MULTIVARIABLE CALCULUS

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INSTRUCTOR: Anar Akhmedov

Name: ____________________

Signature: ____________________

ID #: ____________________

Show all of your work. No credit will be given for an answer without some work or explanation.

Problem	Points
1	
2	
3	
4	
5	
6	
7	
Total (100 points)	

1. Determine if the three vectors $\vec{\mathbf{u}} = < 2, -1, 4 >$, $\vec{\mathbf{v}} = < 1, 4, -7 >$ and $\vec{\mathbf{w}} = < 0, -1, 2 >$ lie in the same plane or not. (12 points)

   Solution:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 2 & -1 & 4 \\ 1 & 4 & -7 \\ 0 & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 4 & -7 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 1 & -7 \\ 0 & 2 \end{vmatrix} + 4 \begin{vmatrix} 1 & 4 \\ 0 & -1 \end{vmatrix} = 2{\cdot}1 + 1{\cdot}2 + 4{\cdot}(-1) = 0$$

   , which says that the volume of the parallelepiped determined by $\vec{u}$,$\vec{v}$ and $\vec{w}$ is equal 0, and thus these vectors do lie in the same plane.

2. Find the equation of the tangent plane to the surface $z = \ln(2x + y)$ at point $(-1, 3, 0)$? (15 points)

   Solution: Since $f(x, y) = \ln(2x + y)$, we have $f_x(x, y) = \frac{2}{2x+y}$ and $f_y(x, y) = \frac{1}{2x+y}$.

   Now plug in $x = -1$ and $y = 3$, we obtain $f_x(-1, 3) = 2$ $f_y(-1, 3) = 1$.

   Thus, the equation of the tangent plane is given by $2(x - (-1)) + (y - 3) = z - 0$

   Simplifying, we obtain $2x + y - z = 1$

3. Determine the equation of the plane perpendicular to the vector $\vec{n} = < 2, 4, 3 >$ which contains the point $P = (3, 2, 1)$. What is the distance of the point $Q = (5, 2, 1)$ from this plane? (15 points)

   Solution: Using the normal vector to the plane $\vec{n} = < 2, 4, 3 >$ and the given point $(3, 2, 1)$, we find the equation of the plane: $2(x - 3) + 4(y - 2) + 3(z - 1) = 0$

   Use the distance formula, we have

   $D = \frac{|2\cdot5+4\cdot2+3\cdot1-17|}{\sqrt{2^2+4^2+3^2}} = \frac{4}{\sqrt{29}}$

4. Let $\mathbf{S}$ be the surface consisting of all points in space whose distance to the point $(0, 0, -1)$ is $\sqrt{2}$ times their distance to xy plane. Find an equation for $\mathbf{S}$ and sketch the surface $\mathbf{S}$. (15 points)

   Solution: Let $P = (x, y, z)$ be an arbitrary point on surface $\mathbf{S}$. Then the distance from P to $(0, 0, -1)$ is $\sqrt{x^2 + y^2 + (z + 1)^2}$ and the distance from P to xy plane is $|z|$.

   So $\sqrt{x^2 + y^2 + (z + 1)^2} = \sqrt{2}|z|$. Which simplifies to $x^2 + y^2 - (z - 1)^2 = -2$. Thus, the surface S is a hyperboloid of two sheets.

5. Find the following limit, if it exists, or show that the limit does not exist. (15 points)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2}$$

   Solution:

   First, we will use the path $y = x = z$. Along this path we have,

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2} = \lim_{(x,x,x)\to(0,0,0)} \frac{x^2 + x^2 + x^2}{x^2 + x^2 + x^2} = \lim_{x\to 0} \frac{3x^2}{3x^2} = 1$$

Now, let's try the path $y = z = 0$. Along this path the limit becomes,

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2} = \lim_{(x,0,0)\to(0,0,0)} \frac{0}{x^2} = 0$$

We have two paths that give different values for the given limit and so the limit doesn't exisit.

6. Verify Clairut's Theorem for $f(x, y) = xe^{-x^2y^2}$. (15 points)

   $f_x = e^{-x^2y^2} - 2x^2y^2e^{-x^2y^2}$, so $f_{xy} = (4y^3x^4 - 6x^2y)e^{-x^2y^2}$. $f_y = -2x^3ye^{-x^2y^2}$, so $f_{yx} = (4y^3x^4 - 6x^2y)e^{-x^2y^2}$. Thus $f_{xy} = f_{yx}$.

7. Find the directional derivative of the function $f(x, y, z) = xe^{\frac{xy}{z}}$ in the direction of vector $\mathbf{\vec{v}} = \ < -1, 2, 2 >$ at the point $P = (3, 0, 1)$. (13 points)

   $f_x = \frac{xy}{z}e^{\frac{xy}{z}}$, $f_y = \frac{x^2}{z}e^{\frac{xy}{z}}$, and $f_z = \frac{-xy}{z^2}e^{\frac{xy}{z}}$. First, we find the unit vector in the direction of vector $\vec{v}$: $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \ < \frac{-1}{3}, \frac{2}{3}, \frac{2}{3} >$.

   $D_{\vec{u}}f(3, 0, 1) = \nabla f(3, 0, 1) \cdot \vec{u} = \ < 0, 9, 0 > \cdot < \frac{-1}{3}, \frac{2}{3}, \frac{2}{3} > = 6$