Homework 8 solutions

MATH 2283 Spring 2015

6.6, 6.7, 6.8 (1pt)

6.9 (4pts)

(a) Here are the first few terms:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_n</td>
<td>2.646</td>
<td>3.106</td>
<td>3.179</td>
<td>3.190</td>
<td>3.192</td>
</tr>
</tbody>
</table>

(b) We show that the sequence is monotonic and bounded then conclude that it is convergent.

Step 1. We use induction to show that the sequence is monotonic increasing.

P(1): Clearly $x_1 \leq x_2$.

Assume $P(n)$: $x_n \leq x_{n+1}$ is true.

$$\Rightarrow x_{n+1} = \sqrt{7 + x_n} \leq \sqrt{7 + x_{n+1}} = x_{n+2}.$$  

This is exactly the statement $P(n+1)$, concluding the induction proof.

Step 2. We now show that the sequence is bounded.

Clearly the sequence is bounded below by $x_1 = \sqrt{7}$.

We use induction to show that the sequence is bounded above by 4.

P(1): $x_1 = \sqrt{7} \leq 4$.

Assume $P(n)$: $x_n \leq 4$.

$$\Rightarrow x_{n+1} = \sqrt{7 + x_n} \leq \sqrt{7 + 4} \leq \sqrt{16} = 4.$$  

This is exactly the statement $P(n+1)$, concluding the induction proof.

Step 3. Since the sequence is monotonic and bounded, then it must converge.

Let $L = \lim_{n \to \infty} x_n$.

Taking limits of both sides of the recursion formula,

$$L = \sqrt{7 + L} \iff L^2 - L - 7 = 0 \iff L = \frac{1 \pm \sqrt{4 + 7\cdot 7}}{2}.$$  

Since the sequence is always positive, the limit must be $L = \frac{1 + \sqrt{29}}{2}$.
6.13 (5pts)

Step 1. We use induction to show that $a_n > \sqrt{c}$ for all $n$.

$P(1)$: By definition, $a_1 > \sqrt{c}$.

Assume $P(n)$: $a_n > \sqrt{c}$.

$$P(n + 1) : a_{n+1} > \sqrt{c} \iff \frac{1}{2} \left( a_n + \frac{c}{a_n} \right) > \sqrt{c}$$

$$\iff \frac{1}{2} \left( a_n + \frac{c}{a_n} - 2\sqrt{c} \right) > 0$$

$$\iff \frac{1}{2} \left( \frac{a_n^2 - 2a_n \sqrt{c} + c}{a_n} \right) > 0$$

$$\iff \frac{(a_n - \sqrt{c})^2}{2a_n} > 0$$

This last inequality is true by induction hypothesis, thus the statement $P(n + 1)$ is true, concluding the induction proof.

Step 2. We show that $a_n$ is monotonically decreasing, that is we show that

$$a_{n+1} \leq a_n \iff \frac{1}{2} \left( a_n + \frac{c}{a_n} \right) \leq a_n$$

$$\iff a_n + \frac{c}{a_n} \leq 2a_n$$

$$\iff a_n^2 + c \leq 2a_n^2$$

$$\iff c \leq a_n^2$$

$$\iff \sqrt{c} \leq a_n$$

This is true for all $n$ by the previous argument, so the sequence is monotonically decreasing.

Step 3. Since the sequence is monotonic and bounded, then it must have a limit: $a_n \to L$. Taking limits of the recursion formula,

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{2} \left( a_n + \frac{c}{a_n} \right)$$

$$\implies L = \frac{1}{2} \left( L + \frac{c}{L} \right)$$

$$\implies 0 = \frac{(L - \sqrt{c})^2}{2L}.$$ 

Since $a_n > \sqrt{c}$ for all $n$, it must be the case that $L = \sqrt{c}$. 