Construction of Lefschetz fibrations via Luttinger surgery

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1. **Symplectic 4-manifolds and Lefschetz fibrations**
   - Symplectic 4-manifolds
   - Lefschetz fibrations
   - Family of Lefschetz fibrations by Y. Matsumoto, M. Korkmaz and Y. Gurtas

2. **Surgery on symplectic 4-manifolds**
   - Luttinger Surgery
   - Symplectic Connected Sum

3. **Construction of Symplectic 4-Manifolds**
   - Luttinger surgeries on product 4-manifolds $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times \mathbb{T}^2$
   - Construction of Lefschetz fibration via Luttinger Surgery
   - Exotic Stein fillings
Symplectic manifolds
Symplectic manifolds

Definition

A (compact) symplectic $2n$-manifold $(X, \omega)$ is a smooth $2n$-manifold with a symplectic form $\omega \in \Omega^2(X)$ (i.e., $\omega$ is closed ($d\omega = 0$) and non-degenerate ($\omega^n = \omega \wedge \cdots \wedge \omega > 0$ everywhere)) 2-form.
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- $X = \mathbb{R}^{2n}$ with linear coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$ and with the 2-form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. 

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- If $(X_1, \omega_1)$ and $(X_2, \omega_2)$ are symplectic manifolds, then $\pi_1^*\omega_1 + \pi_2^*\omega_2$ gives a symplectic structure on $X_1 \times X_2$. $\Sigma_n \times \Sigma_m$ are symplectic 4-manifolds.
- Every Kähler manifold is also a symplectic manifold.
- A closed complex surface $S$ is Kähler iff the first Betti number $b_1(S)$ is even.
Kodaira-Thurston manifold
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- $\gamma_1 : (x_1, x_2, y_1, y_2) \rightarrow (x_1, x_2 + 1, y_1, y_2)$
- $\gamma_2 : (x_1, x_2, y_1, y_2) \rightarrow (x_1, x_2, y_1, y_2 + 1)$
- $\gamma_3 : (x_1, x_2, y_1, y_2) \rightarrow (x_1 + 1, x_2, y_1, y_2)$
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- $\gamma_4 : (x_1, x_2, y_1, y_2) \rightarrow (x_1, x_2 + y_2, y_1 + 1, y_2)$

$M = \mathbb{R}^4/\Gamma$ admits both symplectic structure and complex structures, but non-Kähler.
Example
Kodaira-Thurston manifold continued

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\phi_*(a) &= a + b \\
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$$\pi_1(W) = \langle g_1, g_2, g_3, x; \mid [g_1, g_2] = 1, [g_2, g_3] = 1, [g_1, g_3] = g_2, [x, g_i] = 1 \rangle.$$
Symplectic 4-manifolds via surface bundles over surfaces
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Theorem (W. Thurston (1976))

Assume that $\Sigma_g, \Sigma_h$ are closed, oriented, 2-dimensional surfaces. If $f : X \rightarrow \Sigma_h$ is a bundle with fiber $\Sigma_g$ and the homology class of the fiber is nonzero in $H_2(X; \mathbb{R})$, then $X$ admits a symplectic structure.
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**Theorem (J. Bryan - R. Donagi)**

For any integers $n \geq 2$, there exist smooth algebraic surface $X_n$ that have signature $\sigma(X_n) = 8/3n(n - 1)(n + 1)$ and admit two smooth fibrations $X_n \to B$ and $X_n \to B'$ such that the base and fiber genus are $(3, 3n^3 - n^2 + 1)$ and $(2n^2 + 1, 3n)$ respectively.
Lefschetz fibrations

Definition

Let $X$ be a compact, connected, oriented, smooth 4-manifold. A *Lefschetz fibration* on $X$ is a smooth map $f : X \rightarrow \Sigma_h$, where $\Sigma_h$ is a compact, oriented, smooth 2-manifold of genus $h$, such that $f$ is surjective and each critical point of $f$ has an orientation preserving chart on which $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ is given by $f(z_1, z_2) = z_1^2 + z_2^2$. 
Construction of Lefschetz fibrations via Luttinger surgery

Figure: Lefschetz fibration on $X$ over $S^2$
The genus of the regular fiber of $f$ is defined to be the \textit{genus of the Lefschetz fibration}. 
The genus of the regular fiber of $f$ is defined to be the genus of the Lefschetz fibration. Let $p_1, \ldots, p_s$ denote the critical points of Lefschetz fibration $f : X \to \Sigma_h$.

$$e(X) = e(\Sigma_h)e(\Sigma_g) + s$$

$\sigma(X)$ well understood for fibrations over $\mathbb{S}^2$

(Y. Matsumoto, H. Endo, B. Ozbagci, I. Smith, ...)
Monodromy of Lefschetz fibration

A singular fiber of the genus $g$ Lefschetz fibration can be described by its monodromy, i.e., an element of the mapping class group $\mathcal{M}_g$. This element is a right-handed (or a positive) Dehn twist along a simple closed curve on $\Sigma_g$, called the *vanishing cycle*.
For a genus $g$ Lefschetz fibration over $S^2$, the product of right handed Dehn twists $t_{\alpha_i}$ along the vanishing cycles $\alpha_i$, for $i = 1, \cdots, s$, determines the \textit{global monodromy} of the Lefschetz fibration, the relation $t_{\alpha_1} \cdot t_{\alpha_2} \cdots \cdot t_{\alpha_s} = 1$ in $M_g$. Conversely, such a relation in $M_g$ determines a genus $g$ Lefschetz fibration over $S^2$ with the vanishing cycles $\alpha_1, \cdots, \alpha_s$.

\textbf{Figure:} Lefschetz fibration on $X$ over $S^2$
Example (Genus one Lefschetz fibrations)

$M_1$ be the mapping class group of the torus $\mathbb{T}^2 = a \times b$. 
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$M_1$ be the mapping class group of the torus $\mathbb{T}^2 = a \times b$. $M_1 = \text{SL}(2, \mathbb{Z})$ is generated by

$$t_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$t_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Subject to the relations

$$t_a t_b t_a = t_b t_a t_b$$

$$(t_a t_b)^6 = 1$$
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$(t_a t_b)^{6n} = 1$ in $M_1$. The total space of this fibration is the elliptic surface $E(n)$. $E(1) = \mathbb{CP}^2 \# 9 \overline{\mathbb{CP}}^2$, the complex projective plane blown up at 9 points, and $E(2)$ is $K3$ surface. $E(n)$ also admits a genus $n - 1$ Lefschetz fibration over $S^2$. 
Example (Hyperelliptic Lefschetz fibrations)

Let $\alpha_1, \alpha_2, \ldots, \alpha_{2g}, \alpha_{2g+1}$ denote the collection of simple closed curves given in Figure, and $c_i$ denote the right handed Dehn twists $t_{\alpha_i}$ along the curve $\alpha_i$.

![Vanishing cycles of the genus $g$ Lefschetz fibration given by hyperelliptic involution](image)

The following relations hold in the mapping class group $M_g$:

\begin{align*}
\Gamma_1(g) &= (c_1 c_2 \cdots c_{2g-1} c_{2g} c_{2g+1}^2 c_{2g} c_{2g-1} \cdots c_2 c_1)^2 = 1. \\
\Gamma_2(g) &= (c_1 c_2 \cdots c_{2g-1} c_{2g} c_{2g+1})^{2g+2} = 1. \\
\Gamma_3(g) &= (c_1 c_2 \cdots c_{2g-1} c_{2g})^{2(2g+1)} = 1.
\end{align*}
The monodromy relation $\Gamma_1(g) = 1$, the corresponding genus $g$ Lefschetz fibrations over $S^2$ has total space $X(g, 1) = \mathbb{CP}^2 \# (4g + 5)\overline{\mathbb{CP}^2}$, the complex projective plane blown up at $4g + 5$ points.

It is known that for $g \geq 2$, the above fibration on $X(g, 1)$ admits $4g + 4$ disjoint $(-1)$-sphere sections (proof of this fact using a mapping class group argument is due to S. Tanaka).

The fiber class is of the form $(g + 2)h - ge_1 - e_2 - \cdots - e_{4g+5}$, where $e_i$ denotes the homology class of the exceptional sphere of the $i$-th blow up and $h$ denotes the pullback of the hyperplane class of $\mathbb{CP}^2$. The exceptional spheres represented by the homology classes $e_2, e_3, \ldots, e_{4g+5}$ are sections of the Lefschetz fibration $X(g, 1) \to S^2$. 
**Theorem (S. Donaldson)**

For any symplectic 4-manifold $X$, there exists a non-negative integer $n$ such that the $n$-fold blowup $X \# n\overline{\mathbb{CP}^2}$ of $X$ admits a Lefschetz fibration $f : X \# n\overline{\mathbb{CP}^2} \to \mathbb{S}^2$.

**Theorem (R. Gompf)**

Assume that the closed 4-manifold $X$ admits a genus $g$ Lefschetz fibration $f : X \to \Sigma_h$, and let $[F]$ denote the homology class of the fiber. Then $X$ admits a symplectic structure with symplectic fibers iff $[F] \neq 0$ in $H_2(X; \mathbb{R})$. If $e_1, \ldots, e_n$ is a finite set of sections of the Lefschetz fibration, the symplectic form $\omega$ can be chosen in such a way that all these sections are symplectic.


For any finitely presented group $G$, there exist a Lefschetz fibration $X(G)$ over $\mathbb{S}^2$ with $\pi_1(X(G)) = G$.

$b_2^+(X(G))$ is very large, and depends from the presentation of $G$. 
Main Theorems

Theorem (A. Akhmedov - B. Ozbagci, 2012)

For any finitely presented group $G$, there exist a closed symplectic 4-manifold $X_n(G)$ with $\pi_1(X(G)) = G$, which admits a genus $2g + n - 1$ Lefschetz fibration over $\mathbb{S}^2$ that has at least $4n + 4$ pairwise disjoint sphere sections of self intersection $-2$. Moreover, $X_n(G)$ contains a homologically essential embedded torus of square zero disjoint from these sections which intersects each fiber of the Lefschetz fibrations twice.

Theorem (A. Akhmedov - B. Ozbagci, 2012)

There exist an infinite family of non-holomorphic Lefschetz fibrations $X_n(G, K_i)$ over $\mathbb{S}^2$ with $\pi_1(X_n(G, K_i)) = G$ that can be obtained from $X_n(G)$ via knot surgery along $K_i$, where $K_i$ are an infinite family of genus $g \geq 2$ fibered knots with distinct Alexander polynomials.
Lefschetz fibrations by Y. Matsumoto and M. Korkmaz

Let assume $g = 2k$.

The 4-manifold $Y(1, k) = \Sigma_k \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}}^2$ is the total space of the genus $g$ Lefschetz fibration over $S^2$ with $2g + 4$ singular fibers. This was shown by Yukio Matsumoto for $k = 1$, and in the case $k \geq 2$ by Mustafa Korkmaz, by factorizing the *vertical* involution $\theta$ of the genus $2k$ surface.

**Figure:** The involution $\theta$ of the genus $2k$ surface
Theorem (Y. Matsumoto, M. Korkmaz)

Let $\theta$ denote the vertical involution of the genus $g$ surface with 2 fixed points. In the mapping class group $M_g$, the following relations between right handed Dehn twists hold:

\begin{align*}
  & a) \ (t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_c)^2 = \theta^2 = 1 \text{ if } g \text{ is even}, \\
  & b) \ (t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} (t_a)^2 (t_b)^2)^2 = \theta^2 = 1 \text{ if } g \text{ is odd}.
\end{align*}

$B_k, a, b, c$ are the simple closed curves defined as in Figure.

**Figure:** The vanishing cycles
Yusuf Gurtas generalized the constructions of Matsumoto and Korkmaz even further. He presented the positive Dehn twist expression for a new set of involutions in the mapping class group $M_{2k+n-1}$ of a compact, closed, oriented 2-dimensional surface $\Sigma_{2k+n-1}$. The total space of these genus $g = 2k + n - 1$ Lefschetz fibration over $S^2$ is $Y(n, k) = \Sigma_k \times S^2 \# 4n\mathbb{C}P^2$. 
Figure: The involution $\theta$ of the surface $\Sigma_{2k+n-1}$
A generic horizontal fiber is the double cover of $S^2$, branched over two points. Thus, we have a sphere fibration on $Y(n,k) = \Sigma_k \times S^2 \#4n\overline{CP^2}$. A generic fiber of the vertical fibration is the double cover of $\Sigma_k$, branched over $2n$ points. Thus, a generic fiber of the vertical fibration has genus $n + 2k - 1$.

**Figure:** The branch locus for $\Sigma_k \times S^2 \#4n\overline{CP^2}$
Theorem (Y. Gurtas)

The positive Dehn twist expression for the involution $\theta$ is given by

$$\theta = e_{2i+2} \cdots e_{2n-2} e_{2n-1} e_{2i} \cdots e_1 B_0 e_{2n-1} \cdots e_{2i+2} e_1 \cdots e_{2i} B_1 B_2 \cdots B_{4k-1} B_{4k} e_{2i+1}. $$
Figure: The involution $\theta$ of the surface $\Sigma_{2k+n-1}$
Construction Tools
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**Definition**

Let $X$ be a symplectic 4-manifold with a symplectic form $\omega$, and the torus $\Lambda$ be a Lagrangian submanifold of $X$ with self-intersection 0. Given a simple loop $\lambda$ on $\Lambda$, let $\lambda'$ be a simple loop on $\partial (\nu \Lambda)$ that is parallel to $\lambda$ under the Lagrangian framing. For any integer $m$, the $(\Lambda, \lambda, 1/m)$ Luttinger surgery on $X$ will be $X_{\Lambda, \lambda}(1/m) = (X - \nu(\Lambda)) \cup_{\phi} (S^1 \times S^1 \times D^2)$, the $1/m$ surgery on $\Lambda$ with respect to $\lambda$ under the Lagrangian framing. Here $\phi : S^1 \times S^1 \times \partial D^2 \to \partial (X - \nu(\Lambda))$ denotes a gluing map satisfying $\phi([\partial D^2]) = m[\lambda'] + [\mu_\Lambda]$ in $H_1(\partial (X - \nu(\Lambda)))$, where $\mu_\Lambda$ is a meridian of $\Lambda$.

$X_{\Lambda, \lambda}(1/m)$ possesses a symplectic form that restricts to the original symplectic form $\omega$ on $X \setminus \nu \Lambda$.

Luttinger’s surgery has been very effective tool recently for constructing exotic smooth structures.
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Let $\mathbb{T}^4 = a \times b \times c \times d \cong (c \times d) \times (a \times b)$. Let $K_n$ be an $n$-twist knot. Let $M_{K_n}$ denote the result of performing 0 Dehn surgery on $S^3$ along $K_n$. $S^1 \times M_{K_n}$ is obtained from $\mathbb{T}^4 = (c \times d) \times (a \times b) = c \times (d \times a \times b) = S^1 \times \mathbb{T}^3$ by first performing a Luttinger surgery $(c \times \tilde{a}, \tilde{a}, -1)$ followed by a surgery $(c \times \tilde{b}, \tilde{b}, -n)$. The tori $c \times \tilde{a}$ and $c \times \tilde{b}$ are Lagrangian and the second tilde circle factors in $\mathbb{T}^3$ are as pictured. Use the Lagrangian framing to trivialize their tubular neighborhoods. When $n = 1$ the second surgery is also a Luttinger surgery.

Figure: The 3-torus $d \times a \times b$
Symplectic Connected Sum

Definition
Let $X_1$ and $X_2$ are symplectic 4-manifolds, and $F_i \subset X_i$ are 2-dimensional, smooth, closed, connected symplectic submanifolds in them. Suppose that $[F_1]^2 + [F_2]^2 = 0$ and the genera of $F_1$ and $F_2$ are equal. Take an orientation-preserving diffeomorphism $\psi : F_1 \rightarrow F_2$ and lift it to an orientation-reversing diffeomorphism $\Psi : \partial \nu F_1 \rightarrow \partial \nu F_2$ between the boundaries of the tubular neighborhoods of $\nu F_i$. Using $\Psi$, we glue $X_1 \setminus \nu F_1$ and $X_2 \setminus \nu F_2$ along the boundary. The 4-manifold $X_1 \#_\psi X_2$ is called the (symplectic) connected sum of $X_1$ and $X_2$ along $F_1$ and $F_2$, determined by $\Psi$.

\[
e(X_1 \#_\psi X_2) = e(X_1) + e(X_2) + 4(g - 1), \\
\sigma(X_1 \#_\psi X_2) = \sigma(X_1) + \sigma(X_2),
\]
Lemma

Let \( f : X \to \mathbb{S}^2 \) be a genus \( g \) Lefschetz fibration with global monodromy given by the relation \( t_{\alpha_1} \cdot t_{\alpha_2} \cdot \cdots \cdot t_{\alpha_s} = 1 \). Let \( X \#_{\psi} X \) denote the fiber sum of \( X \) with itself by a self-diffeomorphism \( \psi \) of the generic fiber \( \Sigma \). Then \( X \#_{\psi} X \) has the vanishing cycles \( \alpha_1, \alpha_2, \cdots, \alpha_s, \psi(\alpha_1), \psi(\alpha_2), \cdots, \psi(\alpha_s) \).
Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times \mathbb{T}^2$

Fix integers $n \geq 2$, $p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. Let $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$ denote symplectic 4-manifold obtained by performing the following $2n + 4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$. These $2n + 4$ surgeries comprise of the following 8 surgeries

$$(a'_1 \times c'_1, a'_1, -1), \quad (b'_1 \times c''_1, b'_1, -1),$$
$$(a'_2 \times c'_2, a'_2, -1), \quad (b'_2 \times c''_2, b'_2, -1),$$
$$(a'_2 \times c'_1, c'_1, +1/p_1), \quad (a''_2 \times d'_1, d'_1, +1/q_1),$$
$$(a'_1 \times c'_2, c'_2, +1/p_2), \quad (a'_1 \times d'_2, d'_2, +1/q_2),$$

together with the following $2(n - 2)$ additional Luttinger surgeries

$$(b'_1 \times c'_3, c'_3, -1/p_3), \quad (b'_2 \times d'_3, d'_3, -1/q_3),$$
$$\cdots, \quad \cdots,$$
$$(b'_1 \times c'_n, c'_n, -1/p_n), \quad (b'_2 \times d'_n, d'_n, -1/q_n).$$
Here, $a_i, b_i$ ($i = 1, 2$) and $c_j, d_j$ ($j = 1, \ldots, n$) are the standard loops that generate $\pi_1(\Sigma_2)$ and $\pi_1(\Sigma_n)$.

**Figure:** Lagrangian tori $a_i' \times c_j'$ and $a_i'' \times d_j'$
The Euler characteristic of $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$ is $4n - 4$ and its signature is 0. The fundamental group $\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$ is generated by $a_i, b_i, c_j, d_j$ ($i = 1, 2$ and $j = 1, \ldots, n$) and the following relations hold in $\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$:

\[
\begin{align*}
[b_1^{-1}, d_1^{-1}] &= a_1, \\
[a_1^{-1}, d_1] &= b_1, \\
[b_2^{-1}, d_2^{-1}] &= a_2, \\
[a_2^{-1}, d_2] &= b_2, \\
[d_1^{-1}, b_2^{-1}] &= c_1^{p_1}, \\
[c_1^{-1}, b_2] &= d_1^{q_1}, \\
[d_2^{-1}, b_1^{-1}] &= c_2^{p_2}, \\
[c_2^{-1}, b_1] &= d_2^{q_2}, \\
[a_1, c_1] &= 1, \\
[a_1, c_2] &= 1, \\
[a_1, d_2] &= 1, \\
[b_1, c_1] &= 1, \\
[a_2, c_1] &= 1, \\
[a_2, c_2] &= 1, \\
[a_2, d_1] &= 1, \\
[b_2, c_2] &= 1, \\
[a_1, b_1][a_2, b_2] &= 1, \\
\prod_{j=1}^{n} [c_j, d_j] &= 1, \\
[a_1^{-1}, d_3^{-1}] &= c_3^{p_3}, \\
[a_2^{-1}, c_3^{-1}] &= d_3^{q_3}, \\
[a_1^{-1}, d_n^{-1}] &= c_n^{p_n}, \\
[a_2^{-1}, c_n^{-1}] &= d_n^{q_n}, \\
[b_1, c_3] &= 1, \\
[b_2, d_3] &= 1, \\
[b_1, c_n] &= 1, \\
[b_2, d_n] &= 1.
\end{align*}
\]
The surfaces $\Sigma_2 \times \{\text{pt}\}$ and $\{\text{pt}\} \times \Sigma_n$ in $\Sigma_2 \times \Sigma_n$ descend to surfaces in $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$. They are symplectic submanifolds in $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$. Denote their images by $\Sigma_2$ and $\Sigma_n$. Note that $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$ and $[\Sigma_2] \cdot [\Sigma_n] = 1$. 
Let \( \{p_i, q_i \geq 0 : 1 \leq i \leq g \} \) be a set of nonnegative integers and let \( \overline{p} = (p_1, \ldots, p_g) \) and \( \overline{q} = (q_1, \ldots, q_g) \).
Let \( \{p_i, q_i \geq 0 : 1 \leq i \leq g\} \) be a set of nonnegative integers and let \( \overline{p} = (p_1, \ldots, p_g) \) and \( \overline{q} = (q_1, \ldots, q_g) \). Denote by \( M_g(\overline{p}, \overline{q}) \) the symplectic 4-manifold obtained by performing the following \( 2g \) Luttinger surgeries on the symplectic 4-manifold \( \Sigma_g \times \mathbb{T}^2 \):

\[
\begin{align*}
(a'_1 \times c', a'_1, -1/p_1), & \quad (b'_1 \times c'', b'_1, -1/q_1), \\
(a'_2 \times c', a'_2, -1/p_2), & \quad (b'_2 \times c'', b'_2, -1/q_2), \\
\quad \vdots \quad \vdots \\
(a'_{g-1} \times c', a'_{g-1}, -1/p_{g-1}), & \quad (b'_{g-1} \times c'', b'_{g-1}, -1/q_{g-1}), \\
(a'_g \times c', a'_g, -1/p_g), & \quad (b'_g \times c'', b'_g, -1/q_g).
\end{align*}
\]

Here, \( a_i, b_i \) \( (i = 1, 2, \ldots, g) \) and \( c, d \) denote the standard generators of \( \pi_1(\Sigma_g) \) and \( \pi_1(\mathbb{T}^2) \), respectively.
The fundamental group of $M_g(\overline{p}, \overline{q})$ is generated by $a_i, b_i$ ($i = 1, 2, 3 \cdots, g$) and $c, d$, and the following relations hold in $M_g(\overline{p}, \overline{q})$:

$$[b_1^{-1}, d^{-1}] = a_1^{p_1}, \quad [a_1^{-1}, d] = b_1^{q_1}, \quad [b_2^{-1}, d^{-1}] = a_2^{p_2}, \quad [a_2^{-1}, d] = b_2^{q_2}, \quad (4)$$

$$\cdots, \cdots, \cdots,$$

$$[b_g^{-1}, d^{-1}] = a_g^{p_g}, \quad [a_g^{-1}, d] = b_g^{q_g}, \quad [a_1, c] = 1, \quad [b_1, c] = 1, \quad [a_2, c] = 1, \quad [b_2, c] = 1,$$

$$[a_3, c] = 1, \quad [b_3, c] = 1,$$

$$[a_g, c] = 1, \quad [b_g, c] = 1,$$

$$[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1, \quad [c, d] = 1.$$

Let $\Sigma_g \subset M_g(\overline{p}, \overline{q})$ and $T$ be a genus $g$ and genus 1 surfaces that descend from the surfaces $\Sigma_g \times \{\text{pt}\}$ and $\{\text{pt}\} \times \mathbb{T}^2$ in $\Sigma_g \times \mathbb{T}^2$. 
- $M_g(\overline{p}, \overline{q})$ is a locally trivial genus $g$ bundle over $\mathbb{T}^2$ where $T$ is a section. The $(a'_i \times c', a'_i, -p_i)$ or $(b'_i \times c'', b'_i, -q_i)$ Luttinger surgery in the trivial bundle $\Sigma_g \times \mathbb{T}^2$ preserves the fibration structure over $\mathbb{T}^2$ introducing a monodromy of the fiber $\Sigma_g$ along the curve $c$ in the base. Depending on the type of the surgery the monodromy is either $(t_{a_i})^{p_i}$ or $(t_{b_i})^{q_i}$, where $t$ denotes a Dehn twist.

- $E(n)$ can be obtained as a desingularization of the branched double cover of $\mathbb{S}^2 \times \mathbb{S}^2$ with the branching set being 4 copies of $\{pt\} \times \mathbb{S}^2$ and $2n$ copies of $\mathbb{S}^2 \times \{pt\}$. 
$M_g(\overline{p}, \overline{q})$ is a locally trivial genus $g$ bundle over $\mathbb{T}^2$ where $T$ is a section. The $(a'_i \times c', a'_i, -p_i)$ or $(b'_i \times c'', b'_i, -q_i)$ Luttinger surgery in the trivial bundle $\Sigma_g \times \mathbb{T}^2$ preserves the fibration structure over $\mathbb{T}^2$ introducing a monodromy of the fiber $\Sigma_g$ along the curve $c$ in the base. Depending on the type of the surgery the monodromy is either $(t_{a_i})^{p_i}$ or $(t_{b_i})^{q_i}$, where $t$ denotes a Dehn twist.

$E(n)$ can be obtained as a desingularization of the branched double cover of $S^2 \times S^2$ with the branching set being 4 copies of $\{pt\} \times S^2$ and $2n$ copies of $S^2 \times \{pt\}$. $E(n)$ admits a genus $n - 1$ fibration over $S^2$ and an elliptic fibration over $S^2$. A regular fiber of the elliptic fibration on $E(n)$ intersects every genus $n - 1$ fiber of the other Lefschetz fibration twice.
Construction of Lefschetz fibrations over $\mathbb{S}^2$

- Let $X_{g,n}(\overline{p}, \overline{q})$ denote the symplectic sum of $M_g(\overline{p}, \overline{q})$ along the torus $T = c \times d$ with the elliptic surface $E(n)$ along a regular elliptic fiber.

- The symplectic 4-manifold $X_{g,n}(\overline{p}, \overline{q})$ admits a genus $2g + n - 1$ Lefschetz fibration over $\mathbb{S}^2$ with at least $4n + 4$ pairwise disjoint sphere sections of self intersection $-2$. Moreover, $X_{g,n}(\overline{p}, \overline{q})$ contains a homologically essential embedded torus of square zero disjoint from these sections which intersects each fiber of the Lefschetz fibration twice.

- The symplectic 4-manifold $X_{g,n}(\overline{p}, \overline{q})$ can also be constructed as the twisted fiber sum of two copies of a genus $2g + n - 1$ Lefschetz fibration on $\Sigma_g \times \mathbb{S}^2 \# 4n\overline{\mathbb{C}\mathbb{P}^2}$. This follows from the fact that the symplectic sum of $E(n)$ along a regular elliptic fiber with $\Sigma_g \times \mathbb{T}^2$ along a natural square zero torus is diffeomorphic to the untwisted fiber sum of two copies of the genus $2g + n - 1$ fibration on $\Sigma_g \times \mathbb{S}^2 \# 4n\overline{\mathbb{C}\mathbb{P}^2}$. The gluing $\phi$ diffeomorphism can be described explicitly using the curves along which we perform our Luttinger surgeries.
The fundamental group of the symplectic 4-manifold $X_{g,n}(\overline{p}, \overline{q})$ is generated by the set \{\(a_i, b_i : 1 \leq i \leq g\)\} subject to the relations: 
\[a_i^{p_i} = 1, \ b_i^{q_i} = 1, \text{ for all } 1 \leq i \leq g, \text{ and }\]
\[
\prod_{j=1}^{g}[a_j, b_j] = 1.
\]
By setting $p_i = 1$ and $q_i = 0$, for all $1 \leq i \leq g$, we see that the fundamental group of $X_{g,n}((1,1,\ldots,1),(0,0,\ldots,0))$ is a free group of rank $g$. The gluing diffeomorphism: $\phi = t_{a_1} \cdots t_{a_g}$.

By setting $p_i = 1$ and $q_i = 1$, for all $1 \leq i \leq g$, we see that the fundamental group of $X_{g,n}((1,1,\ldots,1),(1,1,\ldots,1))$ is a trivial. The gluing diffeomorphism: $\phi = t_{a_1} t_{b_1} \cdots t_{a_g} t_{b_g}$.

If we set $p_i = 1$ and $q_i = 0$, for all $1 \leq i \leq k$ and $p_i = 1$ and $q_i = 1$, for all $k + 1 \leq i \leq g$, the fundamental group of $X_{g,n}((1,1,\ldots,1),(1,1,\ldots,0))$ is a free group of rank $k$. The gluing diffeomorphism: $\phi = t_{a_1} \cdots t_{a_k} t_{a_{k+1}} t_{b_{k+1}} \cdots t_{a_g} t_{b_g}$.
Stein fillings from Lefschetz fibrations

Definition

A complex surface $V$ is Stein if it admits a proper holomorphic embedding $f : V \to \mathbb{C}^n$ for some $n$. For a generic point $p \in \mathbb{C}^n$, consider the map $\phi : V \to \mathbb{R}$ defined by $\phi(z) = ||z - p||^2$. For a regular value $a \in \mathbb{R}$, the level set $M = \phi^{-1}(a)$ is a smooth 3-manifold with a distinguished 2-plane field $\xi = TM \cap iTM \subset TV$. $\xi$ defines a contact structure on $M$, and $S = \phi^{-1}([0, a])$ is called a Stein filling of $(M, \xi)$.

Theorem (S. Akbulut - B. Ozbagci)

Let $f : X \to S^2$ be a Lefschetz fibration with a section $\sigma$ and let $\Sigma$ denote a regular fiber of this fibration. Then $S = X \setminus \text{int}(\nu(\sigma \cup \Sigma))$ is a Stein filling of its boundary equipped with the induced (tight) contact structure, where $\nu(\sigma \cup \Sigma)$ denotes a regular neighborhood of $\sigma \cup \Sigma$ in $X$. 
Finiteness Results on Stein Fillings

- The tight contact structure on $S^3$ has a unique Stein filling (Y. Eliashberg, 1989).
- All tight contact structures on lens spaces $\mathbb{L}(p, q)$ have a finite number of Stein fillings (D. McDuff, P. Lisca, 1992).
- Finiteness results also have been verified for simple elliptic singularities (H. Ohta and Y. Ono, 2002).
- Finiteness results on symplectic fillings of Seifert fibered spaces over $S^2$ (L. Starkston, 2013).
Infiniteness Results on Stein Fillings

- B. Ozbagci and A. Stipsicz, and independently I. Smith showed that certain contact structures have an infinite number of Stein fillings (2003). Their examples have non-trivial fundamental groups.
- Infinitely many simply-connected exotic Stein fillings (Akhmedov - Etnyre - Mark - Smith, 2007).
- Exotic Stein fillings with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_n$ (Akhmedov - Ozbagci, 2012).
Theorem (A. Akhmedov - B. Ozbagci, 2012)

For any finitely presented group $G$, there exist an infinite family of exotic Stein 4-manifolds $S_n(G, K_i)$ with $\pi_1(S_n(G, K_i)) = G$, where $K_i$ are inf. family of genus $g \geq 2$ fibered knots with distinct Alexander polynomials.
THANK YOU!