Homework 11 solutions

MATH 2283 Spring 2015

3.5 (1pt)

3.7 (4pts)
(b) By (*), we have

\[ T_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n \leq 1 \]

\[ T_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + \frac{1}{n} - \ln n \geq \frac{1}{n} > 0 \]

Hence \( T_n \) is bounded below by 0 and above by 1.

(c) By the mean value theorem, there exists \( c \in [n, n+1] \) such that

\[ f'(c) = \frac{f(n+1) - f(n)}{n+1 - n} = \ln (n+1) - \ln n. \]

Since \( c < n + 1, \) then \( \frac{1}{c} > \frac{1}{n+1}, \) so

\[ T_{n+1} - T_n = \frac{1}{n+1} - [\ln (n+1) - \ln n] < \frac{1}{c} - [\ln (n+1) - \ln n] = 0 \]

5.12 (5 pts)
(j) Note that \( |\sin(n^2)| \leq 1, \) and \( \sum \frac{1}{n^2} \) converges, then by the comparison test

\[ \sum_{n=1}^{\infty} \frac{|\sin(n^2)|}{n^2} \text{ converges.} \]

(z) We use the comparison test with \( a_n = \frac{(\ln n)^2}{n^2} \) and \( b_n = \frac{1}{n^{3/2}}. \)

\[ \frac{a_n}{b_n} = \frac{(\ln n)^2 n^{3/2}}{n^2} = \frac{(\ln n)^2}{n^{1/2}} \]

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(\ln n)^2}{n^{1/2}} = \lim_{n \to \infty} \frac{2 \ln n}{2n^{-1/2}} = \lim_{n \to \infty} \frac{4 \ln n}{4n^{-1/2}} = \lim_{n \to \infty} \frac{4}{2n^{1/2}} = \lim_{n \to \infty} \frac{8}{2n^{1/2}} = 0 \]

Since \( \sum b_n \) converges by the p-test, then \( \sum a_n \) also converges by the limit comparison test.