Construction of Lefschetz fibrations via Luttinger surgery

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Outline

1. **Symplectic 4-manifolds and Lefschetz fibrations**
   - Symplectic 4-manifolds
   - Lefschetz fibrations
   - Family of Lefschetz fibrations by Y. Matsumoto, M. Korkmaz and Y. Gurtas

2. **Surgery on symplectic 4-manifolds**
   - Luttinger Surgery
   - Symplectic Connected Sum

3. **Construction of Symplectic 4-Manifolds**
   - Luttinger surgeries on product 4-manifolds $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times \mathbb{T}^2$
   - Construction of Lefschetz fibration via Luttinger Surgery
   - Exotic Stein fillings

Symplectic manifolds

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Definition

A (compact) symplectic $2n$-manifold (X, ω) is a smooth $2n$-manifold with a symplectic form $\omega \in \Omega^2(X)$ (i.e., ω is closed ($d\omega = 0$) and non-degenerate ($\omega^n = \omega \wedge \cdots \wedge \omega > 0$ everywhere) 2-form.

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- Every Kähler manifold is also a symplectic manifold.
- A closed complex surface S is Kähler iff the first Betti number $b_1(S)$ is even.

Kodaira-Thurston manifold

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$$\gamma_2 : (x_1, x_2, y_1, y_2) \longrightarrow (x_1, x_2, y_1, y_2 + 1)$$
$$\gamma_3 : (x_1, x_2, y_1, y_2) \longrightarrow (x_1 + 1, x_2, y_1, y_2)$$
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\end{aligned}$$

$M = \mathbb{R}^4/\Gamma$ admits both symplectic structure and complex structures, but non-Kähler.

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Let Z_ϕ denote the mapping torus of ϕ. $W = Z_\phi \times \mathbb{S}^1$ admits $\mathbb{T}^2$ bundle structure over $\mathbb{T}^2$. Presentation for the fundamental group of W:

$$\pi_1(W) = \langle g_1, g_2, g_3, x; \mid [g_1, g_2] = 1, \ [g_2, g_3] = 1, \ [g_1, g_3] = g_2, \ [x, g_i] = 1 \rangle.$$

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Theorem (*W. Thurston (1976)*)

Assume that Σ_g, Σ_h are closed, oriented, 2-dimensional surfaces. If $f : X \to \Sigma_h$ is a bundle with fiber Σ_g and the homology class of the fiber is nonzero in $H_2(X; \mathbb{R})$, then X admits a symplectic structure.

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Theorem (*J. Bryan - R. Donagi*)

For any integers $n \geq 2$, there exisit smooth algebraic surface X_n that have signature $\sigma(X_n) = 8/3n(n-1)(n+1)$ and admit two smooth fibrations $X_n \longrightarrow B$ and $X_n \longrightarrow B'$ such that the base and fiber genus are $(3, 3n^3 - n^2 + 1)$ and $(2n^2 + 1, 3n)$ respectively.

Lefschetz fibrations

Definition

Let X be a compact, connected, oriented, smooth 4-manifold. A *Lefschetz fibration* on X is a smooth map $f : X \longrightarrow \Sigma_h$, where Σ_h is a compact, oriented, smooth 2-manifold of genus h, such that f is surjective and each critical point of f has an orientation preserving chart on which $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is given by $f(z_1, z_2) = {z_1}^2 + {z_2}^2$.

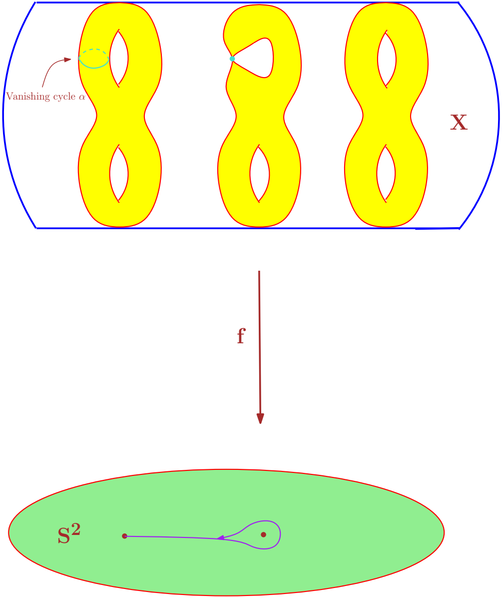

Figure: Lefschetz fibration on X over $\mathbb{S}^2$

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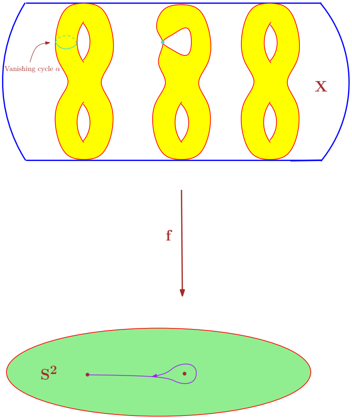
$$e(X) = e(\Sigma_h)e(\Sigma_g) + s$$
$$\sigma(X) \ \ \textit{well understood for fibrations over} \ \ \mathbb{S}^2$$

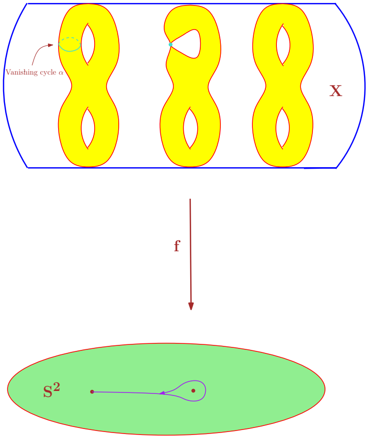
$(Y.Matsumoto, H.Endo, B.Ozbagci, I.Smith, ...)$

Monodromy of Lefschetz fibration

A singular fiber of the genus g Lefschetz fibration can be described by its monodromy, i.e., an element of the mapping class group M_g. This element is a right-handed (or a positive) Dehn twist along a simple closed curve on Σ_g, called the *vanishing cycle*.



For a genus g Lefschetz fibration over $\mathbb{S}^2$, the product of right handed Dehn twists t_{α_i} along the vanishing cycles α_i, for $i = 1, \cdots, s$, determines the *global monodromy* of the Lefschetz fibration, the relation $t_{\alpha_1} \cdot t_{\alpha_2} \cdot \cdots \cdot t_{\alpha_s} = 1$ in M_g. Conversely, such a relation in M_g determines a genus g Lefschetz fibration over $\mathbb{S}^2$ with the vanishing cycles $\alpha_1, \cdots, \alpha_s$.



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M_1 be the mapping class group of the torus $\mathbb{T}^2 = a \times b$. $M_1 = SL(2, \mathbb{Z})$ is generated by

$$t_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$t_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Subject to the relations

$$t_a t_b t_a = t_b t_a t_b$$
$$(t_a t_b)^6 = 1$$

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$(t_a t_b)^{6n} = 1$ in M_1. The total space of this fibration is the elliptic surface $E(n)$. $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$, the complex projective plane blown up at 9 points, and $E(2)$ is $K3$ surface. $E(n)$ also admits a genus $n-1$ Lefschetz fibration over $\mathbb{S}^2$.

Example (Hyperelliptic Lefschetz fibrations)

Let α_1, α_2, ... , α_{2g}, α_{2g+1} denote the collection of simple closed curves given in Figure, and c_i denote the right handed Dehn twists t_{α_i} along the curve α_i.

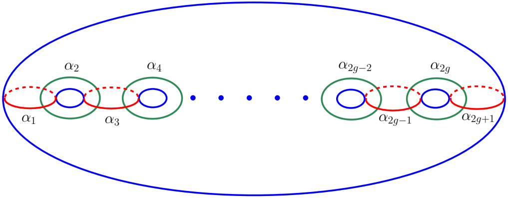


Figure: Vanishing cycles of the genus g Lefschetz fibration given by hyperelliptic involution

The following relations hold in the mapping class group M_g:

$$\begin{aligned} \Gamma_1(g) &= (c_1 c_2 \cdots c_{2g-1} c_{2g} {c_{2g+1}}^2 c_{2g} c_{2g-1} \cdots c_2 c_1)^2 = 1. \\ \Gamma_2(g) &= (c_1 c_2 \cdots c_{2g-1} c_{2g} c_{2g+1})^{2g+2} = 1. \\ \Gamma_3(g) &= (c_1 c_2 \cdots c_{2g-1} c_{2g})^{2(2g+1)} = 1. \end{aligned} \tag{1}$$

The monodromy relation $\Gamma_1(g) = 1$, the corresponding genus g Lefschetz fibrations over $\mathbb{S}^2$ has total space $X(g,1) = \mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}}^2$, the complex projective plane blown up at $4g+5$ points.

It is known that for $g \geq 2$, the above fibration on $X(g,1)$ admits $4g+4$ disjoint (-1)-sphere sections (proof of this fact using a mapping class group argument is due to S. Tanaka).

The fiber class is of the form $(g+2)h - ge_1 - e_2 - \cdots - e_{4g+5}$, where e_i denotes the homology class of the exceptional sphere of the i-th blow up and h denotes the pullback of the hyperplane class of $\mathbb{CP}^2$. The exceptional spheres represented by the homology classes $e_2, e_3, \ldots, e_{4g+5}$ are sections of the Lefschetz fibration $X(g,1) \to \mathbb{S}^2$.

Theorem (*S. Donaldson*)

For any symplectic 4-manifold X, there exists a non-negative integer n such that the n-fold blowup $X\#n\overline{\mathbb{CP}}^2$ of X admits a Lefschetz fibration $f: X\#n\overline{\mathbb{CP}}^2 \to \mathbb{S}^2$.

Theorem (*R. Gompf*)

Assume that the closed 4-manifold X admits a genus g Lefschetz fibration $f: X \to \Sigma_h$, and let $[F]$ denote the homology class of the fiber. Then X admits a symplectic structure with symplectic fibers iff $[F] \neq 0$ in $H_2(X;\mathbb{R})$. If $e_1, \cdots, e_n$ is a finite set of sections of the Lefschetz fibration, the symplectic form ω can be chosen in such a way that all these sections are symplectic.

Theorem (*S. Donaldson and R. Gompf, J. Amoros - F. Bogomolov - L. Katzarkov - T. Pantev*)

For any finitely presented group G, there exist a Lefschetz fibration $X(G)$ over $\mathbb{S}^2$ with $\pi_1(X(G)) = G$.

${b_2}^+(X(G))$ is *very large*, and depends from the presentation of G.

Main Theorems

Theorem (*A. Akhmedov - B. Ozbagci, 2012*)

For any finitely presented group G, there exist a closed symplectic 4-manifold $X_n(G)$ with $\pi_1(X(G)) = G$, which admits a genus $2g + n - 1$ Lefschetz fibration over $\mathbb{S}^2$ that has al least $4n + 4$ pairwise disjoint sphere sections of self intersection -2. Moreover, $X_n(G)$ contains a homologically essential embedded torus of square zero disjoint from these sections which intersects each fiber of the Lefschetz fibrations twice.

Theorem (*A. Akhmedov - B. Ozbagci, 2012*)

There exist an infinite family of non-holomorphic Lefschetz fibrations $X_n(G, K_i)$ over $\mathbb{S}^2$ with $\pi_1(X_n(G, K_i)) = G$ that can be obtained from $X_n(G)$ via knot surgery along K_i, where K_i are inf. family of genus $g \geq 2$ fibered knots with distinct Alexander polynomials.

Lefschetz fibrations by Y. Matsumoto and M. Korkmaz

Let assume $g = 2k$.

The 4-manifold $Y(1,k) = \Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$ is the total space of the genus g Lefschetz fibration over $\mathbb{S}^2$ with $2g+4$ singular fibers. This was shown by Yukio Matsumoto for $k = 1$, and in the case $k \geq 2$ by Mustafa Korkmaz, by factorizing the *vertical* involution θ of the genus $2k$ surface.

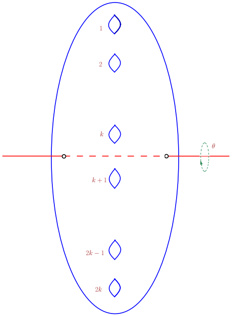


Figure: The involution θ of the genus $2k$ surface

Theorem (*Y. Matsumoto, M. Korkmaz*)

Let θ denote the vertical involution of the genus g surface with 2 fixed points. In the mapping class group M_g, the follwing relations between right handed Dehn twists hold:

a) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_c)^2 = \theta^2 = 1$ *if g is even,*

b) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} (t_a)^2 (t_b)^2)^2 = \theta^2 = 1$ *if g is odd.*

B_k, a, b, c are the simple closed curves defined as in Figure.

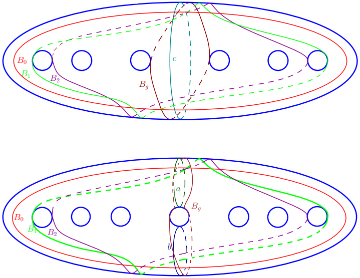


Figure: The vanishing cycles

Lefschetz fibrations by Y. Gurtas

Yusuf Gurtas generalized the constructions of Matsumoto and Korkmaz even further. He presented the positive Dehn twist expression for a new set of involutions in the mapping class group M_{2k+n-1} of a compact, closed, oriented 2-dimensional surface Σ_{2k+n-1}. The total space of these genus $g = 2k + n - 1$ Lefschetz fibration over $\mathbb{S}^2$ is $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$.

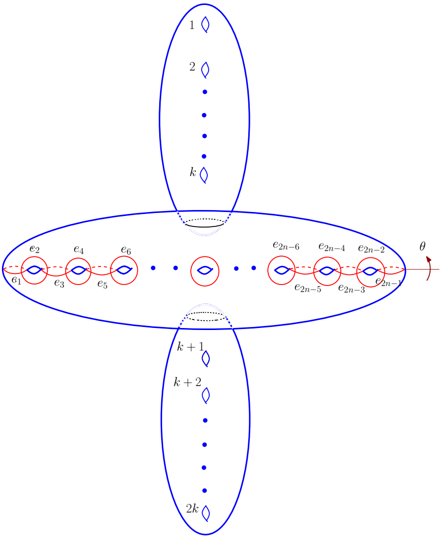


Figure: The involution θ of the surface Σ_{2k+n-1}

The branched-cover description for $\Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$

A generic horizontal fiber is the double cover of $\mathbb{S}^2$, branched over two points. Thus, we have a sphere fibration on $Y(n, k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$. A generic fiber of the vertical fibration is the double cover of Σ_k, branched over $2n$ points. Thus, a generic fiber of the vertical fibration has genus $n + 2k - 1$.

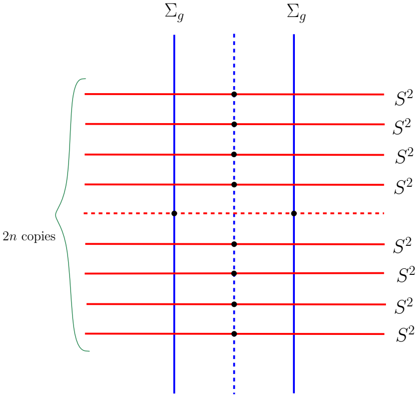
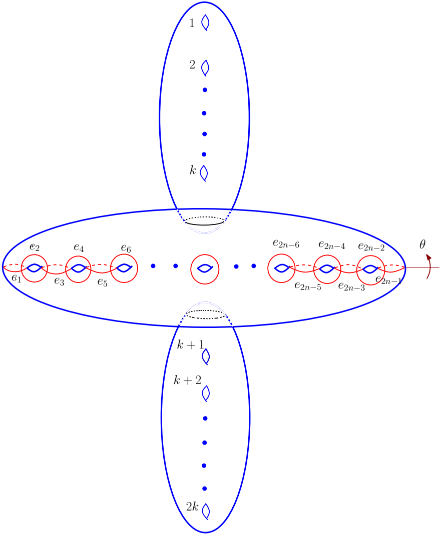


Figure: The branch locus for $\Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$

Theorem (*Y. Gurtas*)

The positive Dehn twist expression for the involution θ is given by

$$\theta = e_{2i+2} \cdots e_{2n-2} e_{2n-1} e_{2i} \cdots e_1 B_0 e_{2n-1} \cdots e_{2i+2} e_1 \cdots e_{2i} B_1 B_2 \cdots B_{4k-1} B_{4k} e_{2i+1}.$$



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Construction Tools

Construction Tools

- Symplectic Connected Sum (1995) (M. Gromov, R. Gompf, J. McCarthy- J. Wolfson)
- Luttinger Surgery (1995) (K. Luttinger, D. Auroux- S. Donaldson- L. Katzarkov)
- Knot Surgery (1998) (R. Fintushel- R. Stern)

Luttinger surgery

Definition

Let X be a symplectic 4-manifold with a symplectic form ω, and the torus Λ be a Lagrangian submanifold of X with self-intersection 0. Given a simple loop λ on Λ, let λ' be a simple loop on $\partial(\nu\Lambda)$ that is parallel to λ under the Lagrangian framing. For any integer m, the $(\Lambda, \lambda, 1/m)$ *Luttinger surgery* on X will be $X_{\Lambda,\lambda}(1/m) = (X - \nu(\Lambda)) \cup_\phi (\mathbb{S}^1 \times \mathbb{S}^1 \times D^2)$, the $1/m$ surgery on Λ with respect to λ under the Lagrangian framing. Here $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \times \partial D^2 \to \partial(X - \nu(\Lambda))$ denotes a gluing map satisfying $\phi([\partial D^2]) = m[\lambda'] + [\mu_\Lambda]$ in $H_1(\partial(X - \nu(\Lambda))$, where μ_Λ is a meridian of Λ.

$X_{\Lambda,\lambda}(1/m)$ possesses a symplectic form that restricts to the original symplectic form ω on $X \setminus \nu\Lambda$.

Luttinger's surgery has been very effective tool recently for constructing exotic smooth structures.

Example

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Let $\mathbb{T}^4 = a \times b \times c \times d \cong (c \times d) \times (a \times b)$. Let K_n be an n-twist knot.

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Let $\mathbb{T}^4 = a \times b \times c \times d \cong (c \times d) \times (a \times b)$. Let K_n be an n-twist knot. Let M_{K_n} denote the result of performing 0 Dehn surgery on $\mathbb{S}^3$ along K_n. $\mathbb{S}^1 \times M_{K_n}$ is obtained from $\mathbb{T}^4 = (c \times d) \times (a \times b) = c \times (d \times a \times b) = \mathbb{S}^1 \times \mathbb{T}^3$ by first performing a Luttinger surgery $(c \times \tilde{a}, \tilde{a}, -1)$ followed by a surgery $(c \times \tilde{b}, \tilde{b}, -n)$. The tori $c \times \tilde{a}$ and $c \times \tilde{b}$ are Lagrangian and the second tilde circle factors in $\mathbb{T}^3$ are as pictured. Use the Lagrangian framing to trivialize their tubular neighborhoods. When $n = 1$ the second surgery is also a Luttinger surgery.

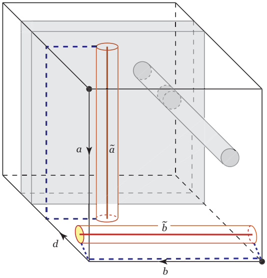

Figure: The 3-torus $d \times a \times b$

Symplectic Connected Sum

Definition

Let X_1 and X_2 are symplectic 4-manifolds, and $F_i \subset X_i$ are 2-dimensional, smooth, closed, connected symplectic submanifolds in them. Suppose that $[F_1]^2 + [F_2]^2 = 0$ and the genera of F_1 and F_2 are equal. Take an orientation-preserving diffemorphism $\psi : F_1 \longrightarrow F_2$ and lift it to an orientation-reversing diffemorphism $\Psi : \partial\nu F_1 \longrightarrow \partial\nu F_2$ between the boundaries of the tubular neighborhoods of νF_i. Using Ψ, we glue $X_1 \setminus \nu F_1$ and $X_2 \setminus \nu F_2$ along the boundary. The 4-manifold $X_1 \#_\Psi X_2$ is called the *(symplectic) connected sum* of X_1 and X_2 along F_1 and F_2, determined by Ψ.

$$\begin{aligned} e(X_1 \#_\Psi X_2) &= e(X_1) + e(X_2) + 4(g-1), \\ \sigma(X_1 \#_\Psi X_2) &= \sigma(X_1) + \sigma(X_2), \end{aligned}$$

The vanishing cycles of twisted fiber sum of Lefschetz fibration

Lemma

Let $f : X \to \mathbb{S}^2$ be a genus g Lefschetz fibration with global monodromy given by the relation $t_{\alpha_1} \cdot t_{\alpha_2} \cdot \cdots \cdot t_{\alpha_s} = 1$. Let $X\#_\psi X$ denote the fiber sum of X with itself by a self-diffemorphism ψ of the generic fiber Σ. Then $X\#_\psi X$ has the vanishing cycles $\alpha_1, \alpha_2, \cdots, \alpha_s, \psi(\alpha_1), \psi(\alpha_2), \cdots, \psi(\alpha_s)$.

Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times \mathbb{T}^2$

Fix integers $n \geq 2$, $p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. Let $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$ denote symplectic 4-manifold obtained by performing the following $2n+4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$. These $2n+4$ surgeries comprise of the following 8 surgeries

$$\begin{array}{ll}(a_1' \times c_1', a_1', -1), & (b_1' \times c_1'', b_1', -1),\\ (a_2' \times c_2', a_2', -1), & (b_2' \times c_2'', b_2', -1),\\ (a_2' \times c_1', c_1', +1/p_1), & (a_2'' \times d_1', d_1', +1/q_1),\\ (a_1' \times c_2', c_2', +1/p_2), & (a_1'' \times d_2', d_2', +1/q_2),\end{array}$$

together with the following $2(n-2)$ additional Luttinger surgeries

$$\begin{array}{ll}(b_1' \times c_3', c_3', -1/p_3), & (b_2' \times d_3', d_3', -1/q_3),\\ \cdots, & \cdots,\\ (b_1' \times c_n', c_n', -1/p_n), & (b_2' \times d_n', d_n', -1/q_n).\end{array}$$

Here, a_i, b_i $(i = 1, 2)$ and c_j, d_j $(j = 1, \ldots, n)$ are the standard loops that generate $\pi_1(\Sigma_2)$ and $\pi_1(\Sigma_n)$.

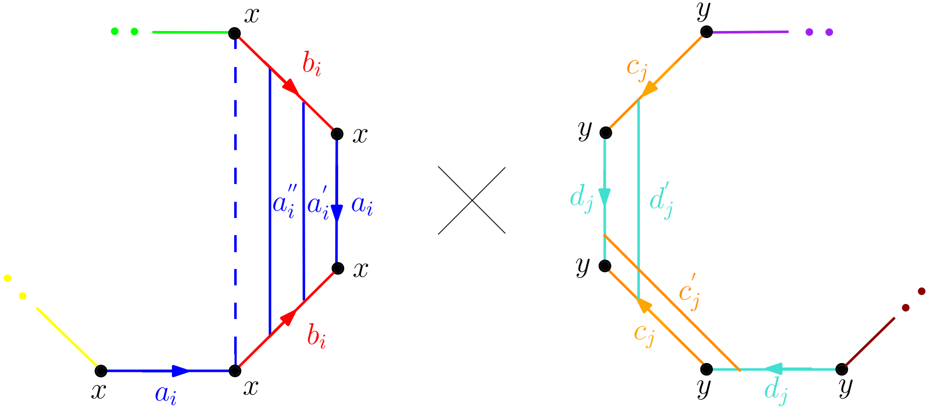

Figure: Lagrangian tori $a_i' \times c_j'$ and $a_i'' \times d_j'$

The Euler characteristic of $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$ is $4n-4$ and its signature is 0. The fundamental group $\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$ is generated by a_i, b_i, c_j, d_j ($i = 1, 2$ and $j = 1, \ldots, n$) and the following relations hold in $\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$:

$$
\begin{gathered}
[b_1^{-1}, d_1^{-1}] = a_1, \ \ [a_1^{-1}, d_1] = b_1, \ \ [b_2^{-1}, d_2^{-1}] = a_2, \ \ [a_2^{-1}, d_2] = b_2, \qquad (2) \\
[d_1^{-1}, b_2^{-1}] = c_1^{p_1}, \ \ [c_1^{-1}, b_2] = d_1^{q_1}, \ \ [d_2^{-1}, b_1^{-1}] = c_2^{p_2}, \ \ [c_2^{-1}, b_1] = d_2^{q_2}, \\
[a_1, c_1] = 1, \ \ [a_1, c_2] = 1, \ \ [a_1, d_2] = 1, \ \ [b_1, c_1] = 1, \\
[a_2, c_1] = 1, \ \ [a_2, c_2] = 1, \ \ [a_2, d_1] = 1, \ \ [b_2, c_2] = 1, \\
[a_1, b_1][a_2, b_2] = 1, \ \ \prod_{j=1}^{n} [c_j, d_j] = 1, \\
[a_1^{-1}, d_3^{-1}] = c_3^{p_3}, \ \ [a_2^{-1}, c_3^{-1}] = d_3^{q_3}, \ \ldots, \ [a_1^{-1}, d_n^{-1}] = c_n^{p_n}, \ \ [a_2^{-1}, c_n^{-1}] = d_n^{q_n}, \\
[b_1, c_3] = 1, \ \ [b_2, d_3] = 1, \ \ldots, \ [b_1, c_n] = 1, \ \ [b_2, d_n] = 1.
\end{gathered}
$$

The surfaces $\Sigma_2 \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \times \Sigma_n$ in $\Sigma_2 \times \Sigma_n$ descend to surfaces in $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$. They are symplectic submanifolds in $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$. Denote their images by Σ_2 and Σ_n. Note that $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$ and $[\Sigma_2] \cdot [\Sigma_n] = 1$.

Let $\{p_i, q_i \geq 0 : 1 \leq i \leq g\}$ be a set of nonnegative integers and let $\overline{p} = (p_1, \ldots, p_g)$ and $\overline{q} = (q_1, \ldots, q_g)$.

Let $\{p_i, q_i \geq 0 : 1 \leq i \leq g\}$ be a set of nonnegative integers and let $\overline{p} = (p_1, \ldots, p_g)$ and $\overline{q} = (q_1, \ldots, q_g)$. Denote by $M_g(\overline{p}, \overline{q})$ the symplectic 4-manifold obtained by performing the following $2g$ Luttinger surgeries on the symplectic 4-manifold $\Sigma_g \times \mathbb{T}^2$:

$$\begin{aligned}
&(a_1' \times c', a_1', -1/p_1), \;\; (b_1' \times c'', b_1', -1/q_1), \\
&(a_2' \times c', a_2', -1/p_2), \;\; (b_2' \times c'', b_2', -1/q_2), \\
&\cdots \quad \cdots \\
&(a_{g-1}' \times c', a_{g-1}', -1/p_{g-1}), \;\; (b_{g-1}' \times c'', b_{g-1}', -1/q_{g-1}), \\
&(a_g' \times c', a_g', -1/p_g), \;\; (b_g' \times c'', b_g, -1/q_g).
\end{aligned} \tag{3}$$

Here, a_i, b_i ($i = 1, 2, \cdots, g$) and c, d denote the standard generators of $\pi_1(\Sigma_g)$ and $\pi_1(\mathbb{T}^2)$, respectively.

The fundamental group of $M_g(\overline{p},\overline{q})$ is generated by a_i, b_i $(i = 1, 2, 3 \cdots, g)$ and c, d, and the following relations hold in $M_g(\overline{p},\overline{q})$:

$$[b_1^{-1}, d^{-1}] = a_1{}^{p_1}, \;\; [a_1^{-1}, d] = b_1{}^{q_1}, \;\; [b_2^{-1}, d^{-1}] = a_2{}^{p_2}, \;\; [a_2^{-1}, d] = b_2{}^{q_2}, \quad (4)$$
$$\cdots, \;\; \cdots, \;\; \cdots,$$
$$[b_g^{-1}, d^{-1}] = a_g{}^{p_g}, \;\; [a_g^{-1}, d] = b_g{}^{q_g}, \;\; [a_1, c] = 1, \;\; [b_1, c] = 1, \;\; [a_2, c] = 1, \;\; [b_2, c] =$$
$$[a_3, c] = 1, \;\; [b_3, c] = 1,$$
$$[a_g, c] = 1, \;\; [b_g, c] = 1,$$
$$[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1, \;\; [c, d] = 1.$$

Let $\Sigma_g \subset M_g(\overline{p},\overline{q})$ and T be a genus g and genus 1 surfaces that desend from the surfaces $\Sigma_g \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \times \mathbb{T}^2$ in $\Sigma_g \times \mathbb{T}^2$.

- $M_g(\overline{p}, \overline{q})$ is a locally trivial genus g bundle over $\mathbb{T}^2$ where T is a section. The $(a_i' \times c', a_i', -p_i)$ or $(b_i' \times c'', b_i', -q_i)$ Luttinger surgery in the trivial bundle $\Sigma_g \times \mathbb{T}^2$ preserves the fibration structure over $\mathbb{T}^2$ introducing a monodromy of the fiber Σ_g along the curve c in the base. Depending on the type of the surgery the monodromy is either $(t_{a_i})^{p_i}$ or $(t_{b_i})^{q_i}$, where t denotes a Dehn twist.
- $E(n)$ can be obtained as a desingularization of the branched double cover of $\mathbb{S}^2 \times \mathbb{S}^2$ with the branching set being 4 copies of $\{pt\} \times \mathbb{S}^2$ and $2n$ copies of $\mathbb{S}^2 \times \{pt\}$.

- $M_g(\overline{p}, \overline{q})$ is a locally trivial genus g bundle over $\mathbb{T}^2$ where T is a section. The $(a_i' \times c', a_i', -p_i)$ or $(b_i' \times c'', b_i', -q_i)$ Luttinger surgery in the trivial bundle $\Sigma_g \times \mathbb{T}^2$ preserves the fibration structure over $\mathbb{T}^2$ introducing a monodromy of the fiber Σ_g along the curve c in the base. Depending on the type of the surgery the monodromy is either $(t_{a_i})^{p_i}$ or $(t_{b_i})^{q_i}$, where t denotes a Dehn twist.
- $E(n)$ can be obtained as a desingularization of the branched double cover of $\mathbb{S}^2 \times \mathbb{S}^2$ with the branching set being 4 copies of $\{pt\} \times \mathbb{S}^2$ and $2n$ copies of $\mathbb{S}^2 \times \{pt\}$. $E(n)$ admits a genus $n-1$ fibration over $\mathbb{S}^2$ and an elliptic fibration over $\mathbb{S}^2$. A regular fiber of the elliptic fibration on $E(n)$ intersects every genus $n-1$ fiber of the other Lefschetz fibration twice.

Construction of Lefschetz fibrations over $\mathbb{S}^2$

- Let $X_{g,n}(\overline{p},\overline{q})$ denote the symplectic sum of $M_g(\overline{p},\overline{q})$ along the torus $T = c \times d$ with the elliptic surface $E(n)$ along a regular elliptic fiber.
- The symplectic 4-manifold $X_{g,n}(\overline{p},\overline{q})$ admits a genus $2g+n-1$ Lefschetz fibration over $\mathbb{S}^2$ with at least $4n+4$ pairwise disjoint sphere sections of self intersection -2. Moreover, $X_{g,n}(\overline{p},\overline{q})$ contains a homologically essential embedded torus of square zero disjoint from these sections which intersects each fiber of the Lefschetz fibration twice.
- The symplectic 4-manifold $X_{g,n}(\overline{p},\overline{q})$ can also be constructed as the twisted fiber sum of two copies of a genus $2g+n-1$ Lefschetz fibration on $\Sigma_g \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$. This follows from the fact that the symplectic sum of $E(n)$ along a regular elliptic fiber with $\Sigma_g \times \mathbb{T}^2$ along a natural square zero torus is diffeomorphic to the untwisted fiber sum of two copies of the genus $2g+n-1$ fibration on $\Sigma_g \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$. The gluing ϕ diffeomorphism can be described explicitly using the curves along which we perform our Luttinger surgeries.

- The fundamental group of the symplectic 4-manifold $X_{g,n}(\overline{p}, \overline{q})$ is generated by the set $\{a_i, b_i : 1 \leq i \leq g\}$ subject to the relations: $a_i^{p_i} = 1, b_i^{q_i} = 1$, for all $1 \leq i \leq g$, and $\Pi_{j=1}^{g}[a_j, b_j] = 1$.
- By setting $p_i = 1$ and $q_i = 0$, for all $1 \leq i \leq g$, we see that the fundamental group of $X_{g,n}((1, 1, \ldots, 1), (0, 0, \ldots, 0))$ is a free group of rank g. The gluing diffeomorphism: $\phi = t_{a_1} \cdots t_{a_g}$.
- By setting $p_i = 1$ and $q_i = 1$, for all $1 \leq i \leq g$, we see that the fundamental group of $X_{g,n}((1, 1, \ldots, 1), (1, 1, \ldots, 1))$ is a trivial. The gluing diffeomorphism: $\phi = t_{a_1} t_{b_1} \cdots t_{a_g} t_{b_g}$.
- If we set $p_i = 1$ and $q_i = 0$, for all $1 \leq i \leq k$ and $p_i = 1$ and $q_i = 1$, for all $k + 1 \leq i \leq g$, the fundamental group of $X_{g,n}((1, 1, \ldots, 1), (1, 1, \ldots, 0))$ is a a free group of rank k. The gluing diffeomorphism: $\phi = t_{a_1} \cdots t_{a_k} t_{a_{k+1}} t_{b_{k+1}} \cdots t_{a_g} t_{b_g}$.

Stein fillings from Lefschetz fibrations

Definition

A complex surface V is Stein if it admits a proper holomorphic embedding $f : V \to \mathbb{C}^n$ for some n. For a generic point $p \in \mathbb{C}^n$, consider the map $\phi : V \to \mathbb{R}$ defined by $\phi(z) = ||z - p||^2$. For a regular value $a \in \mathbb{R}$, the level set $M = \phi^{-1}(a)$ is a smooth 3-manifold with a distinguished 2-plane field $\xi = TM \cap iTM \subset TV$. ξ defines a contact structure on M, and $S = \phi^{-1}([0, a])$ is called a Stein filling of (M, ξ).

Theorem (*S. Akbulut - B. Ozbagci*)

Let $f : X \to \mathbb{S}^2$ be a Lefschetz fibration with a section σ and let Σ denote a regular fiber of this fibration. Then $S = X \setminus int(\nu(\sigma \cup \Sigma))$ is a Stein filling of its boundary equipped with the induced (tight) contact structure, where $\nu(\sigma \cup \Sigma)$ denotes a regular neighborhood of $\sigma \cup \Sigma$ in X.

Finiteness Results on Stein Fillings

- The tight contact structure on $\mathbb{S}^3$ has a unique Stein filling (Y. Eliashberg, 1989).
- All tight contact structures on lens spaces $\mathbb{L}(p, q)$ have a finite number of Stein fillings (D. McDuff, P. Lisca, 1992).
- Finiteness results also have been verified for simple elliptic singularities (H. Ohta and Y. Ono, 2002).
- Finiteness results on symplectic fillings of Seifert fibered spaces over S^2 (L. Starkston, 2013).

Infiniteness Results on Stein Fillings

- B. Ozbagci and A. Stipsicz, and independently I. Smith showed that certain contact structures have an infinite number of Stein fillings (2003). Their examples have non-trivial fundamental groups.
- Infinitely many simply-connected exotic Stein fillings (Akhmedov - Etnyre - Mark - Smith, 2007).
- Small exotic Stein fillings (S. Akbulut - K. Yasui, 2008).
- Exotic Stein fillings with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_n$ (Akhmedov - Ozbagci, 2012).

Theorem (*A. Akhmedov - B. Ozbagci, 2012*)

For any finitely presented group G, there exist an infinite family of exotic Stein 4-manifolds $S_n(G, K_i)$ with $\pi_1(S_n(G, K_i)) = G$, where K_i are inf. family of genus $g \geq 2$ fibered knots with distinct Alexander polynomials.

THANK YOU!