4-MANIFOLDS

SELMAN AKBULUT

0. 4-manifold handlebodies

A smooth manifold M^m is said to be obtained from the smooth manifold N^m by attaching a k-handle and denoted by $M = N \smile_{\varphi} h^k$, if there is an imbedding $\varphi : S^{k-1} \times B^{m-k} \hookrightarrow \partial N$, such that M is obtained from the disjoint union $B^k \times B^{m-k}$ and N by identifying $x \in S^{k-1} \times B^{m-k}$ with $\varphi(x) \in \partial N$. Here $\varphi(S^{k-1} \times 0)$ is called the *attaching sphere*, and $0 \times S^{m-k-1}$ is called the *belt sphere* of this handle.

$$M^m = [B^k \times B^{m-k} \sqcup N]/x \sim \varphi(x)$$

Any Morse function $f : M \to \mathbb{R}$ gives a handle decomposition of $M = \cup_{k=0}^{m} M_k$, with $\phi = M_{-1} \subset M_0 \subset .. \subset M_m = M$, and M_k is obtained from M_{k-1} by attaching k-handles [M] . This is called a handlebody structure of M. In particular, when M^4 is connected and 4-dimensional, then it can be obtained from B^4 (zero handle) by attaching 1-, 2- , and 3-handles and capping with the top B^4 (4-handle).

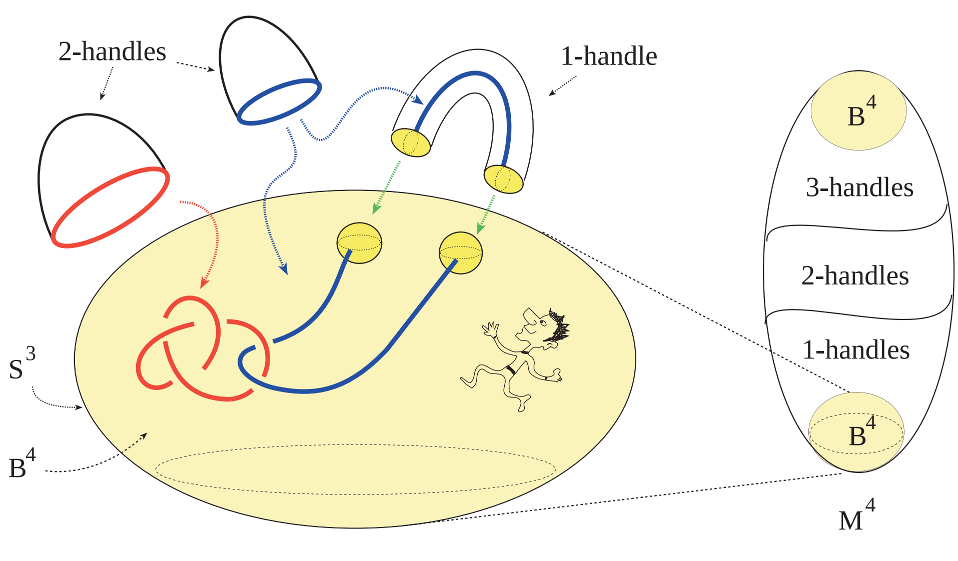

Figure 1. Handlebody of a 4-manifold

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In [S] Smale, and independently in [W] Wallace, defined and studied handlebody structures on smooth manifolds. Smale went further by turning the operations on handlebody structures into a great technical machine to solve difficult problems about smooth manifolds, such as the smooth h-cobordism theorem which implied the proof of the topological Poincare conjecture in dimensions ≥ 5. The two basic techniques which he utilized were the *handle sliding* and *handle canceling* operations [H].

If the manifold M^4 is 4-dimensional and connected, by using its handlebody we can basically see the whole manifold as follows: As shown in Figure 1 we place ourselves on the boundary S^3 of its 0-handle B^4 and watch the feet (the attaching regions) of the 1- and 2-handles. The feet of the 1-handles will look like the pair of balls of same color, and the feet of 2-handles will look like imbedded framed circles (framed knots) which might go over the 1-handles. This is because the 2-handles are attached after the 1-handles.

Fortunately we don't need to visualize the attaching regions of the 3-handles, because an amazing theorem of Laudenbach and Poenaru [LP], which says that any connected smooth 4-manifold is determined by its 1- and 2-handles (i.e. 3-handles are attached uniquely!)

The feet of a 1-handles in S^3 will appear as pair of balls B^3, where on the boundary they are identified by the map $(x, y, z) \mapsto (x, -y, z)$ (with respect to the standard coordinate axis placed at their origins)

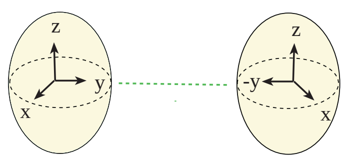

FIGURE 2

Any 2-handle which doesn't go over the 1-handles is attached by an imbedding $\varphi : S^1 \times B^2 \to \mathbb{R}^3 \subset S^3$. This imbedding is determined by the knot $K = \varphi(S^1 \times (0, 0))$ together with an orthonormal framing $\mathbf{e} = \{u, v\}$ of its normal bundle in $\mathbb{R}^3$. This framing $\mathbf{e}$ determines φ by

$$\varphi(x, \lambda, \nu) = (x, \lambda u + \nu v)$$

Also note that by orienting K and using the right hand rule, one normal vector field u determines the other v. Hence any normal vector field of K determines a framing, which can be parametrized by $\pi_1(SO_2) = \mathbb{Z}$.

We make the convention that the normal vector field induced from any Seifert surface of K to be the zero framing u_0. Once this is done, it is clear that the framing $k \in \mathbb{Z}$ corresponds to the vector field u_k which deviates from u_0 by k-full twist. Put another way, u_k is the vector field when we push K along it, we get a copy of K which has linking number k with K. So we denote the framing by an integer. The following exercise gives a useful tool of deciding the k-framing.

Exercise 1. *Orient a diagram of the knot K, and let $C(K)$ be the set of its crossing points, define writhe $w(K)$ of a knot K to be the integer*

$$w(K) = \sum_{p \in C(K)} \epsilon(p)$$

where $\epsilon(p)$ is $+1$ or -1 according to right or left handed crossing at p. Show that the "blackboard framing" of a knot equals to its writhe

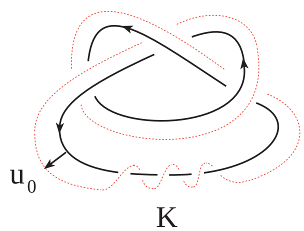

FIGURE 3. $w(K) = 3$ and u_0 is the zero framing

When attaching framed circles of 2-handles go over 1-handles their framing can not be a well defined integer. For example, by the isotopy of Figure 4 the framing can be changed by adding or subtracting 2.

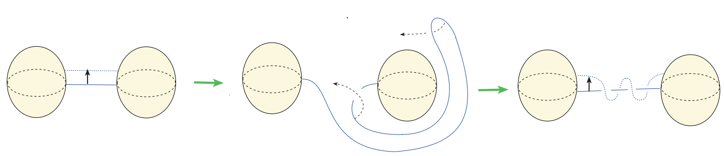

FIGURE 4

One way to prevent this framing changing isotopy is to fix an arc, as shown in Figure 2, connecting the attaching balls of the 1-handle, and make the rule that no isotopy of framed knots may cross this arc. Only after this unnatural rule we can talk about well defined framed circles. Most of the work of [AK1] is based on this convention, used carefully.

0.1. **Carving (Invariant notation of 1-handle).** Carving is based on the following simple observation: If the attaching sphere $\varphi(S^{k-1}\times 0)$ of the k-handle of $M = N \smile_{\varphi} h^k$, bounds a disk in ∂N, then M can be obtained from N by excising (drilling) out an open tubular neighborhood of a properly imbedded disk $D^{m-k-1}\subset M$. In particular if M^4 is connected, attaching a 1-handle to M^4 is equivalent to carving out a properly imbedded 2-disk $D^2 \subset B^4$ from the 0-handle $B^4 \subset M$.

To distinguish the boundary of the carved disk $\partial D \subset S^3$ from the attaching cirlces of the 2-handles, we will denote it by a circle with dot. Any path going through this dotted circle is going over the associated 1-handle. This simple observation has nice consequences (e.g. [A1]).

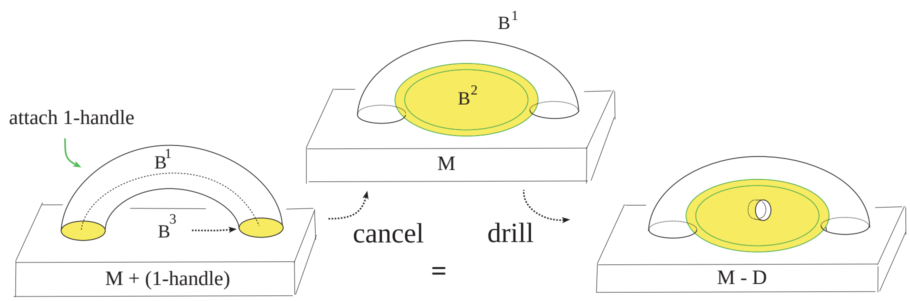

FIGURE 5. Attaching 1-handle is the same as drilling 2-disk

So the observer in Figure 1 standing on the boundary of B^4 will see the 4-manifold as in Figure 6. As discussed before, the framing of the attaching circle of each 2-handle is specified by an integer, so the knots come with integers (in this example we assigned the framings 2 and 3).

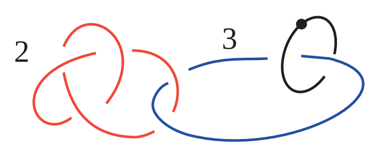

FIGURE 6

This notation of 1-handle has the advantage of not creating ambiguity on the framings of framed knots going through them, as pointed out in Figure 4. This notation can help to construct hard to see diffeomorphisms between the boundaries of 4-manifolds (Section 1.5).

Exercise 2. *Show that $S^1 \times B^3$ and $B^2 \times S^2$ are related to each other by surgeries of their core spheres, correponding to zero and dot exchanges:*

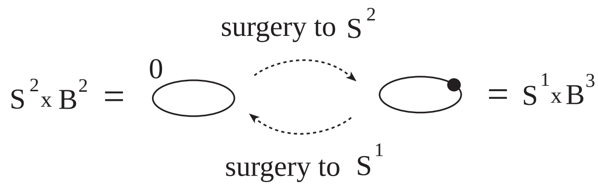

FIGURE 7

Exercise 3. *Show that the manifold W in Figure 8 is a contractible manifold, and by sugeries in its interior (corresponding to zero and dot exchanges on the symmetric link) gives an involution on its boundary $f : \partial W \to \partial W$, where the loops a, b are mapped to each other by f.*

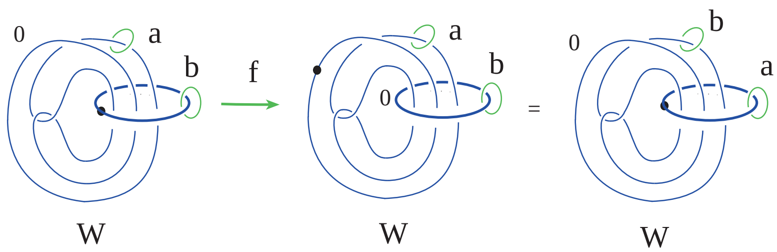

FIGURE 8

0.2. **Sliding handles.** Here we will denote an r-framed knot K by K^r (which will also denote the corresponding 4-manifold). Since any smooth 4-manifold M^4 is determined by its 0, 1-and 2-handles, M can be denoted by a link of framed knots $K_1^{r_1}, K_2^{r_2}, ..$ (2-handles) along with "circle with dots" $C_1, C_2, ..$ (1-handles). Notice here we have the option of denoting the 1-handles either by dotted circles, or pair of balls (each notation has its advantages). Also for convenience, orient this link Λ.

$$\Lambda = \{K_1^{r_1}, .., K_k^{r_k}, C_1, .., C_s\}$$

We notate this by $M = M_\Lambda$. For example, Figure 6 is the 4-manifold M_Λ with $\Lambda = \{K_1^2, K_2^3, C\}$, where K_1 is the trefoil knot, and K_2 is the circle linking K_1 , and C is the 1-handle as shown in the picture.

As explained in [S], [H] we can change a given handlebody of M to another handlebody of M, by sliding any k-handle over other r-handles with $r \leq k$ (Figure 9). In particular in a 4-manifold we can (a) slide a 1-handle over another 1-handle, or (b) slide a 2-handle over a 1-handle or (c) slide a 2-handle over another 2- handle. This process preserves the framing of the attaching circles of 2-handles in a well defined way.

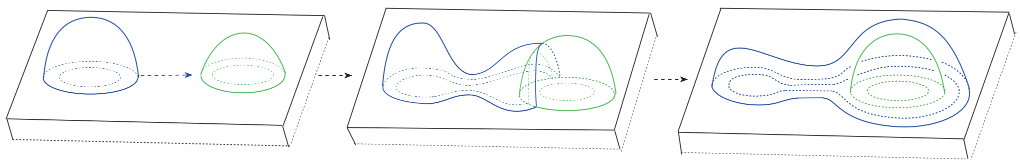

FIGURE 9

Exercise 4. *Let μ_{ij} be the linking number of K_i and C_j, and λ_{ij} be the linking number of K_i and K_j. Also for each i let K'_i denote a parallel copy of K_i (K_i pushed off by the framing r_i), and C'_i be a parallel copy of the circle C_i. Show that the handle slides (a), (b), (c) above corresponds to changing one of the elements of Λ as below ($j \neq s$):*

(a) $C_i \mapsto C_i + C'_j :=$ *The circle obtained, by connected summing C_i to C'_j along an arch which does not go through any of the C's.*

(b) $K_i^{r_i} \mapsto K_i + C'_j := \bar{K}_i^{\bar{r}_i}$ *The framed knot obtained, by connected summing K_i to C'_s along an arc, with framing $\bar{r}_i = r_i + 2\mu_{ij}$*

(c) $K_i^{r_i} \mapsto K_i + K'_s := \bar{K}_i^{\bar{r}_i}$ *The framed knot obtianed, by connected summing K_i to K'_j along an arc, with framing $\bar{r}_i = r_i + r_j + 2\lambda_{ij}$*

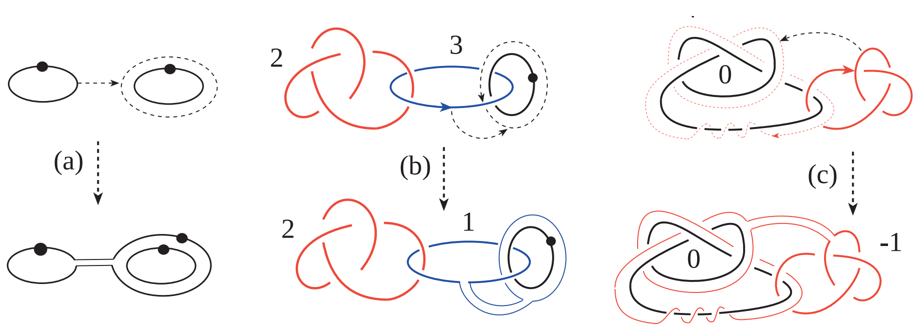

FIGURE 10. Examples of handle slides (a), (b), (c)

0.3. **Canceling handles.** We can cancel a k-handle h^k with a $(k+1)$-handle h^{k+1} provided the attaching sphere of h^{k+1} meets the belt sphere of h^k transversally at a single point (as explained in [S] and [H])

$$N \smile_{\varphi} h^k \smile_{\psi} h^{k+1} \approx N$$

For example, in Figure 5 a canceling 1 and 2-handle pairs was drawn. Any 1-handle, and a 2-handle whose attaching circle (framed knot) goes through the 1-handle geometrically once, forms is a canceling pair. If no other framed knot goes through the 1-handle of a canceling pair, simply erasing the pair from the picture corresponds to canceling operation. Three pictures of Figure 11 are equivalent descriptions of a canceling handle pair, if you want to cancel it just erase it from the picture.

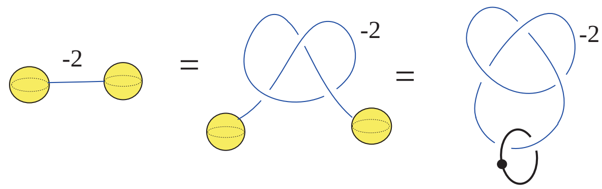

FIGURE 11

It follows from the handle sliding description that, if there are other framed knots going through the 1-handle of a canceling handle pair, those framed knots must be slid over over the 2-handle of the canceling pair, before the canceling operation (erasing the pair from the picture)

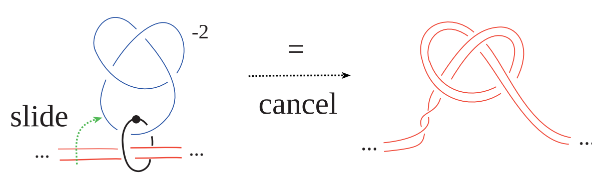

FIGURE 12

Since 3-handles are attached uniquely, introducing a canceling 2- and 3-handle pair is much simpler operation. We just draw its 2-handle as 0-framed unknot, which is $S^2 \times B^2$, and declare that there is a canceling 3-handle on top of it. In the picture of 4-manifold no other handles should go through this 0-framed unknot.

0.4. **Carving ribbons.** In some cases carving operation allows us to slide a 1-handles over a 2-handles, which is in general prohibited (the attaching circle of the 2-handle might go through the 1-handle). But as in the configuration of Figure 13, we can imagine the carved disk (painted yellow inside) as a trough (or a groove) just below the surface, then sliding it over the 2-handle by indicated isotopy makes sense.

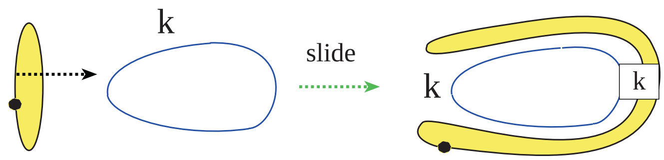

FIGURE 13

Exercise 5. *Justify the isotopy of Figure 13, by first breaking the* 1*-handle into a* 1*-handle and a canceling* 1/2*-handle pair, as in the first picture of Figure 14, then by sliding the new* 2*-handle over the* 2*-handle of Figure 13, then at the end by canceling back the* 1/2*-handle pair.*

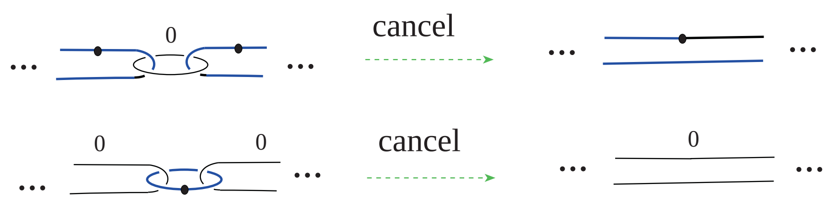

FIGURE 14

This move can turn a 1-handle, which is a carved disk bounding an unknot (in a nonstandard handlebody of B^4) into a carved ribbon bounding a ribbon knot (in the standard handlebody of B^4), as in the example of Figure 15.

We can also consider this operation in the reverse, namely given a ribbon knot, how can we describe complement of the slice disk which ribbon knot bounds in B^4?. In short, what is the ribbon complement obtained by carving this ribbon disk from B^4? Clearly by ribbon moves (i.e. cutting and regluing along bands) we can turn the ribbon knot into disjoint union of unknotted circles, which we can use to carve B^4 along the disks they bound, getting some number of connected sum

$\#_k(S^1 \times B^3)$. Now we can do the reverse of the ribbon moves, which describes a cobordism from the boundary $\#_k(S^1 \times S^2)$ of the carved B^4 to the ribbon complement in S^3. During this cobordism every time two circles coalesce the complement gains a 2-handle as shown in Figure 16.

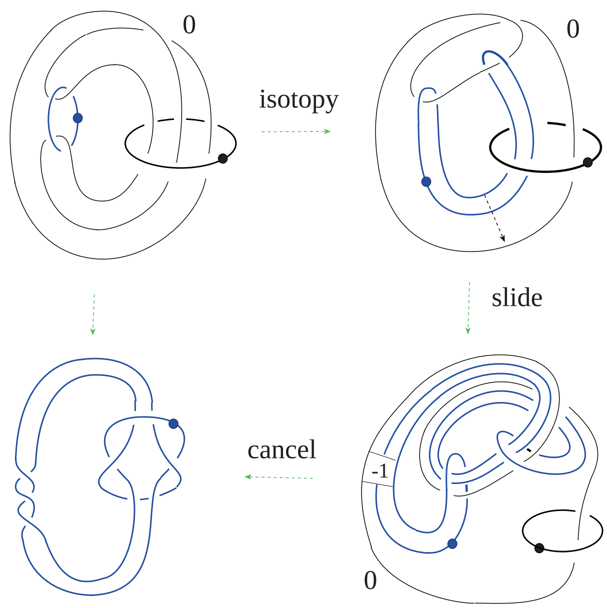

FIGURE 15

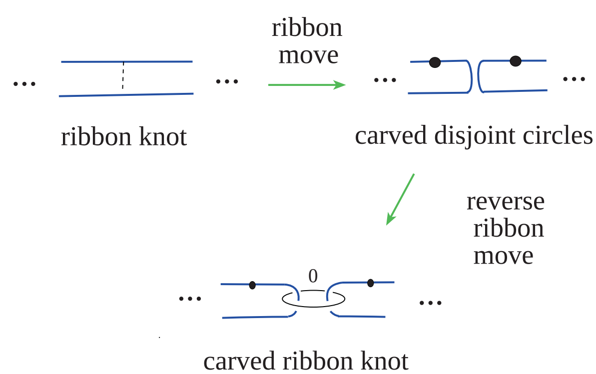

FIGURE 16

For example, applying this process to the ribbon knot of Figure 17 gives the handlebody at the top right picture of Figure 17 (this knot appears in [A2] playing important role in exotic smooth structures of 4-manifolds as we will see later). Note that this process turns carving a ribbon from the standard handlebody picture of B^4 into carving the trivial disks from an non standard handlebody picture of B^4.

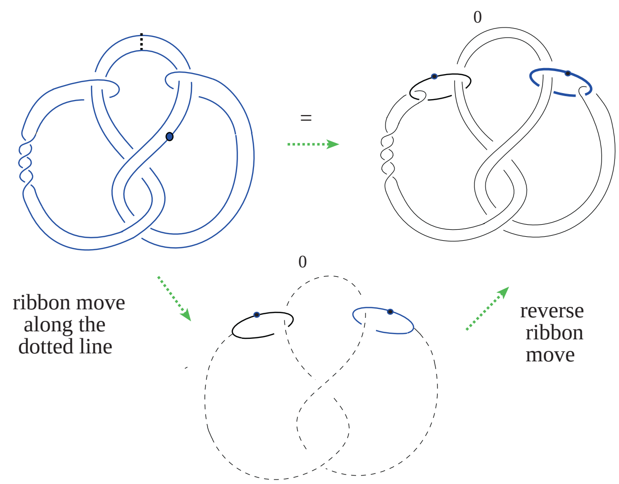

FIGURE 17

Exercise 6. *Show that applying ribbon move to the ribbon knot of Figure 18 gives the handlebody with two* 1*-handles and one* 2*-handle.*

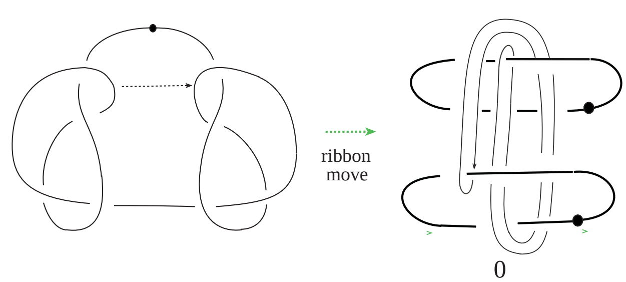

FIGURE 18

0.5. **Non-orientable handles.** If we attach a 1-Handle $B^1 \times B^3$ to B^4 along a pair of balls $\{B^3_-, B^3_{+|}\}$ with the same orientation of ∂B^4 we get a orientation reversing 1-handle, which is the nonoriented B^3-bundle over S^1. So to an observer, located at the boundary of B^4, this will be seen as a pair of balls with their interiors identified by the map $(x, y, z) \mapsto (x, -y, -z)$. To denote this in pictures, we use the notation adapted in [A3], by drawing a pair of balls with little arcs passing through their centers, which means that the usual oriented 1-handle identification is augmented by reflection across the plane which is orthogonal to these arcs (Figure 19).

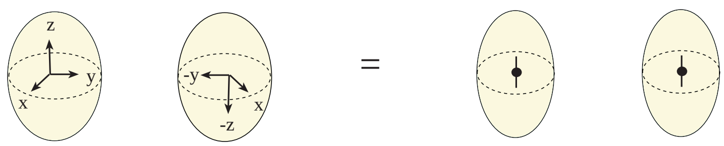

FIGURE 19

When drawing the frame knots going through these handles, one should not forget which points of the spheres ∂B^3_- and ∂B^3_+ are identified. Also framings of the framed knots going through these handles are not well defined, not just because of the isotopies of Figure 4, but also going through an orientation reversing handle the framing changes sign. For that reason when these handles are present, we will conservatively mark the framings by circled integers (with the knowledge that pushing them through the handle that integer changes its sign). For the same reason, any small knot tied to a strand going through this handle will appear as its mirror image when its is pushed though this handle, as indicated in Figure 20.

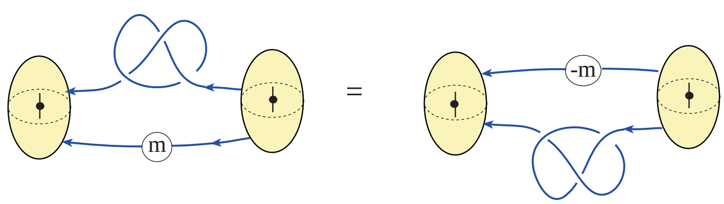

FIGURE 20

To simplify the picture, as in Figure 21 we could place one foot of this 1- handle at the point of infinity ∞ (continue traveling west to ∞ you will appear to be coming back from the east orientation reversed).

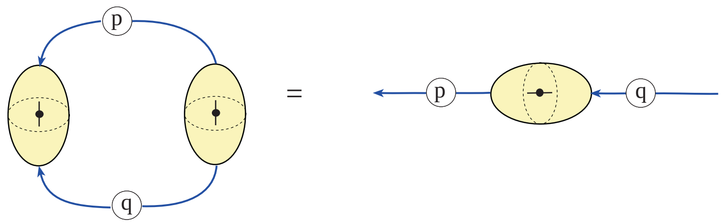

FIGURE 21. M(p,q)

Exercise 7. *Show that the diffeomorphism type of the manifold $M(p,q)$ in Figure 21 depends only on parity of $p+q$, it is $D^2 \times \mathbb{RP}^2$ if the parity is even, or else it is $D^2 \tilde{\times} \mathbb{RP}^2$ (a the twisted D^2-bundle over $\mathbb{RP}^2$). Also prove that there is the identification $\partial(D^2 \tilde{\times} \mathbb{RP}^2) \approx \partial(D^3 \tilde{\times} S^1)$, where $D^3 \tilde{\times} S^1$ is the orientation reversing D^3 bundle over S^1.*

Now if $D^2 \tilde{\times} \mathbb{RP}^2 \subset M^4$ we define *Blowing down* $\mathbb{RP}^2$ operation as:

$$M \mapsto (M - D^2 \tilde{\times} \mathbb{RP}^2) \smile_\partial (D^3 \tilde{\times} S^1)$$

Figure 22 shows this operation in a handlebody. By this in [A3], an exotic copy of $(S^3 \tilde{\times} S^1) \# S^2 \times S^2$ was constructed (Figure 23).

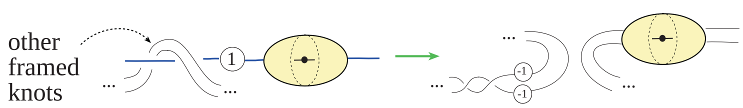

FIGURE 22. Blowing down $\mathbb{RP}^2$ operation

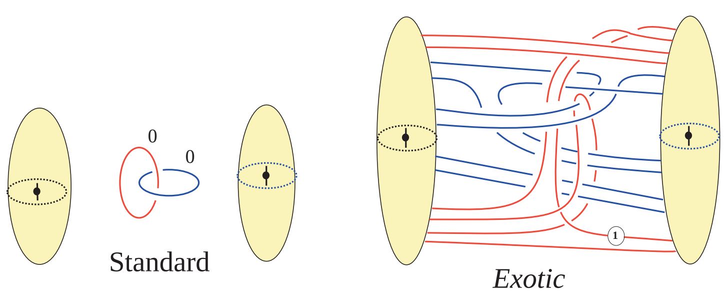

FIGURE 23. Standard and exotic $(S^3 \tilde{\times} S^1) \# S^2 \times S^2$

1. Building low dimensional manifolds

Just as we visualized 4-manifolds by placing ourselves on the boundary of its 0-handle (Figure 1) and observing the feet of 1 and 2-handles, we can visualize 2 and 3 -manifolds in a similar way by placing ourselves on the boundary of their 0 handles. By this way we get analogous handlebody pictures as in Figures 24 and 25, except in this case we don't need to specify the framings of the attaching circles of the 2-handles. The 3-manifold handlebody pictures obtained this way are called *Heegard diagrams*. Clearly we can thicken these handlebodie pictures by crossing them with balls to get higher dimensional handlebodies as indicated in these figures. For example, the middle picture of Figure 24 is a Heegrd diagram of $T^2 \times I$ and the last picture is $T^2 \times B^2$, and the Figure 25 is a Heegard diagram of S^3, and a handlebody of $S^3 \times B^1$.

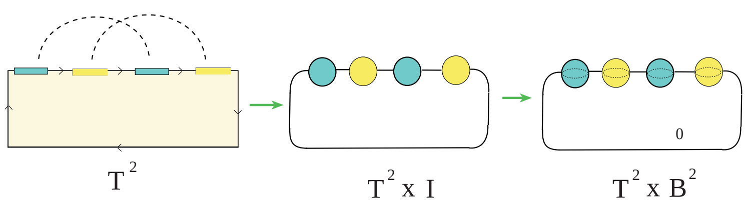

Figure 24

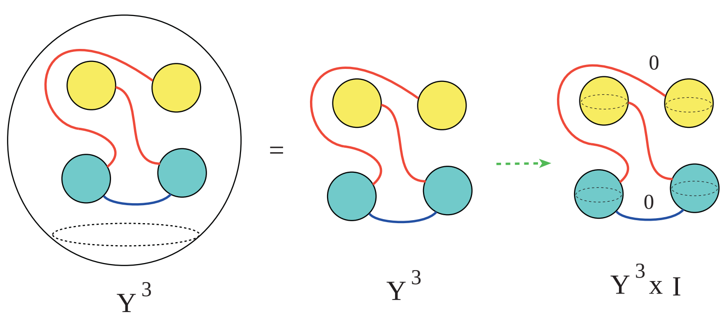

Figure 25

After this process, by changing the 1-handle notations to the circle with dot notation, in Figure 26 we get handlebody pictures of $T^2 \times B^2$ and $F_g \times B^2$ where F_g denotes the surface of genus g.

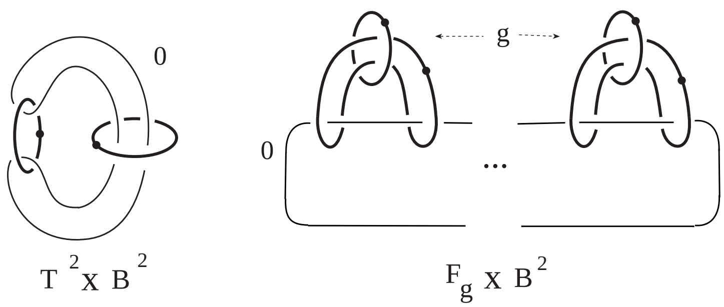

FIGURE 26

Exercise 8. *By using this method verify Exercise 7, and also show that the pictures in Figure 27 are twisted D^2 bundles over S^2, $\mathbb{RP}^2$, and F_g (more specifically Euler class k bundles).*

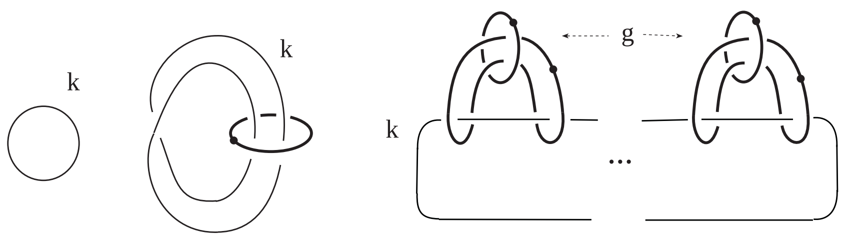

FIGURE 27

1.0.1. *Plumbing.* For $i = 1, 2$ let $D_i \hookrightarrow E_i \to S_i$ be two Euler class k_i disk bundles over the 2-spheres S_i, and let $B_i \subset E_i$ be the disks giving the trivializations $E_i|_{B_i} \stackrel{\approx}{\to} B_i \times D_i$. Plumbing E_1 and E_2 is the process of gluing them together by identifying $D_1 \times B_1$ with $B_2 \times D_2$ (by switching base and fiber directions). Hence, if we locate ourselves on the boundary of $D_1 \times B_1 = B^4$ we will see two 2-handles attached to B^4 along the link of Hopf circles $\{\partial D_1, \partial B_1\}$ with framings k_1, k_2.

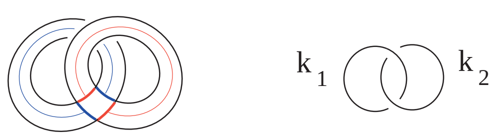

FIGURE 28

This is symbolically denoted by a graph with one edge and two vertices weighted k_1 and k_2. By iterating this process we can similarly construct 4-manifolds corresponding to any graph whose verticies weighted with integers. Also in this construction S_i's don't have to be spheres, they can be any surfaces (orientable or not), but in this case when we abbreviate these plumbings by graphs, to each of its vertex we need to specify specific surface along side an integer weight. Reader can check that the manifolds corresponds to the graphs given in Figure 29.

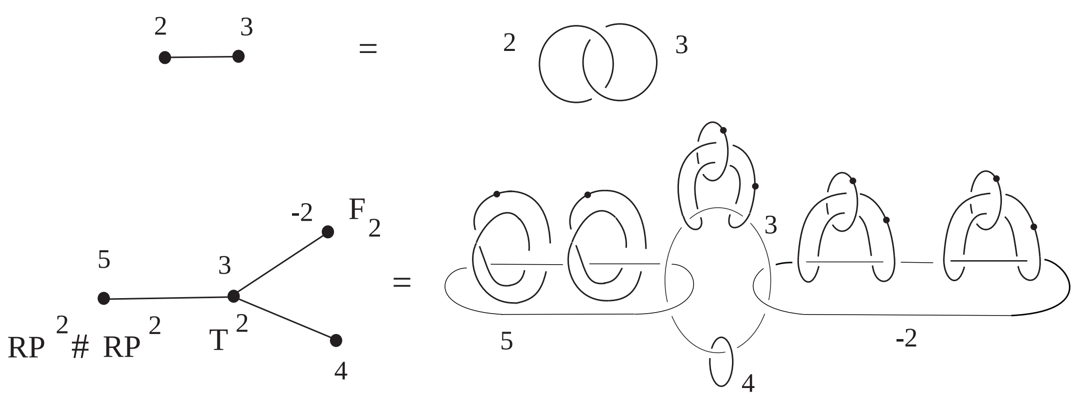

FIGURE 29

The following plumbed manifolds have special names: E_8 and E_{10}.

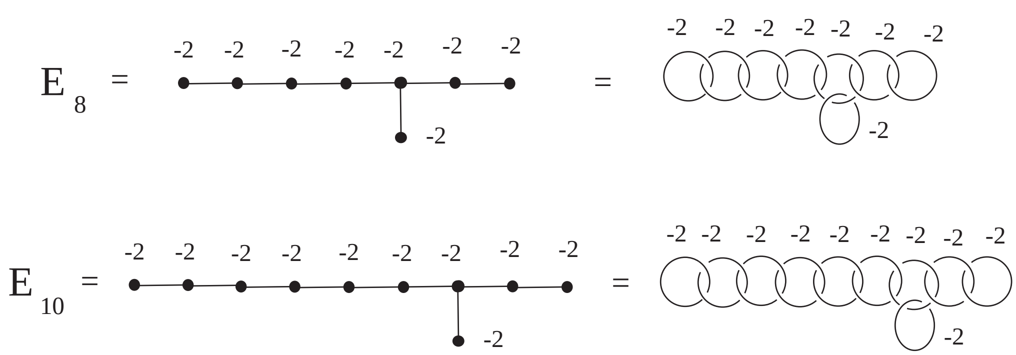

FIGURE 30

An alternative way to denote a plumbing diagram of 2-spheres is to draw the dual plumbing graph, where the vertices are replace by intersecting arc segments, intersecting according to the graph

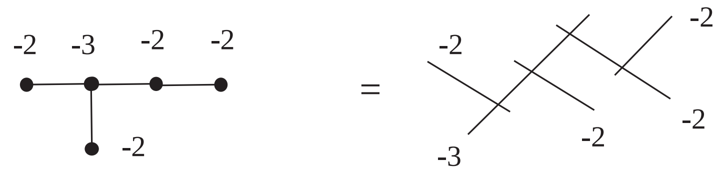

FIGURE 31

1.0.2. *Self plumbing.* By a similar method above, we can plumb a disk bundle $D \hookrightarrow E \to S$ to itself. The only difference is, when we locate ourselves on the boundary of $D_1 \times B_1 = B^4$, we will see a link of thickened Hopf circles being identified with each other by a cylinder $(S^1 \times [0,1]) \times D^2$. This is equivalent to first attaching a 1-handle to B^4 (thickened point $\times[0,1]$), followed by attaching a 2-handle to the connected sum of the two circles over the 1-handle (a tunnel). Figure 32 shows self-plumbings of Euler calls k bundles over S^2 and over T^2.

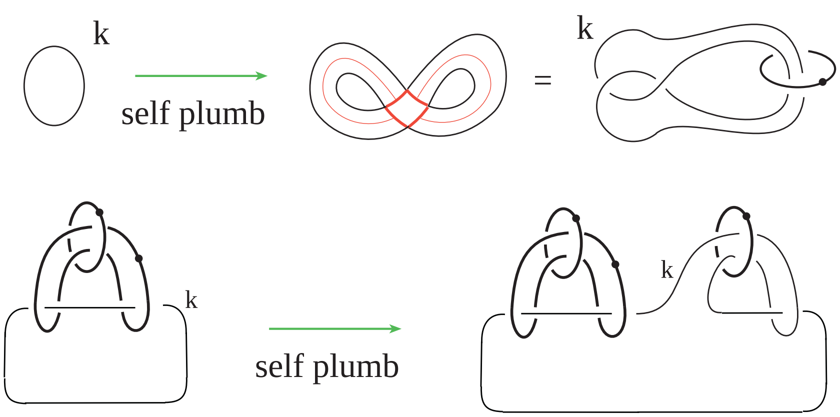

FIGURE 32

1.1. **Some useful diffeomorphisms.** As shown in Figure 33, by introducing a canceling pair of 1 and 2-handles, and sliding new 1-handle over the existing 1-handle, and then canceling the resulting 1 and 2-handle pair, we get an interesting diffeomorphism between the first and last handlebodies of this figure. These diffeomorphisms are used in [AK2] identifying various 3-manifolds.

Exercise 9. *Let $W^{\pm}(l,k)$ be the contractible manifolds given in Figure 34 (where the integer l denotes l-full twist between the stands). By using the above defined diffeomorphisms show that:*

- $W(l,k) \approx W(l+1,k-1)$
- $W^{-}(l,k) \approx -W^{+}(-l,-k+3)$

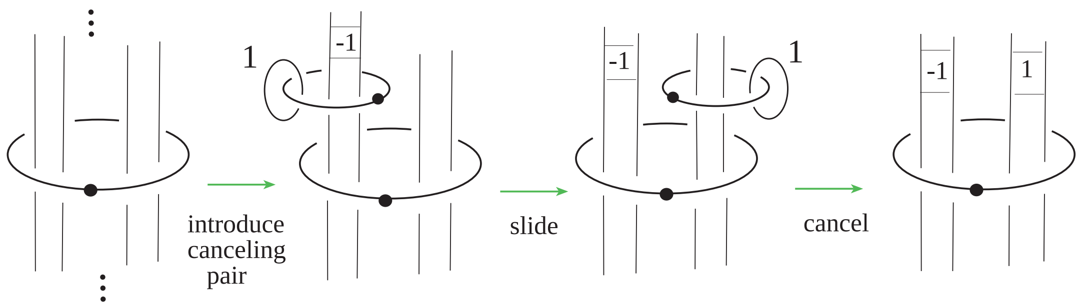

FIGURE 33

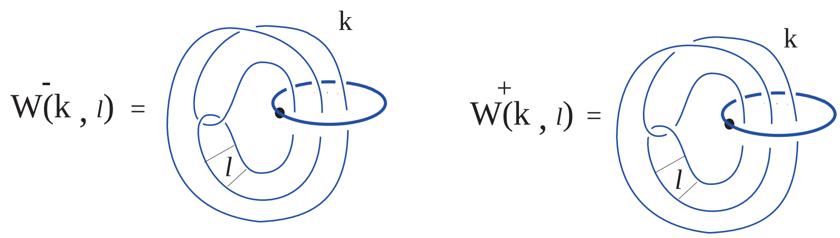

FIGURE 34. $W(l,k)$

Exercise 10. *Verify the diffeomorphisms of Figure 35 by handle slides*

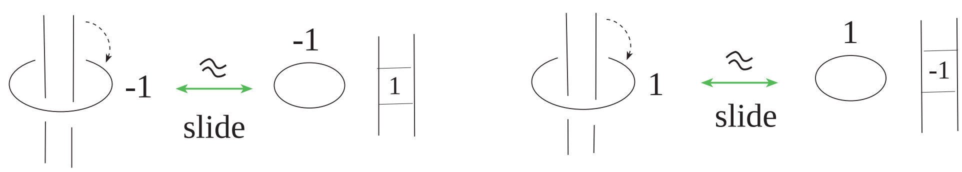

FIGURE 35

Exercise 11. *Show that $S^2 \times S^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ are diffemorphic to each other by sliding operations as shown in Figure 36 (this identification of the 4-manifolds is originally due to Hirzebruch).*

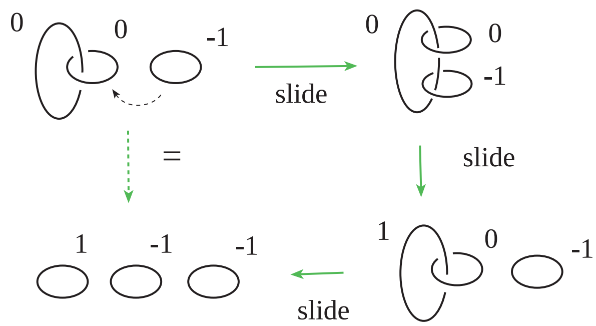

FIGURE 36

1.2. **Algebraic topology.** Algebraic topolology of a 4-manifold M^4 can easily be red off from its handles. For example if $M = M_\Lambda$, where $\Lambda = \{K_1^{r_1}, .., K_k^{r_k}, C_1, .., C_s\}$ and no 3-handle present, to compute $\pi_1(M)$, for each circle-with-dot C_i we introduce a generator x_i, then each loop K_j gives a relation $r_j(x_1, .., x_k)$ recording how it goes through the 1-handles. Also homology groups of M can be computed from the cell complex given by its 1 and 2-handles. For example, to any $\alpha = \sum_j c_j K_j$ with $c_j \in \mathbb{Z}$ we can associate $[\alpha] \in H_1(M)$ (by thinking K_j as loops). Then when $[\alpha] = 0$ and by viewing K_j's as 2-cells, we can think of $\alpha \in H_2(M)$. The intersection form of $H_2(M)$ can be computed from these classes: For example if $\alpha = \sum_i c_i K_i$ and $\beta = \sum_j c'_j K_j$, then $\alpha.\beta = \sum_{i,j} c_i c'_j L(K_i, K_j)$, where $L(K_i, K_j)$ is the linking number of K_i and K_j. So in particular $L(K_j, K_j)$ is given by the framing r_j of K_j. Also changing the orientation of the manifold M_Λ corresponds changing Λ to $-\Lambda = \{-K_1^{-r_1}, .., -K_k^{-r_k}, C_1, .., C_s\}$, where $-K$ denotes the mirror image of the knot K. So in short $-M_\lambda = M_{-\Lambda}$.

Every closed 3-manifold bounds some compact smooth 4-manifold. In particular every oriented 3-manifold is of the form $\partial(M_\Lambda)$, where $\Lambda = \{K_1^{r_1}, .., K_k^{r_k}, C_1, .., C_s\}$. We can assume M has no 3handles (why?). If we are interested only in the boundary 3-manifold, we can replace dotted circles $C_1, .., C_s$ with 0-framed unknots (Exercise 2) turning Λ into collection entirely consisting of framed links. When $\Lambda = \{K^r\}$ the following alternative notations will be used interchangibly:

$$M_\Lambda = B^4(K, r) = K^r$$

So $\partial(K^r)$ will be the 3-manifold obtained by surgering S^3 along K with framing r. If $K \subset Y^3$ is any knot, we will denote he 3-manifold obtained by surgering Y along K with framing r by $Y(K, r)$ or $Y_r(K)$.

1.3. **Kirby calculus.** The operation $M \mapsto M \# \pm \mathbb{CP}^2$ is called *blowing up* operation, and its converse is called *blowing down* operation. If $C^{\pm 1}$ is an unknot with ± 1 framing in a framed link Λ, then by the handle sliding operation of Exercise 10, we get a new framed link Λ' containing $C^{\pm 1}$ such that C does not linking any other elements of Λ'. Write $\Lambda' = \Lambda'' \cup \{C^{\pm 1}\}$, then clearly $M_{\Lambda'} = M_{\Lambda''} \# (\pm \mathbb{CP}^2)$ and hence $\partial M_{\Lambda'} \approx \partial M_{\Lambda''}$. Sometimes the operations $\Lambda' \leftrightarrow \Lambda''$ on framed links are also called blowing up/down operations (of the framed links).

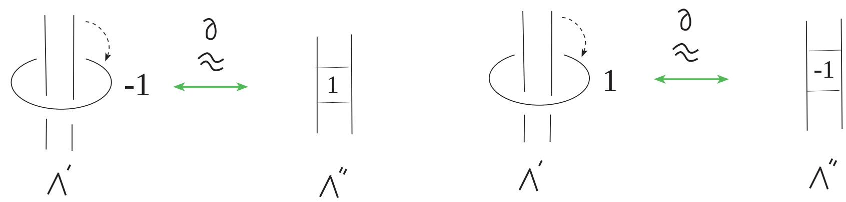

FIGURE 37

A set Λ of framed links encodes handle information on $M = M_\Lambda$, which comes from a Morse function $f : M \to \mathbb{R}$. Cerf theory studies how two different Morse function on a manifold are related to each other [C]. In [K1] by using the Cerf theory Kirby studied the map:

$$\{\text{Framed links}\} \mapsto \{3 - \text{manifolds}\}$$

defined by $\Lambda \mapsto \partial(M_\Lambda)$, and he proved that under this map any two sets of framed links Λ and Λ' are mapped to the same 3-manifold if and only if they are related to each other by handle sliding operation of Excercise 4 (c), and blowing up or down operation. Manipulating framed links by these two operations is usually called "Kirby calculus".

1.4. **Examples.** Links of hypersurfece singularities provide rich class of 3-manifolds, they are obtained by intersecting complex hypersurfaces in $\mathbb{C}^2$ with isolated singularities at the origin with a sphere S^5_ϵ of small radius ϵ, centered at the origin, for example the *Brieskorn* family:

$$\Sigma(a, b, c) := \{(x, y, z) \in \mathbb{C}^3 \mid x^a + y^b + z^c = 0\} \cap S^5_\epsilon$$

These can be identified as boundaries of some interesting 4-manifolds:

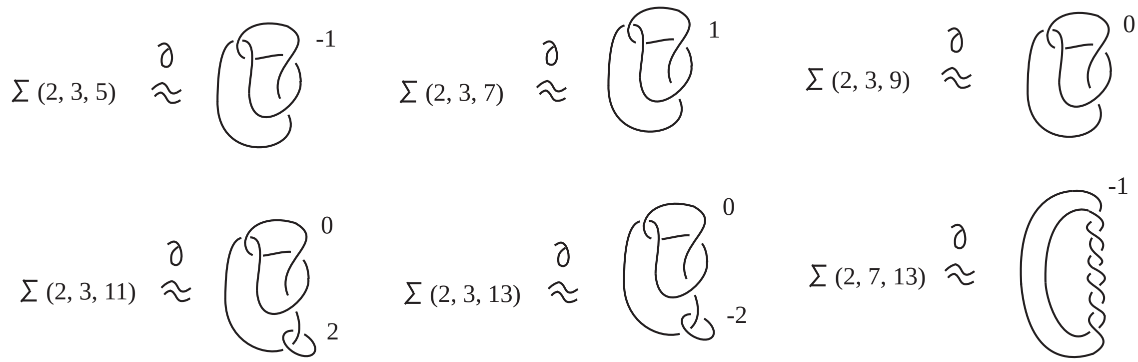

FIGURE 38

$\Sigma(2,3,5)$ is called the *Poincare homology sphere*. There is a useful boundary identification $\Sigma(2,3,5) \approx \partial E_8$ by the steps of Figure 39

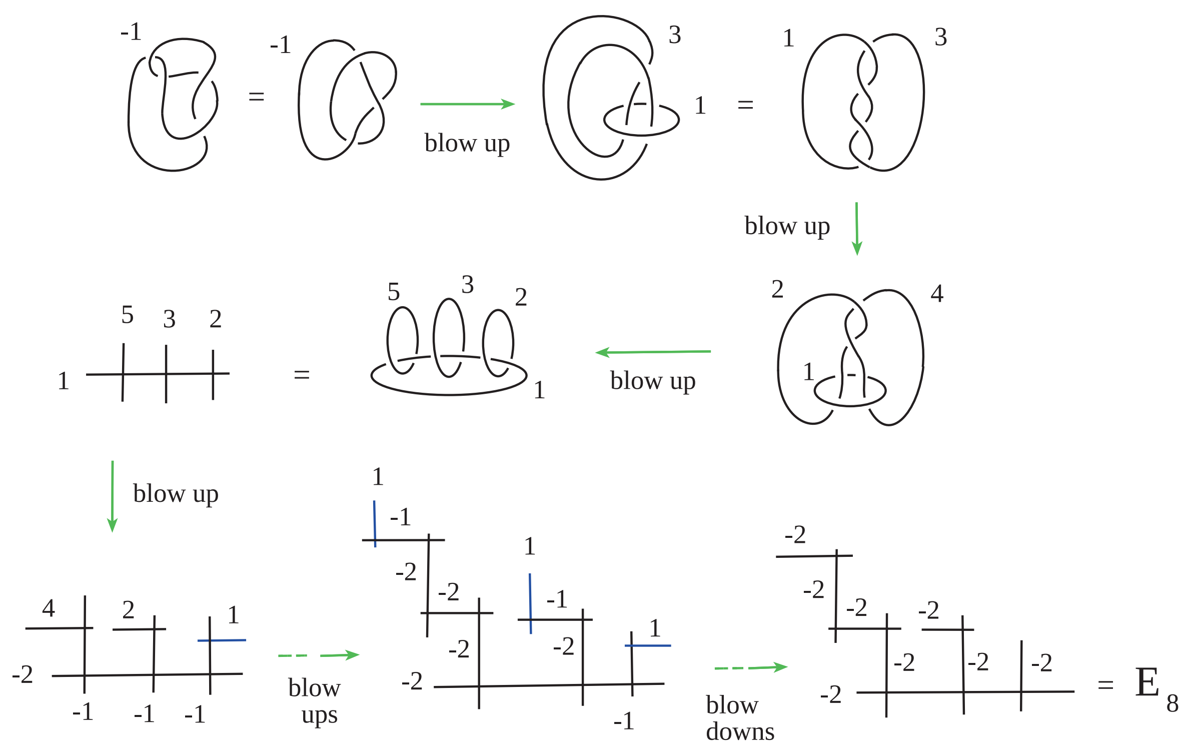

FIGURE 39

Exercise 12. *By imitating the steps of Figure 39 show that* $\Sigma(2,3,7) \approx \partial E_{10}$*. Also by justifying the diffeomorphisms in Figure 39 show that* $\partial E_{10} \approx M^4$*, where* M^4 *is a manifold obtained from* E_8 *by attaching pair of 2-handles, and* M *has the intersection form of* $E_8 \# (S^2 \times S^2)$*.*

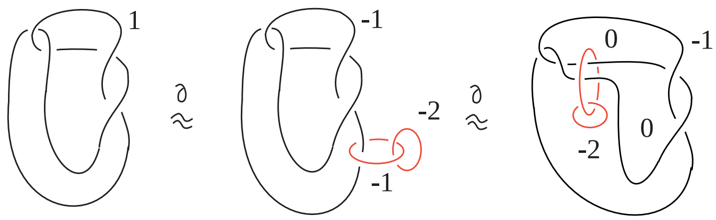

FIGURE 40

Exercise 13. *By justifying steps of Figure 41 construct a closed simply connected smooth manifold with signature* -16 *and the second Betti number 22 (In the figure* $M = A + B$ *means* M *contains each of the handlebodies* A *and* B*, and the handles of* A *and* B *has zero algebraic linking number, in which case we can write* $M = A +$ *handles* $= B +$ *handles, and* M *is homology equivalent to* $A \# B$*).*

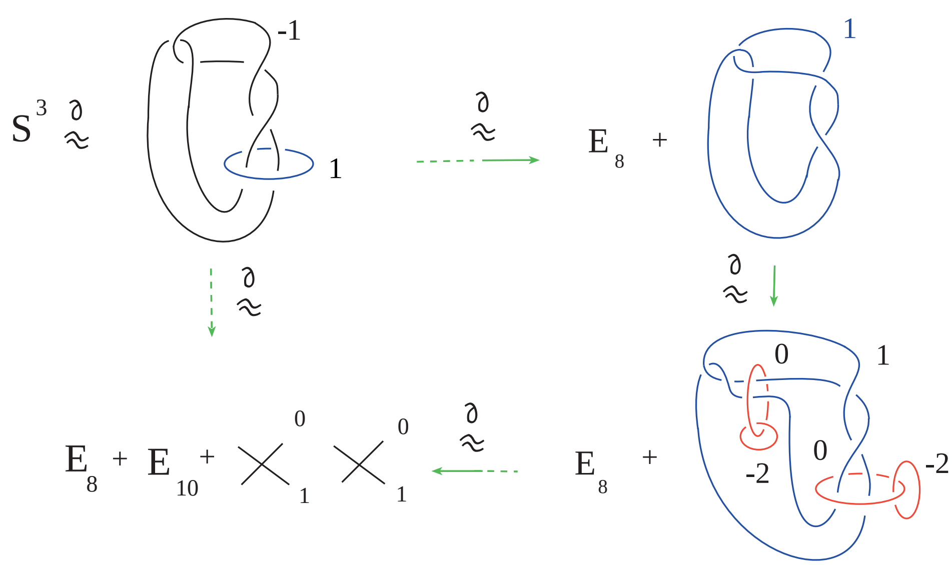

FIGURE 41

$\Sigma(2,3,13)$ bounds a contractible manifold, which can be seen by the identification $\partial\Sigma(2,3,13) \approx \partial W^{+}(1,0)$ as indicated in Figure 42.

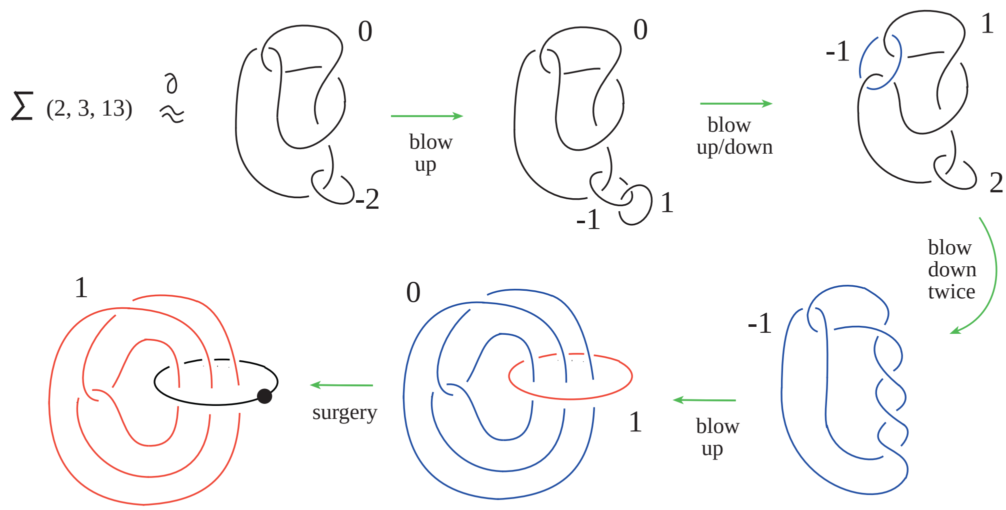

FIGURE 42

Exercise 14. *By verifying the identifications of Figure 43, show that* $\Sigma(2,3,11)$ *bounds a smooth simply connected manifold with signature* -16 *and the second Betti number* 20.

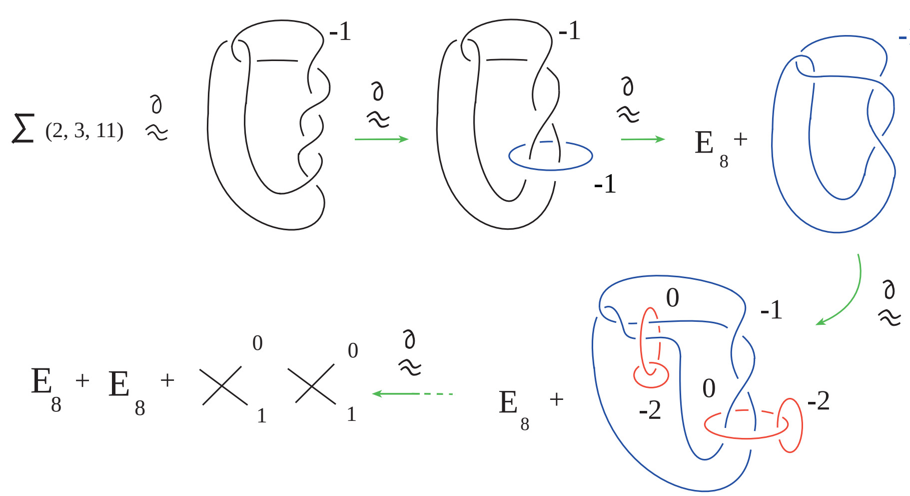

FIGURE 43

1.5. **Constructing diffeomorphisms by carving.** Given a diffeomorphism $f : \partial M \stackrel{\approx}{\to} \partial N$, when does f extend to a diffeomophism inside $F : M \to N$? Some instances carving can provide a solution. One necessary condition is that f must extend to a homotopy equivalence inside, so let us assume this is as a hypothesis. Now let us start $M = M_{\Lambda}$, where $\Lambda = \{K_1^{r_1}, .., K_k^{r_k}, C_1, .., C_s\}$, and let $\{\gamma_1, .., \gamma_k\}$ be the dual circles of the 2-handles. i.e. $\gamma_j = \partial B_j$, where B_j is the co-core of the dual 2-handle of $K_j^{r_j}$. Then if $\{f(\gamma_1), .., f(\gamma_k)\}$ is a slice link in N, that is if each $f(\gamma_j) = \partial D_j$ where $D_j \subset N$ are properly imbedded disjoint disks. Then we can extend f to a diffeomorphism:

$$f' : M' := \partial M_{\epsilon} \smile \cup_j \nu(B_j) \to \partial N_{\epsilon} \smile \cup_j \nu(D_j) := N'$$

where ∂M_{ϵ} and ∂N_{ϵ} are the collar neighborhoods of the boundaries, and $\nu(B_j)$, $\nu(D_j)$ are the tubular neighborhoods of the disks B_j, D_j.

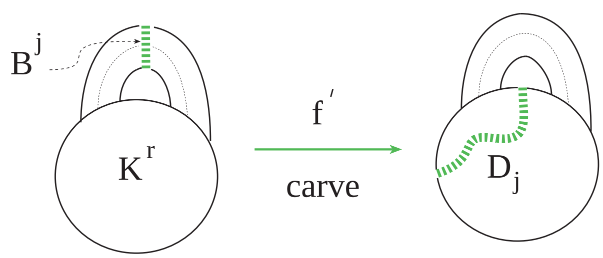

FIGURE 44

This reduces the extension problem to the problem of extending f' to complements $M - M' \to N - N'$. Notice that $M - M' = \#_s(S^1 \times B^3)$ and $N - N'$ is a homotopy equivalent to $\#_s(S^1 \times B^3)$. Since every self diffeomorphism $\#_s(S^1 \times S^2)$ extends to a unique self diffeomorphism of $\#_s(S^1 \times B^3)$, the only way f' doesn't extend to $M - M' \to N - N'$ is $N - N'$ is an exotic copy of $\#_s(S^1 \times B^3)$. The case of $s = 0$ is particularly interesting, since the only way the last extension problem $M - M' \to N - N'$ fails is when the 4-dimensional smooth Poincare conjecture fails (in examples usually this step goes through).

Even in the cases of single 2-handle $M = K^r$ and $N = L^r$ extending a diffeomorphism $f : \partial(K^r) \to \partial(L^r)$ can be difficult task without carving. Because there are no other handles to slide over to construct a diffeomorphisms in a conventional way! Cerf theory says that, if they are diffeomorphic there must be canceling handle pairs but we don't know where they are? Carving at least gives us a way to start.

For example, Figure 45 describes a diffeomorphism [A1].

$$f : \partial(K^1) \xrightarrow{\approx} \partial(L^1)$$

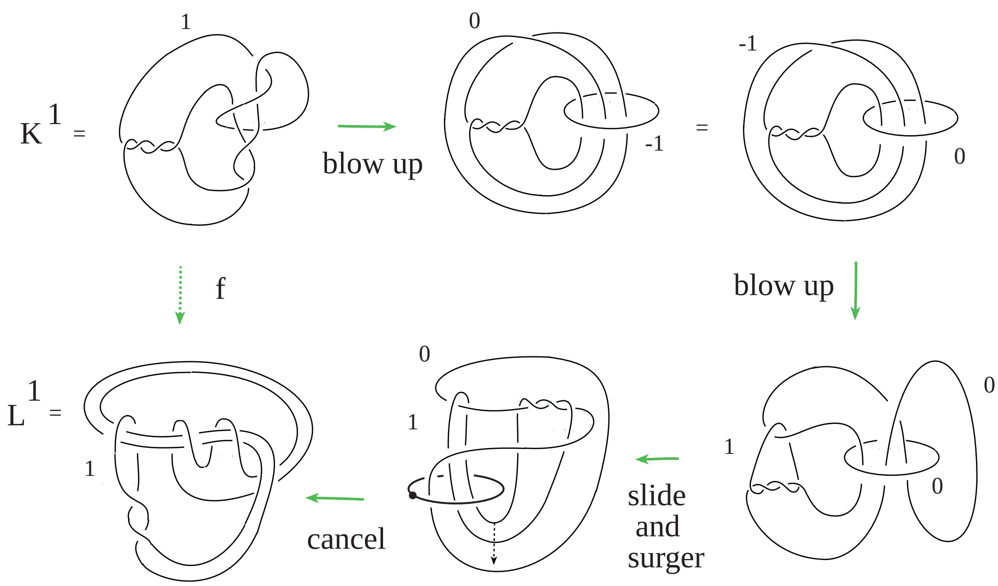

FIGURE 45

Figure 46 shows the dual circle γ and its image $f(\gamma)$. By carving along γ and $f(\gamma)$ we can extend f to a diffeomorphism. To do this we put dots on these circles (making 1-handles). Then we only have to see that the second picture becomes B^4. For this we slide the 2-handle over the 1-handle as indicated in the figure and check that we get B^4.

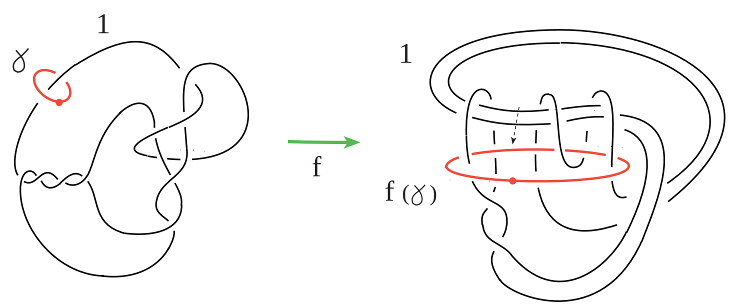

FIGURE 46

The following is a generalization of the above example from [A4]. We claim that for each $r \in \mathbb{Z} - 0$, there are of distinct knot pairs K and L_r (one is slice the other non-slice knots) such that for al $r \neq 0$.

$$K^r \approx L_r^r$$

Previous example gives a hint how to see this by Cerf theory, that is introduce and then cancel 1- and 2-handle pairs as shown in Figure 47.

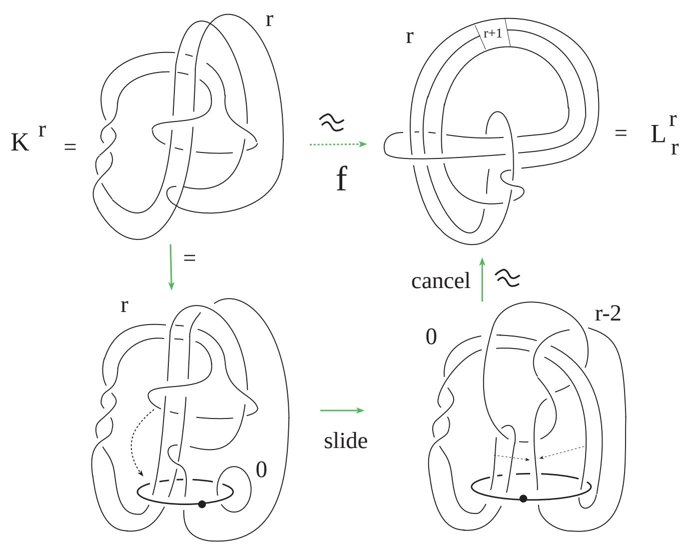

Figure 47

Exercise 15. ([O], [T]) *Show that in ∂M (and ∂N) (Figure 48) the loops a and b are isotopic to each other (Hint: slide over the 2-handle). From this produce distinct knots K_r and L_r with $K_r^0 \approx K_s^0$ and $L_r^4 \approx L_s^4$ for all $r \neq s$ (Hint: consider repeated ± 1 surgeries to a and b).*

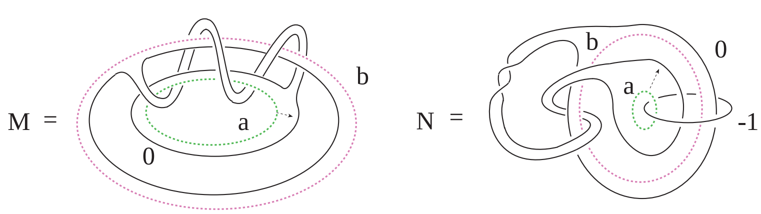

Figure 48

2. Gluing 4 manifolds along their boundaries

Given two connected smooth 4-manifolds with boundary M, N, and an orientation preserving diffeomorphism $f:\partial M\to\partial N$, we can ask how can you draw a handlebody of the oriented manifold?

$$-M\smile_f N$$

Also we can ask when given two copies of codimension zero submanifolds $L\sqcup -L\subset\partial M$, how can we draw the manifold $M(f)$ obtained from M by identifying these two copies by a diffemorphism $f:L\to L$?

$$M(f)=M\sqcup(L\times[0,1])\Big/\begin{array}{rl}(x,0) & \sim x\in L\\ (f(x),1) & \sim x\in -L\end{array}$$

The following provides some ways of doing this:

2.1. Constructing $-M\smile_f N$ by upside down method:

Let $M=M_\Lambda$ with $\Lambda=\{K_1^{r_1},..,K_k^{r_k},C_1,..,C_s\}$ and $N=N_{\Lambda'}$. Let $\{\gamma_1,..,\gamma_k\}$ be the dual zero-framed circles of the 2-handles $K_1,..,K_r$. Then $-M\smile_f N$ is given by $Q_{\Lambda''}$ where $\Lambda''=\Lambda'\smile\{f(\gamma_1),..,f(\gamma_r)\}$. In particular, if we apply this process to $M^0:=-M-\{\text{3-handles}\}$ and any diffeomorphism $f:\partial(M^0)\to\partial\#_p(S^1\times B^3)$ (p is the number of 3-handles), we get the upsidedown handlebody of M. In the special case of when M has no 3-handles, then clearly the framed link $\{f(\gamma_1),..,f(\gamma_r)\}$ in ∂B^4 gives its upside down handlebody of M.

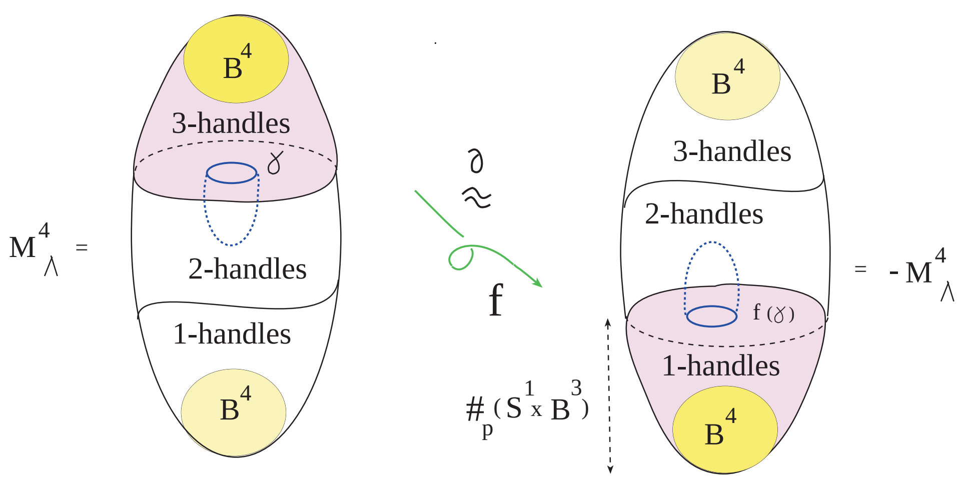

Figure 49

The manifold $-M \smile_{id} M$ is called the double of M and denoted by $D(M)$. For example, from the above description we can can construct handlebody for $D(T^2 \times B^2) = T^2 \times S^2$. Also from we can easily check that the double $D(C)$ of the cusp C is $S^2 \times S^2$ (see Figure 50)

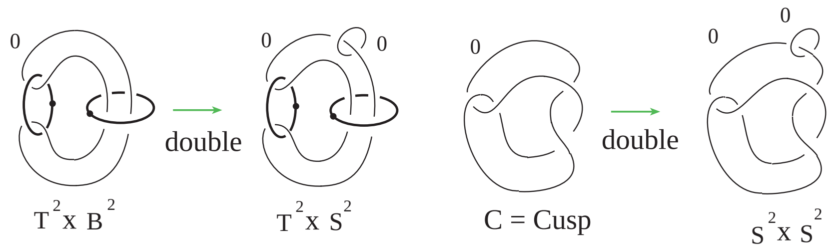

FIGURE 50

Exercise 16. *Let $f : \partial F \to \partial F$ be the diffeomorphism on the boundary of the Fishtail described in Figure 51 (obtained by dot and 0 exchange) Show that $D(F) = S^4$*

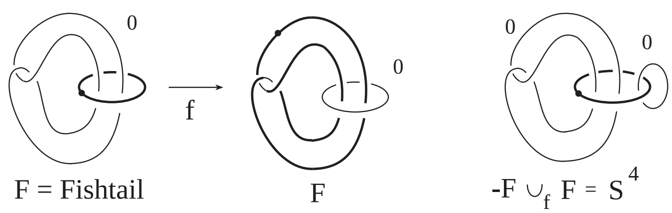

FIGURE 51

Exercise 17. *If $f : \partial M \to \partial E_8$ is the diffeomorphism descibed in Figure 39, show that $-M \smile_f E_8 = \mathbb{CP}^2 \# 9\bar{\mathbb{CP}}^2$ (Hint Figure 52)*

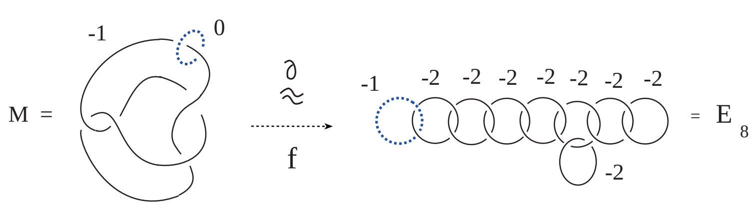

FIGURE 52

2.2. **Constructing $-M \smile_f N$ and $M(f)$ by cylinder method:**

For this we attach a cylinder $\partial M \times [0,1]$ to disjoint union $-M \sqcup N$, where one end of the cylinder is attached by the identity the other end is attached by the map f.

$$-M \smile_f N = -M \smile_{id} (\partial M \times [0,1]) \smile_f N$$

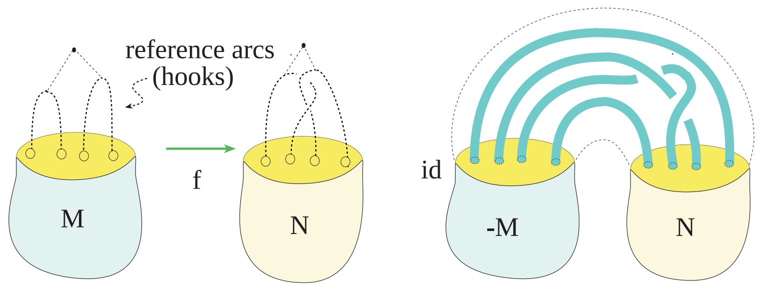

FIGURE 53. $-M \smile_f N$

Similary, for $M(f)$ we glue a cylinder to M running from $-L$ to L

$$M(f) = M \smile_{id \sqcup f} (\partial L \times [0,1])$$

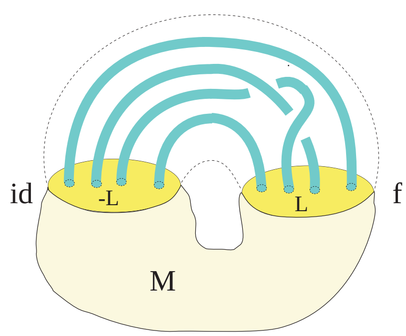

FIGURE 54. $M(f)$

Think of f as a force field hovering over $-M \sqcup N$ carrying points of ∂M to ∂N. To observe the affects of f, we lower ropes (from a central point) with hooks tied at their ends, and the hooks go through the cores γ_j of the 1-handles of M, then we watch where f takes them (Figure 55). To describes the process of gluing the two boundary components by f, we attach 2-handles to $\gamma_j \# f(\gamma_j)$ by using the ropes as guide, as shown in Figure 56, which is $-M \smile_f N$. Put another way, f connects the 0-handle of M with 0-handle of N by a 1-handle (so this 1-handle cancels one of the 0-handles), then by going over this 1- handle it identifies the neighborhoods of γ_j with $f(\gamma_j)$, which amounts to attaching 2-hanles to $\gamma_j \# f(\gamma_j)$ (i.e. creating tunnels).

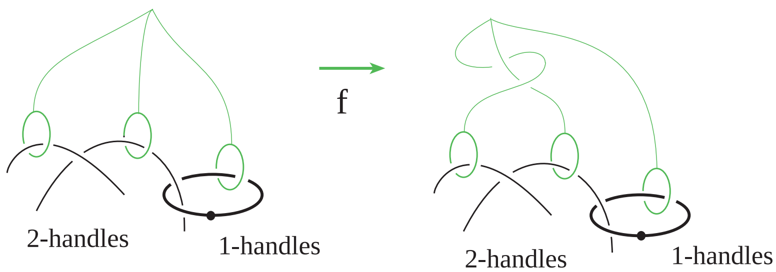

Figure 55. The affect of f

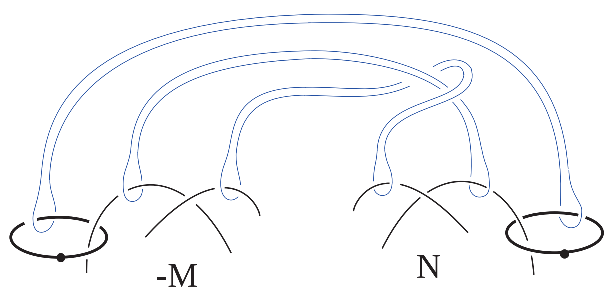

Figure 56. $-M \smile_f N$

We don't need to describe how f identifies the 2- and 3-handles of M with that of N, since by a similar description above, they amount to adding 3- and 4- handles to $-M \sqcup N$. Recall that describing 4-manifold handlebodies we don't need to specify 3- and 4-handles, because they are always attached canonically.

The handlebody of $M(f)$ can be constructed almost the same way, except we must have an extra 1-handle as shown in Figure 57, because in this case the identifying cylinder $L \times [0, 1]$ is attached to a single connected manifold M, as opposed between two disjoint manifolds $-M \sqcup N$. In the former case, this 1-handle was cancelled by one of the 0-handles of $-M \sqcup N$.

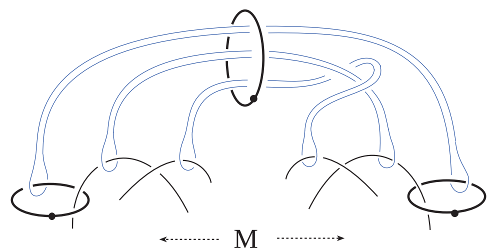

FIGURE 57. $M(f)$

2.3. Codimension zero surgery $M \mapsto M'$:

Let $N \subset M$ is a codimension zero submanifold giving the decomposition $M = N \smile_{\partial} C$, and let $f : \partial N \to \partial N'$ is a diffeomorphism. We call the process $M \mapsto M' := N' \smile_f C$ codimension zero surgery.

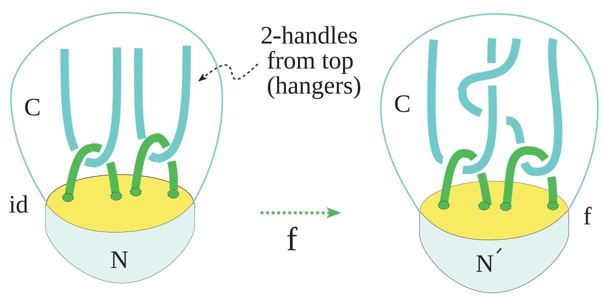

FIGURE 58. Codimension zero surgery $M \mapsto M'$

To visualize this process think of all the handles of C as placed at the top of N. Notice that only the 2-handles of C interacts with the handles of N. So think of N as hanging from the 2-handles of C (Figure 58) like hangers in a dress closet, where f shuffles the hangers below. So while the handles of N are hanging from the top, they look differently according to how f rearranged them below.

More precisely, we apply diffeomoprhism $f : \partial N \to \partial N'$ keeping track of where f throws the 2-handles of C in N'. That is, if $N = N_{\Lambda}$ and $N' = N'_{\Lambda'}$ such that $M = M_{\mathbf{\Lambda}}$ with $\mathbf{\Lambda} = \Lambda \smile \{K_1^{r_1}, .., K_p^{r_p}\}$, then $M' = M'_{\mathbf{\Lambda}'}$ where $\mathbf{\Lambda}' = \Lambda' \smile \{f(K_1^{r_1}), ..f(K_p^{r_p})\}$.

We say *M is obtained from M' by cutting out N' and gluing N via* $f : \partial N \to \partial N'$. So if $M' = M'_{\mathbf{\Lambda}'}$ with $\mathbf{\Lambda}' = \Lambda' \smile \{L_1^{r_1}, .., L_p^{r_p}\}$, then $M = M_{\mathbf{\Lambda}}$ where $\mathbf{\Lambda} = \Lambda \smile \{f^{-1}(L_1^{r_1}), ..f^{-1}(L_p^{r_p})\}$.

3. Surface bundles over surfaces

3.1. $\mathbf{T}^4$. As in Figure 24 we start with T^2 then thicken it $T^2 \times [0,1]$. Then by using the recipe of Section 2.2 we first form T^3 by identifying the front and the back faces of $T^2 \times [0,1]$ (Figure 59), then we construct T^4 by identifying the front and back faces of $T^3 \times [0,1]$ (Figure 60).

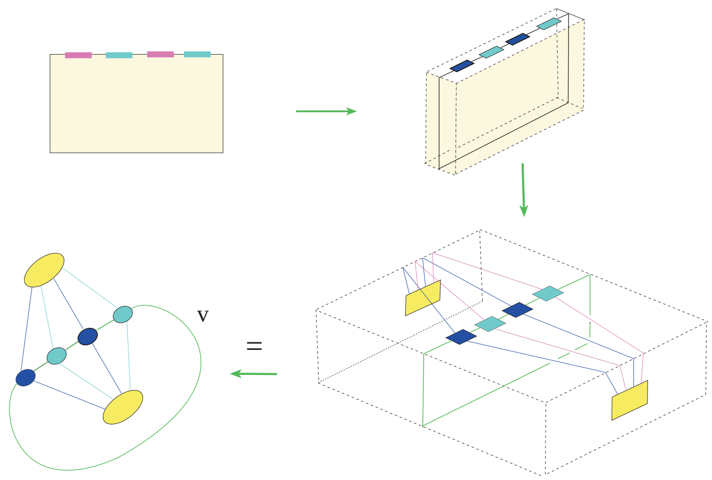

Figure 59. T^3

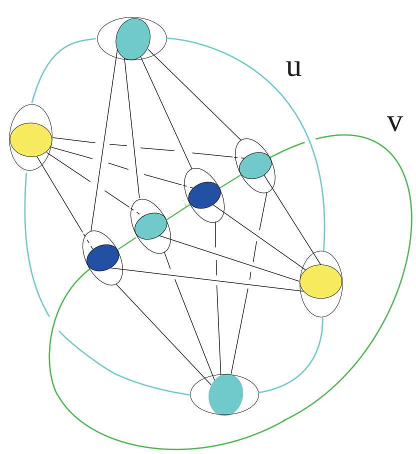

Figure 60. T^4

By converting the 1-handle notation to the circle-with-dot notation (Section 0.1) in Figure 63 we get another handlebody picture of T^4.

For the benefit of the reader: While doing this conversion for the middle 1-handle balls, we first flatten them as in Figure 61 obtaining Figure 62, then convert them to the circle with dot notation with ease.

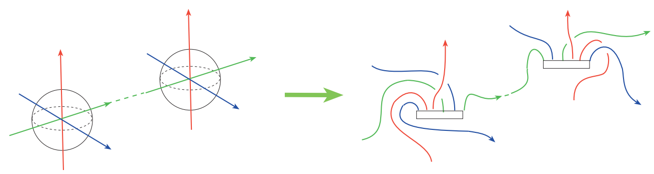

FIGURE 61. T^4

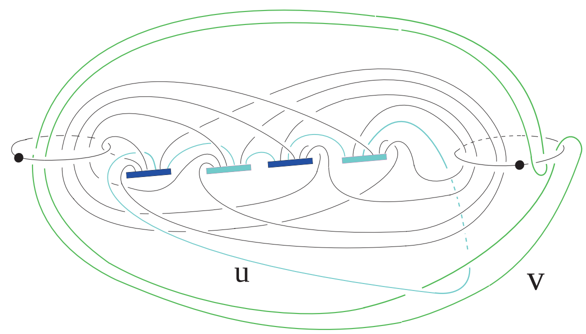

FIGURE 62. T^4

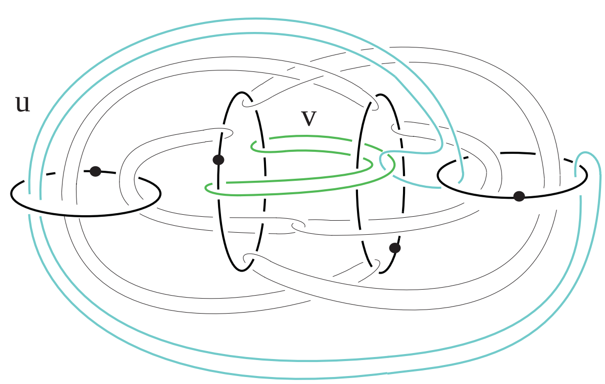

FIGURE 63. T^4

Exercise 18. *Show that without the 2-handle v Figure 60 describes $T^2_0 \times S^1$, and without the 2-handles u and v Figure 63 is just $T^2_0 \times T^2_0$, where $T^2_0 = T^2 - D^2$ is the punctured T^2.*

3.2. **Cacime surface.**

Cacime is a particular surface bundle over a surface, which appears naturally in complex surface theory [CCM]. Understanding this manifold is instructive, and is a good test case for understanding many of the difficulties one encounters constructing handlebodies of surface bundles over surfaces. We will first draw this manifold, and from this drive a recipe for drawing surface bundles over surfaces in general.

Let F_g be the surface of genus g. Let $\tau_2 : F_2 \to F_2$ be the hyperelliptic involution and $\tau_3 : F_3 \to F_3$ be the free involution induced by 180^o rotation (Figure 64). The Cacime surface M is the complex surface obtained by taking the quotient of $F_2 \times F_3$ by the product involution:

$$M = (F_2 \times F_3)/\tau_2 \times \tau_3$$

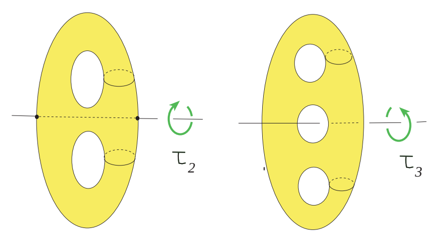

FIGURE 64. T^4

By projecting to the second factor we can describe M as a F_2-bundle over $F_2 = F_3/\tau_3$. Let A denote the twice punctured 2-torus $A = T^2 - D^2_- \sqcup D^2_+$. Then clearly M is obtained by identifying the two boundary components of $F_2 \times A$ by the involution induced by τ_2 (notice that A is a fundamental domain of the action τ_3).

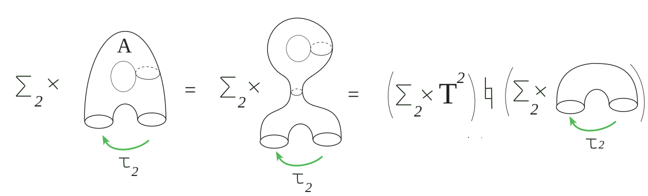

FIGURE 65. T^4

By deforming A as in Figure 65, we see that $M = E \natural E'$ is the fiber summing of two F_2 bundles over T^2, where E is the trivial bundle and

$$E' = \Sigma_2 \times S^1 \times [0,1]/(x, y, 0) \sim (\tau_2(x), y, 1)$$

Now by using the techniques developed in Chapter 2 we will construct the following manifolds, diffeomorphisms and the gluing:

(a) $E_0 := E - F_2 \times D^2 = F_2 \times T^2_0$
(b) $f_1 : \partial E_0 \xrightarrow{\approx} F_2 \times S^1$
(c) $E'_0 = E' - F_2 \times D^2$
(d) $f_2 : \partial E'_0 \xrightarrow{\approx} F_2 \times S^1$
(e) $M = -E_0 \smile_{f_2^{-1} \circ f_1} E'_0$

(a): Very simlar to construction of T^4 in Figures 59 through 63, in Figures 66 through 67 we construct $F \times T^2$

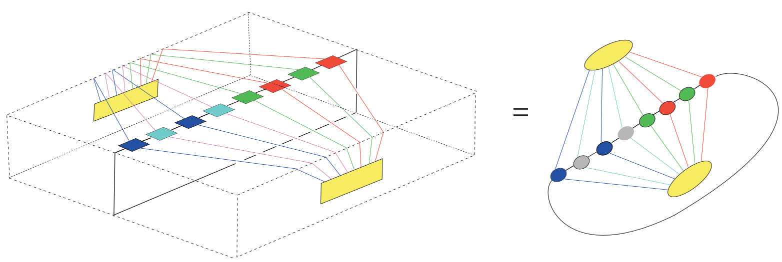

FIGURE 66. $F_2 \times S^1$

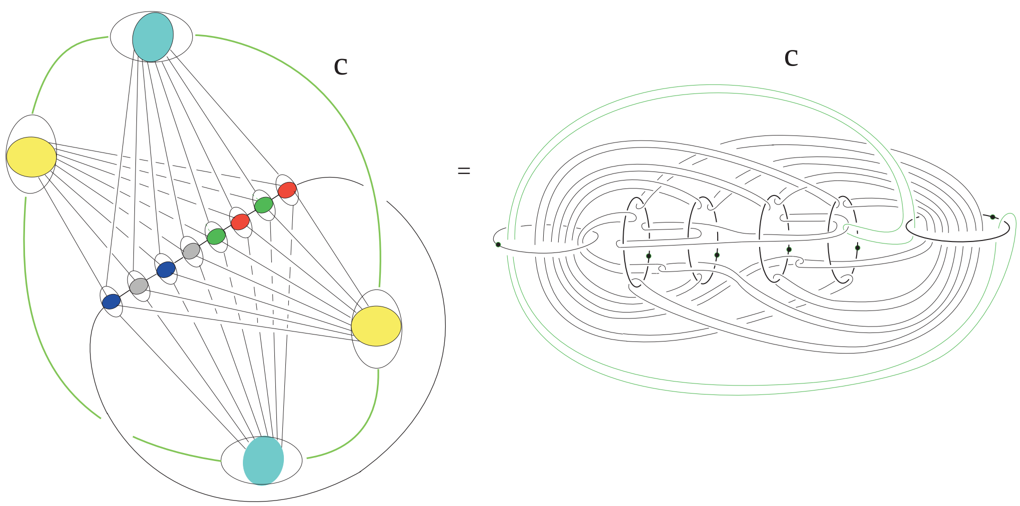

FIGURE 67. $F_2 \times T^2$

Exercise 19. *Show that removing the 2-handle c from the Figure 67 gives a handlebody picture for* $E_0 = F_2 \times T^2_0$

(b): We claim that there is a diffeomorphism $f_1 : \partial E_0 \xrightarrow{\approx} F_2 \times S^1$ which takes the ropes (with hooks) of Figure 67 to the corresponding ropes as indicated in Figure 68.

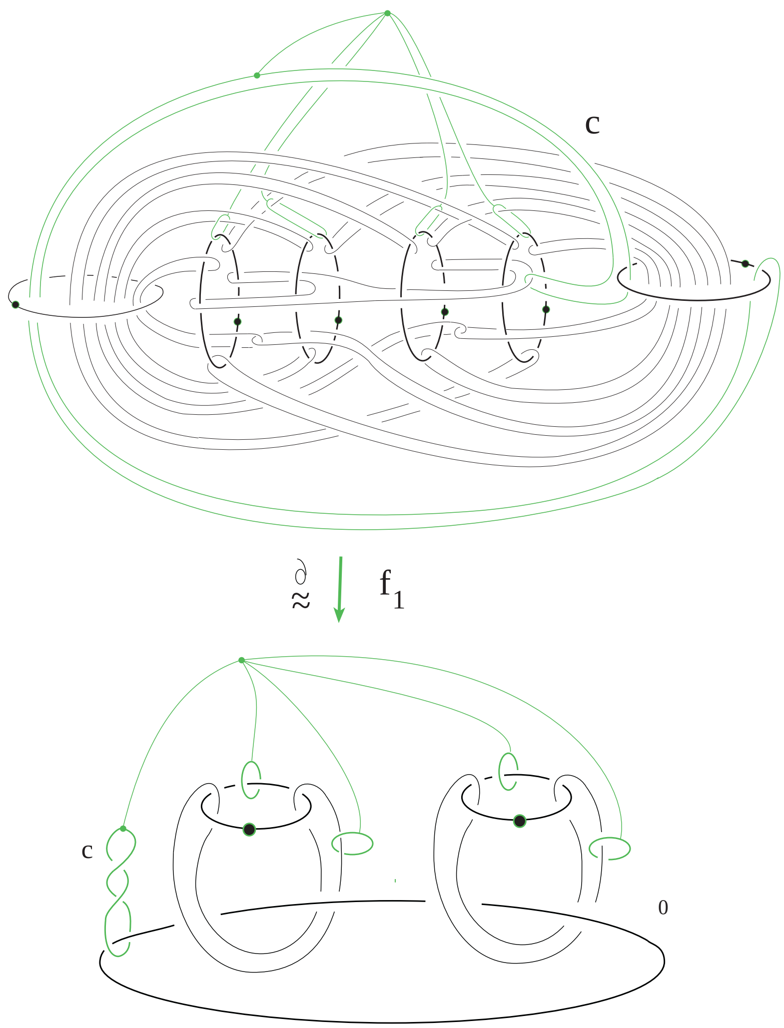

FIGURE 68. Diffeomorphism $\partial E_0 \approx \Sigma_2 \times S^1$ made concrete

Exercise 20. *Show that this diffeomoprphism f_1 can be obtained by the handle slides on the boundary as indicated in Figure 69 (Hint just perform the indicated handle slides while keeping track of the ropes)*

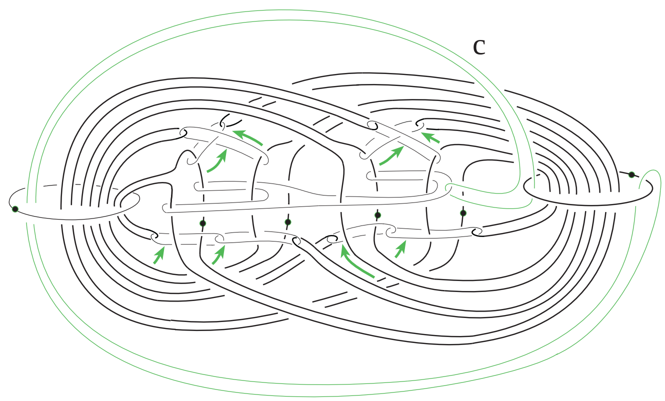

FIGURE 69. Describing f_1

(c): $E_0' = E' - F_2 \times D^2$ is a twisted version of E_0, we proceed as in Figure 66 except that we identify the front and back faces of $F_2 \times [0,1]$ by $\tau : F_2 \to F_2$ and get Figure 71 (a twisted version of Figure 66).

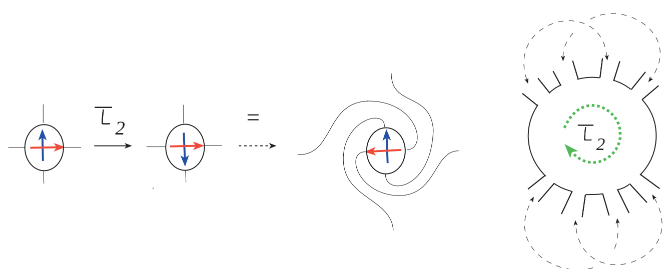

FIGURE 70. The action of $\tau : F_2 \to F_2$

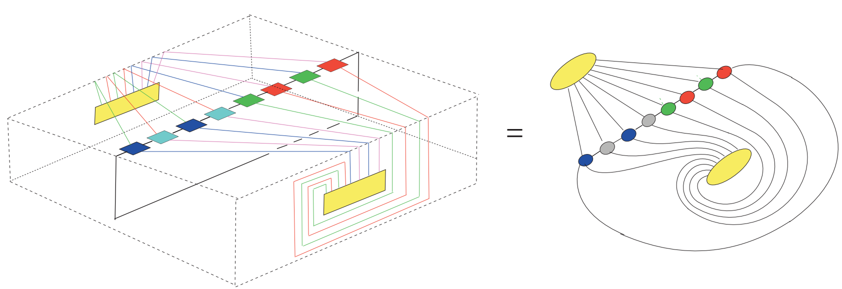

FIGURE 71. $F_2 \times_\tau S^1$

To construct E' we simply cross the twisted F_2 bundle over the circle $F_2 \times_\tau S^1$ with S^1. This gives Figure 72 (drawn in two different 1-handle notations), which is analogous version of Figure 67.

Exercise 21. *Show that without the* 2*-handle* c *Figure 72 describes* $E'_0 = E' - F_2 \times D^2$

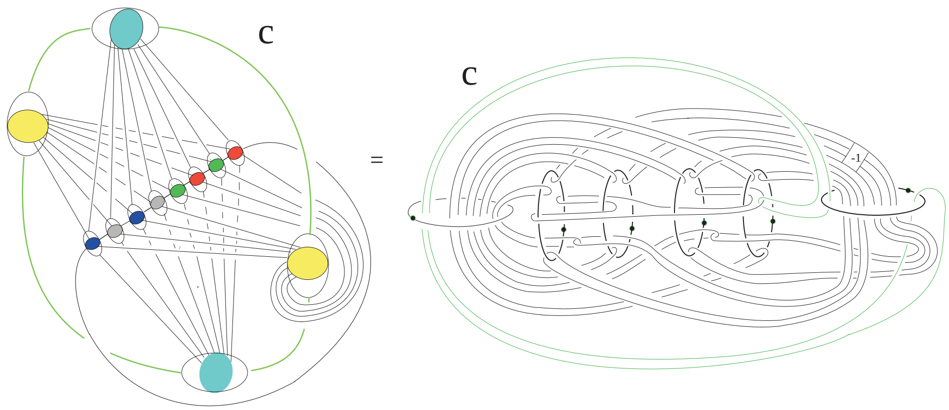

Figure 72

(d): Similar to (b) there is a diffeomorphism $f_2 : \partial E'_0 \xrightarrow{\approx} F_2 \times S^1$ obtained by the handle slides described in Figure 73. As before, to describe this diffeomorphism geometrically in pictures we lower ropes (with hooks) and trace out what f does to these ropes during the diffeomorphism of Figure 73. This gives Figure 74.

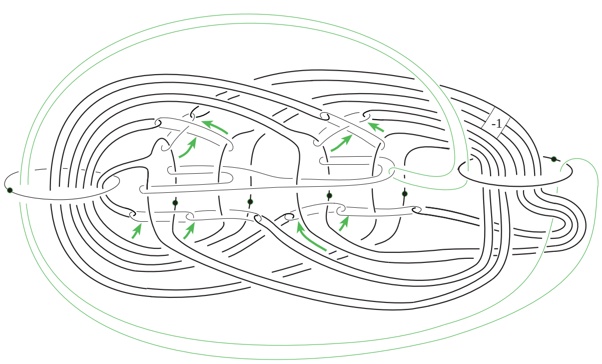

Figure 73. Describing f_2

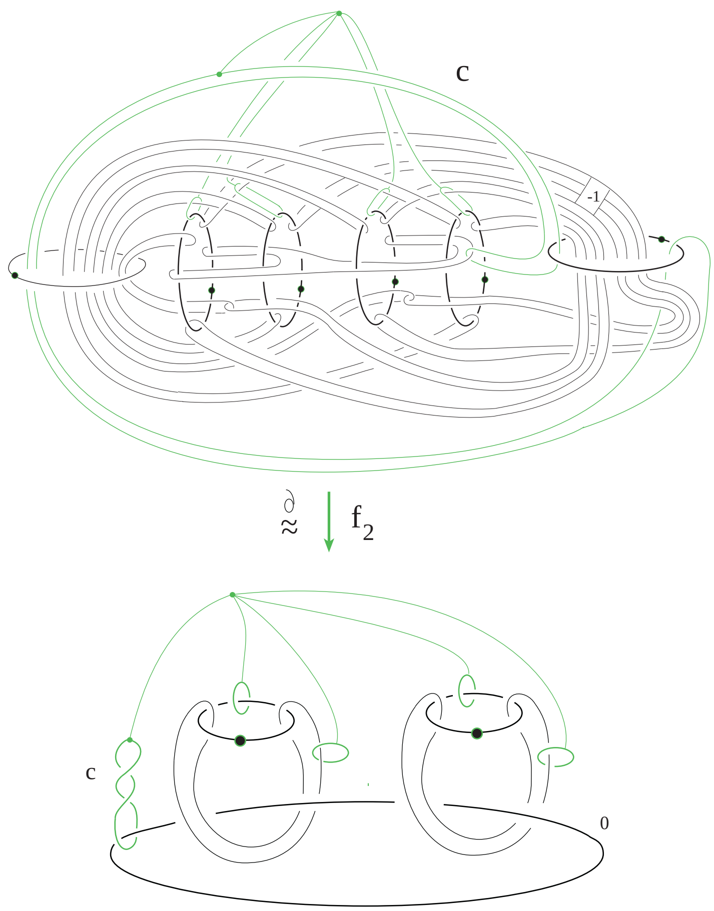

FIGURE 74. Diffeomorphism $\partial E_0' \approx \Sigma_2 \times S^1$ made concrete

(e): Finally by applying the recipe of Section 2.2 we construct the manifold $M = -E_0 \smile_{f_2^{-1} \circ f_1} E_0'$, which is Figure 75. Notice that the ropes in ∂E_0 at top picture of Figure 68 are mapped to the ropes at the top picture of Figure 74 by $f_2^{-1} \circ f_1$.

3.3. **General surface bundles over surfaces.** Now it is clear how to proceed drawing a handlebody picture of a general F_g bundle over F_p $M = F_g \tilde{\times} F_p$: We first decompose M as a fiber sum of F_g bundles over T^2, with monodromies $\tau_j : F_g \to F_g$ $(j = 1, .., p)$, as shown in Figure 75. By removing $F_g \times D^2$ from each, we write $M = \smile_{\partial} E_j$ where each E_j is a F_g bundle over T^2_0, then we perform gluing operations along the boundaries described in Chapter 2.

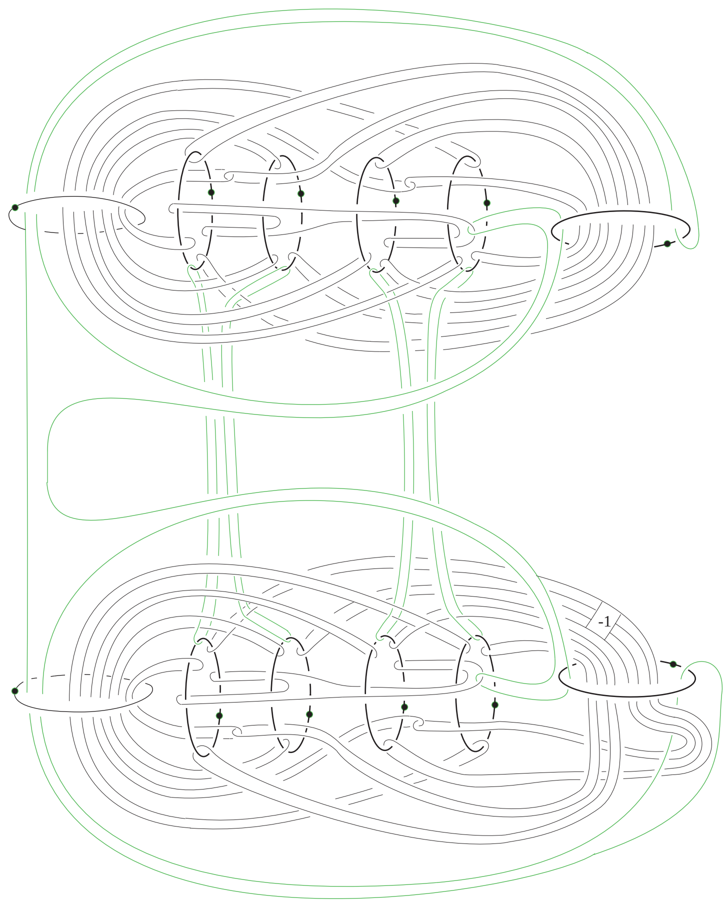

FIGURE 75. Cacime surface

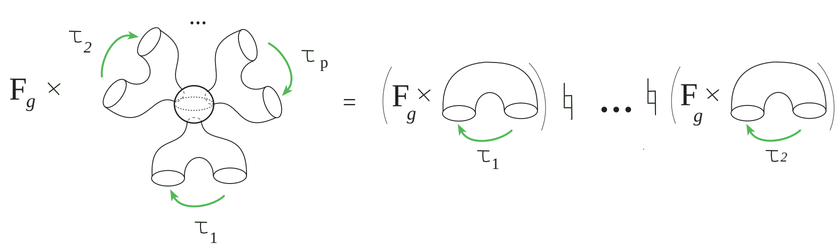

FIGURE 76

3.4. 3-manifold bundles over the circle.

Section 2.2 gave a recipe for drawing the handlebody picture of a 3-manifold Y^3 bundle over the circle with a monodromy $f : Y^3 \to Y^3$:

$$M^4 = M(f) = M \times [0,1]/(x,0) \sim (f(x),1)$$

To put this technique in practice we will apply it to a very interesting example Q^4 of [CS]. Q^4 is an exotic $\mathbb{RP}^4$. Recall that $\mathbb{RP}^4 = N \smile_\partial C$ is the union of codimension zero submanifolds N and C, glued along their common boundaries, where N is a twisted B^2- bundle over $\mathbb{RP}^2$ (Figure 21) and C is the nonorientable B^3- bundle over $\mathbb{RP}^1$. Similarly $Q^4 = N \smile_\partial C_A$ is the union of codimension zero submanifolds N and C_A, glued along their common boundaries, where C_A is a manifold homology equivalent to C, which is obtained as the mapping torus of the diffeomorphism $f_A : T^3_0 \to T^3_0$ induced by the integral matrix A:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

The exotic manifold of Figure 23 [A3] was derived from Q^4. The double covering space of Q is the homotopy sphere $\Sigma = D^2 \times S^2 \smile C_B$, where $B = A^2$ and $f = f_B$. It turns out Σ is the first member Σ_0 of a similarly defined infinite family of homotopy spheres Σ_m, which turned out to be diffeomorphic to the standard 4-sphere ([AK1], [G1], [A5])

$$B = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Let us draw a handlebody picture of Σ (following [AK1]). Notice that $C_B = M(f)$, where $M = T^3 \times [0,1]$ and $f = f_B$. By applying the technique of Section 2.2 we draw $M(f)$. For this we start with a Heegard picture of T^3 as drawn 1- and 2-handles on S^2 (Figure 59).

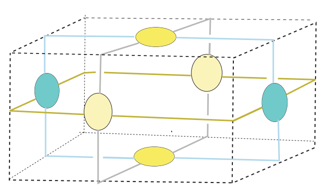

FIGURE 77

We then study $f_B : T^3 \to T^3$ by first checking what it does the coordinate axis (these corresponds to the "ropes" of Section 2.2). Then Figure 79 gives $M(f)$. For the sake of simplicity we put one of the attaching balls of the 1-handle (the companion of the center ball) at ∞

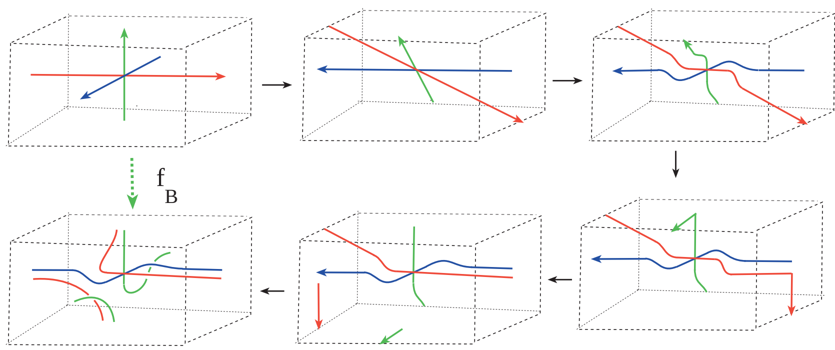

FIGURE 78

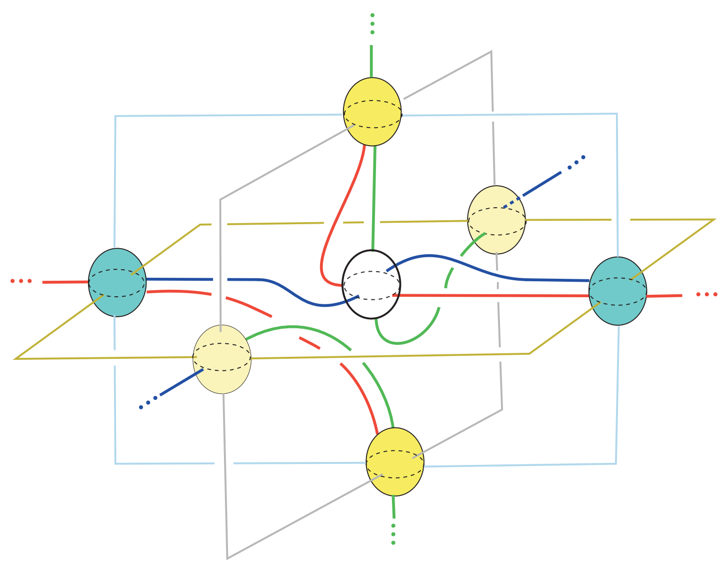

FIGURE 79

Exercise 22. *Convert the* 1*-handles of Figure 79 to the circle-with dots notation. Locate the framed circle corresponding the* 2*-handle of* $D^2 \times S^2$*, and decide the parity of its framing (in* [AK1] *this was mistakenly assumed to be even but it turned out to be odd).*

4. 3-manifolds

4.1. **Dehn surgery.** Dehn surgery of a 3-manifold Y^3 is an operation of taking out an imbedded copy of $S^1 \times B^2$ from Y^3 and regluing it back (by some diffeomorphism of its boundary). Let $K \subset Y^3$ be a null homologous knot (null homologous imbedding of S^1), and $N(K)$ be its tubular neighborhood. Let μ and λ be the meridian and longtitute of $\partial N(K)$. In particular λ is the parallel copy of K which is null homologous in $Y - K$. Let p, q be coprime integers. Fix a trivialization:

$$\phi : B^2 \times S^1 \overset{\approx}{\to} N(K)$$

such that $\phi(1,0) = \mu$ and $\phi(0,1) = \lambda$, where $(1,0)$ and $(0,1)$ are the two generators of $S^1 \times S^1$. For the sake of simplicity we will view ϕ as an identification maps, and parametrize the curves on $\partial N(K)$ by $(p,q) \leftrightarrow p\mu + q\lambda$. Then $r = p/q$ surgery to Y^3 is the manifold.

$$Y(K,r) = [\ Y - N(K)\ ] \smile_{\phi_r} (B^2 \times S^1)$$

where $\phi_r : \partial B^2 \times S^1 \to \partial N(K)$ is the unique diffeomorphism with $\phi_r(1,0) = (p,q)$. With above identification we can express (cf. [Ro])

$$\phi_r = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

where $ps - qr = 1$. In particular $Y(K, \infty) = Y$. For the sake of brevity, when there is no danger of confusion we will abbreviate $K^{p/q} = Y(K,r)$, even though when p/q is an integer we previously used this notation for denoting the 4-manifold obtained B^4 by attaching 2-handle to K.

Exercise 23. *Let $K \subset S^3$ any any knot. By using the identities*

$$\begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} np - q & nr - s \\ p & r \end{pmatrix}$$

justify the 3-manifold diffeomorphism $\cong$ of Figure 80 (Hint: By using Figure 81 reduce this to the case of when $\{K, \mu\}$ is the Hopf link)

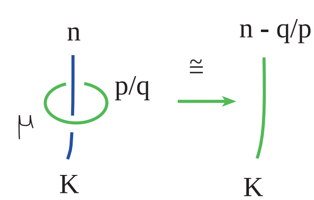

Figure 80

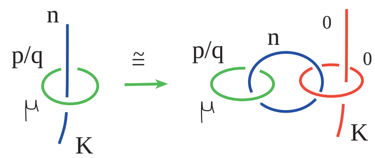

FIGURE 81

Exercise 24. *By iterating the diffeomporphism of Exercise 23 justify the 3-manifold diffeomorphism of Figure 82, where* $p/q = [a_1, a_2, .., a_k]$. *is the continued fraction expension (when* $a_i \geq 2$ *expression is unique).*

$$p/q = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_k}}}$$

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & -1 \\ 1 & 0 \end{pmatrix} .. \begin{pmatrix} a_k & -1 \\ 1 & 0 \end{pmatrix}$$

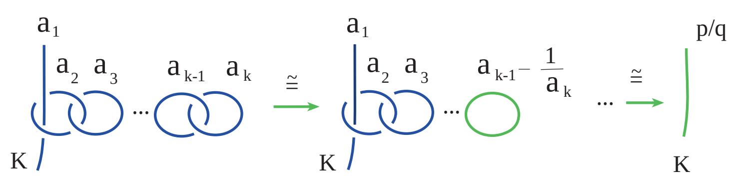

FIGURE 82

4.2. **Constructing Heegard pictures from framed links.**

Given a 4-manfold $M^4 = M_\Lambda$, with $\Lambda = \{K_1^{r_1}, .., K_k^{r_k}\}$, how can we construct the Heegard picture of its boundary $Y^3_\Lambda := \partial M_\Lambda$? If the complement $C(\Lambda) := S^3 - \cup_j N(K_j)$ of the tubular neighborhood of the framed link Λ is a handlebody (i.e. B^3 with 1-handles), then clearly $(C(\Lambda), K_1, .., K_k)$ gives a Heegard decomposition of $Y(\Lambda)$. To use this fact, we first remove solid pipes (i.e. properly imbedded thickened arcs) from $C(\Lambda)$ to turn it to a handlebody $C'(\Lambda) = C(\Lambda) - \cup N(I_j)$. To undo the damage we did by removing pipes I_j, we attach 2-handles γ_j to the meridian of I_j's. Then the end result is a Heegard decomposition, given by the 3-dimensional solid handlebody $C'(\Lambda)$ along with simple disjoint simple closed curves $\{K_1, .., K_k, \gamma_1, ..\gamma_s\}$ lying on its boundary.

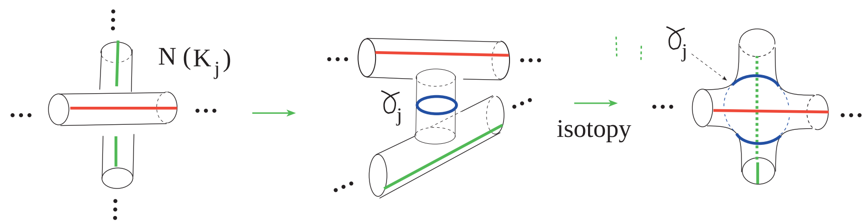

FIGURE 83. Adding solid pipes to the complement

Alternatively this can be described by the surface $F = \partial C'(\Lambda)$ with two sets of disjoint simple closed curves α and β, each describing a solid handlebody. Here $\beta = \{K_1, .., K_k, \gamma_1, ..\gamma_s\}$, and $\alpha = \{\alpha_1, .., \alpha_r\}$ are the obvious collection of curves in F compressing in $C'(\Lambda)$. Applying this process to $\Lambda = \{K^2\}$, where K is the trefoil knot, we get Figure 84.

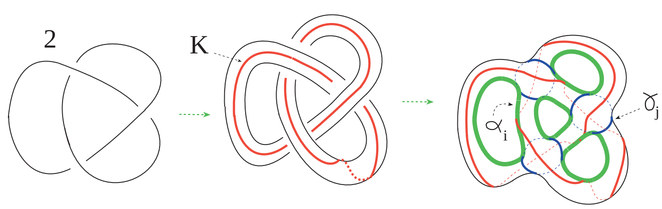

FIGURE 84. Constructing a Heegard picture of $\partial(K^2)$

Exercise 25. *By considering moving pictures (and the above process), identify the complement of a properly imbedded 2-disk in B^4, with a single transverse self intersection, with the Fishtail (Hint: Figure 85). Compare this with the "self plumbing" operation of Figure 32.*

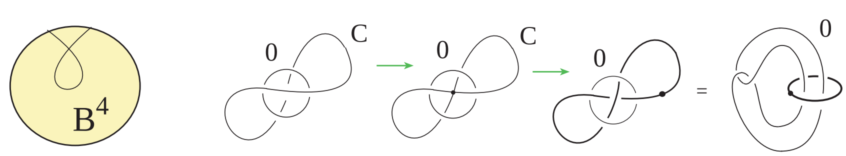

FIGURE 85

4.3. **Gluing knot complements.**

Let $K, L \subset S^3$ two given knots, and let $\phi : \partial N(K) \overset{\approx}{\to} \partial N(L)$ be a diffeomorphism, between the boundaries of the tubular neighborhoods of these knots. We want to construct a 4-manifold $M_\varphi(K, L)$ with

$$\partial M_\varphi(K, L) \approx (S^3 - N(K)) \smile_\phi (S^3 - N(L))$$

Clearly, when K and L are the unknot U and the map $\phi : T^2 \to T^2$ is

$$\phi = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

then $\partial M_\varphi(U, U) = L(p, q)$ (Lens space). Therefore we can take the plumbing $M_\varphi(U, U) = C_{p,q}$ described in Figure 86 (e.g. [Ro])

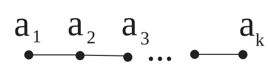

FIGURE 86

where $p/q = [a_1, a_2, ..a_k]$. Write $C = S^3 - N(U) \approx S^1 \times B^2$ so that $L(p, q) = C \smile_\phi C$. Next we will extend this process to any knots.

Let γ and γ' be the dual circles of K and L in S^3, respectively. Let $N(\gamma)$ denote the tubular neighborhood of γ in $\partial(K^0)$. Notice that $\partial(K^0) - N(\gamma) \approx S^3 - N(K)$. We need to identify their boundaries with a cylinder $(S^1 \times S^1) \times [0, 1]$, which is the boundary of the manifod obtained from the disjoint union $K^0 \sqcup C_{p,q} \sqcup L^0$, by identifying $N(\gamma)$ with one copy of C, and identifying the other copy of C with $N(\gamma')$.

$$M_\varphi(K, L) = K^0 \smile U \smile C_{p.q} \smile U' \smile L^0$$

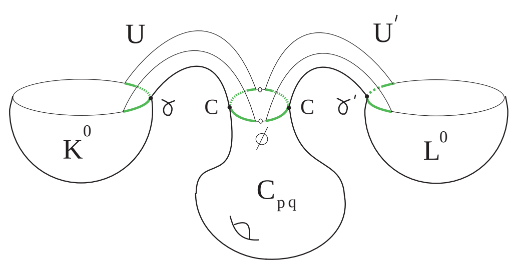

FIGURE 87

Exercise 26. *Show that Figure 88 gives a handlebody for* $M_\phi(K,L)$

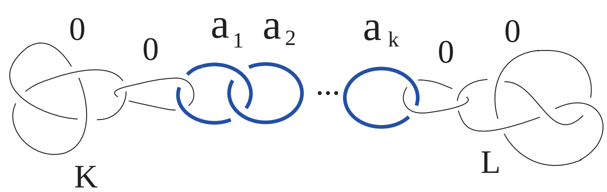

FIGURE 88

Exercise 27. *Show that Figure 89 gives* $\partial M_\phi(K,L)$ *when* $\phi = \begin{pmatrix} p & -1 \\ 1 & 0 \end{pmatrix}$.

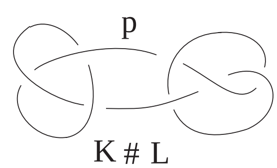

FIGURE 89

Exercise 28. *By modifying above construction verify that Figure 90 describes* $\partial M_\phi(K,L)$ *for some* ϕ, *where* K *is the trefoil knot, and* L *is the figure eight knot.*

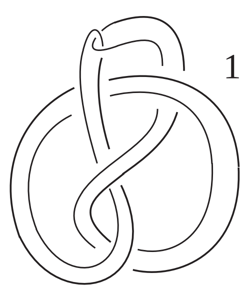

FIGURE 90

5. Operations

5.1. **Gluck twisting.**

It is known that any diffeomorphism $S^2 \times S^1 \stackrel{\approx}{\to} S^2 \times S^1$ is either isotopic to the identity, or isotopic to the map $\varphi : S^2 \times S^1 \to S^2 \times S^1$

$$\varphi(x, \theta) = (\alpha_\theta(x), \theta)$$

where $S^1 \ni \theta \mapsto \alpha_\theta \in SO(3)$ is the nontrivial element of $\pi_1 SO(3) = \mathbb{Z}_2$. Let M be a smooth 4-manifold, and $S \subset X$ be a copy of S^2 imbedded with the trivial norml bundle $S^2 \times B^2 \subset X$. We call the operation

$$X \mapsto X_S := (X - S^2 \times B^2) \smile_\varphi (S^2 \times B^2)$$

Gluck twisting. Best way to understand how a Gluck twisting operation alters the handles of M is draw a handlebody of M as handles attached to the top of this $S^2 \times B^2$ (an unknot with 0-framing). Then clearly the Gluck twisting corresponds to the following operation

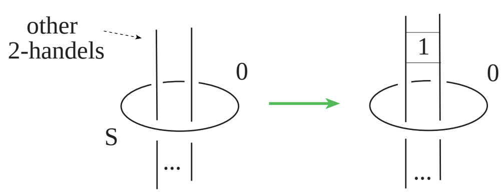

Figure 91

Exercise 29. *Show that the Gluck twisting operation is equivalent to either one the following operations:*

- *The operation described in Figure 92 (where n, m are arbitrary).*
- *The "zero" and "dot" exchange operation of Figure 93.*

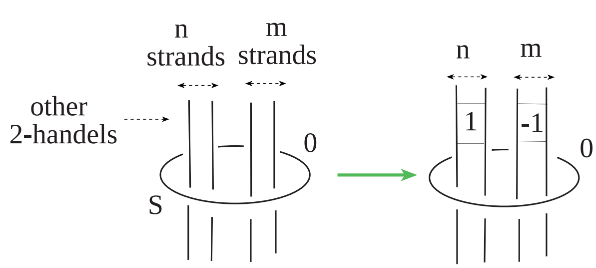

Figure 92

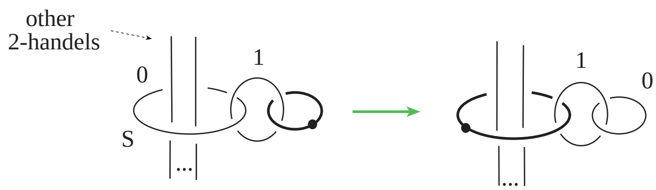

FIGURE 93

When X is simply connected and $S \subset X$ is null homologous, it is clear that the operation $X \mapsto X_S$ does not change the homology groups and the intersection form of X, and also keeps X simply connected, hence X_S is h-cobordant to X and therefore it is homemorphic to X by [F]. It turns out that if the intersection form of X is odd than this operation does not change the smooth structure of X. The following is a special case of a more general theorem of [AY1]

Theorem 1. *(*[AY1]*) If S is null homologous, and the intersection form of X is odd then $X_S \approx X$*

In dimensions ≥ 5 1-handles of simply connected manifolds can be eliminated (this is the first step of the proof of the h-cobordism theorem). In dimension 4 it is not clear if this can be done (at least when the manifold has boundary, it can not be done by Example 7.5 of [AY2]). Nevertheless in dimension 4 there are some useful weaker analogues of this. The following is from [GS] (9.2.17) (which is a somewhat simpler version of [Ma]). Here we use the notation $X^{(i)}$ for the sub-handlebody of X consisting of handles of index $\leq i$.

Proposition 2. *Let X be a 1-connected smooth 4-manifold given as a handlebody $X^{(2)} = M_\Lambda$ where $\Lambda = \{K_1^{r_1}, .., K_k^{r_k}, C_1, .., C_p\}$. Then for each 1-handle, by introducing a canceling pairs of 2 and 3-handles, we can obtain a new enlarged handlebody for X with $X^{(2)} = M_{\Lambda'}$ where $\Lambda' = \{K_1^{r_1}, .., K_k^{r_k}, L_1^{s_1}, .., L_p^{s_p}, C_1, .., C_p\}$, such that each attaching circle of the 2-handle L_j is homotopic in $X^{(1)}$ to the core of the 1-handle C_j.*

Proof. Let γ be the linking circle of a 1-handle C (the core of C). If it bounded a disks in $\partial X^{(2)}$, by attaching a canceling pair of 2- and 3-handles (along γ) we could create canceling 2-handle γ for C. But we don't know this. To prove the proposition we only have to create a 2-handle in X whose attaching circle L is homotopic to γ in $X^{(1)}$. Since X is simply connected we can find an immersed cylinder H in $X^{(2)}$

connecting γ to an unknotted trivial circle $c \subset \partial X^{(2)}$. By the general position H misses the cores of the 1-handles of X, but might meet 2-handles of X along points. By piping these points to γ as shown in Figure 94, we obtain a circle $\gamma' \subset \partial X^{(2)}$, which is homotopic to c in in the complements of the cores of 1- and 2-handles (and also homotopic to γ in $X^{(1)}$). Hence this homotopy H can be pushed to $\partial X^{(2)}$, where H can be viewed as moving pictures starting at γ' occasionally self crossing and ending at c. Therefore by replacing γ' by connected summing with the small dual circles of itself (when passing through self crossing) we obtain L which is isotopic to c, and still homotopic to γ in $X^{(1)}$. So after attaching a canceling 2/3-handle pair to c, we see that L bounds an imbedded disk, which we can viewed as the attaching frame circle of a 2-handle. By repeating this process all the 1-handles we produce the required frame link $\{L_1^{s_1}, .., L_p^{s_p}\}$ □

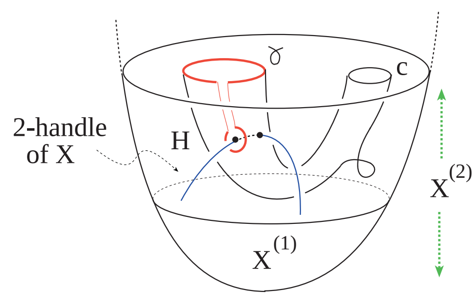

FIGURE 94

Remark 1. *When dim$(X) \geq 5$ we can get a stronger conclusion with less work. In the beginning of the proof, we can make the cylinder H miss all the 2-handles by general position, and imbed into ∂X^2, which we can use to construct the 2-handle canceling the 1-handle.*

proof of Theorem 1: Choose $\alpha \in H_2(X)$ with $\alpha.\alpha =$ odd, and $\alpha.S = 0$. By using the notation of Section 1.2 and by the above proposition, we can represent α by a framed link $K = \sum_{j=1}^{k} c_j K_j$ with odd framing p.

By assumption and Proposition 2, K is null homotopic in $X^{(1)}$, and it links the unknotted zero framed circle representing S algebraically zero times. As indicated in Figure 95 the Gluck twisting operation corresponds by the zero and dot exchanges between the zero framed 2-handle S and the dotted circle T. By sliding the middle 1-framed handle over

K we can make its framing even, and then by sliding it over T we can make its framing zero, and finally by using the "null-homotopy" assumption we can further slide it over T to make it unknotted as shown in the third picture of Figure 95. Now it is clear that the zero and dot exchange between S and T does not change the diffeomorphism type i.e. $X_S \approx X$. □

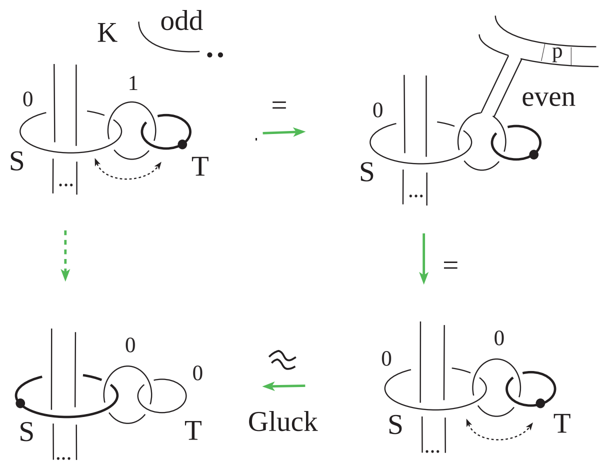

FIGURE 95

5.2. **Logarithmic transform.**

Given a smooth 4-manifold X^4 and an imbedding $T^2 \times B^2 \subset X^4$, the operation of removing this $T^2 \times B^2$ and then gluing it back by a nontrivial diffeomorphism of its boundary $\varphi : T^3 \to T^3$ is called T^2-surgery operation. It is known that any smooth 4-manifold can be obtained from $S^2 \times S^2$ or $\mathbb{CP}^2$ be a sequence of these operations. [I].

A special version of this operation is called *p-log transform*, where $\phi_p : T^2 \times S^1 \to T^2 \times S^1$ is the self-diffeomorphism given by ($p \in \mathbb{Z}$)

$$\phi_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & p \end{pmatrix}$$

This operation is also called *logarithmic transform of order p*, because the degree of the composition of the maps below has order p.

$$S^1 \xrightarrow{\text{inc}} T^2 \times S^1 \xrightarrow{\phi_p} T^2 \times S^1 \xrightarrow{\text{proj}} S^1$$

By using Section 2.3 we can draw a handlebody picture of this operation (compare to [AY2], [GS]). The recipe says that given $T^2 \times D^2 \subset X$, write $X = T^2 \times D^2 \smile$ (other handles), then carry the other handles by the inverse diffeomorphism $\varphi_p := \phi_p^{-1}$ to top of $T^2 \times D^2$.

$$\varphi_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Figure 96 describes this handlebody operation with pictures. This figure gives a picture recipe of how to modify a 4-manifold handlebody, containing a framed torus $T^2 \times D^2$, to get the handlebody of the p-log transformed 4-manifold along this torus (for example Figure 96 describes how the linking loop B is changed by this operation). ± 1 -log transform is usually called a "Luttinger surgery".

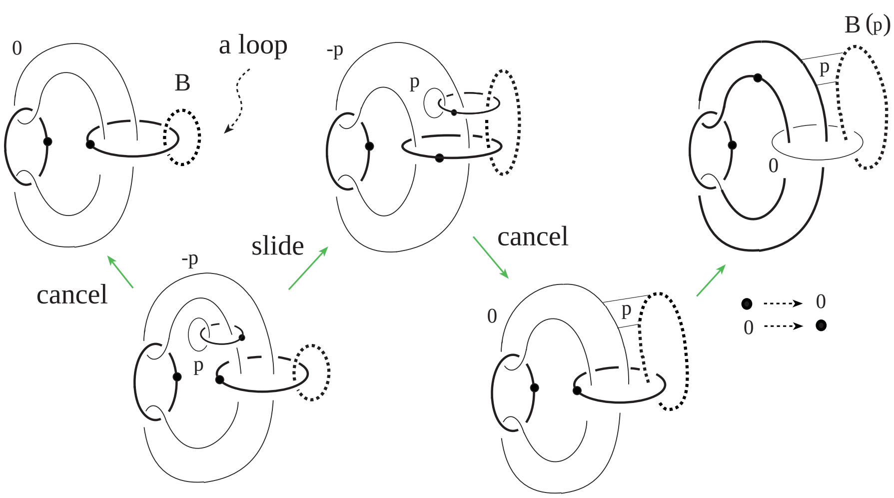

FIGURE 96. p log-transform operation

5.3. **Knot surgery.**

Let X be a smooth 4-manifold, and $T^2 \times B^2 \subset X$ be an imbedded torus with trivial normal bundle, and $K \subset S^3$ be a knot, $N(K)$ be its tubular neighborhood. The Fintushel-Stern *knot surgery operation* is the operation of replacing $T^2 \times B^2$ with $(S^3 - N(K)) \times S^1$, so that the meridian $p \times \partial B^2$ of the torus coincides with the longitude of K [FS1].

$$X \rightsquigarrow X_K = (X - T^2 \times D^2) \cup (S^3 - N(K)) \times S^1$$

A handlebody picture of this operation was constructed in [A6]. Since 4-manifolds are determined by their 1- and 2- handles, to see $(S^3 - N(K)) \times S^1$, which is obtained by identifying the two ends of $(S^3 - N(K)) \times [0,1]$, it suffices to draw $(B^3 - N(K_0)) \times [0,1]$ with its ends identified, where $K_0 \subset B^3$ is a properly imbedded arc with the knot K tied on it (the rest is a 3-handle). The second picture of Figure 97 describes $(B^3 - N(K_0)) \times [0,1]$, identifying its ends (up to 3-handles) corresponds attaching a new 1-handle, and 2-handles, where the new 2-handles are attached along the 1-handles of the two boundary components of $(B^3 - N(K_0)) \times [0,1]$ as shown in Figure 97 (they identifiy the core circles of the knot complements, as described in Section 2.2).

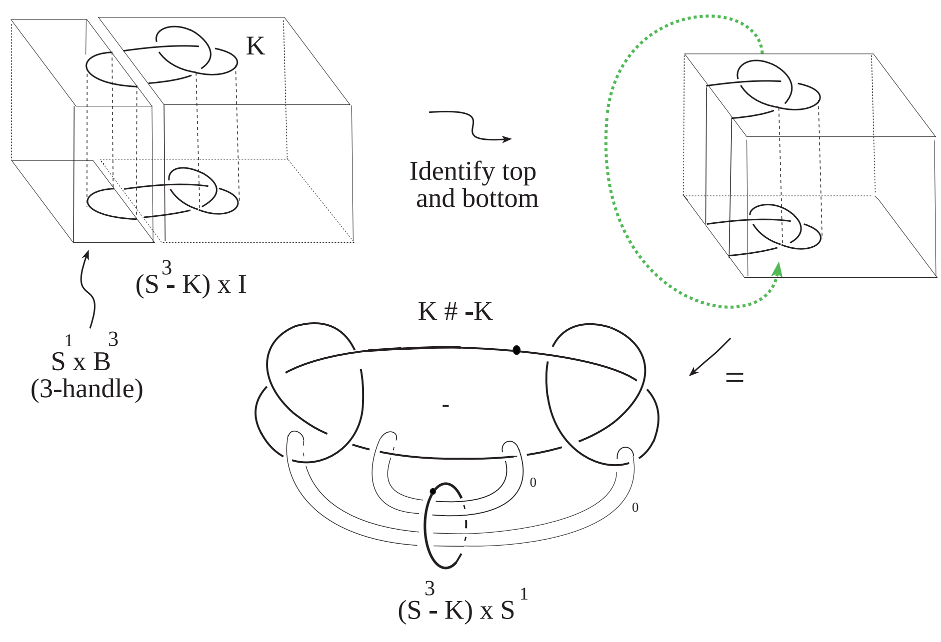

FIGURE 97. $(S^3 - K) \times S^1$, where K is the trefoil knot

Summarizing, Figure 98 gives a recipe of how to modify a 4-manifold handlebody, containing a framed torus $T^2 \times B^2$, to get the handlebody of the knot surgery transfored 4-manifold along this torus by using a knot K (in the Figure K is taken to be trefoil knot). Figure 98 describes how the linking loops a, b, c changes by this operation. For example in this figure, if we attach -1 framed 2-handle to either (both) of the loops a, c we get Fishtail (Cusp) on the left, and the knot surgered Fishtail (Cusp) on the right (recall Figures 50 and 51).

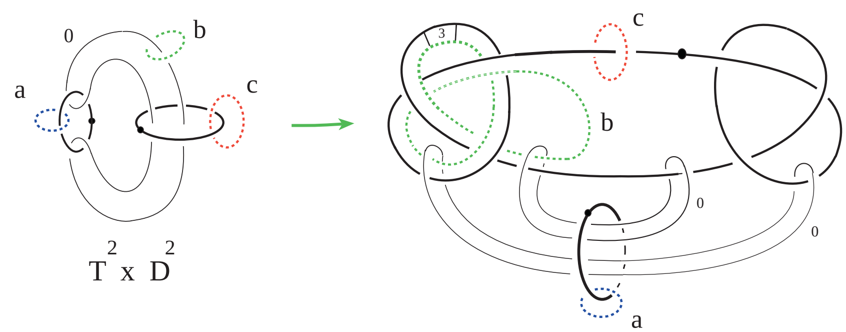

FIGURE 98. The operation $T^2 \times B^2 \mapsto (T^2 \times B^2)_K$

Figure 99 describes the slow evolution of the two pictures of Figure 98, from left to right (i.e. description of the boundary diffeomorphism). In Figure 99, first we introduce a canceling 2/3 handle pair, and replace dot with zero in the middle 1-handle, then do the indicated handle slides then put the dot on the resulting ribbon handle.

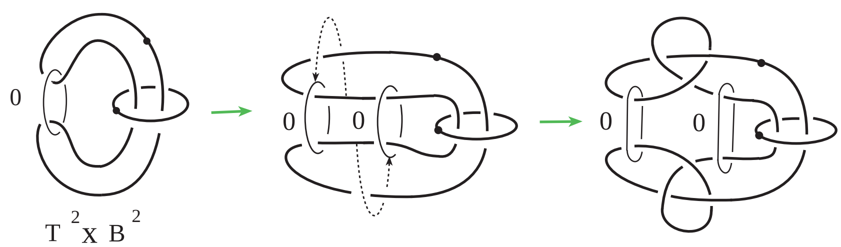

FIGURE 99

As an example, let us take the elliptic surface $E(1)$ (see Chapter 6) and apply a knot surgery operation to the cusp inside. The first picture of Figure 100 is an handlebody of $E(1)$ from [A7], the second is $E(1)_K$

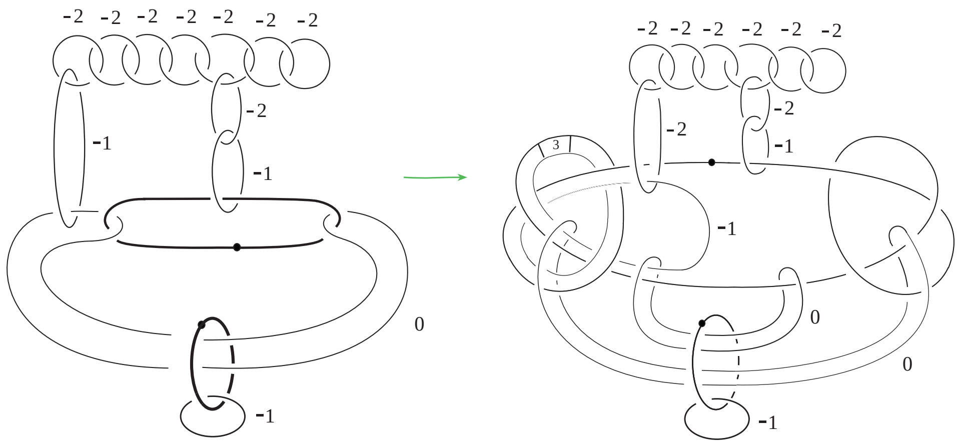

FIGURE 100

Let $[T] \in H_2(X;\mathbb{Z})$ be the homology class of the imbedded torus $T^2 \subset X$ discussed above, and let $t = exp(2[T])$, and $\Delta_K(t)$ be the (symmetric) Alexander polynomial of the knot $K \subset S^3$. Then the Seiberg-Witten invariant (Chaper 13) SW_{X_K} of X_K is given by:

Theorem 3. *(*[FS1]*)* $SW_{X_K} = SW_X.\Delta_K(t)$

5.4. **Rational blowdowns.**

Let $C \subset X^4$ be a negative definite plumbing in a smooth 4-manifold, such that $\partial C \approx \partial B$ for some rational ball B. The operation

$$X \mapsto X^* := (X - C) \smile B$$

is called a *rational blowing down* operation. When ∂C is a Lens space this operation was introduced in [FS2] as a tool in gauge theory. An important special case is when $C_{p,q}$ is the 4-manifold given by the plumbing Figure 101, where each $b_i \geq 2$, and $p^2/pq-1 = [b_k, b_{k-1}, .., b_1]$.

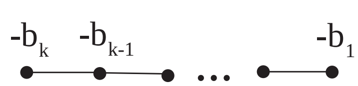

FIGURE 101. $C_{p,q}$

The boundary is the Lens space: $\partial C_{p,q} = L(p^2, pq-1)$, which bounds a rational ball by [CH] (see also [P]). In [LM] it was shown that in fact $L(p^2, pq-1)$ bounds the rational ball $B_{p,q}$ shown in Figure 102.

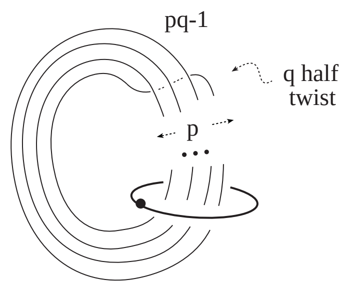

FIGURE 102. $B_{p,q}$

Exercise 30. *Prove that there is a diffeomporphism $\partial C_{p,q} \approx \partial B_{p,q}$, and describe the diffeomorphism.*

In the special case of $q = 1$ the manifold $C_{p,1}$ is the pluming of $p-1$ spheres with Euler numbers $-p-2, -2, -2, .., -2$. Figure 103 demonstrates a concrete diffeomorphism $\partial B_{p,1} \approx \partial C_p$, which allows us to perform the blowing down operation concretely on handlebodies.

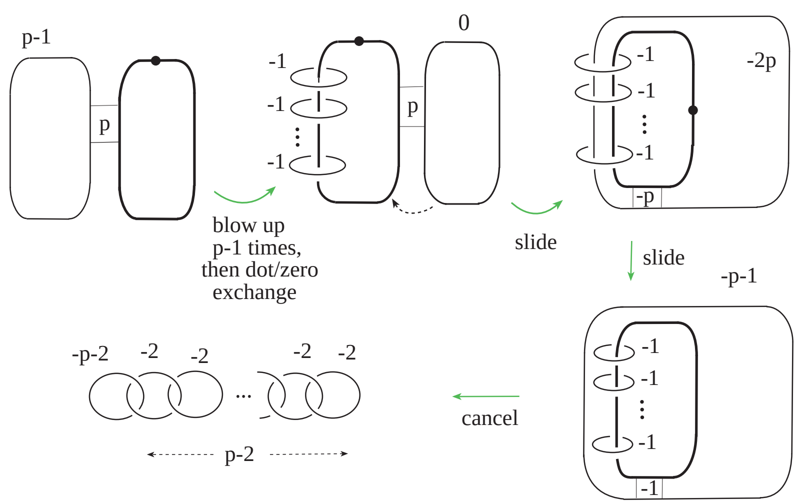

FIGURE 103. Describing a diffeomoprphism $\partial B_p \approx \partial C_p$

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Department of Mathematics, Michigan State University, MI, 48824
E-mail address: akbulut@math.msu.edu