Syllabus for Complex Analysis: Math 8701-02; 2017-2018

A. Marden

August 5, 2017

Texts: *Ahlfors*,) **Complex Analysis** (3rd ed); *Milnor*, **Dynamics in One Complex Variable (3rd ed.), Princeton U. Press.**
Suggested reference: *Gamelin***, Complex Analysis**. This is akin to an encylclopedia.

Some other useful references:

- *Donaldson*, Riemann Surfaces–a new text by a Fields medalist from a more abstract point of view that generalizes to higher dimensions.
- *Rudin*, Real and Complex Analysis (3nd ed)—a classic text treating real and complex analysis together;
- *R. Narasimhan and V. Nievergelt*, Complex Analysis in One Variable (from point of view of several complex variables).
- *E.B.Saff and A.D.Snider*, Fundamentals of Complex Analysis with Appl.
- *J.B.Conway, Functions of One Complex Variable* (simpler treatment).

The above books are on reserve in the math lib.

Course Goal: To provide a solid, classical foundation for the subject while exposing trails leading off in interesting directions. Several of the listed topics and theorems will be presented but not proved. We will want cover the territory and still get to an intro to complex dynamics by the middle of the 2nd term.

Exams: There will be a take-home exam at the end of the Fall term. By the end of the Spring term, each student will have researched a relevant topic, written up their discoveries, and presented them to the class.

1 Introduction

1. Intro to Riemann surfaces; 2D surfaces in 3D space.
2. $\mathbb{C}$omplex numbers, and their basic geometry
3. Lines, circles, de Moivre's formula for roots of equations.
4. Stereographic projection: $\mathbb{C} \cup \{\infty\} \equiv \mathbb{S}^2$; Chordal distance on $\mathbb{S}^2$.

2 Analytic functions

1. Def. of analyticity; Cauchy-Riemann equations and complex derivatives.
2. The basic theorems to be proved.
3. Intro to harmonic functions.

4. Polynomial and rational functions; zeros, poles, and critical points.
5. Partial fraction decomposition.
6. Abel's theorem, convergent power series.
7. Construction of exponential and trig functions.

3 Analytic functions as mappings

1. Parking ramp construction for some Riemann surfaces.
2. Conformal mapping by powers, logs, exponentials, trigs; more Riemann surfaces.
3. Riemann surface of an elliptic curve.
4. Conformal mappings; 4 special mappings.
5. Möbius transformations
   - Classification;
   - Model examples, action in $\mathbb{C}$, quotient spaces;
   - Extension to 3D;
   - Cross ratio;
   - Symmetry;
   - The hyperbolic metric;
   - Application to 2- and 3D- manifolds;
6. Covering surfaces and the Riemann-Hurwitz formula;

4 The Cauchy Theorems

4.1 Preliminaries

1. Line integrals;
2. Useful Lemmas on differentiation;

4.2 Initial Cauchy's Theorem

1. Statement;
2. Cauchy theorem for rectangles and disks; exceptional points;

4.3 Index and homology

1. The winding number (index) of a closed curve;
2. All derivatives of an analytic fctn are analytic;
3. Simple connectivity;
4. Cycles and homology;

4.4 The general case

1. Cauchy's integral theorem;
2. Dixon's proof of CT and CIT, general cases;

5 Local Properties

1. Taylor's theorem with remainder;
2. Singularities: zeros, poles, essential singularities;
3. Finite factorization of zeros and poles;

6 Global Properties

1. Morara's theorem;
2. Liouville's theorem;
3. Maximum principle;
4. Schwarz lemma;
   - Schwarz-Pick lemma;
5. Argument principle;
6. Rouché's theorem; the local inverse;
7. The open mapping theorem;
8. The Bieberbach Conjecture/Theorem;

6.1 Residue Calculus

1. Residues;
2. The residue theorem;
3. Various explicit examples;

6.2 Harmonic Functions

1. Definitions; rectangular and polar coordinate form of laplacian;
2. Properties;
3. du and the conjugate differential $^*du$;
4. Mean values; maximum principle;

6.3 Reflection principle

1. Reflections in lines and circles;
2. Reflection principle;

7 Poisson formulas

1. The Poisson formula and integral;
2. Mean values and Harnak's inequalty;
3. Geometric interpretation;
4. Schwarz's formula;

8 Function representations by series and products

8.1 Pointwise and uniform convergence

1. Weierstrass convergence theorem;
2. Hurwitz theorem;
3. Taylor series with remainder;
4. Laurent series;

8.2 Infinite partial fraction expansions

1. Mittag-Leffler Theorem;
2. Infinite product expansions;
3. Weierstrass Factorization Theorem;
4. Blaschke Products;
5. Jensen's formula;

9 Famous results

9.1 Two famous functions

1. Γ-function; Sterling's formula;
2. ζ-function; Riemann hypothesis;

9.2 Four famous theorems

1. Runge's approximation theorem;
2. Mergelyan's approximation theorem;
3. Holomorphic motion (esp. Sladkowski's theorem);
4. The λ lemma;

10 Picard's big theorem.

1. Equicontinuous and normal families;
    - ....of meromorphic functions;
2. Arzela's theorem;
3. Marty's criterion]
4. Zalcman's criterion;
5. Montel's theorem;
6. Picard's BIG theorem;

11 The Riemann mapping theorem.

1. The proof;
2. Green's function;
3. Discrete Riemann maps;
4. Boundary values;
   - Extension across analytic boundary arcs;
   - Prime ends;

12 Elliptic functions.

1. The Schwarz-Christoffel formula.
2. Rectangle mappings: Intro to elliptic functions.
3. Lattices and tori.
4. The quotient tori T_Γ.
   - The modular group Mod.
5. Elliptic functions;
6. The Weierstrass $\wp$-function;

13 Dirichlet problem

1. Subharmonic functions;
2. Statement of Dirichlet problem;
   - The Perron family;
   - Barriers;
3. Solution of the problem;
4. Dirichlet's principle;

14 Complex dynamics: Iteration of rational fctns

14.1 A. Iteration: Basic Theory

1. Newton's method;
2. Perliminaries, definitions;
3. Critical points, classification of fixed points, periodic points;
4. The Fatou and Julia sets;
5. Grand Orbits;

14.2 B. Properties of the Julia set

1. Examples of smooth Julia sets;
2. The Lattes example: Repelling fixed pts are dense;
3. Blashke products:

14.3 C. Classification of fixed points.

1. Koenig's linearization thm;
2. B otcher's Theorem;
3. Parabolic fixed pts: The Leau-Fatou flower theorem;
4. Irrationally indifferent points: Cremar pts and Siegel disks;
5. Herman rings;
6. Repelling cycles;

14.4 D. Components of the Fatou set.

1. Properties;
2. NO WANDERING DOMAINS (Sullivan's thm);
3. Shishikura's thm: Number of cycles of Fatou components;

14.5 D. Quadratic polynomials

1. The special case of polynomials, quadratic polynomials;
2. The structure of the Fatou and Julia sets;
3. The Mandelbrot set (the parameter space);
    - Definition and properties;
    - Proof that its complement is simply connected;
    - Its interior structure, combinatorics;
4. The big conjectures;