} + \frac{5}{2x^{5/2}} + C$

(D) $\frac{1}{x} + \frac{2}{\sqrt{x}} + C$

(E) $-\frac{1}{x} - \frac{1}{2\sqrt{x}} + C$

10. Let $f(x) = \int_2^x \sqrt{7t^2+8}dt$. Then $f'(2) =$

(A) 0

(B) 2

(C) 6

(D) $\frac{7}{3}$

(E) $\frac{1}{12}$

$f'(x) = \sqrt{7x^2+8}$

$f'(2) = \sqrt{28+8} = 6$

11. The substitution $x = u^2$ turns $\int_2^3 \tan\sqrt{x}dx$ into

(A) $\int_{\sqrt{2}}^{\sqrt{3}} \tan u \; du$

(B) $\int_{\sqrt{2}}^{\sqrt{3}} 2u \tan u \; du$

(C) $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{2} u \tan u \; du$

(D) $\int_4^9 \tan u \; du$

(E) $\int_4^9 2u \; \tan u \; du$

$dx = 2u\,du$

$x \in [2, 3] \Rightarrow u \in [\sqrt{2}, \sqrt{3}]$

12. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

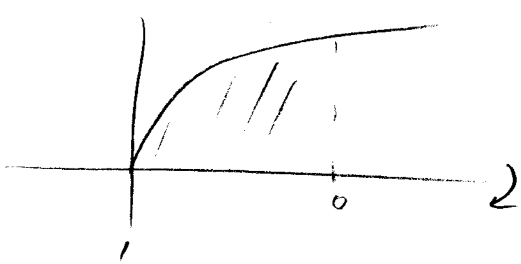
(A) $\frac{\pi}{2}$

(B) π

(C) $\frac{3\pi}{2}$

(D) 2π

(E) $\frac{\pi}{6}$



$$V = \pi \int_0^1 (\sqrt{x})^2 \, dx = \pi \frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

Hand-graded part

13.(16 points) a) If $x^2 + y^2 = 2$, find $\frac{dy}{dx}$.

$$2x + 2y\,y' = 0 \Rightarrow y' = -\frac{x}{y}$$

b) Find an equation of the tangent to the circle $x^2 + y^2 = 2$ at the point $(1, -1)$.

$$y'\big|_{(1,-1)} = 1$$

$$y + 1 = 1\,(x - 1)$$

$$\boxed{y = x - 2}$$

14.(17 points) Show that the equation $5x - 7 - \sin x = 0$ has exactly one real root.

$f(x) = 5x - 7 - \sin x$

- $f(0) = -7 < 0$
- $f(10) = 50 - 7 - \sin 10 > 0$
- f is continuous

$\Rightarrow$ By the Intermediate Value Theorem,

(A) the eq'n $f(x) = 0$ has at least one real root.

$f'(x) = 5 - \cos x > 0$

By the MVT

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$f'(c) \neq 0 \ \forall c \Rightarrow$ there are no two points $a \neq b$ such that $f(a) = f(b)$ $\Rightarrow$

(B) $f = 0$ has at most one real root

A & B $\Rightarrow$ f has exactly one real root.

15.(17 points) A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$. Find the distance traveled during the time period $1 \leq t \leq 4$.

Total distance traveled $= \int_1^4 |t^2 - t - 6|\,dt$

$$t^2 - t - 6 = 0 \iff t_{1,2} = \frac{1 \pm \sqrt{1+24}}{2}$$

$$= \frac{1 \pm 5}{2} = 3, -2$$

+ + + − − − + + +

−2 1 3 4

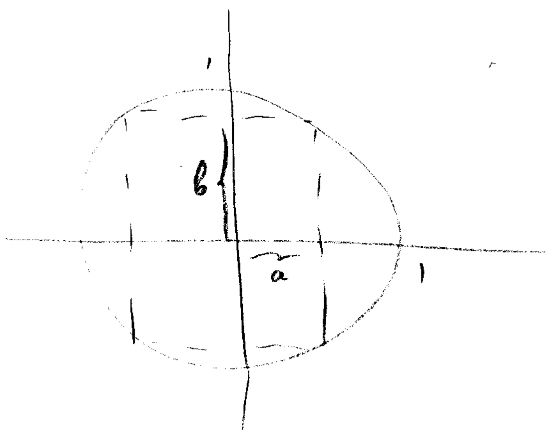
$$\int_1^4 |v(t)|\,dt = \int_1^3 (6 + t - t^2)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

$$= \left(6t + \frac{t^2}{2} - \frac{t^3}{3}\right)\Big|_1^3 + \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Big|_3^4$$

$$= 18 + \frac{9}{2} - 9 - 6 - \frac{1}{2} + \frac{1}{3} + \frac{64}{3} - 8 - 24$$
$$+ 18 + \frac{9}{2} - 9$$

$$= 3 - 14 - \frac{1}{6} + 21 + \frac{1}{3} = 10\frac{1}{6}$$

16.(18 points) Find the area of the largest rectangle that can be inscribed in a semicircle of radius 1. Explain why your answer is an absolute maximum.



By symmetry, restrict attention to I^{st} quadrant.

$a^2 + b^2 = 1$

$b = \sqrt{1-a^2}$

Maximize ab

$f(a) = a\sqrt{1-a^2}$

$f'(a) = \sqrt{1-a^2} + a \frac{1}{2\sqrt{1-a^2}} \cdot (-2a)$

$= \frac{2 - 2a^2 - 2a^2}{2\sqrt{1-a^2}}$

$a = \pm \frac{1}{\sqrt{2}}$ $\quad a > 0 \Rightarrow a = \frac{1}{\sqrt{2}}$

$\Rightarrow b = \frac{1}{\sqrt{2}}$

$\Rightarrow a \cdot b = \frac{1}{2}$

$\Rightarrow$ area of the whole rectangle:

$\boxed{4ab = 2}$

17.(18 points) Consider the function

$$f(x) = \frac{x^2}{x^2 - 4}.$$

We have

$$f'(x) = \frac{-8x}{(x^2-4)^2} \text{ and } f''(x) = \frac{8(4+3x^2)}{(x^2-4)^3}.$$

a) Find the domain of $f(x)$.

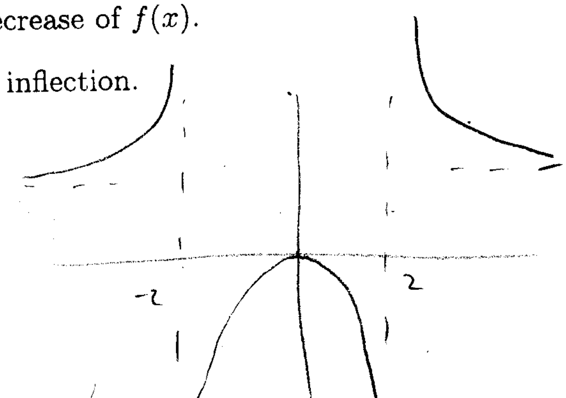
b) Determine the x-intercept and y-intercept of $y = f(x)$.

c) Determine the horizontal and vertical asymptotes of $y = f(x)$.

d) Determine the critical points, intervals of increase or decrease of $f(x)$.

e) Determine the concavity intervals of $f(x)$ and points of inflection.

f) Sketch the curve $y = f(x)$.



a) $x \neq \pm 2$

b) x-intercept:

$f(x) = 0 \iff x = 0 \;\}\; (0,0)$

y-intercept: $f(0) = 0 \;\}\; (0,0)$

c) Horizontal asymptotes:

$\lim_{x \to \pm\infty} f(x) = 1 \Rightarrow \underline{y = 1}$

Vertical asymptotes: $x = \pm 2$,

since $\lim_{x \to 2^+} f(x) = \infty$

$\lim_{x \to -2^-} f(x) = \infty$

e) $+ \quad - \quad +f''$

$-2 \quad 2$

$\cup \quad \cap \quad \cup f$

d) C.P: $x = 0$

$+ \quad + \quad - \quad -$

$-2 \quad 0 \quad 2$

$f \nearrow$ on $(-\infty, -2) \cup (-2, 0)$, $f \searrow$ on $(0, 2) \cup (2, \infty)$