MULTIVARIABLE CALCULUS
Sample Midterm Problems

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1. Convert the point $P = (4, \pi/3, 4)$ from cylindrical to spherical coordinates. Plot the point P using the spherical coordinate system.

   Solution: Notice that the angle θ will not change, so we have $\theta = \pi/3$. Since $\rho^2 = r^2 + z^2 = 4^2 + 4^2 = 32$, we get $\rho = 4\sqrt{2}$. $\tan\phi = \frac{r}{z} = 1$. Since $0 \leq \phi \leq \pi$, we have $\phi = \pi/4$. Thus the spherical coordinates of the point P is given by $(4\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4})$.

2. Use spherical coordinates to evaluate $\int\int\int_E 3e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$ where E is the region inside the sphere $x^2 + y^2 + z^2 = 2$ in the first octant.

   Solution: The region E in spherical coordinates is given by $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq \pi/2$ and $0 \leq \rho \leq \sqrt{2}$. Thus we need to compute the following integral

   $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\sqrt{2}} 3e^{\rho^3}\rho^2\sin(\phi)\, d\rho\, d\theta\, d\phi = \int_0^{\pi/2}\int_0^{\pi/2} e^{\rho^3}\sin(\phi)|_0^{\sqrt{2}}\, d\theta\, d\phi =$
   $\int_0^{\pi/2}\int_0^{\pi/2}(e^{2\sqrt{2}} - 1)\sin(\phi)\, d\theta\, d\phi = \frac{\pi(e^{2\sqrt{2}}-1)}{2}$

3. Let R be the unit square. Use Green's theorem to evaluate the line integral $\oint_{\partial R} y^2dx + x^2dy$

   Solution: By Green's theorem, we have $\oint_{\partial R} y^2dx + x^2dy = \int\int_R \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\, dA = \int\int_R (2x - 2y)\, dA = \int_0^1\int_0^1 (2x - 2y)dy\, dx = 0$

4. Find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside of the cone $z^2 = x^2 + y^2$.

   Solution: We will use the spherical coordinates to compute the volume. The equation of the sphere becomes $\rho = \sqrt{2}$. To convert the equation of the cone, add z^2 to both sides of the equation $z^2 = x^2 + y^2$. We get $2z^2 = x^2 + y^2 + z^2 = \rho^2$. Since $z = \rho\cos(\phi)$, we get $2\rho^2\cos^2(\phi) = \rho^2$. Solving for ϕ, we obtain $\phi = \pi/4$ or $\phi = 3\pi/4$ for the equation of the cone.

   To find the volume, we evaluate $V = \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^{\sqrt{2}} \rho^2\sin(\phi)\, d\rho\, d\theta\, d\phi =$
   $2\pi(\rho^3/3)|_0^{\sqrt{2}}(-cos(\phi))|_{\pi/4}^{3\pi/4} = 8\pi/3.$

5. Consider the double integral $\int\int_D \frac{x-y}{(x+2y)^2}\, dA$ where D is the region bounded by the following four lines: $x - y = 0$, $x - y = 3$, $x + 2y = 2$, and $x + 2y = 6$. Evaulate the integral using an appropriate change of variables.

   Solution: Use the change of variables $u = x + 2y$ and $v = y - x$. Solving for x and y, we obtain the inverse transformation given by $x = \frac{u-2v}{3}$ and $y = \frac{u+v}{3}$. First, note that the image of the region D under the transformation $u = x + 2y$, $v = y - x$ is a rectangle with vertices at $(2, -3)$, $(6, -3)$, $(6, 0)$, and $(2, 0)$. Next, we compute the

Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = 1/3$. Using the change of variable formula for double integrals, we have $\int\int_D \frac{x-y}{(x+2y)^2}\, dA = \int_{-3}^{0} \int_2^6 \frac{-v}{u^2} |\frac{\partial(x,y)}{\partial(u,v)}| du\, dv = 1/2$.

6. Find the area enclosed by the curve $\mathbf{r}(t) = < \cos^2(t), \cos(t)\sin(t) >$, where t is in $[0,\pi]$, using Green's theorem.

   Solution: The area is given by

   $Area = \frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\int_0^\pi (x\frac{dy}{dt} - y\frac{dx}{dt})dt = \frac{1}{2}\int_0^\pi (\cos^2(t)(-\sin(t)\sin(t) + \cos(t)\cos(t)) - \cos(t)\sin(t)(-2\cos(t)\sin(t)))dt = \frac{1}{2}\int_0^\pi \cos^2(t)\, dt = \pi/4$.

7. Explain why the line integral $\int_C xdx + ydy + zdz$ is independent of path. Calculate the integral along two different paths from $(0,0,0)$ to $(1,1,1)$.

   Solution: Note that the vector field $F = < x, y, z >$ is the gradient field of the fuction $f(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2)$. Using the Fundamental Theorem for Line Integrals, we show that the given line integral is independent of path.

   Choose the first path from $(0,0,0)$ to $(1,1,1)$ by $\mathbf{r}(t) = < t, t, t >$. Denote this path by C_1. The line integral over C_1 is

   $\int_{C_1} xdx + ydy + zdz = \int_0^1 (tdt + tdt + tdt) = 3\int_0^1 t\, dt = 3/2$.

   Choose the second path from $(0,0,0)$ to $(1,1,1)$ by $\mathbf{r}(t) = < t, t^2, t^3 >$. Denote this path by C_2. The line integral over C_2 is

   $\int_{C_2} xdx + ydy + zdz = \int_0^1 (tdt + t^2dt^2 + t^3dt^3) = \int_0^1 (t + 2t^3 + 3t^5)\, dt = 3/2$.