① Final 3

1. Which of the following is NOT equal to $\int_0^2 \sqrt{4-x^2}dx$?

A. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{4-(2i)^2/n^2}$

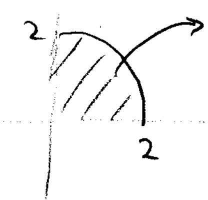
B. π

C. $F(2) - F(0)$ where $F'(x) = \sqrt{4-x^2}$

D. The area of a half circle of radius 2

E. $\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{2}{n}\sqrt{4-(2i)^2/n^2}$

$$\int_0^2 \sqrt{4-x^2}\,dx$$



$$\frac{\pi\cdot(2)^2}{4} = \pi$$

2. Suppose that the function f is continuous and differentiable on the closed interval $[-1, 2]$. Assume that $f(-1) = -5$ and $f(2) = 7$. Which of the following is NOT true of the function f?

A. f achieves an absolute maximum on the interval $[-1, 2]$.

B. There is a point c in the interval $[-1, 2]$ where $f(c) = 0$.

C. There is a point c in the interval $(-1, 2)$ where $f'(c) = 4$.

D. The above statements must all be true for f.

E. The above statements can all be false for f.

$$\frac{f(2) - f(-1)}{2-(-1)} = \frac{7+5}{3} = 4$$

3. Which of the following describes the set of real numbers on which the function $f(x) = \ln(1 + x^2)$ is concave down?

A. $0 < x < 1$

B. $|x| > 1$

C. $-1 < x < 1$

D. $x < -1$

E. $x > 1$

$$f'(x) = \frac{2x}{1+x^2}$$

$$f''(x) = \frac{2(1+x^2) - 4x^2}{1+x^2} = \frac{2(1-x)(1+x)}{1+x^2}$$

$-$ $\quad$ $+$ $\quad$ $-$ $\quad f''$

-1 $\quad$ 1

$\cap$ $\quad$ $\cup$ $\quad$ $\cap$ $\quad f$

4. Evaluate $\int \frac{\sin x}{(\cos x)^{\frac{1}{3}}}\, dx$.

A. $\frac{2}{3}(\cos x)^{-\frac{2}{3}} + C$

B. $\frac{1}{3}(\cos x)^{\frac{3}{2}} + C$

C. $-\frac{3}{2}(\cos x)^{\frac{2}{3}} + C$

D. $-\frac{1}{3}(\cos x)^{\frac{1}{3}} + C$

E. $-\frac{3}{2}(\sin x)^{-\frac{2}{3}} + C$

$u = \cos x$

$du = -\sin x\, dx$

$$\int \frac{\sin x}{(\cos x)^{1/3}}\, dx = -\int \frac{du}{u^{1/3}} =$$

$$= -\frac{3}{2}u^{2/3} + C$$

5. The difference of two numbers is 20. What is the smallest possible value for the product of these two numbers?

A. -100
B. -240
C. -400
D. 100
E. 240

$$x - y = 20 \quad \Rightarrow \quad y = x - 20$$

minimize xy

$$f(x) = x^2 - 20x$$

$$f'(x) = 2x - 20 \qquad x = 10$$

$$y = -10$$

$-$		$+$	f'
	10		
$\searrow$		$\nearrow$	f

6. Evaluate $\displaystyle\lim_{x \to -8} \frac{\sqrt{1-x} - 3}{2 + x^{\frac{1}{3}}}$.

A. -2
B. ∞
C. $-\infty$
D. -4
E. 8

$$\lim_{x \to -8} \frac{\sqrt{1-x} - 3}{2 + x^{1/3}} =$$

$$= \lim_{x \to -8} \frac{\frac{-1}{2\sqrt{1-x}}}{\frac{1}{3}x^{-2/3}} = \lim_{x \to -8} \frac{3x^{2/3}}{2\sqrt{1-x}}$$

$$= \frac{-3 \cdot 4}{2 \cdot 3} = -2$$

7. The area of the region lying between the curves $y = 2x - x^2$ and $y = -x$ is equal to which of the following?

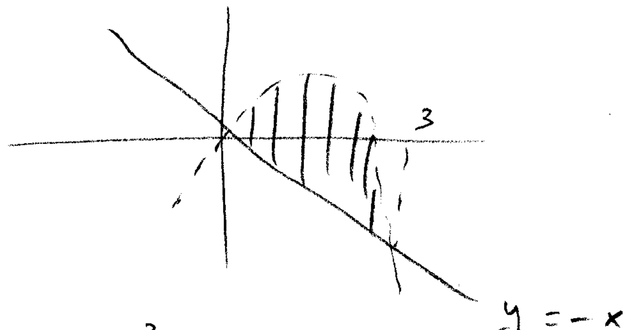
A. 2

B. 9

C. 4

D. 27

E. 9/2



$$2x - x^2 = -x$$

$$3x - x^2 = 0 \qquad x = 0, 3$$

$$\int_0^3 (3x - x^2)\,dx = \left(\frac{3}{2}x^2 - \frac{x^3}{3}\right)\Big|_0^3$$

$$= \frac{27}{2} - \frac{27}{3} = \frac{27}{6} = \frac{9}{2}$$

8. Suppose a particle traveling along a straight line is $\frac{1}{1+t}$ meters from the origin after t seconds. The average velocity of the particle away from the origin between times $t = 0$ and $t = 3$ is:

A. $-\frac{1}{4}$ meters per second

B. $\frac{5}{12}$ meters per second

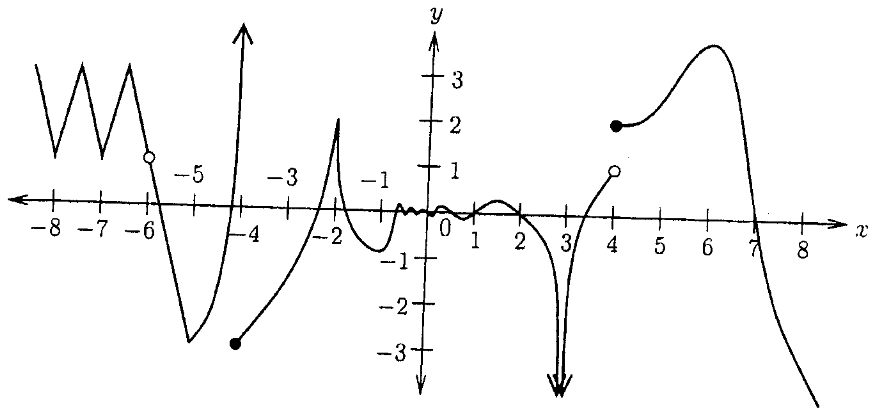
C. $\frac{1}{3}\ln(4)$ meters per second

D. $-\frac{4}{9}$ meters per second

E. $\frac{1}{4}$ meters per second

$$v_{av} = \frac{s(3) - s(0)}{3}$$

$$= \frac{\frac{1}{4} - 1}{3} = -\frac{1}{4}$$



Problems 9 and 10 refer to the graph $y = f(x)$ depicted above. An open circle means that the function's value is not the height of that circle. A solid circle means that the function's value equals the height of that circle.

9. The number of values a in the interval $[-8, 8]$ for which $\lim_{x \to a} f(x)$ does not exist is:

A. 1

B. 2

C. 3

D. 4

E. 5 or more

−4, 4

10. If f' is the derivative of f, then $f'(x) < 0$ when:

A. $x = -6$

B. $1 < x < 2$

C. $x = 4$

D. $x > 6$

E. none of the above

11. If $f(x) = e^{3x} - e^{-3x}$, then what is the 1271$^{\text{th}}$ derivative $f^{(1271)}(x)$?

A. $3^{1271} f(x)$

B. $e^{3813x} - e^{-3813x}$

C. $3^{1271} f(-x)$

D. $1271 f^{(1270)}(x)$

E. $3^{1271}(e^{3x} + e^{-3x})$

$f'(x) = 3e^{x} - (-3)e^{-3x}$

$f''(x) = 3^{2} e^{x} - (-3)^{2} e^{-3x}$

$f^{(n)}(x) = 3^{n} e^{3x} - (-3)^{n} e^{-3x}$

12. The derivative of the function $f(t) = \ln(\ln(\ln t))$ is:

A. $\dfrac{1}{t}$

B. $\dfrac{1}{\ln t}$

C. $\dfrac{1}{t\ln(t)\ln(\ln t)}$

D. $\dfrac{1}{\ln(\ln(\ln t))}$

E. $\dfrac{1}{t\ln(t)^{3}}$

$f'(t) = \dfrac{1}{\ln(\ln t)} \cdot \dfrac{1}{\ln t} \cdot \dfrac{1}{t}$

13. If $f(x) = \int_2^{x+1} t\sqrt{7+t^2}dt$ then what is $f'(2)$?

A. 0

B. $\frac{1}{3}(64 - 11\sqrt{11})$

C. 1

D. $2\sqrt{11}$

E. 12

$$f'(x) = (x+1)\sqrt{7+(x+1)^2}$$

$$f'(2) = 3\sqrt{7+9} = 12$$

14. If $f(x) = \dfrac{\sqrt{x}+1}{\sqrt{x}}$ then $|f(x) - 1| < \frac{1}{10}$ for all $x > N$ if

A. $N = -10.$

B. $N = 10.$

C. $N = -100.$

D. $N = 100.$

E. $N = \frac{1}{10}.$

$$f(x) - 1 < \frac{1}{10}$$

$$f(x) < \frac{11}{10}$$

$$10\sqrt{x} + 10 < 11\sqrt{x}$$

$$\sqrt{x} > 10$$

$$x > 100$$

and

$$f(x) - 1 > -\frac{1}{10}$$

$$f(x) > \frac{9}{10}$$

$$10\sqrt{x} + 10 > 9\sqrt{x}$$

$$\sqrt{x} > -10$$

✓

15. If $e^{x/y} = x + y$, then what is $\frac{dy}{dx}$?

A. $\dfrac{xy}{x^2 + xy + y^2}$

B. $\dfrac{xe^{x/y} + y^2}{ye^{x/y} - y^2}$

C. $\dfrac{xy}{xe^{x/y} - y^2}$

D. $\dfrac{y}{x} - \dfrac{2}{xy^2e^{x/y}}$

E. $\dfrac{ye^{x/y} + y^2}{xe^{x/y} - y^2}$

$$e^{x/y} \cdot \frac{y - xy'}{y^2} = 1 + y'$$

$$ye^{x/y} - xe^{x/y}y' = y^2 + y^2y'$$

$$\boxed{y' = \frac{ye^{x/y} - y^2}{y^2 + xe^{x/y}}}$$

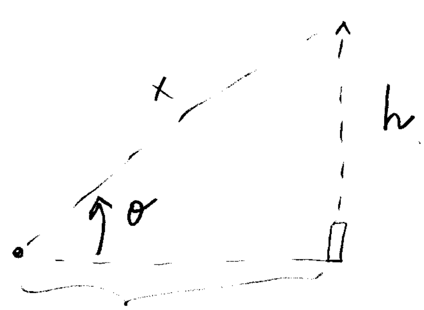
16. A television camera is positioned 4 kilometers from the base of a rocket launching pad. Suppose that the camera is always aimed at a rocket, and that the rocket moves straight upward. If θ is the angle of elevation of the camera, then $\theta = 0$ when the rocket is on the launch pad, and θ increases as the rocket rises.

(i) Express $\tan(\theta)$ in terms of the height of the rocket.

(ii) What is $\cos(\theta)$ when the rocket is 3 kilometers off the ground?

(iii) If the rocket is rising at 500 meters per second when it is 3 kilometers high, then how fast is θ increasing (in radians per second) at that moment? Express your answer as a fraction in lowest terms.

a) $\tan\theta = \frac{h}{4}$ $(*)$



4 km $= 4000\,m$

b) If $h = 3$ $\Rightarrow$

$x = \sqrt{9+16} = 5$

$\Rightarrow \cos\theta = \frac{4}{5}$

$h'(3000) = 500\,m/sec$ // 0.5 km/sec, @t^*

$\theta'(t^*) = ?$

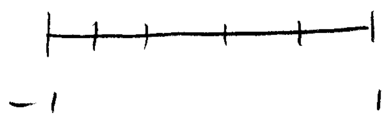
$\frac{d}{dt}(*) \Rightarrow \frac{1}{\cos^2\theta} \cdot \theta'(t) = \frac{1}{4} h'(t)$

$\frac{25}{16} \cdot \theta'(t^*) = \frac{1}{4} \cdot \frac{1}{2} \Rightarrow$

$\boxed{\theta'(t^*) = \frac{2}{25}}$

17. Estimate $\int_{-1}^{1}(5x^2+1)dx$ using a Riemann sum with 5 subintervals of equal length. Use the left endpoints of these subintervals for your sample points. Express your answer as a fraction in lowest terms.

$$\Delta x = \frac{2}{5}$$

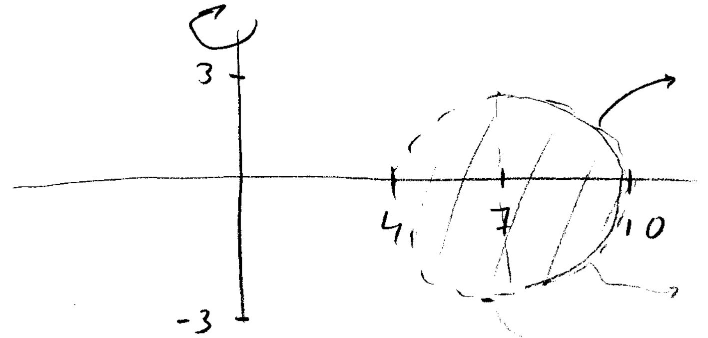


$$x_0 = -1$$
$$x_1 = -1 + \frac{2}{5} = -\frac{3}{5}$$
$$x_2 = -1 + \frac{4}{5} = -\frac{1}{5}$$
$$x_3 = -1 + \frac{6}{5} = \frac{1}{5}$$
$$x_4 = -1 + \frac{8}{5} = \frac{3}{5}$$

$$L_5 = \left(5\cdot(-1)^2+1\right)\cdot\frac{2}{5}$$
$$+\left(5\cdot\left(-\frac{3}{5}\right)^2+1\right)\cdot\frac{2}{5}$$
$$+\left(5\cdot\left(-\frac{1}{5}\right)^2+1\right)\cdot\frac{2}{5}$$
$$+\left(5\cdot\left(\frac{1}{5}\right)^2+1\right)\cdot\frac{2}{5}$$
$$+\left(5\cdot\left(\frac{3}{5}\right)^2+1\right)\cdot\frac{2}{5}$$

$$= 2 + 2(1+9+1+1+9) = 43$$

18. To make a doughnut for breakfast, rotate about the y-axis the disk bounded by $(x-7)^2 + y^2 = 9$, centered at $(7,0)$. Write a definite integral that gives the volume of your breakfast. Evaluate the integral (but don't eat the doughnut).



$$y = \sqrt{9-(x-7)^2}$$

$$x = 7 + \sqrt{9-y^2}$$

$$x = 7 - \sqrt{9-y^2}$$

cyl. shells

$$V = 2\int_4^{10} 2\pi x \sqrt{9-(x-7)^2}\, dx$$

or

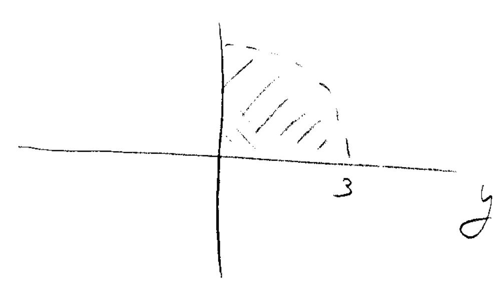
washers

$$V = \pi \int_{-3}^{3} \left\{ \left(7+\sqrt{9-y^2}\right)^2 - \left(7-\sqrt{9-y^2}\right)^2 \right\} dy$$

$$= 2\pi \int_0^3 \left\{ \cancel{49} + 14\sqrt{9-y^2} + \cancel{9} - \cancel{y^2} - \cancel{49} + 14\sqrt{9-y^2} - 9 + y^2 \right\} dy$$

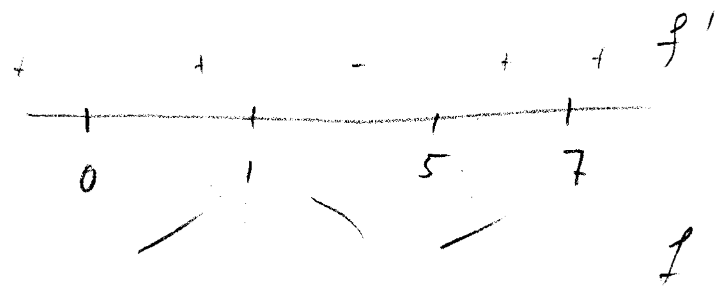
$$= 56\pi \int_0^3 \sqrt{9-y^2}\, dy$$

$$= 56\pi \; \frac{\pi \cdot 9}{4} = \underline{\underline{126\pi^2}} \,//$$



19. Let $f(x) = x(x^2 - 9x + 15)$. Find the absolute maximum and absolute minimum values attained by f on the interval $[0, 7]$. Determine the x-values where each of these extrema is attained.

$$f'(x) = (x^2 - 9x + 15) + x(2x - 9)$$
$$= 3x^2 - 18x + 15$$
$$= 3(x^2 - 6x + 5) = 3(x-1)(x-5)$$



$$f(0) = 0$$
$$f(1) = 1 - 9 + 15 = 7$$
$$f(5) = 5(25 - 45 + 15) = -25$$
$$f(7) = 7(49 - 63 + 15) = 7$$

Abs min $(5, -25)$

Abs max : 7 - achieved at

$x = 1$ & $x = 7$

20. Estimate $\sqrt[3]{66.4}$ by the method of linear approximation, using the fact that $4^3 = 64$. Be clear about which function you are linearly approximating and what its linear approximation is. At the end of your calculations, express your final answer as a single number in decimal notation.

$$f(x) = \sqrt[3]{x} \qquad f(64) = 4$$

$$f'(x) = \frac{1}{3} x^{-2/3} \qquad f'(64) = \frac{1}{3 \cdot 16} = \frac{1}{48}$$

$$, \quad a = 64$$

$$2.4 = \frac{24}{10} = \frac{12}{5}$$

$$L(x) = f(64) + f'(64)(x-64)$$

$$\sqrt[3]{66.4} \approx 4 + \frac{1}{48} \cdot (66.4 - 64) =$$

$$= 4 + \frac{2.4}{48} = 4 + \frac{12}{5 \cdot 48} = 4 + \frac{1}{20} =$$

$$= \underline{\underline{4.05}}$$

21. Write an equation for the tangent line to the graph of $y = x^{\cos(\pi x)}$ when $x = 3$.

$$y = x^{\cos(\pi x)}$$

$$\ln y = \cos(\pi x) \ln x$$

$$\frac{y'}{y} = -\sin(\pi x)\, \pi \ln x + \frac{\cos \pi x}{x}$$

$$y' = x^{\cos \pi x}\left(-\sin(\pi x)\pi \ln x + \frac{\cos \pi x}{x}\right)$$

$$y'(3) = \frac{1}{3} \cdot \left(-\frac{1}{3}\right) = -\frac{1}{9}$$

$$y(3) = \frac{1}{3}$$

$$\boxed{y - \frac{1}{3} = -\frac{1}{9}(x - 3)}$$