MULTIVARIABLE CALCULUS
Sample Midterm Problems

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1. Let $g(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$. Find the critical points of $g(x,y)$. Use the second derivative test to determine the local maximum, local minimum, and saddle points of $g(x,y)$.

   Solution: We have $g_x = 6xy - 6x$ and $g_y = 3x^2 + 3y^2 - 6y$. Critical points will be a common solution of the equations $g_x = 0$ and $g_y = 0$. For $g_x = 0$, we must have $6x(y-1) = 0$, so $x = 0$ or $y = 1$. If $x = 0$, $g_y = 3y^2 - 6y = 0$, thus $y = 0$ or $y = 2$. If $y = 1$, $g_y = 3x^2 - 3 = 0$, thus $x = 1$ or $x = -1$. The second-order partial derivatives are: $g_{xx} = 6y - 6$, $g_{xy} = g_{yx} = 6x$, and $g_{yy} = 6y - 6$. By the Second Derivative Test (see textbook, page 924), we conclude that $g(x,y)$ has a local max at (0,0) , a local min at (0,2), and the saddle points at $(1,1)$ and $(-1,1)$.

2. Use Lagrange multipliers to find the maximum value of $h(x,y,z) = xy^2z^3$ subject to the constraint $x + y + z = 6$, $x > 0$, $y > 0$, $z > 0$.

   Solution: We have $\nabla h = (y^2z^3, 2xyz^3, 3xy^2z^2)$ and $\nabla g = (1,1,1)$, where $g(x,y,z) = x + y + z$. Solve $\nabla h = \lambda \nabla g$: $y^2z^3 = 2xyz^3 = 3xy^2z^2 = \lambda$. Since $x > 0$, $y > 0$, and $z > 0$, we get $y = 2x$ and $z = 3x$. Now, plugging these into $x + y + z = 6$ gives, $x = 1$, $y = 2$, and $z = 3$. By simple checking, we see that h has a maximum at $(1,2,3)$ and the maximum value of h is $h(1,2,3) = 108$.

3. Find the mass and center of mass of the lamina that occupies the region D bounded by the parabolas $y = x^2$ and $x = y^2$ if the density function is $\rho = \sqrt{x}$.

   Solution:

   $m = \int_0^1 \int_{x^2}^{\sqrt{x}} \sqrt{x} dy dx = \int_0^1 \sqrt{x}(\sqrt{x} - x^2) dx = \int_0^1 (x - x^{5/2}) dx = [\frac{1}{2}x^2 - \frac{2}{7}x^{7/2}]_0^1 = \frac{3}{14}$

   $M_y = \int_0^1 \int_{x^2}^{\sqrt{x}} x\sqrt{x} dy dx = \int_0^1 (x^2 - x^{7/2}) dx = [\frac{1}{3}x^3 - \frac{2}{9}x^{9/2}]_0^1 = \frac{1}{9}$

   $M_x = \int_0^1 \int_{x^2}^{\sqrt{x}} y\sqrt{x} dy dx = \int_0^1 (x^{3/2} - x^{9/2}) dx = \frac{1}{2}[\frac{2}{5}x^{5/2} - \frac{2}{11}x^{11/2}]_0^1 = \frac{6}{55}$

   Hence, $(\bar{x}, \bar{y}) = (\frac{\frac{1}{9}}{\frac{3}{14}}, \frac{\frac{6}{55}}{\frac{3}{14}}) = (\frac{14}{27}, \frac{28}{55})$.

4. Sketch the region D of integration in the following double integral $\int_0^3 \int_{y^2}^9 ycos(x^2) dx dy$. Evaluate the given integral by reversing the order of integration.

   Solution:

   $\int_0^3 \int_{y^2}^9 ycos(x^2) dx dy = \int_0^9 \int_0^{\sqrt{x}} ycos(x^2) dy dx = \int_0^9 [\frac{1}{2}y^2 cos(x^2)]_0^{\sqrt{x}} dy = \int_0^9 \frac{1}{2} xcos(x^2) dx = [\frac{1}{4}sin(x^2)]_0^9 = \frac{1}{4}\sin(81)$.

5. Let R be the square $-3 \le x \le 3, -3 \le y \le 3$ in the (x,y)-plane. If $f(x,y)$ is a continuous function, and satisfies $2 \le f(x,y) \le 9 + x + y$, what does this tell you about the value of $\int\int_R f(x,y)\,dA$?

   Solution:

   Since $2 \le f(x,y) \le 9 + x + y$, we have $\int\int_R 2\,dA \le \int\int_R f(x,y)\,dA \le \int\int_R (9+x+y)\,dA$.

   Since R has area 36, and $\int\int_R (9+x+y)\,dA = \int_{-3}^{3}\int_{-3}^{3}(9+x+y)dydx = \int_{-3}^{3}[9y + xy + \frac{1}{2}y^2]_{-3}^{3}\,dx = \int_{-3}^{3}(54+6x)\,dx = [54x+3x^2]_{-3}^{3} = 324$, we have $72 \le \int\int_R f(x,y)\,dA \le 324$.

6. Let D be the circular disk of radius R and center $(0,0)$ in the (x,y)-plane. Find the double integral $\int\int_D e^{x^2+y^2}\,dA$.

   Solution: The region D is more easily described by polar coordinates: $D = \{(r,\theta)|0 \le r \le R, 0 \le \theta \le 2\pi\}$. $\int\int_D e^{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^R e^{r^2}rdrd\theta = 2\pi\int_0^R e^{r^2}rdr = \pi(e^{R^2}-1)$.

7. Let D be the region bounded by $y = \sqrt{x}$ and $y = x^3$. Find the double integral $\int\int_D 16xy - 4y^3\,dA$.

   Solution:

   $\int\int_D 16xy - 4y^3\,dA = \int_0^1\int_{x^3}^{\sqrt{x}}(16xy-4y^3)dydx = \int_0^1[8xy^2-y^4]_{x^3}^{\sqrt{x}}dx = \int_0^1(7x^2-8x^7+x^{12})dx = \frac{55}{39}$.