Symplectic 4-manifolds with non-positive Kodaira dimension

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1. Symplectic Kodaira dimension

2. $\kappa = -\infty$
    - Classification
    - Some open problems

3. $\kappa = 0$
    - Known symplectic CY surfaces
    - Homological classification of symplectic CY surfaces
    - Progress towards smooth classification
        - Speculation
        - Fibred sums, fibrations
        - Surgery

Recall κ $(= \kappa^s)$

(M, ω) is said to be (symplectically) minimal if it doesn't contain any symplectic sphere with self intersection -1.
For a minimal symplectic 4−manifold (M, ω) its Kodaira dimension

$$\kappa(M,\omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } K_\omega^2 < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } K_\omega^2 = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega^2 = 0, \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega^2 > 0. \end{cases}$$

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models.

Seiberg-Witten theory and
existence and almost uniqueness of symplectic minimal model are needed to show that $\kappa(M, \omega)$ is well defined.

Properties of κ

- $\kappa(M, \omega)$ agrees with the usual Kod dim for a Kähler surface
- $\kappa(M, \omega)$ does not depend on the choice of ω, and hence is an oriented diffeomorphism invariant.

We will denote it by $\kappa(M)$.

Additivity

- If $(\tilde{M}, \tilde{\omega})$ is a finite cover of (M, ω), then $\kappa(\tilde{M}, \tilde{\omega}) = \kappa(M, \omega)$
- For a surface bundle $\Sigma \to (M, \omega) \to B$, $\kappa(M, \omega) = \kappa(\Sigma) + \kappa(B)$ if we define

$$\kappa(S^2) = -\infty, \quad \kappa(T^2) = 0, \quad \kappa(\Sigma_{g \geq 2}) = 1$$

Additivity is conjectured to hold for circle bundles or mapping tori with respect to the Kodaira dimension of 3-manifolds introduced by Weiyi Zhang.

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- (Rational manifolds) $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$;
- (Ruled manifolds) S^2-bundle $\# k\overline{\mathbb{C}P^2}$,

These manifolds have $b^+ = 1$ and have Kähler structures
All toric surfaces have $\kappa = -\infty$

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A longstanding conjecture in algebraic geometry: A smooth projective variety is uniruled if and only if it has Kodaira dimension $-\infty$.

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Casini-Panov: On non-minimal irrational ruled manifold, there exists non-Kähler symplectic structures
Not known for rational manifolds with large Euler number.
If every symplectic structure on a rational manifold is Kähler, then the longstanding Nagata conjecture on singularities of plane curves is a consequence

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$\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k = 6, 7, 8$?

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Homology action of SMC generated by Lagrangian Dehn twists (L+Wu).
Homology trivial subgroup of SMC is trivial for up to 4 blow ups.
(McDuff-Abreu, Lalonde+Pinnsonnault+Anjos, J. Li+L+Wu)

Symplectic surfaces

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Well understood when $A \cdot A$ is 'big' via Seiberg-Witten theory. Eg. For up to 8 blowups of $\mathbb{C}P^2$, every class of non-negative square is represented by a connected symplectic surface with respect to some symplectic form.

Uniqueness up to isotopy

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2. T^2-bundles over T^2

- all admit symplectic structure with $\kappa = 0$ (Thurston+Geiges)
- classified up to diffeomorphism (Sakamoto-Fukuhara)
- geometric manifolds modeled on: $\mathrm{Nil}^3 \times \mathbb{R}$, Nil^4, $\mathrm{Sol}^3 \times \mathbb{R}$, $\mathbb{R}^4$ (Ue)
- Sol manifolds
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All fibred by T^2
fibers could be singular, multiple, or homologous trivial

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When $b^+ > 1$, $\mathcal{P}$ is connected and $\pi_{b^+-1}(\mathcal{P}) = \mathbb{Z}$ A winding family is a S^{b^+-1} family of symplectic forms $\phi : S^{b^+-1} \to \Omega$ which represents a generator of $\pi_{b^+-1}(\mathcal{P}) = \mathbb{Z}$

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Question: $\{\omega\}/Diff^+ = \mathcal{C}(M)/D(M)$?
For Enriques and T^2-bundle over T^2 with $b^+ = 1$, each connected component of $\mathcal{M}$ is described this way
If true when $b^+ > 1$, then every symplectic form lies in some winding family

Betti number bounds

Theorem

(L, Bauer)
$b^+(M) \leq 3$ *if* $\kappa(M, \omega) = 0$

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  $b^+ \leq 3 + |comp(K_\omega)|$
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  holds when $\kappa = 0$
- $vb_1(M) \leq 4$, where $vb_1(M)$ is the supremum of $b_1(\tilde{M})$ among all finite covers $\tilde{M}$.

- If M is minimal, then it has the same $\mathbb{Q}-$cohomology ring as K3, Enriques surface or a T^2-bundle over T^2

In fact, $\mathbb{Z}-$homology K3, $\mathbb{Z}-$homology Enriques

The following table list possible homological invariants of $\kappa = 0$ manifolds:

b_1	b_2	b^+	χ	σ	known manifolds
0	22	3	24	-16	K3
0	10	1	12	-8	Enriques surface
4	6	3	0	0	4-torus
3	4	2	0	0	T^2-bundles over T^2
2	2	1	0	0	T^2-bundles over T^2

Virtual 1st Betti number and fundamental group

G a SCY group if $G = \pi_1(M)$ for some SCY surface (M, ω).
If (M, ω) is a SCY surface, then any finite cover is also SCY.
If $b_1(G) = 0$ then $G = 1$ or $\mathbb{Z}_2$ and the corresponding SCY surface unique up to homeomorphism.

- If $G = 1$, homeomorphic to K3 (apply Freedman)
- If $G = \mathbb{Z}_2$, homeo. to Enriques (apply Hambleton+Kreck)

If $b_1(G) > 0$ then $2 \leq vb_1(M) \leq 4$, $\chi(M) = \sigma(M) = 0$

Friedl+Vidussi: Suppose G is SCY with $b_1(G) > 0$.

- If $H^2(G, \mathbb{Z}[G]) = 0$ then the corresponding SCY surfaces are homotopic to $K(G, 1)$ (unique up to homotopy)
- If $G = \pi_1$ of a (Infra)solvable manifold, then the corresponding SCY surfaces are unique up to homeomorphism. In particular, for all known examples, π_1 determines the homeomorphism type.

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Now invoke Taubes' $SW(K_\omega) = \pm 1$

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By the homological classification, a Degree 4 symplectic hypersurface in $\mathbb{C}P^3$ is a homology K3.
Is it diffeomorphic to K3?

Existence of a symplectic torus

Basically, we need to show that if $\kappa(M, \omega) = 0$ then M is T^2-fibred.

Suppose a homology K3 has a winding family,
L-Liu's parametrized SW theory $\Rightarrow$ there is an embedded symplectic torus for some symplectic form in the winding family

Existence of symplectic torus can be proved in some other homology types

Fibred sum

Positive genus sums

Usher: If (M, ω) with $\kappa = 0$ is a non-trivial positive genus fibred sum, then the summands have $\kappa = -\infty$ (rational or ruled), and the sum is along tori representing $-K_\omega$. If (M, ω) is minimal, then M is difeomorphic to K3, Enriques surface or a T^2-bundle over T^2 with $b_1 = 2$.

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Ex: $K3 = E(1)\#_f E(1)$
Enriques $=E(1)\#_f(S^2 \times T^2)$
The sum of two S^2-bundles over T^2 along bi-sections give rise to T^2-bundles over T^2

Genus 0 sums

Dorfmeister: If (M, ω) is minimal and has $\kappa = 0$ is a non-trivial genus 0 fibred sum, then M is diffeomorphic to Enriques surface

Ex: The rational blow down of $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ is Enriques surface

Lefschetz fibrations

If (M, ω) with $\kappa = 0$ is a surface bundle $\Sigma \to M \to B$, then
$\kappa(M, \omega) = \kappa(\Sigma) + \kappa(B)$
$\Rightarrow$ Σ and B are T^2

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If allowing multiple fibers, then M is Enriques surface

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Virtual Betti number of Y with coefficient $\mathbb{Z}$ or $\mathbb{F}_p$ is bounded by 3
Non-torsion Euler class$\Rightarrow b_1(Y) > 1$
McCarthy+Bowdin: Y is irreducible
So Y is Haken with $b_1 > 1$

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- If Y has non-trivial JSJ decomposition, or Seifeit fibred, then it contains an incompressible torus that is not a fiber $vb_1(Y) \leq 3$, together with Kojima, Luecke $\Rightarrow$ a finite cover $\tilde{Y}$ is a T^2-bundle

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- Y is not hyperbolic
  ($vb_1(Y)$ should be ∞ if $b_1(Y) \geq 1$)
  Lubotzky alternative+ Lackenby $\Rightarrow vb_1(Y, \mathbb{F}_p) = \infty$

Question: whether the $\mathbb{F}_p-$virtual Betti number of a symplectic Calabi-Yau surface is bounded by 4.

Mapping tori

L-Ni:
3-manifolds torus bundle
Seifert fibred spaces
irreducible with nontrivial JSJ decomposition $\Rightarrow vb_1(M) = \infty$
hyperbolic

Given a Lagrangian torus L in a symplectic 4-manifold, there is a symplectic surgery along L, depending on an integer k and an embedded curve in L.
It is called a Luttinger surgery. Topologically, it is a generalized log transform of multiplicity 1 and auxiliary multiplicity k
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The following table list possible homological invariants of $\kappa = 0$ manifolds:

b_1	b_2	b^+	χ	σ	known manifolds
0	22	3	24	-16	K3
0	10	1	12	-8	Enriques surface
4	6	3	0	0	4-torus
3	4	2	0	0	T^2−bundles over T^2
2	2	1	0	0	T^2−bundles over T^2

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This implies some framing constraints on Lag tori in $\kappa = 0$ manifolds

Question: Can we obtain any T^2-bundle over T^2 from T^4 via Luttinger surgeries?

True for all Lag T^2-bundles, including all T^2-bundles over T^2 with $b_1 \geq 3$