MULTIVARIABLE CALCULUS
Sample Midterm Problems

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1. Let $P(1,0,-3)$, $Q(0,-2,-4)$ and $R(4,1,6)$ be points.

   (a) Find the equation of the plane through the points P, Q and R.

   (b) Find the area of the triangle with vertices P, Q and R.

   Solution:

   The vector $\vec{PQ} \times \vec{PR} = \ < -1,-2,-1 > \times < 3,1,9 > \ = \ < -17,6,5 >$ is the normal vector of this plane, so equation of the plane is $-17(x-1)+6(y-0)+5(z+3)=0$, which simplifies to $17x-6y-5z=32$.

   $Area = \frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}| < -1,-2,-1 > \times < 3,1,9 > | = \frac{\sqrt{350}}{2}$

2. Let $f(x,y)=(x-y)^3+2xy+x^2-y$. Find the linear approximation $L(x,y)$ near the point $(1,2)$.

   Solution: $f_x = 3x^2-6xy+3y^2+2y+2x$ and $f_y = -3x^2+6xy-3y^2+2x-1$, so $f_x(1,2)=9$ and $f_y(1,2)=-2$. Then the linear approximation of f at $(1,2)$ is given by $L(x,y)=f(1,2)+f_x(1,2)(x-1)+f_y(1,2)(y-2)=2+9(x-1)+(-2)(y-2)$.

3. Find the distance between the parallel planes $x+2y-z=-1$ and $3x+6y-3z=3$.

   Use the following formula to find the distance between the given parallel planes $D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$. Use a point from the second plane (for example $(1,0,0)$) as (x_0,y_0,z_0) and the coefficents from the first plane $a=1$, $b=2$, $c=-1$, and $d=1$. We compute $D=\frac{|1\cdot 1+2\cdot 0+(-1)\cdot 0+1|}{\sqrt{1^2+2^2+(-1)^2}}=\frac{2}{\sqrt{6}}$

4. Find the following limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy+y^2}{x^2+y^2}$$

   Solution:

   First, we will use the path $y=x$. Along this path we have,

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy+y^2}{x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2-x^2+x^2}{x^2+x^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} = 1/2$$

   Now, let's try the path $y=0$. Along this path the limit becomes,

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy+y^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$$

We have two paths that give different values for the given limit and so the limit doesn't exisit.

5. Find the directional derivative of the function $f(x, y, z) = xyz$ in the direction of vector $\mathbf{v} = <\mathbf{5}, \mathbf{-3}, \mathbf{2}>$.

   Solution: $f_x = yz$, $f_y = xz$, and $f_z = xy$. First, we find the unit vector in the direction of vector $\mathbf{v}$: $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} =< \frac{\mathbf{5}}{\sqrt{\mathbf{38}}}, \frac{\mathbf{-3}}{\sqrt{\mathbf{38}}}, \frac{\mathbf{2}}{\sqrt{\mathbf{38}}} >$.

   $\mathbf{D_u f(x, y, z)} = \nabla \mathbf{f} \cdot \mathbf{u} =< \mathbf{yz}, \mathbf{xz}, \mathbf{xy} > \cdot < \frac{\mathbf{5}}{\sqrt{\mathbf{38}}}, \frac{\mathbf{-3}}{\sqrt{\mathbf{38}}}, \frac{\mathbf{2}}{\sqrt{\mathbf{38}}} > = \frac{5}{\sqrt{38}}yz - \frac{3}{\sqrt{38}}xz + \frac{2}{\sqrt{38}}xy.$

6. Find the equation of the tangent plane to the surface $z = 4x^3y^2 + 2y$ at point $(1, -2, 12)$.

   Solution: Since $f(x, y) = 4x^3y^2 + 2y$, we have $f_x(x, y) = 12x^2y^2$ and $f_y(x, y) = 8x^3y + 2$.

   Now plug in $x = 1$ and $y = -2$, we obtain $f_x(1, -2) = 48$ $f_y(1, -2) = -14$.

   Thus, the tangent plane has normal vector $\mathbf{n} =< \mathbf{48}, \mathbf{-14}, \mathbf{-1} >$ at $(1, 2, 12)$ and the equation of the tangent plane is given by $48(x - 1) - 14(y - (-2)) - (z - 12) = 0$

   Simplifying, we obtain $48x - 14y - z = 64$

7. Find all the second order partial derivatives for $f(x, y) = sin(2x) - x^2e^{3y} + y^2$. Verify that the conclusion of Clairut's Theorem holds, that is $f_{xy} = f_{yx}$.

   Solution: $f_x = 2cos(2x) - 2xe^{3y}$, so $f_{xx} = -4sin(2x) - 2e^{3y}$ and $f_{xy} = -6xe^{3y}$. $f_y = -3x^2e^{3y} + 2y$, so $f_{yy} = -9x^2e^{3y} + 2$, and $f_{yx} = -6xe^{3y}$. Thus $f_{xy} = f_{yx}$.

8. Find parametric equations for the line through $A = (1, 2, 3)$ and $B = (0, 2, 2)$. Find the intersection between that line and the sphere of equation $x^2 + y^2 + z^2 = 8$.

   Solution: $\vec{AB} =< 0 - 1, 2 - 2, 2 - 3 >=< -1, 0, -1 >$. Letting $P_0 = (1, 2, 3)$, parametric equations are $x = 1 + (-1)t = 1 - t$, $y = 2$, and $z = 3 + (-1)t = 3 - t$. To find the intersection points of this line and the sphere, plug in $x = 1 - t$, $y = 2$ and $z = 3 - t$ into the equation $x^2 + y^2 + z^2 = 8$. Solving for t, we find that $t = 1$ or $t = 3$. So the intersection points are $(0, 2, 2)$ and $(-2, 2, 0)$.

9. Let $\mathbf{u} = < \mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3} >$ and $\mathbf{v} = < \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} >$ be any two vectors in space. Show the following identity that relates the cross product and the dot product: $|\mathbf{u} \times \mathbf{v}|^\mathbf{2} + |\mathbf{u} \cdot \mathbf{v}|^\mathbf{2} = |\mathbf{u}|^\mathbf{2}|\mathbf{v}|^\mathbf{2}$

   Solution: Using the formulas $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin(\theta)$ and $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$, we have $|\mathbf{u} \times \mathbf{v}|^\mathbf{2} + |\mathbf{u} \cdot \mathbf{v}|^\mathbf{2} = |\mathbf{u}|^\mathbf{2}|\mathbf{v}|^\mathbf{2} \sin^\mathbf{2}(\theta) + |\mathbf{u}|^\mathbf{2}|\mathbf{v}|^\mathbf{2} \cos^\mathbf{2}(\theta) = |\mathbf{u}|^\mathbf{2}|\mathbf{v}|^\mathbf{2}(\sin^\mathbf{2}(\theta) + \cos^\mathbf{2}(\theta)) = |\mathbf{u}|^\mathbf{2}|\mathbf{v}|^\mathbf{2}$

10. Find an equation of the plane that passes through the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$ and passes through the point $(-1, 2, 1)$.

    Solution: A direction vector of this line can be found by calculating the cross product $< 1, 1, -1 > \times < 2, -1, 3 > = < 2, -5, -3 >$. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(-1, 2, 1)$ in the plane. Setting $z = 0$, the equations of the plane reduces to $x + y = 2$ and $2x - y = 1$

with simultaneous solution $x = 1$ and $y = 1$. So a point on the line is $(1,1,0)$ and another vector parallel to the plane is $< -2,1,1 >$. Then a normal vector to the plane is $\mathbf{n} =< 2,-5,-3 > \times < -2,1,1 > \;=\; < -2,4,-8 >$

Using the given point $(-1,2,1)$, we find the equation of the plane: $-2(x+1)+4(y-2)-8(z-1) = 0$

11. Determine if the three vectors $\mathbf{u} = < \mathbf{1}, \mathbf{2}, \mathbf{-3} >$, $\mathbf{v} = < \mathbf{2}, \mathbf{-1}, \mathbf{4} >$ and $\mathbf{w} = < \mathbf{1}, \mathbf{-1}, \mathbf{2} >$ lie in the same plane or not.

   Solution:

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 2 & 3 \\ -2 & -1 & 4 \\ 1 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = 1\cdot 2 - 2\cdot 0 + (-3)\cdot(-1) = 5$$

   , which says that the volume of the parallelepiped determined by $\mathbf{u}$,$\mathbf{v}$ and $\mathbf{w}$ is not equal 0, and thus these vectors don't lie in the same plane.

12. What is the angle between the two planes $\mathbf{x}+\mathbf{y} = \mathbf{0}$ and $\mathbf{y}-\mathbf{z} = \mathbf{2}$?

   Solution:

   The normal vectors for these two planes are $\mathbf{n_1} = < \mathbf{1}, \mathbf{1}, \mathbf{0} >$ and $\mathbf{n_2} = < \mathbf{0}, \mathbf{1}, \mathbf{-1} >$. The angle between them is given by

$$\cos(\theta) = \frac{\mathbf{n_1}\cdot\mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|} = \frac{1}{2}$$

   Thus $\theta = \pi/3$.

13. Identify the quadric surfaces given by equations $\mathbf{x^2}+\mathbf{y^2}-\mathbf{z^2} = \mathbf{1}$ and $\mathbf{x^2}+\mathbf{y^2}+\mathbf{z^2} = \mathbf{10}$. Find the equation and sketch the intersection of these surfaces.

   Solution:

   The quadric surface given by equation $\mathbf{x^2}+\mathbf{y^2}-\mathbf{z^2} = \mathbf{1}$ is a one-sheeted hyperboloid.

   The quadric surface given by equation $\mathbf{x^2}+\mathbf{y^2}+\mathbf{z^2} = \mathbf{10}$ is a sphere of radius $\sqrt{10}$.

   The equation of the intersection: $\mathbf{x^2}+\mathbf{y^2} = \mathbf{11/2}, \mathbf{z} = \frac{\mathbf{3}}{\sqrt{\mathbf{2}}}$ and $\mathbf{x^2}+\mathbf{y^2} = \mathbf{11/2}, \mathbf{z} = -\frac{\mathbf{3}}{\sqrt{\mathbf{2}}}$. The intersectionis the union of two circles.

14. Let $\mathbf{S}$ be the surface consisting of all points in space whose distance to the point $(0,-2,0)$ is same as the distance to the point $(2,2,2)$. Find an equation for $\mathbf{S}$ and sketch the surface $\mathbf{S}$.

   Solution: Let $P = (x,y,z)$ be an arbitrary point equidistant from $(0,-2,0)$ and $(2,2,2)$. Then the distance from P to $(0,-2,0)$ is $\sqrt{x^2+(y+2)^2+z^2}$ and the distance from P to $(2,2,2)$ is $\sqrt{(x-2)^2+(y-2)^2+(z-2)^2}$.

   So $\sqrt{x^2+(y+2)^2+z^2} = \sqrt{(x-2)^2+(y-2)^2+(z-2)^2}$. Which simplifies to $x+2y+z = 2$. Thus, the surface S is the plane $x+2y+z = 2$.