Group Actions on Four-manifolds

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Definition: A finite group action of G on X is called symplectic if there exists a symplectic form ω on X such that $g^*\omega = \omega$ for any $g \in G$; in particular, (X, ω) is a symplectic 4-manifold.

Fact: There is a contractible space of G-invariant, ω-compatible almost complex structures J on X.

Goal: Use G-invariant J-holomorphic curves in X to analyze the G-action.

Outline: (1) Adjunction and intersection formulas for J-holomorphic curves in 4-orbifolds.
(2) Transversality of moduli spaces of J-holomorphic curves in 4-orbifolds and Gromov compactness.
(3) Existence of G-invariant J-holomorphic curves (equivariant/orbifold Seiberg-Witten-Taubes theory).

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Let (M,ω) be a symplectic 4-orbifold, J be a ω-compatible almost complex structure. Let Σ be a compact closed Riemann surface with orbifold points z_i of order m_i. A J-holomorphic map $f:\Sigma\rightarrow M$ is a smooth map of orbifolds such that the local representations between uniformizing systems are equivariant J-holomorphic maps, and moreover, at each orbifold point z_i, let $(f_i,\rho_i):(D_i,Z_{m_i})\rightarrow(V_i,G_i)$ be the local representative, then we require that $\rho_i:Z_{m_i}\rightarrow G_i$ be injective. A J-holomorphic curve C in M is the image of a J-holomorphic map $f:\Sigma\rightarrow M$. We can always assume f is not multiply covered, and we assume that C is not contained entirely in the singular set of M. A J-holomorphic curve $C=Im\, f$ is called embedded if f is injective, its local representations are smooth embeddings, and each $\rho_i:Z_{m_i}\rightarrow G_i$ is isomorphic.

(1) Adjunction and Intersection Formulae.

Orbifold genus of Σ: $g_\Sigma := g_{|\Sigma|} + \sum_i (\frac{1}{2} - \frac{1}{2m_i})$, where $g_{|\Sigma|}$ is the genus of the underlying Riemann surface.

Adjunction Formula: Let $C = Im\, f$, $f : \Sigma \to M$ is J-holomorphic.

$$\frac{1}{2}(C \cdot C + K_M \cdot C) + 1 = g_\Sigma + \sum k_z + \sum k_{[z,z']},$$

where $0 \leq k_z, k_{[z,z']} \in Q$ are contributions from singularities and self-intersections of C; in particular, C is embedded if and only if $k_z = k_{[z,z']} = 0$.

Intersection Formula: Let C, C' be distinct J-holomorphic curves.

$$C \cdot C' = \sum k_{(z,z')},$$

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Example: Let M be the quotient orbifold of CP^2 by a pseudo-free linear Z_p-action. Let $\hat{C} = \{z_1 = 0\}$ in CP^2, and let C be the image of $\hat{C}$ in M. Clearly, $C = Im\ f$ for a $f : \Sigma \rightarrow M$ where Σ is a S^2 with two orbifold points of order p, and C is embedded.

In order to check the adjunction formula, note $C \cdot C = \frac{\hat{C}\cdot\hat{C}}{p} = \frac{1}{p}$, $K_M \cdot C = \frac{K_{CP^2}\cdot\hat{C}}{p} = -\frac{3}{p}$, and $g_\Sigma = 2(\frac{1}{2} - \frac{1}{2p})$. The adjunction formula for C becomes

$$\frac{1}{2}(\frac{1}{p} - \frac{3}{p}) + 1 = 2(\frac{1}{2} - \frac{1}{2p}).$$

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Example: Let $M = P(d_1, d_2, d_3)$ be the weighted projective plane defined by the C^*-action

$$\lambda \cdot (z_1, z_2, z_3) = (\lambda^{d_1} z_1, \lambda^{d_2} z_2, \lambda^{d_3} z_3).$$

The orbifold points of M are $p_1 = [1:0:0]$, $p_2 = [0:1:0]$, $p_3 = [0:0:1]$ of isotropy groups Z_{d_1}, Z_{d_2}, Z_{d_3}. The group of orbifold complex line bundles is generated by $E_0 = (C^3 \setminus \{0\}) \times_{C^*} C$, with $K_M = -(d_1 + d_2 + d_3)E_0$ and $E_0 \cdot E_0 = (d_1 d_2 d_3)^{-1}$.

Consider $C = \{z_1 = 0\}$. Then $C = d_1 E_0$ and $p_2, p_3 \in C$. The left-hand side of the adjunction formula for C is

$$\frac{1}{2}\left(\frac{d_1^2}{d_1 d_2 d_3} + \frac{-(d_1 + d_2 + d_3)d_1}{d_1 d_2 d_3}\right) + 1 = 1 - \frac{1}{2d_2} - \frac{1}{2d_3}.$$

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For the right-hand side, note that the singular points $p_2, p_3 \in C$ will contribute at least $\frac{1}{2} - \frac{1}{2d_2}$, $\frac{1}{2} - \frac{1}{2d_3}$ respectively, so that $C = Im\, f$ for a $f : \Sigma \to M$, where Σ is a S^2 with two orbifold points of orders d_2, d_3 respectively, and C is embedded.

Next consider a family of curves $C_\lambda = \{az_2^{d_3} + bz_3^{d_2} = 0\}$, where $\lambda = [a, b] \in CP^1$, $ab \neq 0$. Clearly, $C_\lambda = d_2 d_3 E_0$ and $p_1 \in C_\lambda$.

The left-hand side of adjunction formula for C_λ is

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For the right-hand side, note that the singular points $p_2, p_3 \in C$ will contribute at least $\frac{1}{2} - \frac{1}{2d_2}$, $\frac{1}{2} - \frac{1}{2d_3}$ respectively, so that $C = Im\ f$ for a $f : \Sigma \rightarrow M$, where Σ is a S^2 with two orbifold points of orders d_2, d_3 respectively, and C is embedded.

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For the right-hand side, let $C_\lambda = Im\, f$ for a $f : \Sigma \to M$. The $p_1 \in C_\lambda$ will contribute at least $\frac{1}{2} - \frac{1}{2d_1}$ to the orbifold genus g_Σ, and contribute at least $\frac{(d_2-1)(d_3-1)}{2d_1}$ to k_z. It is easily see that C_λ is a S^2 with only one singular point p_1 (and Σ is a S^2 with one orbifold point of order d_1).

Finally, consider the intersection of $C_\lambda, C_{\lambda'}$, $\lambda \neq \lambda'$. First, $p_1 \in C_\lambda \cap C_{\lambda'}$ will contribute at least $\frac{d_2 d_3}{d_1}$ to $k_{(z,z')}$ on the right-hand side. On the other hand, $C_\lambda \cdot C_{\lambda'} = \frac{(d_2 d_3)^2}{d_1 d_2 d_3}$ on the left-hand side. It follows from the intersection formula that p_1 is the only intersection point of $C_\lambda, C_{\lambda'}$.

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(2) Moduli spaces and Gromov compactness.

Let $C = Im\, f$ be a J-holomorphic curve, where $f : \Sigma \to M$ is not multiply covered. Here Σ has k orbifold points z_i of orders m_i. Let $(f_i, \rho_i) : (D_i, Z_{m_i}) \to (V_i, G_i)$ be the local representation of f at z_i, let $\mu_{m_i} = \exp(2\pi i/m_i) \in Z_{m_i}$ be the element whose action on the disc D_i is given by rotation of angle $2\pi/m_i$, and let the action of $\rho_i(\mu_{m_i}) \in G_i$ on V_i be given by

$$\rho_i(\mu_{m_i}) \cdot (z_1, z_2) = (\mu_{m_i}^{m_{i,1}} z_1, \mu_{m_i}^{m_{i,2}} z_2),\ 0 < m_{i,1}, m_{i,2} < m_i.$$

Theorem: For generic J, the moduli space of J-holomorphic curves is a finite dimensional smooth manifold, whose dimension at C equals $2d_C$, where

$$d_C = -K_M \cdot C + (g_{|\Sigma|} - 1) + k - \sum_{i=1}^{k} \frac{m_{i,1} + m_{i,2}}{m_i} \in Z.$$

Note: if C exists for generic J, then $d_C \geq 0$.

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Note: if C exists for generic J, then $d_C \geq 0$.

Example: Let X be a symplectic 4-manifold with a symplectic Z_p-action, let J be Z_p-invariant, and let $\hat{C}$ be a Z_p-invariant, embedded J-holomorphic 2-sphere with $\hat{C} \cdot \hat{C} = -2$, containing two isolated fixed points p_1, p_2. (For a generic, non-Z_p-invariant J, such a $\hat{C}$ does not exist.)

Let $M = X/Z_p$, $C = \hat{C}/Z_p$. Then

$$d_C = -K_M \cdot C + (0-1) + 2 - \frac{1+m_1}{p} - \frac{1+m_2}{p},$$

where $0 < m_1, m_2 < p$ are the weights of the Z_p-action at p_1, p_2 in the normal direction to $\hat{C}$. Note that $K_M \cdot C = \frac{K_X \cdot \hat{C}}{p} = 0$, so that $d_C \in Z$ gives

$$m_1 + m_2 + 2 = p \text{ or } m_1 + m_2 + 2 = 2p.$$

Furthermore, $d_C \geq 0$ rules out the latter case.

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Proposition: Let X be a symplectic 4-manifold with a symplectic Z_p-action, let J be Z_p-invariant, and let $\hat{C}$ be a Z_p-invariant J-holomorphic curve such that $\hat{C}\cdot\hat{C}<0$. Let $C=\hat{C}/Z_p$ be the corresponding curve in $M=X/Z_p$, parametrized by $f:\Sigma\rightarrow M$. Then $g_{|\Sigma|}=0$ for generic J.

Sketch of proof: Suppose Σ has k orbifold points of order p. Then the adjunction formula for C gives

$$\frac{1}{2p}(\hat{C}\cdot\hat{C}+K_X\cdot\hat{C})\geq= g_{|\Sigma|}-1+(\frac{1}{2}-\frac{1}{2p})k.$$

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Combing the two inequalities gives

$$\frac{\hat{C}\cdot\hat{C}}{p} \geq -\frac{k}{p} + g_{|\Sigma|} - 1 + \sum_{i=1}^{k} \frac{m_{i,1}+m_{i,2}}{p}.$$

Since $m_{i,1}+m_{i,2} > 1$, $g_{|\Sigma|} > 0$ would imply $\hat{C}\cdot\hat{C} \geq 0$, which is a contradiction. The end of proof.

Theorem (Gromov compactness): The moduli space of J-holomorphic maps in a symplectic orbifold with a fixed homology class is compact, after introducing certain notion of J-holomorphic maps from a nodal orbifold Riemann surface.

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(3) Existence of G-invariant J-holomorphic curves.

Orbifold Version of Taubes Theorem (Chen 2006): Let (M, ω) be a symplectic 4-orbifold, and let E be an orbifold complex line bundle. Suppose there is an unbounded sequence of values for the parameter r such that the corresponding Taubes' deformed Seiberg-Witten equations have a solution (A, α, β). Then for any ω-compatible J, there are J-holomorphic curves C_i with multiplicity m_i such that $c_1(E) = \sum_i m_i C_i$. Moreover, if a closed subset $\Omega \subset M$ is contained in $\alpha^{-1}(0)$ throughout, then $\Omega \subset \cup_i C_i$.

Remarks: There are two typical sources of points in the closed subset Ω. (i) A singular point p is contained in Ω if the isotropy group at p has nontrivial representation on the fiber of E at p. (ii) When the dimension of the Seiberg-Witten moduli space is $2n > 0$, we can specify any n distinct points to be contained in Ω.

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Remarks: There are two typical sources of points in the closed subset Ω. (i) A singular point p is contained in Ω if the isotropy group at p has nontrivial representation on the fiber of E at p. (ii) When the dimension of the Seiberg-Witten moduli space is $2n > 0$, we can specify any n distinct points to be contained in Ω.

Two useful cases (1) $b_2^+(M) > 1$, $E = K_M$. (2) $b_1(M) = 0$, $b_2^+(M) = 1$, and applying wall-crossing argument.

An example for (1): There are no homologically trivial symplectic Z_p-actions on any symplectic 4-manifold with trivial canonical class, $b_2^+ > 1$ and non-zero signature.

Sketch of proof: Let X be the symplectic 4-manifold.

Step 1: $b_2^+(X/Z_p) = b_2^+(X) > 1$, so $K_X = \sum_i m_i C_i$ where $\cup_i C_i$ is Z_p-invariant. K_X is trivial implies that $\cup_i C_i$ is empty, which implies that the representation of Z_p on the fiber of K_X at each fixed-point is trivial, which means the local representation at each fixed point has weights $(a, -a)$, $a \neq 0$. In particular, the fixed points are isolated.

Step 2: K_X is trivial implies that $\chi(X) = -\frac{3}{2} Sign(X)$, so there are $-\frac{3}{2} Sign(X)$ isolated fixed points by the Lefschetz fixed-point theorem and homological triviality.

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Step 3: Computing the signature defect def_m for each fixed point m, local representation has weights $(a, -a)$ implies

$$def_m = \sum_{k=1}^{p-1} \frac{(1+\mu_p^k)((1+\mu_p^{-k})}{(1-\mu_p^k)((1-\mu_p^{-k})} = \frac{1}{3}(p-1)(p-2),$$

where $\mu_p = \exp(2\pi i/p)$.

Step 4: Note that $Sign(X/Z_p) = Sign(X)$. Then the weaker version of G-signature Theorem gives

$$p \cdot Sign(X) = Sign(X) + (-\frac{3}{2} Sign(X)) \cdot \frac{1}{3}(p-1)(p-2),$$

whch implies $Sign(X) = 0$, a contradiction.

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Dimension of Seiberg-Witten moduli space (Chen 2006): Let M be a symplectic 4-orbifold with singular points p_i and singular surfaces Y_j. Then for any orbifold complex line bundle E, the dimension of the Seiberg-Witten moduli space associated to E is

$$d(E) = c_1(E)^2 - c_1(E) \cdot c_1(K_M) + \sum_i I_i + \sum_j I_j,$$

where I_i, I_j are determined from p_i, Y_j as follows.

$$I_i = \frac{1}{|G_{p_i}|} \sum_{g \in G_{p_i} \setminus \{e\}} \frac{2(\rho_{p_i,E}(g) - 1)}{\prod_{k=1}^{2}(1 - \rho_{p_i,k}(g^{-1}))}$$

$$I_j = \frac{1}{|G_{Y_j}|} \sum_{g \in G_{Y_j} \setminus \{e\}} \Big(\frac{2\rho_{Y_j,E}(g) c_1(E)}{1-\rho_{Y_j}(g^{-1})} + \frac{(\rho_{Y_j,E}(g)-1) c_1(TY_j)}{1-\rho_{Y_j}(g^{-1})}$$

$$+ \frac{2\rho_{Y_j}(g^{-1})(\rho_{Y_j,E}(g)-1) c_1(\nu_{Y_j})}{(1-\rho_{Y_j}(g^{-1}))^2} \Big) [Y_j].$$

Let $p > 1$ be a prime number, and let (X, ω) be a symplectic 4-manifold with $b_1 = 0$ equipped with a symplectic Z_p-action. Consider a complex line bundle E such that $c_1(E) \in H^2(X; Z)^{Z_p}$. Denote by $SW_X(E)$ the Seiberg-Witten invariant of X associated to E, which is defined using Taubes chamber when $b_2^+ = 1$.

Theorem (Chen 2013): Assume further that

- $SW_X(E) \neq 0 \pmod{p}$;
- Write $c_1(E)^2 - c_1(K_X) \cdot c_1(E) = 2(lp + k)$, where $l \geq 0$ and $0 \leq k < p$. Then the number of fixed-points $\# X^{Z_p} \geq k$.

Then for any Z_p-invariant ω-compatible J, and for any given subset Ω of X which consists of l distinct free Z_p-orbits and k distinct fixed points, there exist J-holomorphic curves C_i with multiplicity m_i, such that (i) $c_1(E) = \sum_i m_i C_i$, (ii) the set $\cup_i C_i$ is Z_p-invariant, (iii) $m_i = m_j$ if C_i, C_j lie in the same orbit under the Z_p-action on $\cup_i C_i$, and (iv) $\Omega \subset \cup_i C_i$.

The End

Thank You !