Group Actions on Four-manifolds

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Fixed-point set structure and induced action on homology

Let X be a compact closed, oriented 4-manifold. Given an action $\rho : G \times X \to X$, which is locally linear and orientation-preserving, one can associate ρ with the following invariants:

- the singular set $\Sigma X := \{x \in X | G_x \neq \emptyset\}$, where $G_x := \{g \in G | g \cdot x = x\}$, $\forall x \in X$;
- the local representations $\rho_x : G_x \to Aut(T_x X)$, $\forall x \in \Sigma X$;
- the induced action $\rho_* : G \to Aut(H^2(X), \cup)$.

In summary,

- $\rho \Rightarrow |\rho|$ where

$$|\rho| := (G; \Sigma X \subset X; \{\rho_x | x \in \Sigma X\}; \rho_*).$$

- $X \Rightarrow \Gamma_X := \{|\rho| | \rho \text{ is an action on } X\}$.

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Assumption: X is simply connected, $G = Z_p$, p prime. Then

- ΣX: a union of isolated points m and surfaces Y; $G_x = Z_p$.
- $\{\rho_x\}$: (i) $x = m$: $\rho_x(g)$, $g \in G_x$, is given by a pair of integers (a_m, b_m) (i.e., the weights), unordered, unique up to simultaneous change of sign and congruence mod p. (ii) $x \in Y$: $\rho_x(g)$, $g \in G_x$, is given by the weight c_Y in the normal direction.
- ρ_*: $H^2(X) = Z[Z_p]^r \oplus Z^t \oplus Z[\mu_p]^s$, where $Z[Z_p]$ is the regular representation, Z is the trivial one, and $Z[\mu_p]$ is of cyclotomic type, and $r, t, s \geq 0$.

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Constraints on ΣX, $\{\rho_x\}$ and ρ_*.

Proposition: ΣX and ρ_* satisfy the following constraints.

- $b_2(X) = rp + t + s(p-1)$.
- (Lefschetz fixed point formula) $\chi(\Sigma X) = t - s + 2$.
- If $\Sigma X \neq \emptyset$, then $b_1(\Sigma X; Z_p) = s$.
- Suppose $G =< g >= Z_2$. If ΣX is empty or a finite set, then $g^*\alpha \cup \alpha = 0 \pmod 2$ for any $\alpha \in H^2(X)$.

Example: Let g be a locally linear, orientation-preserving involution on $X = CP^2$. Then the fixed point set ΣX is a union of a point and a S^2 if g is homologically trivial, and ΣX is a RP^2 if g is homologically nontrivial. (This is clear when g is a holomorphic or anti-holomorphic involution.)

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G-signature Theorem: (1) Assume each $Y \subset \Sigma X$ is orientable. Set $Sign(g, X) = tr(g)|_{H^{2,+}} - tr(g)|_{H^{2,-}}, \forall g \in G$. Then

$$Sign(g, X) = \sum_{m \in \Sigma X} -\cot(\frac{a_m \pi}{p}) \cdot \cot(\frac{b_m \pi}{p}) + \sum_{Y \subset \Sigma X} \csc^2(\frac{c_Y \pi}{p}) \cdot (Y \cdot Y)$$

where $Y \cdot Y$ is the self-intersection number of Y.

(2) (Weaker version) Denote by *Sign* the signature of a space. Then

$$p \cdot Sign(X/G) = Sign(X) + \sum_{m \in \Sigma X} def_m + \sum_{Y \subset \Sigma X} def_Y,$$

where $def_m = \sum_{1 \neq \lambda \in C, \lambda^p = 1} \frac{(1+\lambda^{a_m})((1+\lambda^{b_m})}{(1-\lambda^{a_m})((1-\lambda^{b_m})}$, $def_Y = \frac{p^2-1}{3} \cdot (Y \cdot Y)$.

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$$1 = Sign(g, X) = -\cot(\frac{\pi}{2}) \cdot \cot(\frac{\pi}{2}) + \csc^2(\frac{\pi}{2}) \cdot (Y \cdot Y),$$

which implies $Y \cdot Y = 1$. Consequently, Y is a generator of $H_2(X)$, and $X \setminus Nd(Y)$ is a homotopy B^4. Furthermore, g is equivalent to a linear action if and only if the involution on $X \setminus Nd(Y)$ is equivalent to a linear action. (The latter is equivalent to whether the cobordism between $Link(m)/g$ and $\partial Nd(Y)/g$ is trivial.)

Answer: topologically, yes (Freedman's surgery); smoothly, unknown; symplectically, yes (Chen 04, via J-holomorphic curves).

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$$\lambda \cdot [z_0 : z_1 : z_2] = [z_0 : \lambda z_1 : \lambda^a z_2], \;\; 1 < a < p.$$

It has 3 fixed-points: $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, with local representations given by $(1,a), (-1,a-1), (-a,1-a)$ for a generator $g \in Z_p$. Clearly $Sign(g,X) = 1$.

(i) $p=3$ and $a=2$: The G-signature Theorem becomes

$$1 = Sign(g,X) = -\cot(\frac{\pi}{3}) \cdot \cot(\frac{2\pi}{3})$$

$$-\cot(\frac{-\pi}{3}) \cdot \cot(\frac{\pi}{3}) - \cot(\frac{-2\pi}{3}) \cdot \cot(\frac{-\pi}{3}),$$

which can be checked easily because $\cot(\frac{\pi}{3}) = -\cot(\frac{2\pi}{3}) = \frac{1}{\sqrt{3}}$.

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(ii) $p > 3$ and a arbitrary: The G-signature Theorem becomes

$$1 = Sign(g, X) = -\cot(\frac{\pi}{p}) \cdot \cot(\frac{a\pi}{p})$$

$$-\cot(\frac{-\pi}{p}) \cdot \cot(\frac{(a-1)\pi}{p}) - \cot(\frac{-a\pi}{p}) \cdot \cot(\frac{(1-a)\pi}{p}).$$

To verify this, note that

$$\cot(\frac{(a-1)\pi}{p}) - \cot(\frac{a\pi}{p}) = \frac{\sin\frac{\pi}{p}}{\sin\frac{a\pi}{p}\sin\frac{(a-1)\pi}{p}}$$

so that the sum of the first two terms in the right-hand side of the G-signature formula becomes

$$\frac{\cos\frac{\pi}{p}}{\sin\frac{a\pi}{p}\sin\frac{(a-1)\pi}{p}}.$$

To continue, note that

$$\cos\frac{\pi}{p}=\cos\frac{a\pi}{p}\cos\frac{(a-1)\pi}{p}+\sin\frac{a\pi}{p}\sin\frac{(a-1)\pi}{p},$$

so that

$$\frac{\cos\frac{\pi}{p}}{\sin\frac{a\pi}{p}\sin\frac{(a-1)\pi}{p}}=\cot(\frac{a\pi}{p})\cdot\cot(\frac{(a-1)\pi}{p})+1,$$

which shows that the RHS of the G-signature formula equals 1.

Theorem (Edmonds-Ewing 89): A locally linear, pseudo-free cyclic action of odd order on a rational homology CP^2 has the same fixed-point set structure as a linear action on CP^2.

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Example: Consider a holomorphic Z_p-action on Hirzebruch surface $X = F_n$, $n > 1$ and odd. Then the action is trivial in homology; in particular, $\chi(\Sigma X) = 4$ and $Sign(g, X) = 0$. Moreover, the action preserves the ruling, and ΣX belongs to one of the following cases:

- the union of the 0-section and ∞-section;
- the union of two fibers;
- the union of one fiber and two points, with $(1, n), (1, -n)$;
- four points, with $(1, m), (-1, m + n), (1, -m), (-1, -m - n)$.

Exercise: (i) Verify the G-signature Theorem.

(ii) How much of the above fixed-point set data can be recovered for a homologically trivial, locally linear Z_p-action using Proposition and G-signature Theorem? What about symplectic actions?

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Construction, Realization of ΣX, $\{\rho_x\}$ and ρ_*.

Theorem (Edmonds-Ewing 92): Let a closed, oriented, simply connected 4-manifold X be given, together with a representation of Z_p on $H^2(X)$ of the form $H^2(X) = Z[Z_p]^r \oplus Z^t$, preserving the cup product, and with a candidate fixed point data for $t+2$ isolated fixed points satisfying the G-signature Theorem and an additional torsion condition (which vanishes for relatively small p). Then there is a locally linear topological Z_p-action on X realizing the given data.

Theorem (Edmonds 87): Let a closed, simply connected 4-manifold X be given. For any prime number $p > 3$, there exists a locally linear, pseudo-free and homologically trivial, topological Z_p-action on X.

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Construction: (1) Attaching 2-handles equivariantly. (2) Capping by a contractible "piece" equivariantly (here need Freedman's theory).

(1) Consider Z_p-action on B^4: $\lambda \cdot (z_1 : z_2) = (\lambda z_1 : \lambda^a z_2)$, and Z_p invariant knot $K = \{z_2 = 0\} \in S^3$. Attach a 2-handle with Z_p-action $\lambda \cdot (z_1 : z_2) = (\lambda^{-1} z_1 : \lambda^{a-1} z_2)$ to K equivariantly, with framing $+1$. (Note: the "weight" $a-1$ is determined by the "weight" a and the framing $+1$.) (2) Cap by B^4 equivariantly.

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Obstructions to Smoothability

(1) G-index Theorem for Dirac operators.

Theorem (Atiyah-Singer): Assume further X is spin and p is odd, and the Z_p-action is smooth. Moreover, it is spin and preserves an almost complex structure. Then the "Spin number" is given by

$$Spin(g,X) = \sum_{m\in\Sigma X} -\epsilon(g,m)\cdot\frac{1}{4}\csc(\frac{a_m\pi}{p})\cdot\csc(\frac{b_m\pi}{p})$$

$$+\sum_{Y\subset\Sigma X}\epsilon(g,Y)\frac{Y\cdot Y}{4}\csc(\frac{c_Y\pi}{p})\cdot\cot(\frac{c_Y\pi}{p}),$$

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Theorem (Atiyah-Singer) (cont.): First, since the Z_p-action preserves an almost complex structure, the weights a_m, b_m, c_Y are uniquely determined (not up to a sign), and we require $0 < a_m, b_m, c_Y < p$. With this understood, $\epsilon(g,m)$, $\epsilon(g,Y)$ are given by

$$\epsilon(g,m) = (-1)^{k(g,m)}, \epsilon(g,Y) = (-1)^{k(g,Y)},$$

where $k(g,m), k(g,Y)$ are determined by

$$k(g,m)\cdot p = 2r_m + a_m + b_m, \;\; k(g,Y)\cdot p = 2r_Y + c_Y$$

for some r_m, r_Y satisfying $0 \leq r_m < p$, $0 < r_Y < p$.

(2) Seiberg-Witten equations.

Theorem (Fang 98): Let M be a closed, oriented smooth 4-manifold with $b_1 = 0$ and $b_2^+ > 1$ which admits a smooth $G = Z_p$-action of prime order such that $b_2^+(M/G) = b_2^+$. Let c be a G-*spin*^{c} structure on M such that the G-index of the Dirac operator $ind_G D = \sum_{k=0}^{p-1} d_k C_k$ satisfies $2d_k \leq b_2^+ - 1$ for all $0 \leq k < p$. Then the Seiberg-Witten invariant associated to *spin*^{c} structure c satisfies $SW_M(c) = 0 \pmod{p}$. Here C_k denotes the complex 1-dimensional representation of Z_p of weight k.

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(3) Kirby-Siebenmann invariant and Rochlin invariant.

Theorem (Freedman-Quinn): Suppose a locally linear action of a finite group G on a closed spin 4-manifold M is spin and pseudo-free. Let N be the spin 4-manifold with boundary obtained from M/G by removing a regular neighborhood of the singular set, and denote by $\partial\eta$ the spin structure on ∂N induced from that of N. Then

$$8 \cdot ks(N) \equiv Sign(N) + roc(\partial N, \partial\eta) \pmod{16},$$

where $ks(N)$ is the Kirby-Siebenmann invariant of N and $roc(\partial N, \partial\eta)$ is the Rochlin invariant of ∂N with respect to the spin structure $\partial\eta$. Note: If the action is smoothable, then $ks(N) = 0$.

Fact: A smooth Z_p-action of odd order on a simply connected spin 4-manifold is always spin; an involution on a spin 4-manifold falls into two types: even, which is spin and has isolated fixed-points, or odd, which is not and has empty or 2-dim'l fixed point set.

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Example: Smoothability of some locally linear, pseudo-free Z_5 and Z_7 actions on $K3$ surface.

- $p = 5$: 14 fixed-points; 2 with $(1, 1)$, 6 with $(-1, 3)$, 2 with $(2, 4)$, and 4 with $(3, 3)$ for local representations.
- $p = 7$: 10 fixed-points; 2 with $(2, 3)$, 2 with $(-1, -1)$, 2 with $(2, 4)$, and 4 with $(-2, 1)$ for local representations.

— Can not rule out that they may be smoothable by the above obstructions.
—Can not rule out that they may be symplectic by J-holomorphic curve techniques.

Problem: Does there exist a symplectic, exotic $K3$ surface such that the above actions can not be realized by a symplectic action?

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The End

Thank You !