MULTIVARIABLE CALCULUS

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Name: ____________________

Signature: ____________________

ID #: ____________________

Show all of your work. No credit will be given for an answer without some work or explanation.

Problem	Points
1	
2	
3	
4	
5	
6	
Total (100 points)	

1. Convert the point $P = (1, 1, -\sqrt{2})$ from Cartesian to spherical coordinates. Plot the point P using the spherical coordinate system. (15 points)

   Solution: We have $x = 1$, $y = 1$, and $z = -\sqrt{2}$. $\rho^2 = x^2 + y^2 + z^2 = 4$, so $\rho = 2$. Next, we find ϕ using $z = \rho\cos(\phi)$, which gives $\cos(\phi) = -\frac{\sqrt{2}}{2}$. Since $0 \leq \phi \leq \pi$, we have $\phi = \frac{3\pi}{4}$. Finally, use the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to find θ. We have $\cos(\theta) = \frac{1}{\sqrt{2}}$ and $\sin(\theta) = \frac{1}{\sqrt{2}}$, we obtain $\theta = \frac{\pi}{4}$.

   Thus, the spherical coordinates of the point P is given by $(2, \frac{\pi}{4}, \frac{3\pi}{4})$.

2. Find the area of the region R bounded by the hyperbolas $xy = 1$ and $xy = 2$, and the curves $xy^2 = 3$ and $xy^2 = 4$. (17 points)

   Solution:

   Use the change of variables $u = xy$ and $v = xy^2$. Solving for x and y, we obtain the inverse transformation given by $x = \frac{u^2}{v}$ and $y = \frac{v}{u}$. First, note that the image of the region R under the transformation $u = xy$, $v = xy^2$ is a square with vertices at $(1, 3)$, $(2, 3)$, $(2, 4)$, and $(1, 4)$. Next, we compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = 1/v$. Using the change of variable formula for double integrals, we have $Area = \int\int_D dA = \int_1^2 \int_3^4 |\frac{\partial(x,y)}{\partial(u,v)}| du\, dv = \int_1^2 \int_3^4 \frac{1}{v} dv\, du = \ln(4) - \ln(3)$.

3. Show that the line integral $\int_C 2x\sin(y)dx + (x^2\cos(y) - 3y^2)dy$ is independent of path and evaluate the given integral for any path from $(2, \pi/6)$ to $(0, -2)$. (16 points)

   Solution: First, we will show the vector field $F(x, y) = < 2x\sin(y), x^2\cos(y) - 3y^2 >$ is a gradient field of some function $f(x, y)$. Using the equations $f_x = 2x\sin(y)$ and $f_y = x^2\cos(y) - 3y^2$, we obtain $f(x, y) = x^2\sin(y) - y^3 + C$. Next, using the Fundamental Theorem for Line Integrals, we show that the given line integral is independent of path.

   Let C be any path from $(2, \pi/6)$ to $(0, -2)$. $\int_C 2x\sin(y)dx + (x^2\cos(y) - 3y^2)dy = f(0, -2) - f(2, \pi/6) = 6 + \frac{\pi^3}{216}$.

4. Find the volume of the solid that lies inside $x^2 + y^2 + z^2 = 2z$ and $z^2 = x^2 + y^2$. (17 points)

   Solution:

   We will use the spherical coordinates to compute the volume. The equation of the sphere becomes $\rho^2 = 2\rho\cos(\phi)$, so $\rho = 2\cos(\phi)$. To convert the equation of the cone, add z^2 to both sides of the equation $z^2 = x^2 + y^2$. We get $2z^2 = x^2 + y^2 + z^2 = \rho^2$. Since $z = \rho\cos(\phi)$, we get $2\rho^2\cos^2(\phi) = \rho^2$. Solving for ϕ, we obtain $\phi = \pi/4$ or $\phi = 3\pi/4$ for the equation of the cone.

   To find the volume inside, we evaluate $V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\cos(\phi)} \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi = \pi$.

5. Compute the integral $\oint_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$

   a) Over any closed curve C not enclosing the origin. (10 points)

   b) Over the circle of radius a centered at $(0, 0)$. (9 points)

   Solution:

a) By Green's theorem, we have $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int\int_R \frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) - \frac{\partial}{\partial y}(\frac{-y}{x^2+y^2})\, dA = \int\int_D (\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2})\, dA = \int\int_D 0\, dA = 0$

b) Notice that we can't apply Green's Theorem here, so we will compute the line integral directly. Let C_a denote the circle radius a centered at $(0,0)$. We have $x = a\cos(t)$, $y = a\sin(t)$, $dx = -a\sin(t)\, dt$ and $dy = a\cos(t)\, dt$.

$\int_{C_a} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} \frac{a^2\sin^2(t)+a^2\cos^2(t)}{a^2}dt = \int_0^{2\pi} 1\, dt = 2\pi.$

6. Find the area enclosed by the curve $\mathbf{r}(t) = <\cos^3(t), \sin^3(t)>$, where t is in $[0, 2\pi]$, using Green's theorem. (16 points)

Solution: The area is given by

$Area = \frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\int_0^{2\pi}(x\frac{dy}{dt} - y\frac{dx}{dt})dt = \frac{1}{2}\int_0^{2\pi}\cos^3(t)(3\sin^2(t)\cos(t)) - \sin^3(t)(-3\cos^2(t)\sin(t))dt$
$= \frac{3}{2}\int_0^{2\pi}\cos^2(t)\sin^2(t)\, dt = \frac{3}{8}\int_0^{2\pi}\sin^2(2t)\, dt = 3\pi/8.$