

SMALL EXOTIC 4-MANIFOLDS

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Dedicated to Professor Ronald J. Stern on the occasion of his sixtieth birthday

ABSTRACT. In this article, we construct the first example of a simply-connected minimal symplectic 4-manifold that is homeomorphic but not diffeomorphic to $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$. We also construct the first exotic minimal *symplectic* $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$.

0. INTRODUCTION

Over the past several years, there has been a considerable amount of progress in the discovery of exotic smooth structures on simply-connected 4-manifolds with small Euler characteristic. In early 2004, Jongil Park [P2] has constructed the first example of exotic smooth structure on $\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$, i.e. 4-manifold homeomorphic but not diffeomorphic to $\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$. Later that year, András Stipsicz and Zoltán Szabó used a similar technique to construct an exotic smooth structure on $\mathbb{C}\mathbb{P}^2 \# 6\overline{\mathbb{C}\mathbb{P}^2}$ [SS1]. Then Fintushel and Stern [FS3] introduced a new technique, the double node surgery, which demonstrated that in fact $\mathbb{C}\mathbb{P}^2 \# k\overline{\mathbb{C}\mathbb{P}^2}$, $k = 6, 7$, and 8 have infinitely many distinct smooth structures. Using the double node surgery technique [FS3], Park, Stipsicz, and Szabó constructed infinitely many smooth structures on $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$ [PSS]. The examples in [PSS] are not known if symplectic. Based on similar ideas, Stipsicz and Szabó constructed the exotic smooth structures on $3\mathbb{C}\mathbb{P}^2 \# k\overline{\mathbb{C}\mathbb{P}^2}$ for $k = 9$ [SS2] and Park for $k = 8$ [P3]. In this article, we construct an exotic smooth structure on $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$. We also construct an exotic *symplectic* $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$, the first known such symplectic example.

Our approach is different from the above constructions in the sense that we do not use any rational-blowdown surgery ([FS1], [P1]). Also,

Date: November 15, 2006.

2000 Mathematics Subject Classification. Primary 57R55; Secondary 57R17, 57M05.

in contrary to the previous constructions, we use non-simply connected building blocks ([A], [M]) to produce the simply-connected examples. The main surgery technique used in our construction is the symplectic fiber sum operation ([G], [MW]) along the genus two surfaces. Our results can be stated as follows.

Theorem 0.1. *There exist a smooth closed simply-connected minimal symplectic 4-manifold X that is homeomorphic but not diffeomorphic to $3\mathbb{C}\mathbb{P}^2\#7\overline{\mathbb{C}\mathbb{P}^2}$.*

Theorem 0.2. *There exist a smooth closed simply-connected minimal symplectic 4-manifold Y which is homeomorphic but not diffeomorphic to the rational surface $\mathbb{C}\mathbb{P}^2\#5\overline{\mathbb{C}\mathbb{P}^2}$.*

This article is organized as follows. The first two sections give a quick introduction to Seiberg-Witten invariants and a fiber sum operation. In the third section, we review the symplectic building blocks for our construction. Finally, in sections 4 and 5, we construct minimal symplectic 4-manifolds X and Y homeomorphic but not diffeomorphic to $3\mathbb{C}\mathbb{P}^2\#7\overline{\mathbb{C}\mathbb{P}^2}$ and $\mathbb{C}\mathbb{P}^2\#5\overline{\mathbb{C}\mathbb{P}^2}$, respectively.

Acknowledgments: I would like to thank John Etnyre, Ron Stern and András Stipsicz for their interest in this work, and for their encouragement. Also, I am grateful to B. Doug Park for the comments on the first draft of this article, kindly pointing out some errors in the fundamental group computations, and for the corrections. Finally, I wish to thank the referee for many helpful suggestions which improved the exposition of this article. This work is partially supported by NSF grant FRG-0244663.

1. SEIBERG-WITTEN INVARIANTS

In this section, we briefly recall the basics of Seiberg-Witten invariants introduced by Seiberg and Witten. Seiberg-Witten invariant of a smooth closed oriented 4-manifold X with $b_2^+(X) > 1$ is an integer valued function which is defined on the set of $spin^c$ structures over X [W]. For simplicity, we assume that $H_1(X, \mathbf{Z})$ has no 2-torsion. Then there is a one-to-one correspondence between the set of $spin^c$ structures over X and the set of characteristic elements of $H^2(X, \mathbf{Z})$.

In this set up, we can view the Seiberg-Witten invariant as an integer valued function

$$SW_X : \{k \in H^2(X, \mathbf{Z}) \mid k \equiv w_2(TX) \pmod{2}\} \longrightarrow \mathbf{Z}.$$

The Seiberg-Witten invariant SW_X is a diffeomorphism invariant. We call β a *basic class* of X if $\text{SW}_X(\beta) \neq 0$. It is a fundamental fact that the set of basic classes is finite. Also, if β is a basic class, then so is $-\beta$ with

$$\text{SW}_X(-\beta) = (-1)^{(e+\sigma)(X)/4} \text{SW}_X(\beta)$$

where $e(X)$ is the Euler characteristic and $\sigma(X)$ is the signature of X .

Theorem 1.1. [T] *Suppose that (X, ω) is a closed symplectic 4-manifold with $b_2^+(X) > 1$ and the canonical class K_X . Then $\text{SW}_X(\pm K_X) = \pm 1$.*

2. FIBER SUM

Definition 2.1. Let X and Y be closed, oriented, smooth 4-manifolds each containing a smoothly embedded surface Σ of genus $g \geq 1$. Assume Σ represents a homology class of infinite order and has self-intersection zero in X and Y , so that there exist a tubular neighborhood, say $\nu\Sigma \cong \Sigma \times D^2$, in both X and Y . Using an orientation-reversing and fiber-preserving diffeomorphism $\psi : S^1 \times \Sigma \rightarrow S^1 \times \Sigma$, we can glue $X \setminus \nu\Sigma$ and $Y \setminus \nu\Sigma$ along the boundary $\partial(\nu\Sigma) \cong \Sigma \times S^1$. This new oriented smooth 4-manifold $X \#_\psi Y$ is called a *generalized fiber sum* of X and Y along Σ , determined by ψ .

Definition 2.2. Let $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of a closed oriented smooth 4-manifold X , respectively. We define

$$c_1^2(X) := 2e(X) + 3\sigma(X), \quad \chi_h(X) := \frac{e(X) + \sigma(X)}{4}.$$

In the case that X is a complex surface, then $c_1^2(X)$ and $\chi_h(X)$ are the self-intersection of the first Chern class $c_1(X)$ and the holomorphic Euler characteristic, respectively.

Lemma 2.3. *Let X and Y be closed, oriented, smooth 4-manifolds containing an embedded surface Σ of self-intersection 0. Then*

$$\begin{aligned} c_1^2(X \#_\psi Y) &= c_1^2(X) + c_1^2(Y) + 8(g-1), \\ \chi_h(X \#_\psi Y) &= \chi_h(X) + \chi_h(Y) + (g-1), \end{aligned}$$

where g is the genus of the surface Σ .

Proof. The above simply follows from the well-known formulas

$$e(X \#_\psi Y) = e(X) + e(Y) - 2e(\Sigma), \quad \sigma(X \#_\psi Y) = \sigma(X) + \sigma(Y).$$

□

If X, Y are symplectic manifolds and Σ is an embedded symplectic submanifold in X and Y , then according to theorem of Gompf [G] $X \#_{\psi} Y$ admits a symplectic structure.

We will use the following theorem of M. Usher [U1] to show that the symplectic manifolds constructed in Sections 4 and 5 are minimal. Here we slightly abuse the above notation for the fiber sum.

Theorem 2.4. [U1] (**Minimality of Symplectic Sums**) *Let $Z = X_1 \#_{F_1=F_2} X_2$ be symplectic fiber sum of manifolds X_1 and X_2 . Then:*

(i) *If either $X_1 \setminus F_1$ or $X_2 \setminus F_2$ contains an embedded symplectic sphere of square -1 , then Z is not minimal.*

(ii) *If one of the summands X_i (say X_1) admits the structure of an S^2 -bundle over a surface of genus g such that F_i is a section of this fiber bundle, then Z is minimal if and only if X_2 is minimal.*

(iii) *In all other cases, Z is minimal.*

3. BUILDING BLOCKS

The building blocks for our construction will be:

(i) The manifold $T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ equipped with the genus two Lefschetz fibration of Matsumoto [M].

(ii) The symplectic manifolds X_K and Y_K [A]. For the convenience of the reader, we recall the construction in [A].

3.1. Matsumoto fibration. First, recall that the manifold $Z = T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ can be described as the double branched cover of $S^2 \times T^2$ where the branch set $B_{2,2}$ is the union of two disjoint copies of $S^2 \times \{\text{pt}\}$ and two disjoint copies of $\{\text{pt}\} \times T^2$. The branch cover has 4 singular points, corresponding to the number of the intersections points of the horizontal lines and the vertical tori in the branch set $B_{2,2}$. After desingularizing the above singular manifold, one obtains $T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$. The vertical fibration of $S^2 \times T^2$ pulls back to give a fibration of $T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ over S^2 . A generic fiber of the vertical fibration is the double cover of T^2 , branched over 2 points. Thus a generic fiber will be a genus two surface. According to Matsumoto [M], this fibration can be perturbed to be a Lefschetz fibration over S^2 with the global monodromy $(\beta_1\beta_2\beta_3\beta_4)^2 = 1$, where the curves $\beta_1, \beta_2, \beta_3$, and β_4 are shown in Figure 1.

FIGURE 1. Dehn Twists for Matsumoto fibration

Let us denote the regular fiber by Σ'_2 and the images of standard generators of the fundamental group of Σ'_2 as a_1 , b_1 , a_2 , and b_2 . Using the homotopy exact sequence for a Lefschetz fibration,

$$\pi_1(\Sigma'_2) \longrightarrow \pi_1(Z) \longrightarrow \pi_1(S^2)$$

we have the following identification of the fundamental group of Z [OS]:

$$\pi_1(Z) = \pi_1(\Sigma'_2) / \langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle.$$

- (1) $\beta_1 = b_1 b_2,$
- (2) $\beta_2 = a_1 b_1 a_1^{-1} b_1^{-1} = a_2 b_2 a_2^{-1} b_2^{-1},$
- (3) $\beta_3 = b_2 a_2 b_2^{-1} a_1,$
- (4) $\beta_4 = b_2 a_2 a_1 b_1,$

Hence $\pi_1(Z) = \langle a_1, b_1, a_2, b_2 \mid b_1 b_2 = [a_1, b_1] = [a_2, b_2] = a_1 a_2 = 1 \rangle.$

Note that the fundamental group of $T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ is $\mathbf{Z} \oplus \mathbf{Z}$, generated by two of these standard generators (say a_1 and b_1). The other two generators a_2 and b_2 are the inverses of a_1 and b_1 in the fundamental group. Also, the fundamental group of the complement of $\nu\Sigma'_2$ is $\mathbf{Z} \oplus \mathbf{Z}$. It is generated by a_1 and b_1 . The normal circle $\lambda' = pt \times \partial D^2$ to Σ'_2 can be deformed using one of the exceptional spheres, thus is trivial in $\pi_1(T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2} \setminus \nu\Sigma'_2) = \mathbf{Z} \oplus \mathbf{Z}$.

Lemma 3.1. $c_1^2(Z) = -4$, $\sigma(Z) = -4$, and $\chi_h(Z) = 0$.

Proof. We have $c_1^2(Z) = c_1^2(T^2 \times S^2) - 4 = -4$, $\sigma(Z) = \sigma(T^2 \times S^2) - 4 = -4$, and $\chi_h(Z) = \chi_h(T^2 \times S^2) = 0$. \square

Note that this Lefschetz fibration can be given a symplectic structure. This means that Z admits a symplectic structure such that the regular fibers are symplectic submanifolds. We consider such a symplectic structure on Z .

3.2. Symplectic 4-manifolds cohomology equivalent to $S^2 \times S^2$. Our second building block will be X_K , the symplectic cohomology $S^2 \times S^2$ [A], or the symplectic manifold Y_K , an intermediate building block in that construction [A], (see also [FS2]). For the sake of completeness, the details of this construction are included below. We refer the reader to [A] for more details, and for the generalization of these symplectic building blocks.

Let K be a fibered knot of genus one (i.e., the trefoil or the figure eight knot) in S^3 and m be a meridional circle to K . We perform 0-framed surgery on K and denote the resulting 3-manifold by M_K . Since K is fibered and has genus one, it follows the 3-manifold M_K is a torus bundle over S^1 ; hence the 4-manifold $M_K \times S^1$ is a torus bundle over a torus. Furthermore, $M_K \times S^1$ admits a symplectic structure, and both the torus fiber and the torus section $T_m = m \times S^1 = m \times x$ are symplectically embedded and have a self-intersection zero. The first homology of $M_K \times S^1$ is generated by the standard first homology generators m and x of the torus section. On the other hand, the classes of circles γ_1 and γ_2 of the fiber F , coming from the Seifert surface, are trivial in homology. In addition, $M_K \times S^1$ is minimal symplectic, i.e., it does not contain symplectic -1 sphere.

We form a twisted fiber sum of two copies of the manifold $M_K \times S^1$, we identify the fiber F' of one fibration to the section T_m of other. Let Y_K denote the mentioned twisted fiber sum $Y_K = M_K \times S^1 \#_{F=T_m} M_K \times S^1$

S^1 . It follows from Gompf's theorem [G] that Y_K is symplectic, and by Usher's theorem in Section 2 that Y_K is minimal symplectic.

Let T_1 be the section of the first copy of $M_K \times S^1$ and T_2 be the fiber in the second copy. Then the genus two surface $\Sigma_2 = T_1 \# T_2$ symplectically embeds into Y_K and has self-intersection zero. Let X_K be a symplectic 4-manifold constructed as follows: Take two copies of Y_K , and form the fiber sum along the genus two surface Σ_2 using the special gluing diffeomorphism ϕ , the vertical involution of Σ_2 with two fixed points. Thus $X_K := Y_K \#_{\phi} Y_K$. Let m, x, γ_1 , and γ_2 denote the generators of $\pi_1(\Sigma_2)$ under the inclusion. The diffeomorphism $\phi : T_1 \# T_2 \rightarrow T_1 \# T_2$ of Σ_2 maps on the generators as follows: $\phi_*(m') = \gamma_1$, $\phi_*(x') = \gamma_2$, $\phi_*(\gamma_1') = m$, and $\phi_*(\gamma_2') = x$. In [A] we show that the manifold X_K has first Betti number zero, and has the integral cohomology of $S^2 \times S^2$. Furthermore, $H_2(X_K, \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$, where the basis for the second homology are the classes of $\Sigma_2 = S$ and the new genus two surface T resulting from the last fiber sum operation (two punctured genus one surfaces glue to form a genus two surface). Also, $S^2 = T^2 = 0$ and $S \cdot T = 1$. Furthermore, $c_1^2(X_K) = 8$, $\sigma(X_K) = 0$ and $\chi_h(X_K) = 1$. Since Y_K is minimal symplectic, it follows from Theorem 2.4 that X_K is minimal symplectic as well.

3.2.1. Fundamental Group of $M_K \times S^1$. We will assume that K is the trefoil knot. Let a, b , and c denote the Wirtinger generators of the trefoil. The knot group of the trefoil has the following presentations: $\pi_1(K) = \langle a, b, c \mid ab = bc, ca = ab \rangle = \langle a, b \mid aba = bab \rangle = \langle u, v \mid u^2 = v^3 \rangle$ where $u = bab$ and $v = ab$. The homotopy classes of the meridian and the longitude of the trefoil are given as follows: $m = uv^{-1} = b$ and $l = u^2(uv^{-1})^{-6} = ab^2ab^{-4}$ [BZ]. Also, the homotopy classes of γ_1 and γ_2 are given as follows: $\gamma_1 = a^{-1}b$ and $\gamma_2 = b^{-1}aba^{-1}$. Notice that the fundamental group of M_K , 0-surgery on the trefoil, is obtained from the knot group of the trefoil by adjoining the relation $l = u^2(uv^{-1})^{-6} = ab^2ab^{-4} = 1$. Thus, we have $\pi_1(M_K) = \langle u, v \mid u^2 = v^3, u^2(uv^{-1})^{-6} = 1 \rangle = \langle a, b \mid aba = bab, ab^2a = b^4 \rangle$, and $\pi_1(M_K \times S^1) = \langle a, b, x \mid aba = bab, ab^2a = b^4, [x, a] = [x, b] = 1 \rangle$.

3.2.2. Fundamental Group of Y_K . The next step is to take two copies of the manifold $M_K \times S^1$, and perform the fiber sum along symplectic tori. In the first copy of $M_K \times S^1$, we take a tubular neighborhood of the torus section T_m , remove it from $M_K \times S^1$, and denote the resulting manifold by C_S . In the second copy, we remove a tubular neighborhood of the fiber F and denote it by C_F . Notice that $C_S = M_K \times S^1 \setminus \nu T_m = (M_K \setminus \nu(m)) \times S^1$. We have

$\pi_1(C_S) = \pi_1(K) \oplus \langle x \rangle$ where x is the generator corresponding to the S^1 copy. Also using the above computation, we easily derive: $H_1(C_S) = H_1(M_K) = \langle m \rangle \oplus \langle x \rangle$.

To compute the fundamental group of the C_F , we will use the following observation: νF is the preimage of the small disk on $T_{m'} = m' \times y$. The elements y and $m' = d$ of the $\pi_1(C_F)$ do not commute anymore, but y still commutes with generators γ'_1 and γ'_2 . The fundamental group and the first homology of the C_F will be isomorphic to the following: $\pi_1(C_F) = \langle d, y, \gamma'_1, \gamma'_2 \mid [y, \gamma'_1] = [y, \gamma'_2] = [\gamma'_1, \gamma'_2] = 1, d\gamma'_1 d^{-1} = \gamma'_1 \gamma'_2, d\gamma'_2 d^{-1} = (\gamma'_1)^{-1} \rangle$ and $H_1(C_F) = \langle d \rangle \oplus \langle y \rangle$.

We use the Van Kampen's Theorem to compute the fundamental group of Y_K .

$$\begin{aligned} \pi_1(Y_K) &= \pi_1(C_F) *_{\pi_1(T^3)} \pi_1(C_S) = \langle d, y, \gamma'_1, \gamma'_2 \mid [y, \gamma'_1] = [y, \gamma'_2] = \\ &[\gamma'_1, \gamma'_2] = 1, d\gamma'_1 d^{-1} = \gamma'_1 \gamma'_2, d\gamma'_2 d^{-1} = (\gamma'_1)^{-1} \rangle_{\langle \gamma'_1=x, \gamma'_2=b, \lambda=\lambda \rangle} \langle a, b, x \mid aba = \\ &bab, [x, a] = [x, b] = 1 \rangle = \langle a, b, x, \gamma'_1, \gamma'_2, d, y \mid aba = bab, [x, a] = \\ &[x, b] = [y, \gamma'_1] = [y, \gamma'_2] = [\gamma'_1, \gamma'_2] = 1, d\gamma'_1 d^{-1} = \gamma'_1 \gamma'_2, d\gamma'_2 d^{-1} = \\ &(\gamma'_1)^{-1}, \gamma'_1 = x, \gamma'_2 = b, [\gamma'_1, \gamma'_2] = [d, y] \rangle. \end{aligned}$$

Inside Y_K , we can find a genus 2 symplectic submanifold Σ_2 which is the internal sum of a punctured fiber in C_S and a punctured section in C_F . The inclusion-induced homomorphism maps the standard generators of $\pi_1(\Sigma_2)$ to $a^{-1}b, b^{-1}aba^{-1}, d$ and y in $\pi_1(Y_K)$.

Lemma 3.2. [A] *There are nonnegative integers m and n such that*

$$\begin{aligned} \pi_1(Y_K \setminus \nu\Sigma_2) &= \langle a, b, x, d, y; g_1, \dots, g_m \mid aba = bab, \\ &[y, x] = [y, b] = 1, dxd^{-1} = xb, dbd^{-1} = x^{-1}, \\ &ab^2ab^{-4} = [d, y], r_1 = \dots = r_n = 1, r_{n+1} = 1 \rangle, \end{aligned}$$

where the generators g_1, \dots, g_m and relators r_1, \dots, r_n all lie in the normal subgroup N generated by the element $[x, b]$, and the relator r_{n+1} is a word in x, a and elements of N . Moreover, if we add an extra relation $[x, b] = 1$, then the relation $r_{n+1} = 1$ simplifies to $[x, a] = 1$.

Proof. This follows from Van Kampen's Theorem. Note that $[x, b]$ is a meridian of Σ_2 in Y_K . Hence setting $[x, b] = 1$ should turn $\pi_1(Y_K \setminus \nu\Sigma_2)$ into $\pi_1(Y_K)$. Also note that $[x, a]$ is the boundary of a punctured section in $C_S \setminus \nu(\text{fiber})$, and is no longer trivial in $\pi_1(Y_K \setminus \nu\Sigma_2)$. By setting $[x, b] = 1$, the relation $r_{n+1} = 1$ is to turn into $[x, a] = 1$.

It remains to check that the relations in $\pi_1(Y_K)$ other than $[x, a] = [x, b] = 1$ remain the same in $\pi_1(Y_K \setminus \nu\Sigma_2)$. By choosing a suitable point $\theta \in S^1$ away from the image of the fiber that forms part of Σ_2 , we obtain an embedding of the knot complement $(S^3 \setminus \nu K) \times \{\theta\} \hookrightarrow C_S \setminus \nu(\text{fiber})$.

This means that $aba = bab$ holds in $\pi_1(Y_K \setminus \nu\Sigma_2)$. Since $[\Sigma_2]^2 = 0$, there exists a parallel copy of Σ_2 outside $\nu\Sigma_2$, wherein the identity $ab^2ab^{-4} = [d, y]$ still holds. The other remaining relations in $\pi_1(Y_K)$ are coming from the monodromy of the torus bundle over a torus. Since these relations will now describe the monodromy of the punctured torus bundle over a punctured torus, they hold true in $\pi_1(Y_K \setminus \nu\Sigma_2)$. \square

3.2.3. Fundamental Group of X_K . Finally, we carry out the computations of the fundamental group and the first homology of X_K . Suppose that e, f, z, s , and t are the generators of the fundamental group in the second copy of Y_K corresponding to the generators a, b, x, d , and y as in above discussion. Our gluing map ϕ maps the generators of $\pi_1(\Sigma_2)$ as follows:

$$\phi_*(a^{-1}b) = s, \phi_*(b^{-1}aba^{-1}) = t, \phi_*(d) = e^{-1}f, \phi_*(y) = f^{-1}efe^{-1}.$$

By Van Kampen's Theorem and Lemma 3.2, we have

$$\begin{aligned} \pi_1(X_K) = \langle & a, b, x, d, y; e, f, z, s, t; g_1, \dots, g_m; h_1, \dots, h_m \mid \\ & aba = bab, [y, x] = [y, b] = 1, \\ & dxd^{-1} = xb, dbd^{-1} = x^{-1}, ab^2ab^{-4} = [d, y], \\ & r_1 = \dots = r_{n+1} = 1, r'_1 = \dots = r'_{n+1} = 1, \\ & efe = fef, [t, z] = [t, f] = 1, \\ & szs^{-1} = zf, sfs^{-1} = z^{-1}, ef^2ef^{-4} = [s, t], \\ & d = e^{-1}f, y = f^{-1}efe^{-1}, a^{-1}b = s, b^{-1}aba^{-1} = t, \\ & [x, b] = [z, f] \rangle, \end{aligned}$$

where the elements g_i, h_i ($i = 1, \dots, m$) and r_j, r'_j ($j = 1, \dots, n+1$) all are in the normal subgroup generated by $[x, b] = [z, f]$.

Notice that it follows from our gluing that the images of standard generators of the fundamental group of Σ_2 are $a^{-1}b, b^{-1}aba^{-1}, d$ and y in $\pi_1(X_K)$. By abelianizing $\pi_1(X_K)$, we easily see that $H_1(X_K, \mathbf{Z}) = 0$.

4. CONSTRUCTION OF AN EXOTIC $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$

In this section, we construct a simply-connected minimal symplectic 4-manifold X homeomorphic but not diffeomorphic to $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$. Using Seiberg-Witten invariants, we will distinguish X from $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$.

Our manifold X will be the symplectic fiber sum of X_K and $Z = T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ along the genus two surfaces Σ_2 and Σ'_2 . Recall that $a^{-1}b$, $b^{-1}aba^{-1}$, d , y , and $\lambda = \{\text{pt}\} \times S^1 = [x, b][z, f]^{-1}$ generate the inclusion-induced image of $\pi_1(\Sigma_2 \times S^1)$ inside $\pi_1(X_K \setminus \nu\Sigma_2)$. Let a_1 , b_1 , a_2 , b_2 and $\lambda' = 1$ be generators of $\pi_1(Z \setminus \nu\Sigma'_2)$ as in Section 3.1. We choose the gluing diffeomorphism $\psi : \Sigma_2 \times S^1 \rightarrow \Sigma'_2 \times S^1$ that maps the fundamental group generators as follows:

$$\psi_*(a^{-1}b) = a_2, \quad \psi_*(b^{-1}aba^{-1}) = b_2, \quad \psi_*(d) = a_1, \quad \psi_*(y) = b_1, \quad \psi_*(\lambda) = \lambda'.$$

λ and λ' above denote the meridians of Σ and Σ'_2 in X_K and Z , respectively.

It follows from Gompf's theorem [G] that $X = X_K \#_\psi (T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2})$ is symplectic.

Lemma 4.1. *X is simply connected.*

Proof. By Van Kampen's theorem, we have

$$\pi_1(X) = \frac{\pi_1(X_K \setminus \nu\Sigma_2) * \pi_1(Z \setminus \nu\Sigma'_2)}{\langle a^{-1}b = a_2, b^{-1}aba^{-1} = b_2, d = a_1, y = b_1, \lambda = 1 \rangle}.$$

Since λ' is nullhomotopic in $Z \setminus \nu\Sigma'_2$, the normal circle λ of $\pi_1(X_K \setminus \nu\Sigma_2)$ becomes trivial in $\pi_1(X)$. Also, using the relations $b_1b_2 = [a_1, b_1] = [a_2, b_2] = b_2a_2b_2^{-1}a_1 = a_1a_2 = 1$ in $\pi_1(Z \setminus \nu\Sigma'_2)$, we get the following relations in the fundamental group of X : $a^{-1}bd = [a^{-1}b, b^{-1}aba^{-1}] = [d, y] = [d, b^{-1}aba^{-1}] = yb^{-1}aba^{-1} = 1$. Note that the fundamental group of Z is an abelian group of rank two. In addition, we have the following relations in $\pi_1(X)$ coming from the fundamental group of $\pi_1(X_K \setminus \nu\Sigma_2)$: $aba = bab$, $efe = fef$, $[y, b] = [t, f] = 1$, $dbd^{-1} = x^{-1}$, $dxd^{-1} = xb$, $sfs^{-1} = z^{-1}$, $szs^{-1} = zf$, $a^{-1}b = s$, $b^{-1}aba^{-1} = t$, $y = f^{-1}efe^{-1}$ and $e^{-1}f = d$. These set of relations give rise to the following identities:

$$(5) \quad a = bd,$$

$$(6) \quad yb = by,$$

$$(7) \quad aba = bab,$$

$$(8) \quad yab = ba,$$

Next, multiply the relation (8) by a from the right and use $aba = bab$. We have $yaba = ba^2 \implies ybab = ba^2$. By cancelling the element b ,

we obtain $yab = a^2$. Finally, applying the relation (8) again, we have $ba = a^2$. The latter implies that $b = a$. Since $a = bd$, $dbd^{-1} = x^{-1}$, $dxd^{-1} = xb$, $aba = bab$ and $yb^{-1}aba^{-1} = 1$, we obtain $d = y = x = b = a = 1$. Furthermore, using the relations $a^{-1}b = s$, $b^{-1}aba^{-1} = t$, $efe = fef$, $e^{-1}f = d$, $sfs^{-1} = z^{-1}$ and $szs^{-1} = zf$, we similarly have $s = t = z = f = e = 1$. Thus, we can conclude that the elements a , b , x , d , y , e , f , z , s and t are all trivial in the fundamental group of X . Since we identified $a^{-1}b$ and $b^{-1}aba^{-1}$ with generators a_2 and b_2 of the group $\pi_1(Z \setminus \nu\Sigma'_2) = \mathbf{Z} \oplus \mathbf{Z}$, it follows that a_2 and b_2 are trivial in the fundamental group of X as well. This proves that X is simply connected. \square

Lemma 4.2. $c_1^2(X) = 12$, $\sigma(X) = -4$ and $\chi_h(X) = 2$

Proof. We have $c_1^2(X) = c_1^2(X_K) + c_1^2(T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}) + 8$, $\sigma(X) = \sigma(X_K) + \sigma(T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2})$, and $\chi_h(X) = \chi_h(X_K) + \chi_h(T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}) + 1$. Since $c_1^2(X_K) = 8$, $\sigma(X_K) = 0$, and $\chi_h(X_K) = 1$, the result follows from the lemmas 2.3 and 3.1. \square

By Freedman's theorem [F] and the lemmas 4.1 and 4.2, X is homeomorphic to $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$. It follows from Taubes theorem of Section 1 that $\text{SW}_X(K_X) = \pm 1$. Next we apply the connected sum theorem for the Seiberg-Witten invariant and show that SW function is trivial for $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$. Since the Seiberg-Witten invariants are diffeomorphism invariants, we conclude that X is not diffeomorphic to $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$. Notice that case (i) of Theorem 2.4 does not apply and X_K is a minimal symplectic manifold. Thus, we can conclude that X is minimal. Since symplectic minimality implies irreducibility for simply-connected 4-manifolds with $b_2^+ > 1$ [K], it follows that X is also smoothly irreducible.

5. CONSTRUCTION OF AN EXOTIC SYMPLECTIC $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$

In this section, we construct a simply-connected minimal symplectic 4-manifold Y homeomorphic but not diffeomorphic to $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$. Using Usher's Theorem [U1], we will distinguish Y from $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$.

The manifold Y will be the symplectic fiber sum of Y_K and $T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ along the genus two surfaces Σ_2 and Σ'_2 . Let us choose the gluing diffeomorphism $\varphi : \Sigma_2 \times S^1 \rightarrow \Sigma'_2 \times S^1$ that maps the generators

$a^{-1}b$, $b^{-1}aba^{-1}$, d , y and μ of $\pi_1(Y_K \setminus \nu\Sigma_2)$ to the generators a_1 , b_1 , a_2 , b_2 and μ' of $\pi_1(Z \setminus \nu\Sigma'_2)$ according to the following rule:

$$\varphi_*(a^{-1}b) = a_2, \varphi_*(b^{-1}aba^{-1}) = b_2, \varphi_*(d) = a_1, \varphi_*(y) = b_1, \varphi_*(\mu) = \mu'.$$

Here, μ and μ' denote the meridians of Σ and Σ'_2 .

Again, by Gompf's theorem [G], $Y = Y_K \#_\varphi (T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2})$ is symplectic.

Lemma 5.1. *Y is simply connected.*

Proof. By Van Kampen's theorem, we have

$$\pi_1(Y) = \frac{\pi_1(Y_K \setminus \nu\Sigma_2) * \pi_1(Z \setminus \nu\Sigma'_2)}{\langle a^{-1}b = a_2, b^{-1}aba^{-1} = b_2, d = a_1, y = b_1, \lambda = 1 \rangle}.$$

The following set of relations hold in $\pi_1(Y)$.

$$(9) \quad a = \overline{bd},$$

$$(10) \quad yb = by,$$

$$(11) \quad aba = bab,$$

$$(12) \quad yab = ba,$$

Using the same argument as in proof of Lemma 4.1, we have $a = b = x = d = y = 1$. Thus $\pi_1(Y) = 0$. \square

Lemma 5.2. $c_1^2(Y) = 4$, $\sigma(Y) = -4$ and $\chi_h(Y) = 1$

Proof. We have $c_1^2(Y) = c_1^2(Y_K) + c_1^2(T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}) + 8$, $\sigma(Y) = \sigma(Y_K) + \sigma(T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2})$, and $\chi_h(Y) = \chi_h(Y_K) + \chi_h(T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}) + 1$. Since $c_1^2(Y_K) = 0$, $\sigma(Y_K) = 0$, and $\chi_h(Y_K) = 0$, the result follows from lemmas 2.3 and 3.1. \square

By Freedman's classification theorem [F] and lemmas 5.1 and 5.2 above, Y is homeomorphic to $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$. Notice that Y is a fiber sum of non minimal manifold $Z = T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ with minimal manifold Y_K . All 4 exceptional spheres E_1 , E_2 , E_3 and E_4 in Z meet with the genus two fiber $2T + S - E_1 - E_2 - E_3 - E_4$. Also, any embedded symplectic -1 sphere in $T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ is of the form $mS \pm E_i$, thus intersect non-trivially with the fiber class $2T + S - E_1 - E_2 - E_3 - E_4$. It follows from Theorem 2.4. that Y is a minimal symplectic manifold.

Since symplectic minimality implies irreducibility for simply-connected 4-manifolds for $b_2^+ = 1$ [HK], it follows that Y is also smoothly irreducible. We conclude that Y is not diffeomorphic to $\mathbb{C}\mathbb{P}^2 \# 5 \overline{\mathbb{C}\mathbb{P}^2}$.

Remark. *Alternatively, one can apply the concept of symplectic Kodaira dimension to prove the exoticness of X and Y . We refer the reader to the articles [LY] and [U2] for a detailed treatment of how the Kodaira dimension behaves under symplectic fiber sums.*

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