Group Actions on Four-manifolds

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The central theme: The problem of group actions is about studying automorphism groups of geometric structures from a topological point of view.

The basic question: Let $X$ be a manifold and $\xi$ be a given geometric structure. Denote by $\text{Aut}(X, \xi)$ the corresponding group of automorphisms. Then, for any finite group $G \subset \text{Homeo}(X)$ (or $G \subset \text{Diffeo}(X)$), is $G$ conjugate to a subgroup of $\text{Aut}(X, \xi)$?

Weaker version: How much does a finite subgroup of $\text{Homeo}(X)$ or $\text{Diffeo}(X)$ resemble a finite subgroup of $\text{Aut}(X, \xi)$?
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Lecture 1: Introduction

The basic example: Let $X = S^n \subset R^{n+1}$, and $\xi$ be the linear structure or the structure of constant sectional curvature metric. Then $Aut(X, \xi) = O(n+1)$. In this case, the basic question asks: if a finite group $G$ acts continuously or smoothly on $S^n$ (i.e., $G \subset Homeo(X)$ or $G \subset Diffeo(X)$), is the $G$-action equivalent to a linear action (i.e., is $G$ conjugate to a subgroup of $Aut(X, \xi)$)?

The weaker version asks: if a finite group $G$ acts on $S^n$ continuously or smoothly, how much does the $G$-action resemble a linear action on $S^n$? In particular, is $G$ necessarily isomorphic to a subgroup of $O(n+1)$? The same fixed-point set structure?

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Review of results in low-dimensions \((n = 2, 3, 4)\):

1. \(n = 2\): Theorem (Brouwer 1919, Kerekjarto 1921, Eilenberg 1934): Every finite group action by orientation-preserving homeomorphisms on \(S^2\) is equivalent to a linear action.

2. \(n = 3\): (1) Bing’s examples and local linearity condition. Theorem (Bing 1952, 1964): There exist finite group actions on \(S^3\) with “wildly" embedded fixed-point sets (Alexander horned sphere or an untamed knot).

**Remarks:** Such actions can not be equivalent to a linear action.

**Definition:** An action is **locally linear** if it is locally equivalent to a linear action.

**Remarks:** The fixed-point set of a locally linear action is a flatly embedded submanifold. It’s known that **local linearity=smooth in dim 3** (Kwasik-Lee).
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2. $n = 3$ (cont.) (2) Smith theory and Smith Conjecture.

Theorem (P.A. Smith 1940): The fixed-point set of an orientation-preserving cyclic action of prime order on a homology $S^n$ is a homology $S^r$, where $n - r$ is even and $-1 \leq r < n$, with $r = -1$ meaning the fixed-point set is empty.

**Remarks:** The fixed-point set of an orientation-preserving cyclic action of prime order on $S^3$, when nonempty, is a knot in $S^3$.

**Smith Conjecture:** The fixed-point set of an orientation-preserving cyclic action of prime order on $S^3$ must be an unknot if it’s nonempty.

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2. \( n = 3 \) (cont.) (3) Free actions on \( S^3 \): the list of possible finite groups which can act freely on \( S^3 \) (J. Milnor 1957, R. Lee 1973).

Finite subgroups of \( SO(4) \): there is a 2 to 1 covering homomorphism

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\rho : SO(4) \to SO(3) \times SO(3).
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Finite subgroups of \( SO(3) \):
- a cyclic group \( C_n \) of order \( n \);
- a dihedral group \( D_{2n} \) of order \( 2n \)
- the tetrahedral group \( T \cong A_4 \) of order 12;
- the octahedral group \( O \cong S_4 \) of order 24;
- the icosahedral group \( I \cong A_5 \) of order 60.
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Fact: if a finite subgroup of $SO(4)$ acts on $S^3$ freely, then it must be a subgroup of $U(2)$. (Such a list was obtained by Brieskorn, in classification of certain complex surface singularities.)

Milnor’s list of exotic groups $Q(8, a, b, c)$: Let $a, b, c$ be odd, pairwise co-prime integers. There is a split extension

$$1 \rightarrow Z_a \times Z_b \times Z_c \rightarrow Q(8, a, b, c) \rightarrow Q(8) \rightarrow 1,$$

where $Q(8) = \{ x, y | x^4 = 1, y^2 = x^2, xyxy^{-1} = x^{-1} \}$, and if $Z_a = \langle u \rangle$, $Z_b = \langle v \rangle$, $Z_c = \langle w \rangle$, then

$$xux^{-1} = u^{-1}, xvx^{-1} = v, xwx^{-1} = w^{-1},$$

$$yuy^{-1} = u, yyv^{-1} = v^{-1}, ywy^{-1} = w^{-1}.$$

Fact: $Q(8, a, b, c)$ is not a subgroup of $O(n)$ for $n \leq 7$. 
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Fact: \( Q(8, a, b, c) \) is not a subgroup of \( O(n) \) for \( n \leq 7 \).
2. $n = 3$ (cont.) (4) Ricci flow and geometrization.

Theorem: Every finite group action on $S^3$ by orientation-preserving diffeomorphisms is equivalent to a linear action.

Corollary: $Q(8, a, b, c)$ can not act freely on $S^3$.

Problem: (i) Find an alternative proof (gauge theoretic?) of the fact that $Q(8, a, b, c)$ can not act freely on $S^3$.

(ii) It’s known that some of the $Q(8, a, b, c)$’s act freely on a homology 3-sphere $\Sigma$ (existence of $\Sigma$ coming from surgery theory). What can we say about $\Sigma$ (particularly, $\pi_1 \Sigma$)?
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Theorem (Poenaru 1960, Mazur 1961, de Rham 1962) There exist smooth involutions on $S^4$ whose fixed-point set is a homology 3-sphere $\Sigma \neq S^3$.

Theorem (Giffen 1966) There exist smooth $\mathbb{Z}_p$-actions, for odd $p$, on $S^4$ whose fixed-point set is a knotted 2-sphere.

Theorem (Fintushel-Stern 1981) There exists a free involution on $S^4$ which is not smoothly equivalent to the antipodal map.

(2) Smooth s-cobordisms of elliptic 3-manifolds.

Conjecture: If a smooth, pseudo-free finite group action on $S^4$ has an isolated fixed-point, then it must be equivalent to a linear action.

Theorem (Chen 2004, 2006) A symplectic s-cobordism of elliptic 3-manifolds is smoothly a product.

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3. \( n = 4 \) (cont.) (3) Fixed-points and local representations.

Theorem (Furuta 1989, DeMichelis 1989) There is no smooth (or locally linear) finite group action on \( S^4 \) with one fixed-point.

Theorem (Hambleton-Lee 92, Braam-Matic 93, DeMichelis 89) If a finite group acts smoothly on \( S^4 \) with two isolated fixed-points, then the representations on the tangent space of the two fixed-points are equivalent.

Remarks: (1) The proofs of Furuta, Hambleton-Lee, and Braam-Matic were gauge theoretic (Yang-Mills) and are for smooth actions. The proofs of DeMichelis were traditional and work for locally linear actions on homology \( S^4 \), after learning of the gauge theoretic attempts.

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3. $n = 4$ (cont.) (4) Finite groups which act on $S^4$.

Theorem A (Chen-Kwasik-Schultz 2013): If a finite group $G$ acts locally linearly and orientation-preservingly on a homology $S^4$, then $G$ is isomorphic to a subgroup of $SO(5)$.

Note: Mecchia and Zimmermann showed earlier that either $G$ is isomorphic to a subgroup of $SO(5)$, or an index 2 subgroup of $G$ is isomorphic to a subgroup of $SO(4)$.

Theorem B (Chen-Kwasik-Schultz 2013): There are finite groups $G$ acting topologically and orientation-reversingly on $S^4$, which are NOT isomorphic to a subgroup of $O(5)$.

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Other model spaces in dimension 4: \((X, \xi)\), where \(X\) is a Kahler surface, and \(\xi\) is a complex structure. There are two main classes which have been extensively studied by the algebraic geometers: rational surfaces and K3 surfaces.

(1) \(X = \mathbb{CP}^2\): \(\text{Aut}(X, \xi) = \text{PU}(3)\).

Theorem (Wilczynski 87, Hambleton-Lee 88) If a finite group acts locally linearly and homologically trivially on \(\mathbb{CP}^2\), then it is a subgroup of \(\text{PU}(3)\).

Theorem (Wilczynski 91) A locally linear, pseudo-free action of a finite cyclic group on \(\mathbb{CP}^2\) is equivalently to an action by projective transformations.
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(1) $X = CP^2$ (cont.):

Theorem (Hambleton-Lee): There are smooth $Z_p \times Z_p$-actions on $CP^2$ which are not equivalent to an action by projective transformations.

(Such actions have relatively knotted 2-dim’l singular set; can’t be symplectic.)

Theorem (Chen 2006): A symplectic finite group action of $G$ on $CP^2$ is smoothly equivalent to an action by projective transformations, if $G$ is cyclic or metacyclic with non-empty fixed-point set, or there is a fixed-point whose induced $G$-action on the link $S^3$ is free.
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Other model spaces in dimension 4: (2) \( X = \) a rational surface.

**Definition:** A rational \( G \)-surface is a rational surface with a finite automorphism group \( G \).

**Remarks:** Minimal rational \( G \)-surfaces played the key role in the modern treatment of the classical problem of classifying finite subgroups of the plane Cremona group, i.e., the group of birational transformations of \( CP^2 \).

**Problem:** (i) Are there symplectic actions on a rational surface which are not equivalent to a holomorphic action?

(ii) Let \( X \) be a rational surface, \( G \) be a finite group acting on \( X \) pseudo-freely, preserving a symplectic structure. Consider any smooth resolution \( \tilde{X} \rightarrow X/G \) of the quotient orbifold. Is \( \tilde{X} \) diffeomorphic to a rational surface?
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(2) $X = \text{a rational surface (cont.)}: \text{Consider the minimal rational } G\text{-surface } X, \text{ where } X = F_n \text{ is a Hirzebruch surface with } n > 1 \text{ odd, } G = \mathbb{Z}_p, p \text{ prime.}

Theorem (Chen 2013) Assume the $G$-action is pseudo-free with the weights of the local representations given by
$(1, m), (-1, m + n), (1, -m), (-1, -m - n)$, where $0 < m < p$. Let $\omega$ be any $G$-invariant symplectic form on $X$. Then $X$ contains a $G$-invariant, $\omega$-symplectically embedded $(-1)$-sphere if and only if it contains a $G$-invariant, smoothly embedded $(-1)$-sphere, provided that $p - 7 \leq 2m \leq p + 5$. 

(2) $X = a$ rational surface (cont.): Consider the minimal rational $G$-surface $X$, where $X = F_n$ is a Hirzebruch surface with $n > 1$ odd, $G = \mathbb{Z}_p$, $p$ prime.

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**Corollary**: For $G = \mathbb{Z}_p$ where $p \geq 5$, there exist symplectic rational $G$-manifolds such that

- the $G$-actions are homologically trivial;
- the underlying symplectic (resp. smooth) 4-manifolds are not minimal; but
- the symplectic (resp. smooth) $G$-manifolds are minimal.

**Theorem (Chen 2013)**: For any prime number $p > 1$, there exists a smooth $\mathbb{Z}_p$-action on $CP^2 \# CP^2$ such that

- there are infinitely many distinct complex structures on $CP^2 \# CP^2$ which are invariant under the $\mathbb{Z}_p$-action, and
- for any cohomologous symplectic forms $\omega_1, \omega_2$ on $CP^2 \# CP^2$ which are invariant under the $\mathbb{Z}_p$-action, there is an equivariant diffeomorphism $f$ such that $f^* \omega_2 = \omega_1$. 
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Other model spaces in dimension 4: (3) $X = \text{a K3 surface.}$

(A) **Homological Rigidity**: An automorphism of a K3 surface must be trivial if it acts as identity on the homology.

**Problem**: Are there any nontrivial smooth $\mathbb{Z}_p$-actions on the K3 surface which are homologically trivial?

**Theorem (Ruberman, Matumoto)**: There are no nontrivial locally linear $\mathbb{Z}_2$-actions on the K3 surface which are homologically trivial.

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Definition: A finite group is called a K3 group if it can be realized as a symplectic automorphism group of a K3 surface, i.e., an automorphism group which fixes a nonvanishing holomorphic 2-form.

Remarks: K3 groups are studied extensively by Nikulin, Mukai, and others. There are 11 maximal K3 groups:

\[ L_2(7), A_6, S_5, M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}. \]

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Lecture 1: Introduction

(3) $X = \text{K3 surface (cont.)}: \text{(B) K3 groups and smooth structures.} $

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Theorem (Chen-Kwasik 2011): Let \( X \) be a symplectic K3 surface with trivial canonical bundle. If a finite group \( G \) acts on \( X \) symplectically such that \( b_2^+(X/G) = 3 \). Then \( G \) must be isomorphic to a K3 group, with the same fixed-point set structure of a symplectic holomorphic \( G \)-action.
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Other model spaces in dim 4: (4) $X = $ a surface of general type.

**Bound of Automorphisms.**

Theorem (Hurwitz): Let $\Sigma$ be a compact Riemann surface of genus $g > 1$. Then

$$|Aut(\Sigma)| \leq 84(g - 1) = 42 \text{deg}K_{\Sigma}.$$

Theorem (G. Xiao, 1994): Let $X$ be a minimal surface of general type. Then

$$|Aut(X)| \leq (42)^2 c_1(K_X)^2.$$

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(4) $X = \text{a surface of general type (cont.)}: \text{Symplectic version.}$

**Problem:** Let $(X, \omega)$ be a minimal symplectic 4-manifold with $c_1(K_X) \cdot [\omega] > 0$ and $c_1(K_X)^2 > 0$. Let $G$ be a finite group acting on $X$ preserving $\omega$. Is there a universal constant $c > 0$ such that

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Note: The 3-dimensional version of the above question is affirmative. However, in higher dimensions, there are compact closed manifolds admitting no smooth $S^1$-actions but admitting smooth $Z_p$-actions for infinitely many primes. So no such a bound $C$ exists.

Definition: If such a bound $C$ exists, we will call it a Hurwitz-type bound for smooth $Z_p$-actions on $X$. Note: it’s known that no such a bound exists for locally linear actions in dimension 4.
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**Theorem (Chen, 2011)** Let $X$ be a compact complex surface with $b_2^+ > 0$ which does not admit any smooth $S^1$-actions. Then for any holomorphic $\mathbb{Z}_p$-actions of prime order on $X$, the order $p$ satisfies

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where $c > 0$ is a universal constant.

**Theorem (Chen, 2011)** For any prime number $p > 3$, there is a symplectic 4-manifold $X_p$ with the following properties:

- $X_p$ is homeomorphic to $\mathbb{CP}^2 \# 9(-\mathbb{CP}^2)$;
- $X_p$ supports no smooth $S^1$-actions;
- $X_p$ admits a symplectic $\mathbb{Z}_p$-action.

**Remarks:** From the construction, the multiplicity of $c_1(K_{X_p})$ grows linearly with $p$. 
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Theorem (Chen, 2012) For any integer \( n > 1 \), there is a 4-manifold \( X_n \) with \( b_2^+ > 1 \) and non-zero Seiberg-Witten invariant, such that

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**Note:** From the construction, the \( \pi_1 \) of \( X_n \) gets more and more complicated as \( n \to \infty \).

**Problem:** Let \( X \) be a simply connected smoothable 4-manifold with even intersection form and non-zero signature. Is there a constant \( C > 0 \) depending only on the homeomorphism type of \( X \), such that for any smoothable \( Z_p \)-actions of prime order on \( X \), the order \( p \) satisfies \( p \leq C \)?
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