TRIANGLE INEQUALITY

For all $x \in \mathbf{R}$, we define the *absolutely value* $|\cdot|$ by

$$|x| = \begin{cases} x & x > 0 \\ -x & x \leq 0 \end{cases}.$$

1. *Prove triangle inequality: for all* $x, y \in \mathbf{R}$,

$$|x + y| \leq |x| + |y|.$$

Hint: Use cases.

Proof. There are a couple ways to do it, depending on how you want to divide up cases. This may be the simplest:

First, we note that for all $x \in \mathbf{R}$,

$$x \leq |x|. \tag{1}$$

To verify this, we need to check two cases. If $x > 0$, then $x \leq |x|$. If $x \leq 0$, then $x \leq 0 \leq |x|$. Similarly, by applying the inequality (1) to $-x$, we obtain

$$-x \leq |x|$$

as well. (Here, we have used that $|x| = |-x|$.)

Now, let us use the inequality (1) to solve the problem. To analyze $|x+y|$, we must *take away the absolute value.* We will need to use cases. If $x+y > 0$, then

$$|x + y| = x + y \leq |x| + |y|. \tag{2}$$

On the other hand, if $x + y \leq 0$, then

$$|x + y| = -(x + y) = -x - y \leq |x| + |y|. \tag{3}$$

This completes the proof. ■

It is possible to do a different case analysis, e.g. using case 1) $x, y \geq 0$, and case 2) $x \geq 0$, $y \leq 0$. Then one would further break up into the cases 2a) $|x| \geq |y|$, and case 2b) $|x| \leq |y|$.

There is actually an elegant and more general proof of the triangle inequality. We will discuss this later when we talk about Cauchy-Schwarz.

2. *Prove the reverse triangle inequality: for all* $x, y \in \mathbf{R}$,

$$\big||x| - |y|\big| \leq |x - y|.$$

Hint: "Add zero" and use the triangle inequality.

Proof. Exercise. ■

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