

Exotic Lefschetz Fibrations and Stein Fillings with Arbitrary Fundamental Group

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Lefschetz fibrations

Definition

Let X be a compact, connected, oriented, smooth 4-manifold. A *Lefschetz fibration* on X is a smooth map $f : X \rightarrow \Sigma_h$, where Σ_h is a compact, oriented, smooth 2-manifold of genus h , such that f is surjective and each critical point of f has an orientation preserving chart on which $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ is given by $f(z_1, z_2) = z_1^2 + z_2^2$.

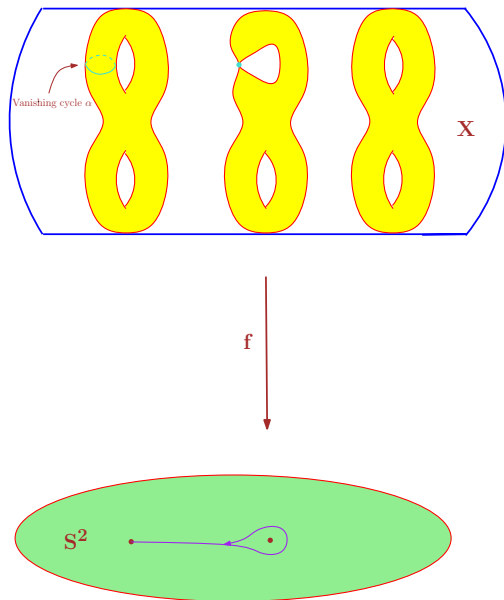


Figure : Lefschetz fibration on X over S^2

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$$e(X) = e(\Sigma_h)e(\Sigma_g) + s$$

The signature $\sigma(X)$ well understood for fibrations over \mathbb{S}^2 .

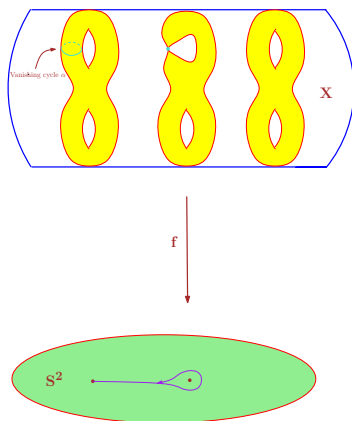
There are formulas by Y. Matsumoto, H. Endo, B. Ozbagci, I. Smith, ...)

$$\sigma(X) = -\frac{g+1}{2g+1}n_0 + \sum_{h=1}^{\lfloor \frac{g}{2} \rfloor} \left(\frac{4h(g-h)}{2g+1} - 1 \right) n_h$$

Monodromy of Lefschetz fibration

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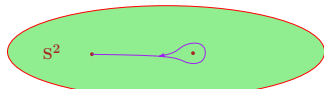
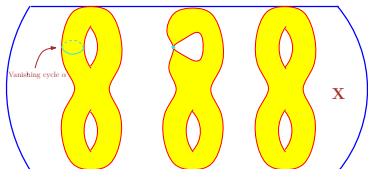
A singular fiber of the genus g Lefschetz fibration can be described by its monodromy, i.e., an element of the mapping class group M_g . This element is a right-handed (or a positive) Dehn twist along a simple closed curve on Σ_g , called the *vanishing cycle*.



Global monodromy of Lefschetz fibration

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For a genus g Lefschetz fibration over \mathbb{S}^2 , the product of right handed Dehn twists t_{α_i} along the vanishing cycles α_i , for $i = 1, \dots, s$, determines the *global monodromy* of the Lefschetz fibration, the relation $t_{\alpha_1} \cdot t_{\alpha_2} \cdot \dots \cdot t_{\alpha_s} = 1$ in M_g . Conversely, such a relation in M_g determines a genus g Lefschetz fibration over \mathbb{S}^2 with the vanishing cycles $\alpha_1, \dots, \alpha_s$.



Example (Genus one Lefschetz fibrations)

M_1 be the mapping class group of the torus $\mathbb{T}^2 = a \times b$.

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$$t_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$t_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Subject to the relations

$$\begin{aligned} t_a t_b t_a &= t_b t_a t_b \\ (t_a t_b)^6 &= 1 \end{aligned}$$

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$(t_a t_b)^{6n} = 1$ in M_1 . The total space of this fibration is the elliptic surface $E(n)$. $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$, the complex projective plane blown up at 9 points, and $E(2)$ is $K3$ surface. $E(n)$ also admits a genus $n - 1$ Lefschetz fibration over \mathbb{S}^2 .

Example (Families of hyperelliptic Lefschetz fibrations)

Let $\alpha_1, \alpha_2, \dots, \alpha_{2g}, \alpha_{2g+1}$ denote the collection of simple closed curves given in Figure, and c_j denote the right handed Dehn twists t_{α_j} along the curve α_j .

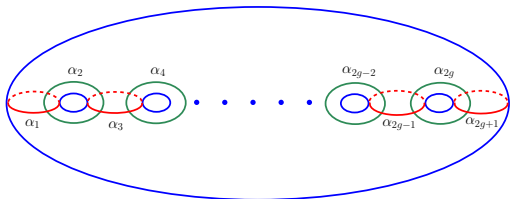


Figure : Vanishing cycles of the genus g Lefschetz fibration given by hyperelliptic involution

The following relations hold in the mapping class group M_g :

$$\begin{aligned} \Gamma_1(g) &= (c_1 c_2 \cdots c_{2g-1} c_{2g} c_{2g+1}^2 c_{2g} c_{2g-1} \cdots c_2 c_1)^2 = 1. \\ \Gamma_2(g) &= (c_1 c_2 \cdots c_{2g-1} c_{2g} c_{2g+1})^{2g+2} = 1. \\ \Gamma_3(g) &= (c_1 c_2 \cdots c_{2g-1} c_{2g})^{2(2g+1)} = 1. \end{aligned} \tag{1}$$

The monodromy relation $\Gamma_1(g) = 1$, the corresponding genus g Lefschetz fibrations over \mathbb{S}^2 has total space $X(g, 1) = \mathbb{C}\mathbb{P}^2 \# (4g + 5)\overline{\mathbb{C}\mathbb{P}^2}$, the complex projective plane blown up at $4g + 5$ points.

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It is known that for $g \geq 2$, the above fibration on $X(g, 1)$ admits $4g + 4$ disjoint (-1) -sphere sections (proof of this fact using a mapping class group argument is due to S. Tanaka).

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The fiber class is of the form $(g + 2)h - ge_1 - e_2 - \cdots - e_{4g+5}$, where e_i denotes the homology class of the exceptional sphere of the i -th blow up and h denotes the pullback of the hyperplane class of $\mathbb{C}\mathbb{P}^2$. The exceptional spheres represented by the homology classes $e_2, e_3, \dots, e_{4g+5}$ are sections of the Lefschetz fibration $X(g, 1) \rightarrow \mathbb{S}^2$.

Important Theorems on Lefschetz fibrations

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Theorem (S. Donaldson)

For any symplectic 4-manifold X , there exists a non-negative integer n such that the n -fold blowup $X\#n\overline{\mathbb{C}P^2}$ of X admits a Lefschetz fibration $f : X\#n\overline{\mathbb{C}P^2} \rightarrow \mathbb{S}^2$.

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Theorem (R. Gompf)

Assume that the closed 4-manifold X admits a genus g Lefschetz fibration $f : X \rightarrow \Sigma_h$, and let $[F]$ denote the homology class of the fiber. Then X admits a symplectic structure with symplectic fibers iff $[F] \neq 0$ in $H_2(X; \mathbb{R})$. If e_1, \dots, e_n is a finite set of sections of the Lefschetz fibration, the symplectic form ω can be chosen in such a way that all these sections are symplectic.

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Theorem (S. Donaldson and R. Gompf, J. Amorós - F. Bogomolov - L. Katzarkov - T. Pantev, M. Korkmaz)

For any finitely presented group G , there exist a Lefschetz fibration $X(G)$ over \mathbb{S}^2 with $\pi_1(X(G)) = G$.

$b_2^+(X(G))$ is very large. These examples don't contain essential tori

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For any finitely presented group G , there exist a closed symplectic 4-manifold $X_n(G)$ with $\pi_1(X(G)) = G$, which admits a genus $2g + n - 1$ Lefschetz fibration over \mathbb{S}^2 that has at least $4n + 4$ pairwise disjoint sphere sections of self intersection -2 . Moreover, $X_n(G)$ contains a homologically essential embedded torus of square zero disjoint from these sections which intersects each fiber of the Lefschetz fibrations twice.

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Theorem (A. Akhmedov - B. Ozbagci, 2012)

There exist an infinite family of non-holomorphic Lefschetz fibrations $X_n(G, K_i)$ over \mathbb{S}^2 with $\pi_1(X_n(G, K_i)) = G$ that can be obtained from $X_n(G)$ via knot surgery along K_i , where K_i are inf. family of genus $g \geq 2$ fibered knots with distinct Alexander polynomials.

Lefschetz fibrations by Y. Matsumoto and M. Korkmaz

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Let assume $g = 2k$.

The 4-manifold $Y(1, k) = \Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{C}P}^2$ is the total space of the genus g Lefschetz fibration over \mathbb{S}^2 with $2g + 4$ singular fibers. This was shown by Yukio Matsumoto for $k = 1$, and in the case $k \geq 2$ by Mustafa Korkmaz, by factorizing the *vertical* involution θ of the genus $2k$ surface.

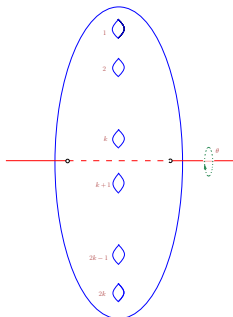


Figure : The involution θ of the genus $2k$ surface

Theorem (Y. Matsumoto, M. Korkmaz)

Let θ denote the vertical involution of the genus g surface with 2 fixed points. In the mapping class group M_g , the following relations between right handed Dehn twists hold:

- a) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_c)^2 = \theta^2 = 1$ if g is even,
 b) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} (t_a)^2 (t_b)^2)^2 = \theta^2 = 1$ if g is odd.

B_k , a , b , c are the simple closed curves defined as in Figure.

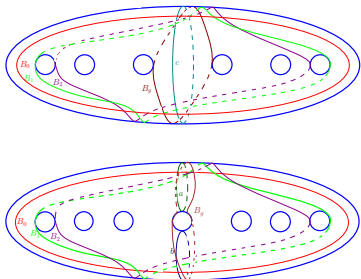


Figure : The vanishing cycles

Lefschetz fibrations by Y. Gurtas

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Yusuf Gurtas generalized the constructions of Matsumoto and Korkmaz even further. He presented the positive Dehn twist expression for a new set of involutions in the mapping class group M_{2k+n-1} of a compact, closed, oriented 2-dimensional surface Σ_{2k+n-1} . The total space of these genus $g = 2k + n - 1$ Lefschetz fibration over \mathbb{S}^2 is $Y(n, k) = \Sigma_k \times \mathbb{S}^2 \# 4n \overline{\mathbb{C}P}^2$.

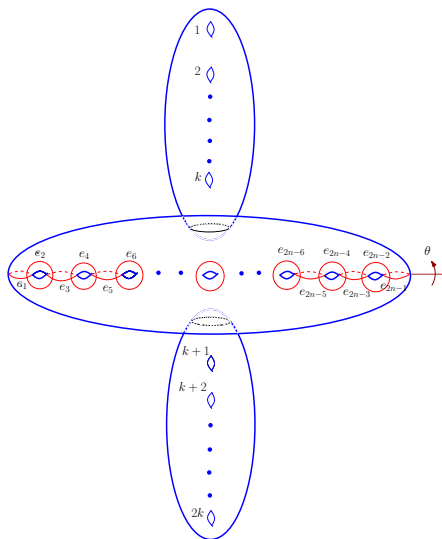


Figure : The involution θ of the surface Σ_{2k+n-1}

The branched-cover description for $\Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{C}\mathbb{P}^2}$

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A generic horizontal fiber is the double cover of \mathbb{S}^2 , branched over two points. Thus, we have a sphere fibration on $Y(n, k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{C}\mathbb{P}^2}$. A generic fiber of the vertical fibration is the double cover of Σ_k , branched over $2n$ points. Thus, a generic fiber of the vertical fibration has genus $n + 2k - 1$.

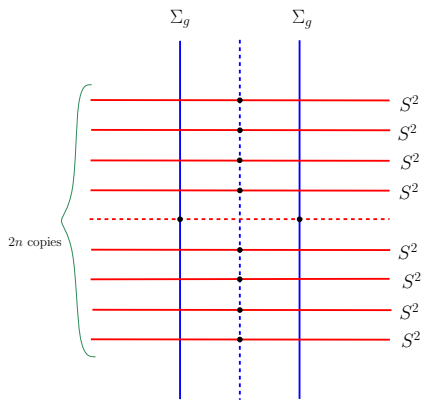


Figure : The branch locus for $\Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{C}\mathbb{P}^2}$

Theorem (Y. Gurtas)

The positive Dehn twist expression for the involution θ is given by

$$\theta = e_{2i+2} \cdots e_{2n-2} e_{2n-1} e_{2i} \cdots e_1 B_0 e_{2n-1} \cdots e_{2i+2} e_1 \cdots e_{2i} B_1 B_2 \cdots B_{4k-1} B_{4k} e_{2i+1}.$$

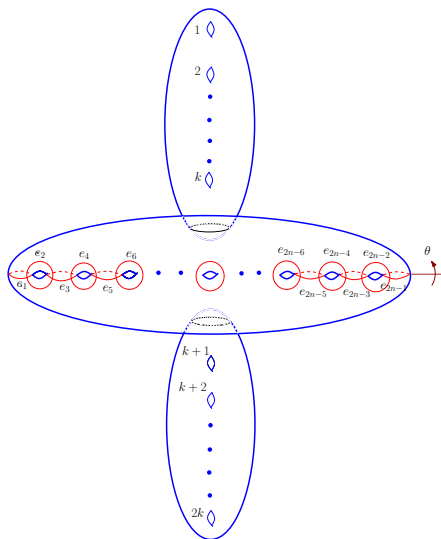


Figure : The involution θ of the surface Σ_{2k+n-1}

Construction Tools



Construction Tools

- Symplectic Connected Sum (1995) (M. Gromov, R. Gompf, J. McCarthy- J. Wolfson)
- Luttinger Surgery (1995) (K. Luttinger)
- Knot Surgery (1998) (R. Fintushel- R. Stern)

Luttinger surgery

Definition

Let X be a symplectic 4-manifold with a symplectic form ω , and the torus Λ be a Lagrangian submanifold of X with self-intersection 0. Given a simple loop λ on Λ , let λ' be a simple loop on $\partial(\nu\Lambda)$ that is parallel to λ under the Lagrangian framing. For any integer m , the $(\Lambda, \lambda, 1/m)$ *Luttinger surgery* on X will be $X_{\Lambda, \lambda}(1/m) = (X - \nu(\Lambda)) \cup_{\phi} (\mathbb{S}^1 \times \mathbb{S}^1 \times D^2)$, the $1/m$ surgery on Λ with respect to λ under the Lagrangian framing. Here

$\phi : \mathbb{S}^1 \times \mathbb{S}^1 \times \partial D^2 \rightarrow \partial(X - \nu(\Lambda))$ denotes a gluing map satisfying $\phi([\partial D^2]) = m[\lambda'] + [\mu_{\Lambda}]$ in $H_1(\partial(X - \nu(\Lambda)))$, where μ_{Λ} is a meridian of Λ .

$X_{\Lambda, \lambda}(1/m)$ possesses a symplectic form that restricts to the original symplectic form ω on $X \setminus \nu\Lambda$.

Luttinger's surgery has been very effective tool recently for constructing exotic smooth structures.

Example

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Let $\mathbb{T}^4 = a \times b \times c \times d \cong (c \times d) \times (a \times b)$. Let K_n be an n -twist knot.

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Let $\mathbb{T}^4 = a \times b \times c \times d \cong (c \times d) \times (a \times b)$. Let K_n be an n -twist knot. Let M_{K_n} denote the result of performing 0 Dehn surgery on \mathbb{S}^3 along K_n . $\mathbb{S}^1 \times M_{K_n}$ is obtained from $\mathbb{T}^4 = (c \times d) \times (a \times b) = c \times (d \times a \times b) = \mathbb{S}^1 \times \mathbb{T}^3$ by first performing a Luttinger surgery $(c \times \tilde{a}, \tilde{a}, -1)$ followed by a surgery $(c \times \tilde{b}, \tilde{b}, -n)$. The tori $c \times \tilde{a}$ and $c \times \tilde{b}$ are Lagrangian and the second tilde circle factors in \mathbb{T}^3 are as pictured. Use the Lagrangian framing to trivialize their tubular neighborhoods. When $n = 1$ the second surgery is also a Luttinger surgery.

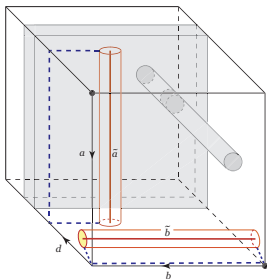


Figure : The 3-torus $d \times a \times b$

Symplectic Connected Sum

Definition

Let X_1 and X_2 be symplectic 4-manifolds, and $F_i \subset X_i$ are 2-dimensional, smooth, closed, connected symplectic submanifolds in them. Suppose that $[F_1]^2 + [F_2]^2 = 0$ and the genera of F_1 and F_2 are equal. Take an orientation-preserving diffeomorphism $\psi : F_1 \rightarrow F_2$ and lift it to an orientation-reversing diffeomorphism $\Psi : \partial\nu F_1 \rightarrow \partial\nu F_2$ between the boundaries of the tubular neighborhoods of νF_i . Using Ψ , we glue $X_1 \setminus \nu F_1$ and $X_2 \setminus \nu F_2$ along the boundary. The 4-manifold $X_1 \#_{\psi} X_2$ is called the *(symplectic) connected sum* of X_1 and X_2 along F_1 and F_2 , determined by Ψ .

$$\begin{aligned} e(X_1 \#_{\psi} X_2) &= e(X_1) + e(X_2) + 4(g - 1), \\ \sigma(X_1 \#_{\psi} X_2) &= \sigma(X_1) + \sigma(X_2), \end{aligned}$$

The vanishing cycles of twisted fiber sum of Lefschetz fibration

Lemma

Let $f : X \rightarrow \mathbb{S}^2$ be a genus g Lefschetz fibration with global monodromy given by the relation $t_{\alpha_1} \cdot t_{\alpha_2} \cdot \cdots \cdot t_{\alpha_s} = 1$. Let $X \#_{\psi} X$ denote the fiber sum of X with itself by a self-diffeomorphism ψ of the generic fiber Σ . Then $X \#_{\psi} X$ has the vanishing cycles $\alpha_1, \alpha_2, \cdots, \alpha_s, \psi(\alpha_1), \psi(\alpha_2), \cdots, \psi(\alpha_s)$.

Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times \mathbb{T}^2$

Fix integers $n \geq 2$, $p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. Let $Y_n(1/p_1, 1/q_1, \dots, 1/p_n, 1/q_n)$ denote symplectic 4-manifold obtained by performing the following $2n + 4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$. These $2n + 4$ surgeries comprise of the following 8 surgeries

$$\begin{aligned} & (a'_1 \times c'_1, a'_1, -1), \quad (b'_1 \times c''_1, b'_1, -1), \\ & (a'_2 \times c'_2, a'_2, -1), \quad (b'_2 \times c''_2, b'_2, -1), \\ & (a'_2 \times c'_1, c'_1, +1/p_1), \quad (a''_2 \times d'_1, d'_1, +1/q_1), \\ & (a'_1 \times c'_2, c'_2, +1/p_2), \quad (a''_1 \times d'_2, d'_2, +1/q_2), \end{aligned}$$

together with the following $2(n - 2)$ additional Luttinger surgeries

$$\begin{aligned} & (b'_1 \times c'_3, c'_3, -1/p_3), \quad (b'_2 \times d'_3, d'_3, -1/q_3), \\ & \quad \quad \quad \dots, \quad \dots, \\ & (b'_1 \times c'_n, c'_n, -1/p_n), \quad (b'_2 \times d'_n, d'_n, -1/q_n). \end{aligned}$$

Here, a_i, b_i ($i = 1, 2$) and c_j, d_j ($j = 1, \dots, n$) are the standard loops that generate $\pi_1(\Sigma_2)$ and $\pi_1(\Sigma_n)$.

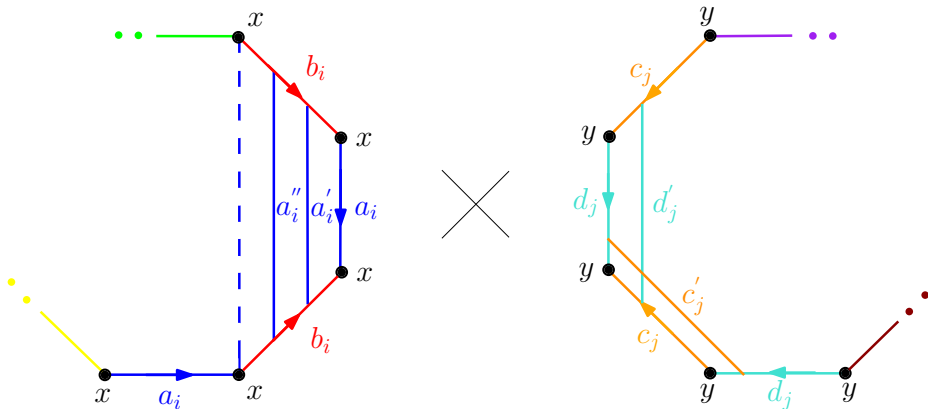


Figure : Lagrangian tori $a_i' \times c_j'$ and $a_i'' \times d_j'$

The Euler characteristic of $Y_n(1/p_1, 1/q_1, \dots, 1/p_n, 1/q_n)$ is $4n - 4$ and its signature is 0. The fundamental group $\pi_1(Y_n(1/p_1, 1/q_1, \dots, 1/p_n, 1/q_n))$ is generated by a_i, b_i, c_j, d_j ($i = 1, 2$ and $j = 1, \dots, n$) and the following relations hold in $\pi_1(Y_n(1/p_1, 1/q_1, \dots, 1/p_n, 1/q_n))$:

$$\begin{aligned}
 [b_1^{-1}, d_1^{-1}] &= a_1, & [a_1^{-1}, d_1] &= b_1, & [b_2^{-1}, d_2^{-1}] &= a_2, & [a_2^{-1}, d_2] &= b_2, & (2) \\
 [d_1^{-1}, b_2^{-1}] &= c_1^{p_1}, & [c_1^{-1}, b_2] &= d_1^{q_1}, & [d_2^{-1}, b_1^{-1}] &= c_2^{p_2}, & [c_2^{-1}, b_1] &= d_2^{q_2}, \\
 [a_1, c_1] &= 1, & [a_1, c_2] &= 1, & [a_1, d_2] &= 1, & [b_1, c_1] &= 1, \\
 [a_2, c_1] &= 1, & [a_2, c_2] &= 1, & [a_2, d_1] &= 1, & [b_2, c_2] &= 1, \\
 [a_1, b_1][a_2, b_2] &= 1, & \prod_{j=1}^n [c_j, d_j] &= 1, \\
 [a_1^{-1}, d_3^{-1}] &= c_3^{p_3}, & [a_2^{-1}, c_3^{-1}] &= d_3^{q_3}, & \dots, & [a_1^{-1}, d_n^{-1}] &= c_n^{p_n}, & [a_2^{-1}, c_n^{-1}] &= d_n^{q_n}, \\
 [b_1, c_3] &= 1, & [b_2, d_3] &= 1, & \dots, & [b_1, c_n] &= 1, & [b_2, d_n] &= 1.
 \end{aligned}$$

The surfaces $\Sigma_2 \times \{\text{pt}\}$ and $\{\text{pt}\} \times \Sigma_n$ in $\Sigma_2 \times \Sigma_n$ descend to surfaces in $Y_n(1/p_1, 1/q_1, \dots, 1/p_n, 1/q_n)$. They are symplectic submanifolds in $Y_n(1/p_1, 1/q_1, \dots, 1/p_n, 1/q_n)$. Denote their images by Σ_2 and Σ_n . Note that $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$ and $[\Sigma_2] \cdot [\Sigma_n] = 1$.

Let $\{p_i, q_i \geq 0 : 1 \leq i \leq g\}$ be a set of nonnegative integers and let $\bar{p} = (p_1, \dots, p_g)$ and $\bar{q} = (q_1, \dots, q_g)$.

Let $\{p_i, q_i \geq 0 : 1 \leq i \leq g\}$ be a set of nonnegative integers and let $\bar{p} = (p_1, \dots, p_g)$ and $\bar{q} = (q_1, \dots, q_g)$. Denote by $M_g(\bar{p}, \bar{q})$ the symplectic 4-manifold obtained by performing the following $2g$ Luttinger surgeries on the symplectic 4-manifold $\Sigma_g \times \mathbb{T}^2$:

$$\begin{aligned}
 & (a'_1 \times c', a'_1, -1/p_1), \quad (b'_1 \times c'', b'_1, -1/q_1), \\
 & (a'_2 \times c', a'_2, -1/p_2), \quad (b'_2 \times c'', b'_2, -1/q_2), \\
 & \dots \quad \dots \\
 & (a'_{g-1} \times c', a'_{g-1}, -1/p_{g-1}), \quad (b'_{g-1} \times c'', b'_{g-1}, -1/q_{g-1}), \\
 & (a'_g \times c', a'_g, -1/p_g), \quad (b'_g \times c'', b_g, -1/q_g).
 \end{aligned} \tag{3}$$

Here, a_i, b_i ($i = 1, 2, \dots, g$) and c, d denote the standard generators of $\pi_1(\Sigma_g)$ and $\pi_1(\mathbb{T}^2)$, respectively.

The fundamental group of $M_g(\bar{\rho}, \bar{q})$ is generated by a_i, b_i ($i = 1, 2, 3 \dots, g$) and c, d , and the following relations hold in $M_g(\bar{\rho}, \bar{q})$:

$$[b_1^{-1}, d^{-1}] = a_1^{p_1}, [a_1^{-1}, d] = b_1^{q_1}, [b_2^{-1}, d^{-1}] = a_2^{p_2}, [a_2^{-1}, d] = b_2^{q_2}, \quad (4)$$

$\dots, \dots, \dots,$

$$[b_g^{-1}, d^{-1}] = a_g^{p_g}, [a_g^{-1}, d] = b_g^{q_g}, [a_1, c] = 1, [b_1, c] = 1, [a_2, c] = 1, [b_2, c] = 1,$$

$$[a_3, c] = 1, [b_3, c] = 1,$$

$$[a_g, c] = 1, [b_g, c] = 1,$$

$$[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1, [c, d] = 1.$$

Let $\Sigma_g \subset M_g(\bar{\rho}, \bar{q})$ and T be a genus g and genus 1 surfaces that descend from the surfaces $\Sigma_g \times \{\text{pt}\}$ and $\{\text{pt}\} \times \mathbb{T}^2$ in $\Sigma_g \times \mathbb{T}^2$.

- $M_g(\bar{p}, \bar{q})$ is a locally trivial genus g bundle over \mathbb{T}^2 where T is a section. The $(a'_i \times c', a'_i, -p_i)$ or $(b'_i \times c'', b'_i, -q_i)$ Luttinger surgery in the trivial bundle $\Sigma_g \times \mathbb{T}^2$ preserves the fibration structure over \mathbb{T}^2 introducing a monodromy of the fiber Σ_g along the curve c in the base. Depending on the type of the surgery the monodromy is either $(t_{a_i})^{p_i}$ or $(t_{b_i})^{q_i}$, where t denotes a Dehn twist.
- $E(n)$ can be obtained as a desingularization of the branched double cover of $\mathbb{S}^2 \times \mathbb{S}^2$ with the branching set being 4 copies of $\{pt\} \times \mathbb{S}^2$ and $2n$ copies of $\mathbb{S}^2 \times \{pt\}$.

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- $E(n)$ can be obtained as a desingularization of the branched double cover of $\mathbb{S}^2 \times \mathbb{S}^2$ with the branching set being 4 copies of $\{pt\} \times \mathbb{S}^2$ and $2n$ copies of $\mathbb{S}^2 \times \{pt\}$. $E(n)$ admits a genus $n - 1$ fibration over \mathbb{S}^2 and an elliptic fibration over \mathbb{S}^2 . A regular fiber of the elliptic fibration on $E(n)$ intersects every genus $n - 1$ fiber of the other Lefschetz fibration twice.

Exotic Non-holomorphic Lefschetz fibrations over \mathbb{S}^2

- Let $X_{g,n}(\bar{p}, \bar{q})$ denote the symplectic sum of $M_g(\bar{p}, \bar{q})$ along the torus $T = c \times d$ with the elliptic surface $E(n)$ along a regular elliptic fiber.
- The symplectic 4-manifold $X_{g,n}(\bar{p}, \bar{q})$ admits a genus $2g + n - 1$ Lefschetz fibration over \mathbb{S}^2 with at least $4n + 4$ pairwise disjoint sphere sections of self intersection -2 . Moreover, $X_{g,n}(\bar{p}, \bar{q})$ contains a homologically essential embedded torus of square zero disjoint from these sections which intersects each fiber of the Lefschetz fibration twice.
- The symplectic 4-manifold $X_{g,n}(\bar{p}, \bar{q})$ can also be constructed as the twisted fiber sum of two copies of a genus $2g + n - 1$ Lefschetz fibration on $\Sigma_g \times \mathbb{S}^2 \# 4n\overline{\mathbb{C}\mathbb{P}^2}$. This follows from the fact that the symplectic sum of $E(n)$ along a regular elliptic fiber with $\Sigma_g \times \mathbb{T}^2$ along a natural square zero torus is diffeomorphic to the untwisted fiber sum of two copies of the genus $2g + n - 1$ fibration on $\Sigma_g \times \mathbb{S}^2 \# 4n\overline{\mathbb{C}\mathbb{P}^2}$. The gluing ϕ diffeomorphism can be described explicitly using the curves along which we perform our Luttinger surgeries.

- The fundamental group of the symplectic 4-manifold $X_{g,n}(\bar{p}, \bar{q})$ is generated by the set $\{a_i, b_i : 1 \leq i \leq g\}$ subject to the relations:
 $a_i^{p_i} = 1, b_i^{q_i} = 1$, for all $1 \leq i \leq g$, and
 $\prod_{j=1}^g [a_j, b_j] = 1$.
- By setting $p_i = 1$ and $q_i = 0$, for all $1 \leq i \leq g$, we see that the fundamental group of $X_{g,n}((1, 1, \dots, 1), (0, 0, \dots, 0))$ is a free group of rank g . The gluing diffeomorphism: $\phi = t_{a_1} \cdots t_{a_g}$.
- By setting $p_i = 1$ and $q_i = 1$, for all $1 \leq i \leq g$, we see that the fundamental group of $X_{g,n}((1, 1, \dots, 1), (1, 1, \dots, 1))$ is a trivial. The gluing diffeomorphism: $\phi = t_{a_1} t_{b_1} \cdots t_{a_g} t_{b_g}$.
- If we set $p_i = 1$ and $q_i = 0$, for all $1 \leq i \leq k$ and $p_i = 1$ and $q_i = 1$, for all $k + 1 \leq i \leq g$, the fundamental group of $X_{g,n}((1, 1, \dots, 1), (1, 1, \dots, 0))$ is a free group of rank k . The gluing diffeomorphism:
 $\phi = t_{a_1} \cdots t_{a_k} t_{a_{k+1}} t_{b_{k+1}} \cdots t_{a_g} t_{b_g}$.

Stein fillings from Lefschetz fibrations

Definition

A complex surface V is Stein if it admits a proper holomorphic embedding $f : V \rightarrow \mathbb{C}^n$ for some n . For a generic point $p \in \mathbb{C}^n$, consider the map $\phi : V \rightarrow \mathbb{R}$ defined by $\phi(z) = \|z - p\|^2$. For a regular value $a \in \mathbb{R}$, the level set $M = \phi^{-1}(a)$ is a smooth 3-manifold with a distinguished 2-plane field $\xi = TM \cap iTM \subset TV$. ξ defines a contact structure on M , and $S = \phi^{-1}([0, a])$ is called a Stein filling of (M, ξ) .

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Theorem (S. Akbulut - B. Ozbagci)

Let $f : X \rightarrow \mathbb{S}^2$ be a Lefschetz fibration with a section σ and let Σ denote a regular fiber of this fibration. Then $S = X \setminus \text{int}(\nu(\sigma \cup \Sigma))$ is a Stein filling of its boundary equipped with the induced (tight) contact structure, where $\nu(\sigma \cup \Sigma)$ denotes a regular neighborhood of $\sigma \cup \Sigma$ in X .

Finiteness Results on Stein Fillings

- The tight contact structure on \mathbb{S}^3 has a unique Stein filling (Y. Eliashberg, 1989).
- All tight contact structures on lens spaces $\mathbb{L}(p, q)$ have a finite number of Stein fillings (D. McDuff, P. Lisca, 1992).
- Finiteness results also have been verified for simple elliptic singularities (H. Ohta and Y. Ono, 2002).
- Finiteness results on Stein fillings of Seifert fibered spaces over \mathbb{S}^2 (L. Starkston, 2013).

Infiniteness Results on Stein Fillings

- B. Ozbagci and A. Stipsicz, and independently I. Smith showed that certain contact structures have an infinite number of Stein fillings (2003). Their examples have non-trivial fundamental groups and the fillings are pairwise non homotopy equivalent.
- Infinitely many simply-connected exotic Stein fillings (Akhmedov - Etnyre - Mark - Smith, 2007).
- Small exotic Stein fillings (S. Akbulut - K. Yasui, 2008).
- Infinitely many exotic Stein fillings with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_n$ (Akhmedov - Ozbagci, 2012).

Theorem (A. Akhmedov - B. Ozbagci, 2012)

For any finitely presented group G , there exist an infinite family of exotic Stein 4-manifolds $S_n(G, K_i)$ with $\pi_1(S_n(G, K_i)) = G$, where K_i are inf. family of genus $g \geq 2$ fibered knots with distinct Alexander polynomials.

THANK YOU!