Fisher-KPP propagation driven by a line of fast diffusion, influence of nonlocal exchanges

ANR Nonlocal, réunion de lancement

Antoine Pauthier

Institut de Mathématique de Toulouse

Thèse dirigée par Henri Berestycki (EHESS) et Jean-Michel Roquejoffre (IMT)
Supportée par le projet ERC ReaDi

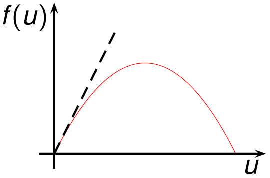
Nonlocal model

Model under study

$$\begin{cases} \partial_t u - D\partial_{xx} u = -\overline{\mu} u + \int \nu(y) v(t,x,y) dy & x \in \mathbb{R},\ t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu(y) u(t,x) - \nu(y) v(t,x,y) & (x,y) \in \mathbb{R}^2,\ t > 0 \end{cases} \quad (1)$$

Hypothesis :

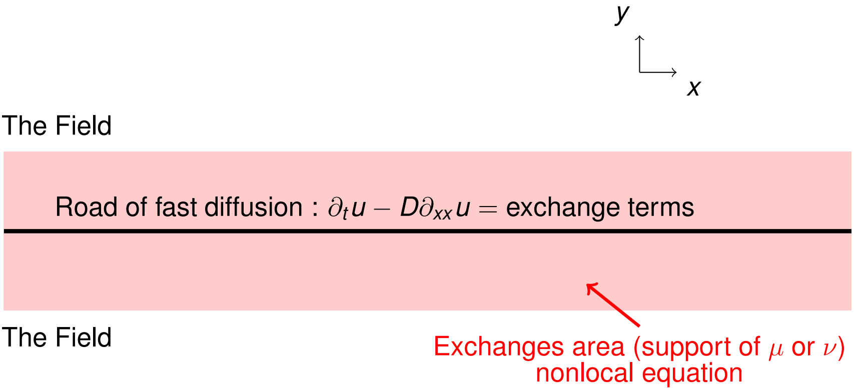
- f is of KPP-type.
- $\nu, \mu \geq 0$, continuous, even and compactly supported. $\overline{\mu} = \int \mu$, $\overline{\nu} = \int \nu$.



The functions ν and μ model exchanges of densities between the road and the field $\rightarrow$ **exchange functions.**

Initial question

Enhancement of biological invasion by heterogeneities: effect of a line of fast diffusion.



KPP Reaction-Diffusion

$$\partial_t v - d\Delta v = f(v)$$

Initial model

Introduced in 2012 by H. Berestycki, J.-M. Roquejoffre, and L. Rossi.

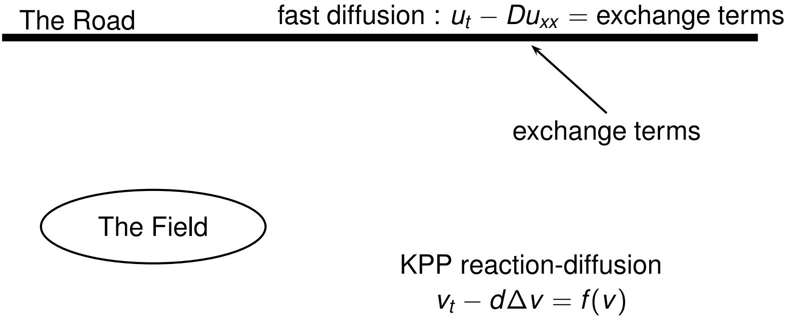


Figure : Road with fast diffusion : exchanges by boundary conditions

Our model deals with nonlocal exchange terms.

Initial model

Berestycki-Roquejoffre-Rossi

$$\begin{cases} \partial_t u - D\partial_{xx} u = \overline{\nu} v(t,x,0) - \overline{\mu} u & x \in \mathbb{R},\ t > 0 \\ \partial_t v - d\Delta v = f(v) & (x,y) \in \mathbb{R} \times \mathbb{R}^*,\ t > 0 \end{cases} \tag{2}$$

$$\begin{cases} v(t,x,0^+) = v(t,x,0^-), & x \in \mathbb{R},\ t > 0 \\ -d\left\{\partial_y v(t,x,0^+) - \partial_y v(t,x,0^-)\right\} = \overline{\mu} u(t,x) - \overline{\nu} v(t,x,0) & x \in \mathbb{R},\ t > 0. \end{cases} \tag{3}$$

Initial question

Does the road enhance the spreading ?

Results of Berestycki-Roquejoffre-Rossi

Theorem

There exists $c^* = c^*(\mu, d, D) > 0$ *such that:*

- *for all* $c > c^*$, $\lim_{t\to\infty} \sup_{|x|\geq ct}(u(x,t), v(x,y,t)) = (0,0)$*;*
- *for all* $c < c^*$, $\lim_{t\to\infty} \inf_{|x|\leq ct}(u(x,t), v(x,y,t)) = (\nu/\mu, 1)$*.*

Moreover:

- *if* $D \leq 2d$*, then* $c^*(\mu, d, D) = c_{KPP} := 2\sqrt{df'(0)}$ *;*
- *if* $D > 2d$*, then* $c^*(\mu, d, D) > c_{KPP}$ *and* $\lim_{D\to\infty} c^*(\mu, d, D)/\sqrt{D}$ *exists and is positive.*

- Enhancement of the spreading in the direction of the road.
- Threshold $D > 2d$.

Questions

1. Can we retrieve the same kind of results for the nonlocal model ?
2. How do nonlocal exchanges modify the spreading speed ?

Plan

Stationary solutions

Proposition

(1) admits a unique nonnegative bounded stationary solution $(U_s, V_s(y)) \not\equiv (0,0)$. This solution is $x-$invariant.

$$\begin{cases} U_s &= \frac{1}{\mu}\int \nu(y)V_s(y)dy \\ -dV_s''(y) &= f(V_s(y)) + U_s\mu(y) - V_s(y)\nu(y) \end{cases}$$

$V_s(\pm\infty) = 1.$

Reminder: in the initial BRR-case, $(U_s, V_s) = \left(\frac{\overline{\nu}}{\overline{\mu}}, 1\right).$

Robustness of the BRR-result

Theorem

there exists $c^* = c^*(\mu, \nu, d, D, f'(0)) > 0$ *such that:*

- *for all* $c > c^*$, $\lim_{t\to\infty} \sup_{|x|\geq ct} (u(t,x), v(t,x,y)) = (0,0)$;
- *for all* $c < c^*$, $\lim_{t\to\infty} \inf_{|x|\leq ct} (u(t,x), v(t,x,y)) = (U_s, V_s)$.

Moreover, c^* *satisfies:*

- *if* $D \leq 2d$, $c^* = c_{KPP} := 2\sqrt{df'(0)}$;
- *if* $D > 2d$, $c^* > c_{KPP}$.

Remark

The threshold is still $D = 2d$.

Main tool: construction of planar waves

They serve as supersolutions ($f(v) \leq f'(0)v$).

Linearised system

$$\begin{cases} \partial_t u - D\partial_{xx} u = -\overline{\mu} u + \int \nu(y) v(t,x,y) dy & x \in \mathbb{R}, \\ \partial_t v - d\Delta v = f'(0)v + \mu(y)u(t,x) - \nu(y)v(t,x,y) & (x,y) \in \mathbb{R}^2, \end{cases} \tag{4}$$

Exponential solutions of the form

$$\begin{pmatrix} u(t,x) \\ v(t,x,y) \end{pmatrix} = e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\ \phi(y) \end{pmatrix}, \tag{5}$$

With nonnegative $\lambda, c, \phi \in H^1(\mathbb{R})$.

- In the BRR model, they were given by an algebraic computation.
- Here we are led to a nonlinear eigenvalue problem.

Equivalent system in λ, ϕ, c

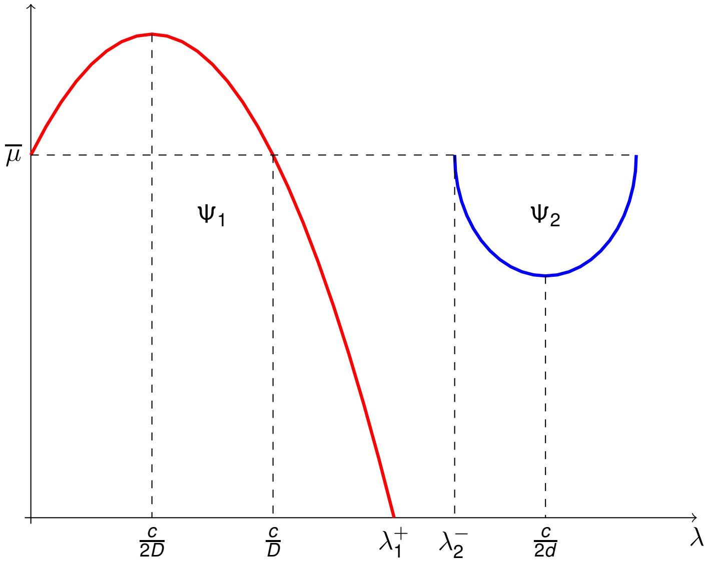
$$\begin{cases} -D\lambda^2 + \lambda c + \overline{\mu} = \int \nu(y)\phi(y)dy \\ -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y). \end{cases}$$

- First equation gives a map $\lambda \mapsto \Psi_1(\lambda, c) := -D\lambda^2 + \lambda c + \overline{\mu}$.
- Second equation: at most one solution $\phi = \phi(y; \lambda, c)$. Then set $\Psi_2(\lambda, c) := \int \nu(y)\phi(y)dy$.

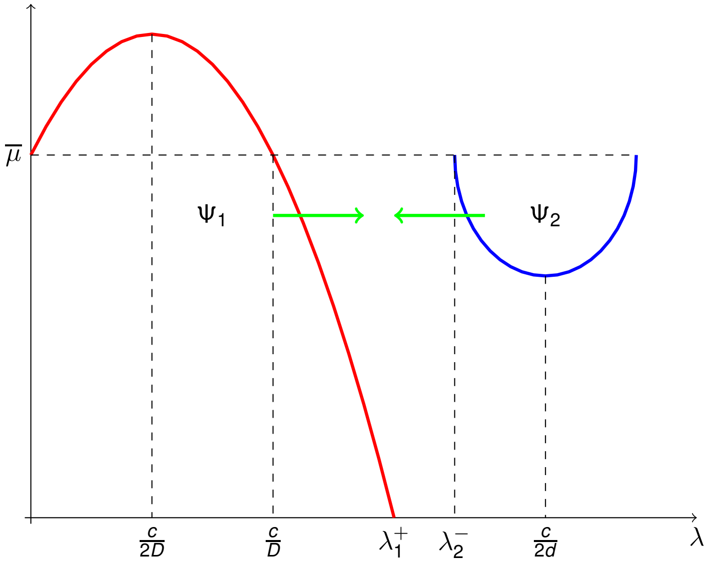
Goal

Find λ, c such that the graphs of $\lambda \mapsto \Psi_1(\lambda)$ and $\lambda \mapsto \Psi_2(\lambda)$ intersect.

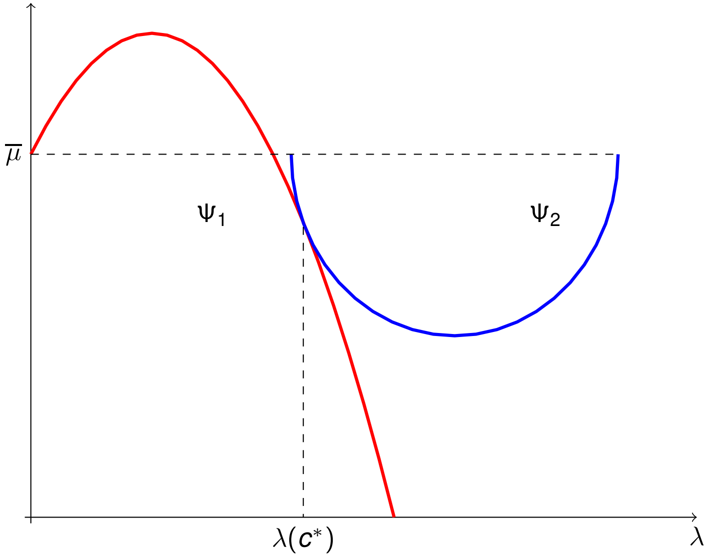
The two curves Ψ_1 and Ψ_2



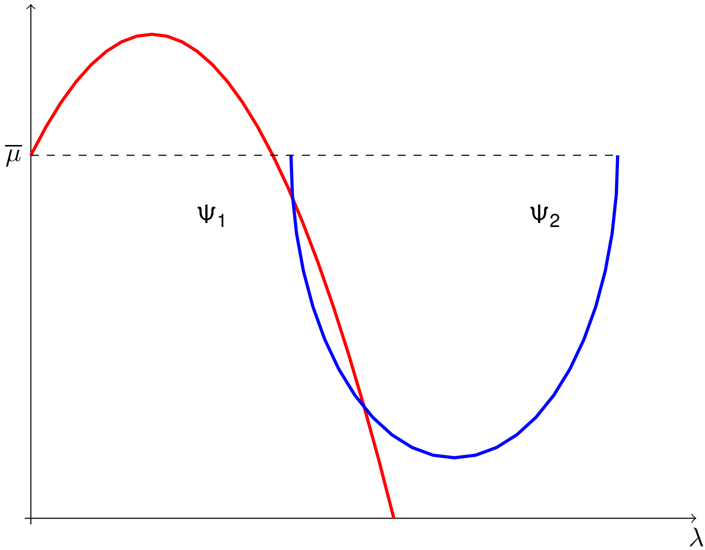
Behaviour as c increases



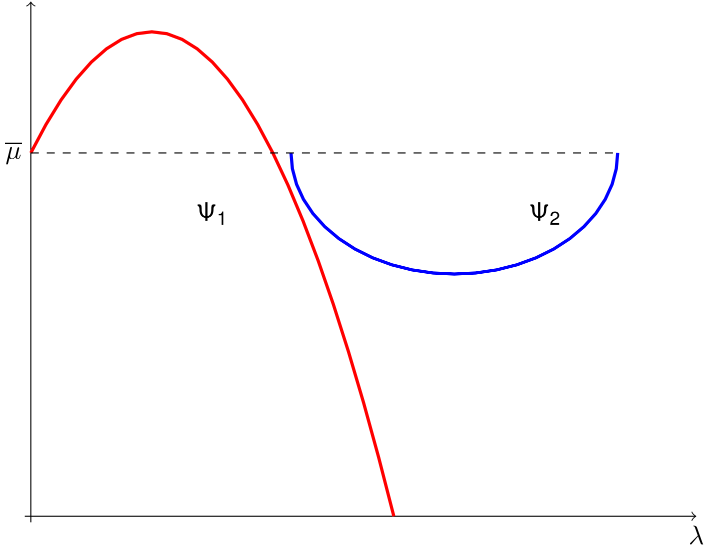
Intersection for $c = c^*$



Slower spreading



Faster spreading



Natural question (due to G. Nadin): influence of nonlocal exchanges on the spreading speed

For fixed parameters $d, D, f'(0), \overline{\mu}, \overline{\nu}$ we consider the set of admissible exchanges

$$\Lambda_{\overline{\mu}} = \{\mu \in C_0(\mathbb{R}), \mu \geq 0, \int \mu = \overline{\mu}, \mu \text{ even}\}.$$

For $\mu \in \Lambda_{\overline{\mu}}$ and $\nu \in \Lambda_{\overline{\nu}}$, there exists a spreading speed $c^*(\mu, \nu)$. Let c_0^* be the spreading speed for the initial BRR model (that is, $c_0^* = c^*(\overline{\mu}\delta_0, \overline{\nu}\delta_0)$).

Questions

- Can we compare $c^*(\mu, \nu)$ with c_0^* ?
- $\inf\{c^*(\mu, \nu),\ \mu \in \Lambda_{\overline{\mu}}, \nu \in \Lambda_{\overline{\nu}}\}$?
- $\sup\{c^*(\mu, \nu),\ \mu \in \Lambda_{\overline{\mu}}, \nu \in \Lambda_{\overline{\nu}}\} = c_0^*$?

Long range exchange terms: new threshold

For fixed parameters $d, D, f'(0), \overline{\mu}, \overline{\nu}$ we can get the infimum with

$$\nu_R(y) = \frac{1}{R}\nu\left(\frac{y}{R}\right), \text{ or } \mu_R(y) = \frac{y}{R}\mu\left(\frac{1}{R}\right), \; R \to +\infty$$

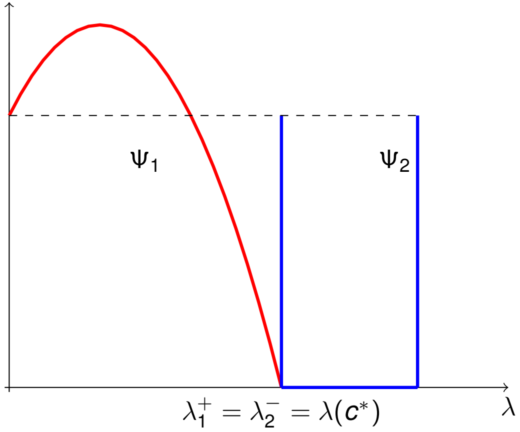
Theorem

Let us consider the nonlocal system (1) with fixed exchange masses $\overline{\mu}$ and $\overline{\nu}$. Let $c^* = c^*(\mu, \nu)$ be the spreading speed given by Theorem 2.1, depending on the repartition of μ or ν.

1. *If $D \in \left[2d, d\left(2 + \frac{\overline{\mu}}{f'(0)}\right)\right]$, $\inf\{c^*, \; \mu, \nu \in \Lambda_{\overline{\mu},\overline{\nu}}\} = 2\sqrt{df'(0)}$.*
2. *Fix $D > d\left(2 + \frac{\overline{\mu}}{f'(0)}\right)$, then $\inf\{c^*, \; \mu, \nu \in \Lambda_{\overline{\mu},\overline{\nu}}\} > 2\sqrt{df'(0)}$.*

Moreover, in both cases, minimizing sequences can be given by long range exchange terms of the form $\mu_R(y) = \frac{1}{R}\mu\left(\frac{y}{R}\right)$ or $\nu_R(y) = \frac{1}{R}\nu\left(\frac{y}{R}\right)$ with $R \to \infty$.

limit curve: infimum for the spreading speed



Ψ_1 is fixed. The extremal points of Ψ_2 do not depend on the repartition of μ and ν.

First intermediate model

$$\begin{cases} \partial_t u - D\partial_{xx} u = -\overline{\mu} u + \overline{\nu} v(t,x,0) & x \in \mathbb{R},\ t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu(y)u(t,x) & (x,y) \in \mathbb{R}\times\mathbb{R}^*,\ t > 0 \\ v(t,x,0^+) = v(t,x,0^-), & x \in \mathbb{R},\ t > 0 \\ -d\left\{\partial_y v(t,x,0^+) - \partial_y v(t,x,0^-)\right\} = -\overline{\nu} v(t,x,0) & x \in \mathbb{R},\ t > 0. \end{cases} \tag{6}$$

- Exchanges field $\rightarrow$ road by boundary condition, *id est* $\nu = \overline{\nu}\delta_0$.
- Exchanges road $\rightarrow$ field by a function μ with nontrivial support.
- We get the same general results (existence, spreading, minimal speed, ...).

Maximum spreading speed

Parameters $d, D, f'(0), \overline{\nu}, \overline{\mu}$ are fixed. We consider the set of admissible exchanges

$$\Lambda_{\overline{\mu}} = \{\mu \in C_0(\mathbb{R}), \mu \geq 0, \int \mu = \overline{\mu}, \mu \text{ even}\}.$$

For $\mu \in \Lambda_{\overline{\mu}}$ there exists $c^*(\mu)$ spreading speed. Let c_0^* be the BRR spreading speed.

Proposition

$$c_0^* = \sup\{c^*(\mu),\ \mu \in \Lambda_{\overline{\mu}}\}.$$

Fastest spreading for localised exchanges from the road to the field.
The proof is an explicit computation.

Second intermediate model

$$
\begin{cases}
\partial_t u - D\partial_{xx} u = -\overline{\mu} u + \int \nu(y) v(t,x,y)dy & x \in \mathbb{R},\ t > 0 \\
\partial_t v - d\Delta v = f(v) - \nu(y) v(t,x,y) & (x,y) \in \mathbb{R} \times \mathbb{R}^*,\ t > 0 \\
v(t,x,0^+) = v(t,x,0^-), & x \in \mathbb{R},\ t > 0 \\
-d\left\{\partial_y v(t,x,0^+) - \partial_y v(t,x,0^-)\right\} = \overline{\mu} u(t,x) & x \in \mathbb{R},\ t > 0
\end{cases}
\tag{7}
$$

- Exchanges field $\rightarrow$ road by function ν with nontrivial support.
- Exchanges road $\rightarrow$ field by boundary condition (*id est* $\mu = \overline{\mu}\delta_0$).
- General theorems are preserved (existence, spreading, ...).

Do we get the same kind of results ?

First case: selfsimilar exchanges

For fixed parameters $d, D, f'(0), \overline{\mu}$ we consider the set of admissible exchanges

$$\Lambda_{\overline{\nu}} = \{\nu \in C_0(\mathbb{R}), \nu \geq 0, \int \nu = \overline{\nu}, \nu \text{ even}\}.$$

For a given function ν, we set

$$\nu_\varepsilon(y) = \frac{1}{\varepsilon}\nu\left(\frac{y}{\varepsilon}\right) \implies c^*(\varepsilon).$$

Proposition

Let us consider c^* as a function of the ε variable. Then there exists ε_0,

$$\forall \varepsilon < \varepsilon_0, \; c^*(\varepsilon) > c_0^*.$$

- Localised exchanges seem to be a local minimizer for the spreading speed.
- It does not depend of the function ν.

Is it a general result, that is, are localised exchange terms a local minimizer for the spreading speed ?

Second case: perturbation of a Dirac

Mixed exchanges field → road: boundary condition + small nonlocal contribution.

$$\nu(y) = (1-\varepsilon)\delta_0 + \varepsilon\upsilon(y), \qquad \upsilon \in \Lambda_1 \longrightarrow c^*(\varepsilon).$$

Theorem

There exists $m_1 > 2$ depending on $f'(0)$, M_1 depending on $\overline{\mu}$ such that:

1. *If $D < m_1$ there exists ε_0 and $\upsilon \in \Lambda_1$ such that $\forall\varepsilon < \varepsilon_0$, $c_0^* < c^*(\varepsilon)$;*
2. *if $\overline{\mu} > 4$ and $D, f'(0) > M_1$ there exists ε_0 such that $\forall\upsilon \in \Lambda_1$, $\forall\varepsilon < \varepsilon_0$, $c_0^* > c^*(\varepsilon)$.*

- No general result for this model.
- Various behaviours may happen even in the neighborhood of localised exchanges.

Conclusion

- Persistence of the initial results of Berestycki, Roquejoffre and Rossi.
- A new threshold for infinitely supported exchanges.
- Differences between the two exchange functions and their influence on the dynamics.

Thank you for your attention.