Show all of your work. No credit will be given for an answer without some work or explanation.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
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<tr>
<td>3</td>
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<td>5</td>
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<td>7</td>
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<tr>
<td>Total (100 points)</td>
<td></td>
</tr>
</tbody>
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1. Determine if the three vectors \( \mathbf{u} = < 2, -1, 4 >, \mathbf{v} = < 1, 4, -7 > \) and \( \mathbf{w} = < 0, -1, 2 > \) lie in the same plane or not. (12 points)

Solution:

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & -1 & 4 \\ 1 & 4 & -7 \\ 0 & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 4 & -7 \\ -1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 4 & -7 \\ 0 & 2 \end{vmatrix} + 4 \begin{vmatrix} 4 & -7 \\ 0 & -1 \end{vmatrix} = 2 \cdot 1 + 1 \cdot 2 + 4 \cdot (-1) = 0
\]

, which says that the volume of the parallelepiped determined by \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) is equal 0, and thus these vectors do lie in the same plane.

2. Find the equation of the tangent plane to the surface \( z = \ln(2x + y) \) at point \((-1, 3, 0)\)? (15 points)

Solution: Since \( f(x, y) = \ln(2x + y) \), we have \( f_x(x, y) = \frac{2}{2x+y} \) and \( f_y(x, y) = \frac{1}{2x+y} \).

Now plug in \( x = -1 \) and \( y = 3 \), we obtain \( f_x(-1, 3) = 2 \) \( f_y(-1, 3) = 1 \).

Thus, the equation of the tangent plane is given by \( 2(x - (-1)) + (y - 3) = z - 0 \)

Simplifying, we obtain \( 2x + y - z = 1 \)

3. Determine the equation of the plane perpendicular to the vector \( \mathbf{n} = < 2, 4, 3 > \) which contains the point \( P = (3, 2, 1) \). What is the distance of the point \( Q = (5, 2, 1) \) from this plane? (15 points)

Solution: Using the normal vector to the plane \( \mathbf{n} = < 2, 4, 3 > \) and the given point \( (3, 2, 1) \), we find the equation of the plane: \( 2(x - 3) + 4(y - 2) + 3(z - 1) = 0 \)

Use the distance formula, we have

\[
D = \frac{|2 \cdot 5 + 4 \cdot 2 + 3 \cdot 1 - 17|}{\sqrt{2^2 + 4^2 + 3^2}} = \frac{4}{\sqrt{29}}
\]

4. Let \( S \) be the surface consisting of all points in space whose distance to the point \( (0, 0, -1) \) is \( \sqrt{2} \) times their distance to \( xy \) plane. Find an equation for \( S \) and sketch the surface \( S \). (15 points)

Solution: Let \( P = (x, y, z) \) be an arbitrary point on surface \( S \). Then the distance from \( P \) to \( (0, 0, -1) \) is \( \sqrt{x^2 + y^2 + (z + 1)^2} \) and the distance from \( P \) to \( xy \) plane is \( |z| \).

So \( \sqrt{x^2 + y^2 + (z + 1)^2} = \sqrt{2} |z| \). Which simplifies to \( x^2 + y^2 - (z - 1)^2 = -2 \). Thus, the surface \( S \) is a hyperboloid of two sheets.

5. Find the following limit, if it exists, or show that the limit does not exist. (15 points)

\[
\lim_{(x,y,z) \to (0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2}
\]

Solution:

First, we will use the path \( y = x = z \). Along this path we have,

\[
\lim_{(x,y,z) \to (0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2} = \lim_{(x,x,x) \to (0,0,0)} \frac{x^2 + x^2 + x^2}{x^2 + x^2 + x^2} = \lim_{x \to 0} \frac{3x^2}{3x^2} = 1
\]
Now, let’s try the path \( y = z = 0 \). Along this path the limit becomes,

\[
\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2} = \lim_{(x,0,0)\to(0,0,0)} \frac{0}{x^2} = 0
\]

We have two paths that give different values for the given limit and so the limit doesn’t exist.

6. Verify Clairut’s Theorem for \( f(x, y) = xe^{-x^2y^2} \). (15 points)

\[
f_x = e^{-x^2y^2} - 2x^2y^2e^{-x^2y^2}, \quad f_{xy} = (4y^3x^4 - 6x^2y)e^{-x^2y^2}, \quad f_y = -2x^3ye^{-x^2y^2}, \quad f_{yx} = (4y^3x^4 - 6x^2y)e^{-x^2y^2}.
\]
Thus \( f_{xy} = f_{yx} \).

7. Find the directional derivative of the function \( f(x, y, z) = xe^{\frac{xy}{z}} \) in the direction of vector \( \vec{v} = \langle -1, 2, 2 \rangle \) at the point \( P = (3, 0, 1) \). (13 points)

\[
f_x = \frac{ze^{\frac{xy}{z}}}{z}, \quad f_y = \frac{xe^{\frac{xy}{z}}}{z}, \quad f_z = \frac{-xe^{\frac{xy}{z}}}{z^2}. \text{ First, we find the unit vector in the direction of vector } \vec{v}: \quad \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle -\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle.
\]
\[
D_y f(3, 0, 1) = \nabla f(3, 0, 1) \cdot \vec{u} = \langle 0, 9, 0 \rangle \cdot \langle -\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle = 6
\]