Group Actions on Four-manifolds

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The central theme: The problem of group actions is about studying automorphism groups of geometric structures from a topological point of view.

The basic question: Let X be a manifold and ξ be a given geometric structure. Denote by $Aut(X,\xi)$ the corresponding group of automorphisms. Then, for any finite group $G \subset Homeo(X)$ (or $G \subset Diffeo(X)$), is G conjugate to a subgroup of $Aut(X,\xi)$?

Weaker version: How much does a finite subgroup of $Homeo(X)$ or $Diffeo(X)$ resemble a finite subgroup of $Aut(X,\xi)$?

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The basic example: Let $X = S^n \subset R^{n+1}$, and ξ be the linear structure or the structure of constant sectional curvature metric. Then $Aut(X, \xi) = O(n+1)$. In this case, the basic question asks: if a finite group G acts continuously or smoothly on S^n (i.e., $G \subset Homeo(X)$ or $G \subset Diffeo(X)$), is the G-action equivalent to a linear action (i.e., is G conjugate to a subgroup of $Aut(X, \xi)$)?

The weaker version asks: if a finite group G acts on S^n continuously or smoothly, how much does the G-action resemble a linear action on S^n? In particular, is G necessarily isomorphic to a subgroup of $O(n+1)$? The same fixed-point set structure?

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Review of results in low-dimensions ($n = 2, 3, 4$):
1. $n = 2$: Theorem (Brouwer 1919, Kerekjarto 1921, Eilenberg 1934): Every finite group action by orientation-preserving homeomorphisms on S^2 is equivalent to a linear action.

2. $n = 3$: (1) Bing's examples and local linearity condition. Theorem (Bing 1952,1964): There exist finite group actions on S^3 with "wildly" embedded fixed-point sets (Alexander horned sphere or an untamed knot).

Remarks: Such actions can not be equivalent to a linear action.

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2. $n = 3$ (cont.) (2) Smith theory and Smith Conjecture.
Theorem (P.A. Smith 1940): The fixed-point set of an orientation-preserving cyclic action of prime order on a homology S^n is a homology S^r, where $n - r$ is even and $-1 \leq r < n$, with $r = -1$ meaning the fixed-point set is empty.

Remarks: The fixed-point set of an orientation-preserving cyclic action of prime order on S^3, when nonempty, is a knot in S^3.

Smith Conjecture: The fixed-point set of an orientation-preserving cyclic action of prime order on S^3 must be an unknot if it's nonempty.

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2. $n = 3$ (cont.) (3) Free actions on S^3: the list of possible finite groups which can act freely on S^3 (J. Milnor 1957, R. Lee 1973).

Finite subgroups of $SO(4)$: there is a 2 to 1 covering homomorphism

$$\rho : SO(4) \rightarrow SO(3) \times SO(3).$$

Finite subgroups of $SO(3)$:

- a cyclic group C_n of order n;
- a dihedral group D_{2n} of order $2n$
- the tetrahedral group $T \cong A_4$ of order 12;
- the octahedral group $O \cong S_4$ of order 24;
- the icosahedral group $I \cong A_5$ of order 60.

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Fact: if a finite subgroup of $SO(4)$ acts on S^3 freely, then it must be a subgroup of $U(2)$. (Such a list was obtained by Brieskorn, in classification of certain complex surface singularities.)

Milnor's list of exotic groups $Q(8, a, b, c)$: Let a, b, c be odd, pairwise co-prime integers. There is a split extension

$$1 \rightarrow Z_a \times Z_b \times Z_c \rightarrow Q(8, a, b, c) \rightarrow Q(8) \rightarrow 1,$$

where $Q(8) = \{x, y | x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1}\}$, and if $Z_a =< u >$, $Z_b =< v >$, $Z_c =< w >$, then

$$xux^{-1} = u^{-1}, xvx^{-1} = v, xwx^{-1} = w^{-1},$$

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2. $n = 3$ (cont.) (4) Ricci flow and geometrization.

Theorem: Every finite group action on S^3 by orientation-preserving diffeomorphisms is equivalent to a linear action.

Corollary: $Q(8, a, b, c)$ can not act freely on S^3.

Problem: (i) Find an alternative proof (gauge theoretic?) of the fact that $Q(8, a, b, c)$ can not act freely on S^3.

(ii) It's known that some of the $Q(8, a, b, c)$'s act freely on a homology 3-sphere Σ (existence of Σ coming from surgery theory). What can we say about Σ (particularly, $\pi_1\Sigma$)?

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3. $n = 4$. (1) Existence of nonlinear actions.

Theorem (Poenaru 1960, Mazur 1961, de Rham 1962) There exist smooth involutions on S^4 whose fixed-point set is a homology 3-sphere $\Sigma \neq S^3$.

Theorem (Giffen 1966) There exist smooth Z_p-actions, for odd p, on S^4 whose fixed-point set is a knotted 2-sphere.

Theorem (Fintushel-Stern 1981) There exists a free involution on S^4 which is not smoothly equivalent to the antipodal map.

(2) Smooth s-cobordisms of elliptic 3-manifolds.
Conjecture: If a smooth, pseudo-free finite group action on S^4 has an isolated fixed-point, then it must be equivalent to a linear action.

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Theorem (Furuta 1989, DeMichelis 1989) There is no smooth (or locally linear) finite group action on S^4 with one fixed-point.

Theorem (Hambleton-Lee 92, Braam-Matic 93, DeMichelis 89) If a finite group acts smoothly on S^4 with two isolated fixed-points, then the representations on the tangent space of the two fixed-points are equivalent.

Remarks: (1) The proofs of Furuta, Hambleton-Lee, and Braam-Matic were gauge theoretic (Yang-Mills) and are for smooth actions. The proofs of DeMichelis were traditional and work for locally linear actions on homology S^4, after learning of the gauge theoretic attempts.

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3. $n = 4$ (cont.) (4) Finite groups which act on S^4.
Theorem A (Chen-Kwasik-Schultz 2013): If a finite group G acts locally linearly and orientation-preservingly on a homology S^4, then G is isomorphic to a subgroup of $SO(5)$.

Note: Mecchia and Zimmermann showed earlier that either G is isomorphic to a subgroup of $SO(5)$, or an index 2 subgroup of G is isomorphic to a subgroup of $SO(4)$.

Theorem B (Chen-Kwasik-Schultz 2013): There are finite groups G acting topologically and orientation-reversingly on S^4, which are NOT isomorphic to a subgroup of $O(5)$.

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Other model spaces in dimension 4: (X, ξ), where X is a Kahler surface, and ξ is a complex structure. There are two main classes which have been extensively studied by the algebraic geometers: rational surfaces and K3 surfaces.

(1) $X = CP^2$: $Aut(X, \xi) = PU(3)$.

Theorem (Wilczynski 87, Hambleton-Lee 88) If a finite group acts locally linearly and homologically trivially on CP^2, then it is a subgroup of $PU(3)$.

Theorem (Wilczynski 91) A locally linear, pseudo-free action of a finite cyclic group on CP^2 is equivalently to an action by projective transformations.

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(Such actions have relatively knotted 2-dim'l singular set; can't be symplectic.)

Theorem (Chen 2006): A symplectic finite group action of G on CP^2 is smoothly equivalent to an action by projective transformations, if G is cyclic or metacyclic with non-empty fixed-point set, or there is a fixed-point whose induced G-action on the link S^3 is free.

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Other model spaces in dimension 4: (2) $X =$ a rational surface.

Definition: A rational G-surface is a rational surface with a finite automorphism group G.

Remarks: Minimal rational G-surfaces played the key role in the modern treatment of the classical problem of classifying finite subgroups of the plane Cremona group, i.e., the group of birational transformations of CP^2.

Problem: (i) Are there symplectic actions on a rational surface which are not equivalent to a holomorphic action?

(ii) Let X be a rational surface, G be a finite group acting on X pseudo-freely, preserving a symplectic structure. Consider any smooth resolution $\widetilde{X} \rightarrow X/G$ of the quotient orbifold. Is $\widetilde{X}$ diffeomorphic to a rational surface?

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(2) X = a rational surface (cont.): Consider the minimal rational G-surface X, where $X = F_n$ is a Hirzebruch surface with $n > 1$ odd, $G = Z_p$, p prime.

Theorem (Chen 2013) Assume the G-action is pseudo-free with the weights of the local representations given by $(1, m), (-1, m+n), (1, -m), (-1, -m-n)$, where $0 < m < p$. Let ω be any G-invariant symplectic form on X. Then X contains a G-invariant, ω-symplectically embedded (-1)-sphere if and only if it contains a G-invariant, smoothly embedded (-1)-sphere, provided that $p - 7 \leq 2m \leq p + 5$.

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Corollary: For $G = Z_p$ where $p \geq 5$, there exist symplectic rational G-manifolds such that

- the G-actions are homologically trivial;
- the underlying symplectic (resp. smooth) 4-manifolds are not minimal; but
- the symplectic (resp. smooth) G-manifolds are minimal.

Theorem (Chen 2013): For any prime number $p > 1$, there exists a smooth Z_p-action on $CP^2 \# \overline{CP^2}$ such that

- there are infinitely many distinct complex structures on $CP^2 \# \overline{CP^2}$ which are invariant under the Z_p-action, and
- for any cohomologous symplectic forms ω_1, ω_2 on $CP^2 \# \overline{CP^2}$ which are invariant under the Z_p-action, there is an equivariant diffeomorphism f such that $f^*\omega_2 = \omega_1$.

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Other model spaces in dimension 4: (3) $X =$ a K3 surface.

(A) Homological Rigidity: An automorphism of a K3 surface must be trivial if it acts as identity on the homology.

Problem: Are there any nontrivial smooth Z_p-actions on the K3 surface which are homologically trivial?

Theorem (Ruberman, Matumoto): There are no nontrivial locally linear Z_2-actions on the K3 surface which are homologically trivial.

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(3) $X =$ K3 surface (cont.): (B) K3 groups and smooth structures.

Definition: A finite group is called a K3 group if it can be realized as a symplectic automorphism group of a K3 surface, i.e., an automorphism group which fixes a nonvanishing holomorphic 2-form.

Remarks: K3 groups are studied extensively by Nikulin, Mukai, and others. There are 11 maximal K3 groups:

$$L_2(7), A_6, S_5, M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}.$$

Theorem (Chen-Kwasik 2008): There are infinitely many distinct exotic smooth K3 surfaces such that none of the following maximal K3 groups can act smoothly on them

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(3) $X =$ K3 surface (cont.): (B) K3 groups and smooth structures.

Definition: A finite group is called a K3 group if it can be realized as a symplectic automorphism group of a K3 surface, i.e., an automorphism group which fixes a nonvanishing holomorphic 2-form.

Remarks: K3 groups are studied extensively by Nikulin, Mukai, and others. There are 11 maximal K3 groups:

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Other model spaces in dim 4: (4) $X =$ a surface of general type.

Bound of Automorphisms.

Theorem (Hurwitz): Let Σ be a compact Riemann surface of genus $g > 1$. Then

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Problem: Let (X, ω) be a minimal symplectic 4-manifold with $c_1(K_X) \cdot [\omega] > 0$ and $c_1(K_X)^2 > 0$. Let G be a finite group acting on X preserving ω. Is there a universal constant $c > 0$ such that

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Question: Let X be a smooth, orientable 4-manifold which does not admit any smooth S^1-action. Is there a constant $C > 0$ such that for any smooth Z_p-actions of prime order, the order p satisfies $p \leq C$?

Note: The 3-dimensional version of the above question is affirmative. However, in higher dimensions, there are compact closed manifolds admitting no smooth S^1-actions but admitting smooth Z_p-actions for infinitely many primes. So no such a bound C exists.

Definition: If such a bound C exists, we will call it a Hurwitz-type bound for smooth Z_p-actions on X. Note: it's known that no such a bound exists for locally linear actions in dimension 4.

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where $c > 0$ is a universal constant.

Theorem (Chen, 2011) For any prime number $p > 3$, there is a symplectic 4-manifold X_p with the following properties:

- X_p is homeomorphic to $CP^2 \# 9(-CP^2)$;
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Remarks: From the construction, the multiplicity of $c_1(K_{X_p})$ grows linearly with p.

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Note: From the construction, the π_1 of X_n gets more and more complicated as $n \to \infty$.

Problem: Let X be a simply connected smoothable 4-manifold with even intersection form and non-zero signature. Is there a constant $C > 0$ depending only on the homeomorphism type of X, such that for any smoothable Z_p-actions of prime order on X, the order p satisfies $p \leq C$?

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The End

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