EXOTIC SMOOTH STRUCTURES ON SMALL 4-MANIFOLDS

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Dedicated to Ronald J. Stern on the occasion of his sixtieth birthday

Abstract. Let M be either $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ or $3\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$. We construct the first example of a simply-connected irreducible symplectic 4-manifold that is homeomorphic but not diffeomorphic to M.

1. Introduction

Given a a smooth 4-manifold X, an *exotic* smooth structure on X is another smooth 4-manifold X' such that X' is homeomorphic but not diffeomorphic to X. Given two 4-manifolds X and Y, we denote their connected sum by $X\#Y$. For a positive integer $m \geq 2$, the connected sum of m copies of X will be denoted by mX for short. Let $\mathbb{CP}^2$ denote the complex projective plane and let $\overline{\mathbb{CP}}^2$ denote the underlying smooth 4-manifold $\mathbb{CP}^2$ equipped with the opposite orientation. There is an extensive and growing literature on the construction of exotic smooth structures on the closed 4-manifolds $\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$ and $3\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ for some small positive integers m and n. In the next two paragraphs, we briefly highlight some of the papers that have appeared.

The existence of an exotic smooth structure on a 4-manifold was first proved by Donaldson in [Do]. Using SU(2) gauge theory, he showed that a Dolgachev surface $E(1)_{2,3}$ is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. Infinitely many irreducible smooth structures on $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ were first constructed by Friedman and Morgan in [FM1], and many more infinite families were later constructed by Fintushel, Stern and Szabó in [FS2, Sz2]. Kotschick showed in [Ko1] that the Barlow surface in [Ba] is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}}^2$. More recently, J. Park was able to construct an exotic $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ in [Pa2] using the rational blowdown technique of Fintushel and Stern in [FS1]. Using a more general blowdown technique in [Pa1], Stipsicz and Szabó constructed an exotic $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$ in [SS1]. In [FS3], Fintushel and Stern constructed infinitely many irreducible smooth structures on $\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ for $6 \leq n \leq 8$. Using Fintushel and Stern's technique in [FS3], J. Park, Stipsicz and Szabó were able to construct infinitely many irreducible non-symplectic smooth structures on $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ in [PSS]. The first *symplectic* exotic irreducible smooth structure on $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ was constructed in [A2] by the first author.

The existence of infinitely many irreducible smooth structures on $3\mathbb{CP}^2\#19\overline{\mathbb{CP}}^2$ was first shown by Friedman and Morgan (see Corollary 4 in [FM2] for announcement and [FM3] for proof). Exotic smooth structures on $3\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$ for $14 \leq$

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$m \leq 18$ were first constructed by Gompf in [Go]. The existence of infinitely many irreducible smooth structures on $3\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$ for $14 \leq m \leq 18$ were shown by Stipsicz, Szabó and Yu in [St1, St2, Sz1, Yu]. Infinitely many irreducible smooth structures on $3\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$ for $10 \leq m \leq 13$ were constructed by the second author in [P1, P2, P3, P4]. Using rational blowdown techniques, Stipsicz and Szabó constructed infinitely many irreducible smooth structures on $3\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ in [SS2], and J. Park constructed infinitely many irreducible smooth structures on $3\mathbb{CP}^2\#8\overline{\mathbb{CP}}^2$ in [Pa3]. The first exotic irreducible smooth structure on $3\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ was constructed by the first author in [A2].

In this paper, we use the techniques and constructions in [A1] and [A2] to prove the following.

Theorem 1. *Let M be either $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ or $3\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$. There exists a smooth closed simply-connected irreducible symplectic 4-manifold that is homeomorphic but not diffeomorphic to M.*

A new idea in our construction is that it is sometimes possible to obtain a simply-connected 4-manifold by gluing together two 4-manifolds both of which have nontrivial fundamental group. Accordingly the main technical difficulty in the proof of Theorem 1 is the computation of the fundamental groups of our exotic 4-manifolds. Even though we cannot always completely pin down the fundamental groups at all stages of our construction, we are able to identify enough of the generators (up to conjugacy) and determine enough relations among them so that we are able to deduce, after careful choices of gluing, that the resulting 4-manifolds are simply-connected.

Shortly after the appearance of this paper, alternative constructions of exotic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ have been given in [BK] and [FPS]. In the follow-up paper [ABP] with R. İnanç Baykur, we present some alternative constructions of an irreducible symplectic smooth structure on M. In [ABP], we also construct infinitely many irreducible non-symplectic smooth structures on M, $3\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ and other small 4-manifolds. These smooth structures are distinguished by comparing their Seiberg-Witten invariants.

Here is how our paper is organized. Section 2 contains some definitions and formulas that will be important throughout the paper. Section 3 quickly reviews the 4-manifolds that were constructed in [A1]. These 4-manifolds will then serve as some of the building blocks in constructing an exotic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ in Section 4 and an exotic $3\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ in Section 5.

2. Generalized fiber sum

We first present a few standard definitions that will be used throughout the paper.

Definition 2. Let X and Y be closed oriented smooth 4-manifolds each containing a smoothly embedded surface Σ of genus $g \geq 1$. Assume Σ represents a homology class of infinite order and has self-intersection zero in X and Y, so that there exists a product tubular neighborhood, say $\nu\Sigma \cong \Sigma \times D^2$, in both X and Y. Using an orientation-reversing and fiber-preserving diffeomorphism $\psi : \Sigma \times S^1 \to \Sigma \times S^1$, we can glue $X \setminus \nu\Sigma$ and $Y \setminus \nu\Sigma$ along the boundary $\partial(\nu\Sigma) \cong \Sigma \times S^1$. The resulting closed oriented smooth 4-manifold, denoted $X\#_\psi Y$, is called a *generalized fiber sum* of X and Y along Σ.

Definition 3. Let $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of a closed oriented smooth 4-manifold X, respectively. We define

$$c_1^2(X) := 2e(X) + 3\sigma(X), \quad \chi_h(X) := \frac{e(X) + \sigma(X)}{4}.$$

If X is a complex surface, then $c_1^2(X)$ and $\chi_h(X)$ are the square of the first Chern class $c_1(X)$ and the holomorphic Euler characteristic, respectively. Note that these can be used as the coordinates for the "geography problem" for complex surfaces or irreducible smooth 4-manifolds (cf. [GS]).

For the simply-connected 4-manifolds $m\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$, we have $e = 2+m+n$ and $\sigma = m-n$. Hence we get

$$c_1^2(m\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2) = 5m - n + 4, \quad \chi_h(m\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2) = \frac{m+1}{2}.$$

Lemma 4. *Let X and Y be closed, oriented, smooth 4-manifolds containing an embedded surface Σ of self-intersection 0. Then*

$$\begin{aligned} c_1^2(X\#_\psi Y) &= c_1^2(X) + c_1^2(Y) + 8(g-1), \\ \chi_h(X\#_\psi Y) &= \chi_h(X) + \chi_h(Y) + (g-1), \end{aligned}$$

where g is the genus of the surface Σ.

Proof. The above formulas simply follow from the well-known formulas

$$e(X\#_\psi Y) = e(X) + e(Y) - 2e(\Sigma), \quad \sigma(X\#_\psi Y) = \sigma(X) + \sigma(Y). \qquad \square$$

If X and Y are symplectic 4-manifolds and Σ is a symplectic submanifold in both, then according to a theorem of Gompf (cf. [Go]), $X\#_\psi Y$ admits a symplectic structure. In such a case, we will call $X\#_\psi Y$ a *symplectic sum.*

3. Building blocks

We review the main construction in [A1]. From now on, let K denote a left-handed trefoil knot in S^3. Let νK denote the tubular neighborhood of K in S^3. It is well-known (see Example 3.7 in [BZ]) that

$$\pi_1(S^3 \setminus \nu K) = \langle a, b \mid aba = bab \rangle,$$

where the generators a and b are both represented by meridians of K. We choose a peripheral system $(\mu(K), \lambda(K))$, where $\mu(K)$ and $\lambda(K)$ are preferred meridian and longitude of K, respectively. We choose an orientation on $\lambda(K)$ and then orient $\mu(K)$ according to the right-hand rule. We orient the boundary torus $\partial(\nu K)$ so that the cup product of the Poincaré duals is $\mu(K) \cdot \lambda(K) = +1$. If we choose $\mu(K) = b$, then by Remark 3.13 of [BZ] we may choose

$$\lambda(K) = a^{-1}b^{-1}(ab^{-1}a^{-1})b^3 = a^{-1}b^{-1}(b^{-1}a^{-1}b)b^3 = a^{-1}b^{-2}a^{-1}b^4.$$

Here, we have used the fact that $aba = bab$ implies $ab^{-1}a^{-1} = b^{-1}a^{-1}b$. It is also well-known (see Subsection 5.15 of [BZ]) that $\gamma_1 = a^{-1}b$ and $\gamma_2 = b^{-1}aba^{-1}$ generate the image of the fundamental group of the genus one Seifert surface of K under the inclusion-induced homomorphism.

Since $aba = bab$, we must have $ba^{-1}b^{-1} = a^{-1}b^{-1}a$, $ab^{-1}a^{-1} = b^{-1}a^{-1}b$, and

$$\begin{aligned}[\gamma_1, \gamma_2] &= \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} = a^{-1}bb^{-1}aba^{-1}b^{-1}aab^{-1}a^{-1}b \\ &= (ba^{-1}b^{-1})a^2b^{-1}a^{-1}b = (a^{-1}b^{-1}a)a^2b^{-1}a^{-1}b \\ &= a^{-1}b^{-1}a^2(ab^{-1}a^{-1})b = a^{-1}b^{-1}a^2(b^{-1}a^{-1}b)b \\ &= a^{-1}b^{-1}a(ab^{-1}a^{-1})b^2 = a^{-1}b^{-1}a(b^{-1}a^{-1}b)b^2 \\ &= a^{-1}b^{-1}(ab^{-1}a^{-1})b^3 = a^{-1}b^{-1}(b^{-1}a^{-1}b)b^3 \\ &= a^{-1}b^{-2}a^{-1}b^4 = \lambda(K)\end{aligned}$$

in $\pi_1(S^3 \setminus \nu K)$. Let M_K denote the result of 0-surgery on K. Clearly, we have

$$\begin{aligned}\pi_1(S^1 \times M_K) &= \mathbb{Z} \oplus \pi_1(M_K) \\ &= \langle a, b, x \mid aba = bab,\ a^{-1}b^{-2}a^{-1}b^4 = 1,\ [x, a] = [x, b] = 1\rangle.\end{aligned} \tag{1}$$

Since K is a genus one fibered knot, M_K is a T^2 fiber bundle over S^1 with a section $\tau : S^1 \to M_K$ whose image represents the homotopy class b. Thus $S^1 \times M_K$ is a T^2 fiber bundle over T^2, with a section $\mathrm{id} \times \tau : S^1 \times S^1 \to S^1 \times M_K$ whose image is the torus $S = S^1 \times \mu(K) = x \times b$.

From [Th], we know that there exists a symplectic form on $S^1 \times M_K$ with respect to which both S and a torus fiber $F = \gamma_1 \times \gamma_2$ are symplectic submanifolds. We choose a symplectic form such that the cup products of the Poincaré duals in these submanifolds satisfy $x \cdot b = +1$ and $\gamma_1 \cdot \gamma_2 = +1$ with respect to the induced orientations on S and F, respectively. Note that the orientation of x is determined from the orientation of the meridian b since S is oriented as a symplectic submanifold and we require $x \cdot b = +1$.

Lemma 5. *Let $C_S = (S^1 \times M_K) \setminus \nu S$ be the complement of a tubular neighborhood of a section S in $S^1 \times M_K$. Then we have*

$$\pi_1(C_S) = \langle a, b, x \mid aba = bab,\ [x, a] = [x, b] = 1\rangle.$$

Proof. Our claim follows immediately from the fact that $C_S = S^1 \times (M_K \setminus \nu b) = S^1 \times (S^3 \setminus \nu K)$. □

Lemma 6. *Let $C_F = (S^1 \times M_K) \setminus \nu F$ be the complement of a tubular neighborhood of a fiber F in $S^1 \times M_K$. Then we have*

$$\begin{aligned}\pi_1(C_F) = \langle \gamma_1', \gamma_2', d, y \mid\ & [\gamma_1', \gamma_2'] = [y, \gamma_1'] = [y, \gamma_2'] = 1, \\ & d\gamma_1'd^{-1} = \gamma_1'\gamma_2',\ d\gamma_2'd^{-1} = (\gamma_1')^{-1}\rangle.\end{aligned}$$

Proof. In later applications, we will need to distinguish the generators of $\pi_1(C_F)$ from the generators of $\pi_1(C_S)$. Hence it will be convenient to replace the previous notation $\gamma_1, \gamma_2, a, b, x$ in $\pi_1(S^1 \times M_K)$ by $\gamma_1', \gamma_2', c, d, y$, respectively. In particular, d is a meridian of K and is oriented accordingly. C_F is homotopy equivalent to a T^2 fiber bundle over a wedge of two circles. The monodromy along the circle y is trivial whereas the monodromy along the circle d is the same as the monodromy of M_K. Recall that $\gamma_1' = c^{-1}d$ and $\gamma_2' = d^{-1}cdc^{-1}$. Thus we have

$$\begin{aligned}d\gamma_1'd^{-1} &= dc^{-1}dd^{-1} = dc^{-1} = \gamma_1'\gamma_2', \\ d\gamma_2'd^{-1} &= dd^{-1}cdc^{-1}d^{-1} = (cdc^{-1})d^{-1} \\ &= (d^{-1}cd)d^{-1} = d^{-1}c = (\gamma_1')^{-1}.\end{aligned}$$

Note that $cdc = dcd$ implies $cdc^{-1} = d^{-1}cd$. □

Let Y_K be the symplectic sum of two copies of $S^1 \times M_K$, identifying a section S in the first copy with a fiber F in the second copy. We choose the gluing map $\psi : \partial(\nu S) = S^1 \times \mu(K) \times \lambda(K) \to \partial(\nu F)$ that comes from an orientation preserving diffeomorphism $S \to F$ which sends the generators of π_1 as follows:

$$\psi_*(x) = \gamma_1', \quad \psi_*(b) = \gamma_2'.$$

It follows from $S \cdot F = +1$ that $[\gamma_1, \gamma_2] = \lambda(K) = a^{-1}b^{-2}a^{-1}b^4$ is a meridian of section $S = S^1 \times \mu(K)$ in the first copy of $S^1 \times M_K$. Similarly, the commutator $[y, d]$ (which is $[x, b]$ in the original notation) is a meridian of $F = \gamma_1' \times \gamma_2'$ in the second copy of $S^1 \times M_K$. We require ψ_* to map $[\gamma_1, \gamma_2]$ to the inverse of $[y, d]$ so that ψ is indeed an orientation reversing diffeomorphism of the boundary 3-tori.

Lemma 7. *Let $Y_K = (S^1 \times M_K)\#_\psi(S^1 \times M_K)$ be the above symplectic sum. Then*

$$\pi_1(Y_K) = \langle a, b, x, d, y \mid aba = bab,\ [x, a] = [x, b] = [y, x] = [y, b] = 1, \\ dxd^{-1} = xb,\ dbd^{-1} = x^{-1},\ a^{-1}b^{-2}a^{-1}b^4 = [y, d]^{-1}\rangle. \tag{2}$$

Proof. By Seifert-Van Kampen Theorem, we have

$$\begin{aligned}\pi_1(Y_K) &= (\pi_1(C_S) * \pi_1(C_F))/\pi_1(T^3)\\ &= \langle a, b, x, \gamma_1', \gamma_2', d, y \mid aba = bab,\ [x, a] = [x, b] = 1,\\ &\quad [\gamma_1', \gamma_2'] = [y, \gamma_1'] = [y, \gamma_2'] = 1,\ d\gamma_1'd^{-1} = \gamma_1'\gamma_2',\ d\gamma_2'd^{-1} = (\gamma_1')^{-1},\\ &\quad x = \gamma_1',\ b = \gamma_2',\ a^{-1}b^{-2}a^{-1}b^4 = [y, d]^{-1}\rangle.\end{aligned}$$

One circle factor of T^3 is identified with the longitude of K on the C_S side and the inverse of the meridian of a torus fiber $\gamma_1' \times \gamma_2'$ on the C_F side. This gives the last relation. □

Inside Y_K, we can find a genus two surface Σ_2 which is the internal sum of a punctured fiber F_0 in C_S and a punctured section S_0 in C_F. Σ_2 can be made into a symplectic submanifold of Y_K by Theorem 1.4 in [Go]. The inclusion-induced homomorphism sends the standard generators of $\pi_1(\Sigma_2)$ to $\{y, d, a^{-1}b, b^{-1}aba^{-1}\}$ in $\pi_1(Y_K)$ in that particular order. We can check that the product of corresponding commutators is indeed trivial in $\pi_1(Y_K)$:

$$[y, d][a^{-1}b, b^{-1}aba^{-1}] = [y, d][\gamma_1, \gamma_2] = [y, d]a^{-1}b^{-2}a^{-1}b^4 = 1.$$

4. Construction of an exotic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$

In this section, we construct a simply-connected symplectic 4-manifold U that is homeomorphic but not diffemorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. Using Usher's Theorem in [Us], we will show that U is irreducible and thereby distinguish U from $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$.

The manifold U will be the symplectic sum of the 4-manifold Y_K in Section 3 and $Q = (S^1 \times M_K)\#2\overline{\mathbb{CP}}^2$ along the genus two surfaces $\Sigma_2 \subset Y_K$ and $\Sigma_2' \subset Q$. The symplectic genus two submanifold $\Sigma_2' \subset Q$ is obtained by symplectically resolving the intersection of a torus fiber F and a torus section S in $S^1 \times M_K$ (see Section 3) and then blowing up at two points.

Let g, h, z be generators of $\pi_1(Q) \cong \pi_1(S^1 \times M_K)$ corresponding to generators a, b, x in (1) respectively so that the inclusion-induced homomorphisms send the generators of $\pi_1(S)$ to z and h, and the generators of $\pi_1(F)$ to $g^{-1}h$ and $h^{-1}ghg^{-1}$. Then the inclusion-induced homomorphism sends the standard generators of $\pi_1(\Sigma_2')$ to $\{z, h, g^{-1}h, h^{-1}ghg^{-1}\}$ in that particular order.

Let us choose the gluing diffeomorphism $\varphi : \partial(\nu\Sigma_2) \to \partial(\nu\Sigma'_2)$ that comes from an orientation preserving diffeomorphism $\Sigma_2 \to \Sigma'_2$ which sends the generators of π_1 as follows:

$$\varphi_*(y) = g^{-1}h,\ \varphi_*(d) = h^{-1}ghg^{-1},\ \varphi_*(a^{-1}b) = z,\ \varphi_*(b^{-1}aba^{-1}) = h. \tag{3}$$

If μ and μ' denote the meridians of Σ and Σ'_2 respectively, then we require $\varphi_*(\mu) = (\mu')^{-1}$.

With respect to ordered bases $\{y, d, a^{-1}b, b^{-1}aba^{-1}\}$ and $\{z, h, g^{-1}h, h^{-1}ghg^{-1}\}$, the homomorphism $H_1(\Sigma_2;\mathbb{Z}) \to H_1(\Sigma'_2;\mathbb{Z})$ induced by φ has matrix representative

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{4}$$

which commutes with the following matrix representing the skew-symmetric bilinear form on H_1 given by the Poincaré dual of the cup product pairing

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Note that (4) can be realized by an orientation preserving involution of a genus two surface with two fixed points. For example, we can think of a genus two surface as a regular octagon with the standard identification of the sides, and our involution is just the 180 degree counterclockwise rotation about the center of the octagon. Thus (3) is realized by an orientation preserving diffeomorphism $\Sigma_2 \to \Sigma'_2$. It now follows from Gompf's theorem (cf. [Go]) that $U := Y_K \#_\varphi ((S^1 \times M_K)\#2\overline{\mathbb{CP}}^2)$ is a symplectic 4-manifold.

Lemma 8. *U is simply-connected.*

Proof. By Seifert-Van Kampen Theorem, we have

$$\pi_1(U) = \frac{\pi_1(Y_K \setminus \nu\Sigma_2) * \pi_1(Q \setminus \nu\Sigma'_2)}{\langle y = g^{-1}h,\ d = h^{-1}ghg^{-1}, a^{-1}b = z,\ b^{-1}aba^{-1} = h,\ \mu = (\mu')^{-1}\rangle}.$$

Since Σ'_2 intersects each exceptional sphere of the blow-up once in Q, meridian μ' of Σ'_2 bounds a disk in $Q \setminus \nu\Sigma'_2$ that is a punctured exceptional sphere. Hence $\mu' = 1$ in $\pi_1(Q \setminus \nu\Sigma'_2)$. It follows that $\pi_1(Q \setminus \nu\Sigma'_2) \cong \pi_1(Q) \cong \pi_1(S^1 \times M_K)$. Since $\mu = (\mu')^{-1} = 1$ after gluing, we conclude that

$$\pi_1(U) = \frac{\pi_1(Y_K) * \pi_1(S^1 \times M_K)}{\langle y = g^{-1}h,\ d = h^{-1}ghg^{-1}, a^{-1}b = z,\ b^{-1}aba^{-1} = h\rangle}.$$

It follows that $\pi_1(U)$ is generated by

$$\{a,\ b,\ x,\ d,\ y;\ g,\ h,\ z\}. \tag{5}$$

The following relations hold in $\pi_1(U)$:

$$\begin{aligned} & aba = bab,\ [x, a] = [x, b] = [y, x] = [y, b] = 1, \\ & dxd^{-1} = xb,\ dbd^{-1} = x^{-1},\ a^{-1}b^{-2}a^{-1}b^4 = [y, d]^{-1}, \\ & ghg = hgh,\ g^{-1}h^{-2}g^{-1}h^4 = 1,\ [z, g] = [z, h] = 1, \\ & y = g^{-1}h,\ d = h^{-1}ghg^{-1},\ a^{-1}b = z,\ b^{-1}aba^{-1} = h. \end{aligned} \tag{6}$$

First note that y commutes with b. y also commutes with z since $y = g^{-1}h$ and z commutes with both g and h. It follows that y commutes with $a = bz^{-1}$, which in turn implies that y commutes with $h = b^{-1}aba^{-1}$. Since $g = hy^{-1}$, we must have

$$d = h^{-1}ghg^{-1} = h^{-1}(hy^{-1})h(hy^{-1})^{-1} = y^{-1}hyh^{-1} = 1$$

because y and h commute. Plugging $d = 1$ into (6), we easily deduce that all other generators in (5) are trivial. Thus we have shown that $\pi_1(U)$ is trivial. □

Lemma 9. *$e(U) = 6$, $\sigma(U) = -2$, $c_1^2(U) = 6$, and $\chi_h(U) = 1$.*

Proof. Let $Q = (S^1 \times M_K)\#2\overline{\mathbb{CP}}^2$. We have $e(U) = e(Y_K) + e(Q) + 4$, $\sigma(U) = \sigma(Y_K)+\sigma(Q)$, $c_1^2(U) = c_1^2(Y_K)+c_1^2(Q)+8$, and $\chi_h(U) = \chi_h(Y_K)+\chi_h(Q)+1$. Since $e(Q) = 2$, $\sigma(Q) = -2$, $c_1^2(Q) = -2$, $\chi_h(Q) = 0$, and $e(Y_K) = \sigma(Y_K) = c_1^2(Y_K) = \chi_h(Y_K) = 0$ (cf. [A1]), our results follow. □

From the above lemmas and Freedman's theorem on the classification of simply-connected topological 4-manifolds (cf. [Fr]), we immediately deduce that U is homeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. Since $\pi_2(S^1 \times M_K) = 0$, we conclude that $S^1 \times M_K$ is a minimal symplectic 4-manifold. Usher's theorem (see Theorem 1.1 in [Us]) then implies that the symplectic sum Y_K is minimal as well. Note that U is a symplectic sum of a minimal 4-manifold Y_K with a non-minimal 4-manifold $Q = (S^1 \times M_K)\#2\overline{\mathbb{CP}}^2$. However, the only -1 spheres in Q are the exceptional spheres E_1 and E_2 of the two blow-ups (see Corollary 3 in [Li]). Both E_1 and E_2 transversely intersect the genus two surface Σ_2' once in Q since $[\Sigma_2'] = [S] + [F] - [E_1] - [E_2] \in H_2(Q;\mathbb{Z})$. It follows that there is no -1 sphere in $Q \setminus \nu\Sigma_2'$, and hence Usher's Theorem can again be applied to show that U is a minimal symplectic 4-manifold. Since symplectic minimality implies irreducibility for simply-connected 4-manifolds (see [HK] for $b_2^+ = 1$ case), U is also smoothly irreducible. We conclude that U cannot be diffeomorphic to the reducible 4-manifold $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$.

5. Construction of an exotic $3\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$

In this section, we construct a simply-connected, symplectic 4-manifold V that is homeomorphic but not diffemorphic to $3\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$. The construction is very similar to the one in Section 4, and in particular V is a symplectic sum of certain 4-manifolds R and X_K along some genus two surfaces. Because of the similarity, we will be succinct in our presentation of the fundamental group calculation. We will use the Seiberg-Witten invariant of V to distinguish V from $3\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$.

Our first building block is $R = T^4\#2\overline{\mathbb{CP}}^2$, the 4-torus blown up twice. We find a symplectically embedded genus two surface Σ_2'' in R as follows. First we introduce the notation $T_{i,j}$ $(1 \le i < j \le 4)$ for the 2-torus inside T^4 that has nontrivial ith and jth circle factors. For example, $T_{1,2} = S^1 \times S^1 \times \{\text{pt}\} \times \{\text{pt}'\}$. Let $p_{i,j} : T^4 \to T_{i,j}$ be the projection map. Let $\omega_{i,j}$ be a standard product volume form on $T_{i,j}$.

Next we fix a factorization $T^4 = T^2 \times T^2$ and endow T^4 with a corresponding product symplectic form $\omega = p_{1,2}^*(\omega_{1,2})+p_{3,4}^*(\omega_{3,4})$. Consider one copy of a horizontal torus $T_{1,2}$, and one copy of a vertical torus $T_{3,4}$. They are both symplectically embedded in T^4 with respect to ω. We symplectically resolve their intersection and obtain a symplectic genus two surface of self-intersection 2. Next we blow up twice to get a symplectic surface Σ_2'' of self-intersection 0 in $R = T^4\#2\overline{\mathbb{CP}}^2$.

Note that $\pi_1(R) \cong \pi_1(T^4) \cong \mathbb{Z}^4$. Let α_i $(i = 1, \dots, 4)$ denote the generator of $\pi_1(R)$ corresponding to the ith circle factor. Both exceptional spheres of the blow-ups intersect Σ_2'' transversely once in R. Hence a meridian of Σ_2'' is trivial in $\pi_1(R \setminus \nu\Sigma_2'')$ since it bounds a disk which is a punctured exceptional sphere. It follows that $\pi_1(R \setminus \nu\Sigma_2'') \cong \pi_1(R) \cong \mathbb{Z}^4$ and $\pi_1(R \setminus \nu\Sigma_2'')$ is generated by the α_i's. Note that the inclusion-induced homomorphism sends the standard generators of $\pi_1(\Sigma_2'')$ to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ in that particular order.

Our second building block X_K is the symplectic sum of two copies of Y_K along the genus two surface $\Sigma_2 \subset Y_K$ that was constructed in Section 3. Let e, f, z, s, t denote the generators of $\pi_1(Y_K)$ for the second copy of Y_K that correspond to the generators a, b, x, d, y for the first copy in (2), respectively. Then the inclusion-induced homomorphism in the second copy of Y_K sends the standard generators of $\pi_1(\Sigma_2)$ to $\{t, s, e^{-1}f, f^{-1}efe^{-1}\}$ in that particular order. Let X_K be the symplectic sum $Y_K \#_\psi Y_K$, where the gluing map $\psi : \partial(\nu\Sigma_2) \to \partial(\nu\Sigma_2)$ sends the generators of $\pi_1(\Sigma_2)$ as follows:

$$\psi_*(y) = e^{-1}f,\ \psi_*(d) = f^{-1}efe^{-1},\ \psi_*(a^{-1}b) = t,\ \psi_*(b^{-1}aba^{-1}) = s.$$

As observed in Section 4, the above homomorphism on the fundamental group does indeed come from an orientation preserving diffeomorphism $\Sigma_2 \to \Sigma_2$ corresponding to matrix (4). As before, $\psi_*(\mu) = \widetilde{\mu}^{-1}$, where μ and $\widetilde{\mu}$ are meridians of Σ_2 in the first and the second copy of Y_K respectively.

By Seifert-Van Kampen Theorem, we have

$$\pi_1(X_K) = \frac{\pi_1(Y_K \setminus \nu\Sigma_2) * \pi_1(Y_K \setminus \nu\Sigma_2)}{\langle y = e^{-1}f,\ d = f^{-1}efe^{-1},\ a^{-1}b = t,\ b^{-1}aba^{-1} = s,\ \mu = \widetilde{\mu}^{-1}\rangle}.$$

Let N be the normal subgroup of $\pi_1(X_K)$ generated by

$$\{a, b, x, d, y;\ e, f, z, s, t\},$$

where a, b, x, d, y are elements of the first copy of $\pi_1(Y_K \setminus \nu\Sigma_2)$ and e, f, z, s, t are elements of the second copy.

Lemma 10. *We have $N = \pi_1(X_K)$. Moreover, the following relations hold in $\pi_1(X_K)$:*

$$\begin{aligned} &aba = bab,\ [y, x] = [y, b] = 1,\ dxd^{-1} = xb,\ dbd^{-1} = x^{-1}, \\ &efe = fef,\ [t, z] = [t, f] = 1,\ szs^{-1} = zf,\ sfs^{-1} = z^{-1}, \\ &y = e^{-1}f,\ d = f^{-1}efe^{-1},\ a^{-1}b = t,\ b^{-1}aba^{-1} = s. \end{aligned} \tag{7}$$

Proof. Note that the first copy of $\pi_1(Y_K \setminus \nu\Sigma_2)$ is generated by $\{a, b, x, d, y\}$ (or $\{e, f, z, s, t\}$ for the second copy) and a finite number of meridians of Σ_2. Recall from Section 3 that Σ_2 is the union of a punctured fiber F_0 in C_S and a punctured section S_0 in C_F. Hence we can write $Y_K \setminus \nu\Sigma_2 = (C_S \setminus \nu F_0) \cup (C_F \setminus \nu S_0)$. Every meridian of Σ_2 is conjugate to the commutator $[x, b]$ (or $[z, f]$ for the second copy) which is represented by the boundary of a parallel punctured torus section in $C_S \setminus \nu F_0 \subset Y_K \setminus \nu\Sigma_2$. Hence we have

$$\frac{\pi_1(Y_K \setminus \nu\Sigma_2)}{\langle a = b = x = d = y = 1\rangle} = \frac{\pi_1(Y_K)}{\langle a = b = x = d = y = 1\rangle} = 1,$$

and it follows that $\pi_1(X_K)/N = 1$.

Next we show that the first line of relations in (7) hold in the first copy of $\pi_1(Y_K \setminus \nu\Sigma_2)$. Let $\mathrm{pr} : S^1 \times M_K \to S^1$ denote the projection map onto the first factor. By choosing a suitable point $\theta \in S^1$ away from $\mathrm{pr}|_{C_S}(F_0)$, the projection of the punctured fiber F_0 that forms a part of Σ_2, we obtain an embedding of the knot complement $\{\theta\} \times (S^3 \setminus \nu K) \hookrightarrow C_S \setminus \nu F_0$. This implies that $aba = bab$ holds in $\pi_1(Y_K \setminus \nu\Sigma_2)$. The other four remaining relations are holdovers from $\pi_1(Y_K)$ corresponding to the monodromy of a torus fiber bundle over a torus. Since these four relations will now describe the monodromy of a punctured torus fiber bundle over a punctured torus in $Y_K \setminus \nu\Sigma_2$, they still hold true in $\pi_1(Y_K \setminus \nu\Sigma_2)$.

Replacing $\{a, b, x, d, y\}$ by $\{e, f, z, s, t\}$, the second line of relations in (7) hold in the second copy of $\pi_1(Y_K \setminus \nu\Sigma_2)$ for the same reasons. The last line of relations are the consequence of our choice of the gluing map ψ. $\square$

Now let $\bar{\Sigma}_2$ be a symplectic genus two submanifold of X_K that is a parallel copy of Σ_2 lying on the boundary of $Y_K \setminus \nu\Sigma_2$. We form a symplectic sum V of $R = T^4\#2\overline{\mathbb{CP}}^2$ and X_K along the genus two surfaces Σ_2'' and $\bar{\Sigma}_2$. Recall that the inclusion-induced homomorphism sends the standard generators of $\pi_1(\Sigma_2'')$ to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ in $\pi_1(R)$ in that particular order. The inclusion-induced homomorphism sends the standard generators of $\pi_1(\bar{\Sigma}_2)$ to $\{y, d, a^{-1}b, b^{-1}aba^{-1}\}$ (or $\{e^{-1}f, f^{-1}efe^{-1}, t, s\}$) in $\pi_1(X_K)$ in that particular order. We choose the gluing diffeomorphism $\varphi : \partial(\nu\Sigma_2'') \to \partial(\nu\bar{\Sigma}_2)$ that sends the generators of $\pi_1(\Sigma_2'')$ as follows:

$$\varphi_*(\alpha_1) = y,\ \varphi_*(\alpha_2) = d,\ \varphi_*(\alpha_3) = a^{-1}b,\ \varphi_*(\alpha_4) = b^{-1}aba^{-1}. \tag{8}$$

Clearly, (8) is realized by an orientation preserving diffeomorphism of genus two surfaces. With respect to bases $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\{y, d, a^{-1}b, b^{-1}aba^{-1}\}$, the induced homomorphism $H_1(\Sigma_2''; \mathbb{Z}) \to H_1(\bar{\Sigma}_2; \mathbb{Z})$ is represented by the identity matrix. We require $\varphi_*(\mu'') = \bar{\mu}^{-1}$, where μ'' and $\bar{\mu}$ are meridians of Σ_2'' and $\bar{\Sigma}_2$ respectively. By Gompf's theorem (cf. [Go]), $V = R\#_\varphi X_K$ is a symplectic 4-manifold.

Lemma 11. *V is simply-connected.*

Proof. By Seifert-Van Kampen Theorem, we have

$$\begin{aligned}
\pi_1(V) &= \frac{\pi_1(R \setminus \nu\Sigma_2'') * \pi_1(X_K \setminus \nu\bar{\Sigma}_2)}{\langle \alpha_1 = y,\ \alpha_2 = d,\ \alpha_3 = a^{-1}b,\ \alpha_4 = b^{-1}aba^{-1},\ \mu'' = \bar{\mu}^{-1}\rangle} \\
&= \frac{\pi_1(R) * \pi_1(X_K)}{\langle \alpha_1 = y,\ \alpha_2 = d,\ \alpha_3 = a^{-1}b,\ \alpha_4 = b^{-1}aba^{-1}\rangle} \\
&= \pi_1(X_K)/\langle y,\ d,\ a^{-1}b,\ b^{-1}aba^{-1} \text{ commute}\rangle.
\end{aligned}$$

The second isomorphism holds because $\mu'' = 1$ and so the meridian $\bar{\mu}$ of $\bar{\Sigma}_2$ becomes trivial in $\pi_1(V)$. The last isomorphism holds because α_i's commute with one another. From Lemma 10, we conclude that $\pi_1(V)$ is normally generated by

$$\{a, b, x, d, y;\ e, f, z, s, t\}. \tag{9}$$

The following relations hold in $\pi_1(V)$:

$$\begin{aligned}
&aba = bab,\ [y, x] = [y, b] = 1,\ dxd^{-1} = xb,\ dbd^{-1} = x^{-1}, \\
&efe = fef,\ [t, z] = [t, f] = 1,\ szs^{-1} = zf,\ sfs^{-1} = z^{-1}, \\
&y = e^{-1}f,\ d = f^{-1}efe^{-1},\ a^{-1}b = t,\ b^{-1}aba^{-1} = s, \\
&[y, d] = [y, t] = [y, s] = [d, t] = [d, s] = [t, s] = 1.
\end{aligned} \tag{10}$$

The above relations imply the following three identities in $\pi_1(V)$:

$$d = (d^{-1}b^{-1}d)(bdb^{-1}), \tag{11}$$

$$t = (btb^{-1})(b^{-1}tb), \tag{12}$$

$$(bdb^{-1})t = t(bdb^{-1}). \tag{13}$$

From presentation (10), we get $x = db^{-1}d^{-1}$. Substituting this into the relation $dxd^{-1} = xb$, we get $d^2b^{-1}d^{-2} = db^{-1}d^{-1}b$, which can be rearranged to $d^{-2} = bd^{-1}b^{-1}d^{-1}b$. Thus $d^2 = b^{-1}dbdb^{-1}$, which can be rearranged to (11).

Next note that $a = bt^{-1}$, and thus $aba = bab$ implies that $bt^{-1}b^2t^{-1} = b^2t^{-1}b$. Hence $b^{-1}t^{-1}b^2t^{-1}b^{-1} = t^{-1}$, and by taking the inverse, we obtain (12).

Finally, since $ds = sd$ and $s = b^{-1}aba^{-1} = b^{-1}(bt^{-1})b(bt^{-1})^{-1} = t^{-1}btb^{-1}$, we have $dt^{-1}btb^{-1} = t^{-1}btb^{-1}d$. Since d commutes with t, we have

$$d(btb^{-1}) = (btb^{-1})d.$$

From (12), we also know that d commutes with the product $(btb^{-1})(b^{-1}tb)$. It follows that d commutes with $b^{-1}tb$, i.e.

$$d(b^{-1}tb) = (b^{-1}tb)d,$$

which can be rearranged to (13).

Since t commutes with d and bdb^{-1}, we conclude from (11) that t also commutes with $d^{-1}b^{-1}d$, i.e. $t(d^{-1}b^{-1}d) = (d^{-1}b^{-1}d)t$. Since t commutes with d, we must have $tb^{-1} = b^{-1}t$. Hence t and b commute, and (12) implies that $t = 1$. From $t = 1$, we can easily deduce that all other generators in (9) are trivial. Thus $\pi_1(V)$ is trivial. □

Lemma 12. *$e(V) = 10$, $\sigma(V) = -2$, $c_1^2(V) = 14$, and $\chi_h(V) = 2$.*

Proof. Let $R = T^4\#2\overline{\mathbb{CP}}^2$ as before. We easily compute that $e(R) = 2$, $\sigma(R) = -2$, $c_1^2(R) = -2$, and $\chi_h(R) = 0$. Since $e(X_K) = 4$, $\sigma(X_K) = 0$, $c_1^2(X_K) = 8$ and $\chi_h(X_K) = 1$ (cf. [A1]), our results follow. □

From Freedman's theorem (cf. [Fr]), we conclude that V is homeomorphic to $3\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$. It follows from Taubes's theorem (cf. [Ta]) that $\mathrm{SW}_V(K_V) = \pm 1$, where K_V is the canonical class of V. Next we apply the connected sum theorem (cf. [Wi]) for the Seiberg-Witten invariant to deduce that the Seiberg-Witten invariant is trivial for $3\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$. Since the Seiberg-Witten invariant is a diffeomorphism invariant, we conclude that V is not diffeomorphic to $3\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$.

Since Y_K is minimal, Usher's theorem (cf. [Us]) implies that the symplectic sum $X_K = Y_K\#_\psi Y_K$ is minimal as well. By Corollary 3 in [Li], the only -1 spheres in R are the exceptional spheres E_1 and E_2 of the blow-ups, both of which intersect Σ_2'' once since $[\Sigma_2''] = [T_{1,2}] + [T_{3,4}] - [E_1] - [E_2] \in H_2(R;\mathbb{Z})$. Thus there is no -1 sphere in $R \setminus \nu\Sigma_2''$, and Usher's theorem once again implies that the symplectic sum $V = R\#_\varphi X_K$ is a minimal symplectic 4-manifold. Since symplectic minimality implies irreducibility for simply-connected 4-manifolds (see [Ko2] for $b_2^+ > 1$ case), V is also smoothly irreducible.

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